FLOER HOMOLOGY FOR FAMILIES - REPORT OF A PROJECT IN PROGRESS -

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0. INTRODUCTION[1] - 1 -

In this article, we propose a project to construct a Floer cohomologies for family of Lagrangian submanifolds. The author believes that to realize this project will be an important step toward a proof of homological mirror symmetry conjecture by Kontsevitch [Ko1],[Ko2]. See [Fu4] §0 the statement of the version of the homological mirror symmetry conjecture the author is working on to prove.

We first review a construction in complex geometry which is expected to be a mirror of Floer cohomologies for family of Lagrangian submanifolds. Actually the mirror object is classical in complex geometry.

Let $\pi : M \to N$ be a proper holomorphic map from a complex manifold M to another complex manifold N. Let $\mathcal{E}$ be a holomorphic vector bundle on M. (More generally $\mathcal{E}$ may be an object of the derived category of coherent $\mathcal{O}_M$ module sheaves.) The direct image sheaf $R\pi_*\mathcal{E}$ is defined as an object of derived category of coherent $\mathcal{O}_N$ module sheaves on N.

Let us recall how $R\pi_*\mathcal{E}$ is regarded as a family of sheaf cohomology groups. Let $p \in N$. We consider the fiber $\pi^{-1}(p) \subset M$ and restrict $\mathcal{E}$ there. Then the sheaf cohomology group

$$H^k(\pi^{-1}(p); \mathcal{E}) \tag{0.1}$$

is well defined. On a subset U of N where the rank of (0.1) is constant, we obtain a holomorphic vector bundle whose fiber at $p \in U$ is given by (0.1).

In general, the rank of the vector space (0.1) jumps. However the alternative sum

$$\bigoplus_k (-1)^k H^k(\pi^{-1}(p); \mathcal{E}) \tag{0.2}$$

is well defined as an object of derived category of coherent $\mathcal{O}_N$ module sheaves on N. This is our $R\pi_*\mathcal{E}$. This construction goes back to Grotendieck's idea on

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Grotendieck-Riemann-Roch theorem and was translated by Atiyah-Singer to C^∞ category (index theory of family of elliptic operators).

Let us consider the following variant of this construction. We fix a complex manifold X. Let N be a subset of the moduli space of holomorphic vector bundles on X. In an ideal situation, we have a universal bundle $\mathcal{E}$ on $M \to N$. Namely $\mathcal{E}$ is a vector bundle on M and $M \to N$ is a locally trivial fiber bundle in complex analytic category, such that the fiber of $M \to N$ is isomorphic to X and the restriction of $\mathcal{E}$ to the fiber of $p \in N$ is isomorphic to the vector bundle of X corresponding to p.

Let N' be another subset of the moduli space of holomorphic vector bundles on X. We define $M' \to N'$ and $\mathcal{E}' \to M'$ in a similar way.

Now let $(p,p') \in N \times N'$. We have two holomorphic vector bundles $\mathcal{E}|_{\pi^{-1}(p)}$, $\mathcal{E}'|_{\pi^{-1}(p')}$ on X. We then have graded vector spaces

$$\mathrm{Ext}^*(\mathcal{E}|_{\pi^{-1}(p)}, \mathcal{E}'|_{\pi^{-1}(p')}). \tag{0.3}$$

The rank of (0.3) jumps when we move $(p,p') \in M \times M'$. However, in a way similar to (0.2), the family (0.3) is well defined as an object of derived category of coherent sheaves on $N \times N'$.

One may try to extend this object to one on an appropriate compactification of the moduli space $N \times N'$.

1. Introduction - 2 -

The target of our project is to construct a mirror of the construction we reviewed in section 0. Let us make it a bit more precise.

Let (M,ω) be a symplectic manifold. We take a closed two form B, which is called the *B field*, and put $\Omega = \omega - 2\pi\sqrt{-1}B$. We let $\mathcal{LAG}^\sim(M,\Omega)$ be the set of all triples $(L,\mathcal{L},\nabla)$ satisfying the conditions below.

Condition 1.1.
(1.1.1) $\dim L = \dim M/2$.
(1.1.2) $\mathcal{L}$ is a line bundle on L, ∇ is a unitary connection on it.
(1.1.3) The curvature F_∇ of ∇ satisfies

$$F_\nabla = 2\pi\sqrt{-1}B.$$

The space $\mathcal{LAG}^\sim(M,\Omega)$ is of infinite dimension. We will define an equivalence relation on it so that the quotient space is of finite dimension.

We first recall the definition of Hamiltonian isotopy. Let $f : M \times [0,1] \to \mathbb{R}$ be a smooth function. We put $f_t(x) = f(x,t)$. Let X_{f_t} be a unique vector field on M such that

$$\omega(X_{f_t}, V) = df_t(V) \tag{1.2}$$

holds for any vector field V on M. X_{f_t} is called the *Hamiltonian vector field* generated by f_t. We then obtain a family of diffeomorphisms $\Phi_t : M \to M$ such that :

$$\begin{cases} \Phi_0(x) = x \\ \dfrac{\partial \Phi_t(x)}{\partial t} = X_{f_t}(\Phi_t(x)). \end{cases} \tag{1.3}$$

It is well known that Φ_t preserves the symplectic form ω. We call Φ_t the *Hamiltonian isotopy* generated by f.

Definition 1.4. Let $(L_0,\mathcal{L}_0,\nabla_0)$, $(L_1,\mathcal{L}_1,\nabla_1) \in \mathcal{LAG}^{\sim}(M,\Omega)$. We say that they are *Hamiltonian equivalent* to each other and write $(L_0,\mathcal{L}_0,\nabla_0) \sim (L_1,\mathcal{L}_1,\nabla_1)$ if there exists a function $f : M \times [0,1] \to \mathbb{R}$, a (complex) line bundle $\mathcal{L} \to L \times [0,1]$ and its connection ∇ with the following properties. Let Φ_t be the Hamiltonian isotopy generated by f.

(1.5.1) $\Phi_1(L_0) = L_1$.
(1.5.2) $(\mathcal{L},\nabla)|_{L\times\{0\}} = (\mathcal{L}_0,\nabla_0)$, $(\mathcal{L},\nabla)|_{L\times\{1\}} = \Phi_1^*(\mathcal{L}_1,\nabla_1)$.
(1.5.3) $F_\nabla = 2\pi\sqrt{-1}\Phi^*B$. Here $\Phi : L_0\times[0,1] \to M$ is defined by $\Phi(x,t) = \Phi_t(x)$.

We denote the quotient space $\mathcal{LAG}^{\sim}(M,\Omega)/\sim$ by $\mathcal{LAG}(M,\Omega)$.

We do not know appropriate definition of stability to modify the construction of the moduli space $\mathcal{LAG}(M,\Omega)$ and obtain a Hausdorff space. (We remark here that to define a Hausdorff moduli space of holomorphic vector bundles, we need to restrict ourselves to stable or semi-stable bundles.) We do not discuss this point here. (See [Fu5] §2.)

If we forget the difficulty related to stability and Hausdorffness of moduli space, we can define a complex structure on $\mathcal{LAG}(M,\Omega)$ as follows. Let $(L,\mathcal{L},\nabla) \in \mathcal{LAG}^{\sim}(M,\Omega)$. We define

$$I_1 : \Gamma(L;TM|_L) \to \Gamma(L;T^*M|_L)\otimes\mathbb{C}$$

by

$$X \mapsto i_X(\Omega).$$

Let $(L_t,\mathcal{L}_t,\nabla_t)$ be a smooth family of elements of $\mathcal{LAG}^{\sim}(M,\Omega)$. We fix a family of diffeomorphisms $L \cong L_t$, which determine the embedding $i_t : L \to M$. We define $V \in \Gamma(L,TM|_L)$ by

$$V = \frac{\partial i_t}{\partial t}. \tag{1.6}$$

We next fix a family of bundle isomorphisms $\mathcal{L} \cong \Phi^{t*}\mathcal{L}_t$. We pull back the connections ∇_t by this isomorphism and denote the pull back by the same symbol ∇_t. We define

$$2\pi\sqrt{-1}u = \frac{\partial\nabla_t}{\partial t} \in \Gamma(L,T^*L\otimes u(1)).$$

Here we identify $u(1) \cong 2\pi\sqrt{-1}\mathbb{R}$. We now put

$$I\left(\frac{\partial}{\partial t}(L_t,\mathcal{L}_t,\nabla_t)\right) = I_1(V) + 2\pi\sqrt{-1}u \in \Gamma(L;T^*L)\otimes\mathbb{C}. \tag{1.7}$$

Lemma 1.8. *(1.7) induces an isomorphism*

$$I : T_{[L_0,\mathcal{L}_0,\nabla_0]}\mathcal{LAG}(M,\Omega) \to H^1_{Dr}(L;\mathbb{C}).$$

Proof. Since L is a Lagrangian submanifold, it follows that $\operatorname{Re} I_1(V) = 0$ if and only if V is tangent to L. Hence the real part of (1.7) is independent of the choice of the family of diffeomorphisms $L \cong L_t$.

If we change the family of isomorphisms $\mathcal{L} \cong i_t^*\mathcal{L}_t$ then u changes by an exact one form. So we may take any isomorphism $\mathcal{L} \cong i_t^*\mathcal{L}_t$ to prove Theorem 1.8.

Let us study the imaginary part of (1.7). Let V be a vector field on a neighborhood of L. We assume that V is tangent to L. We denote by $\exp(tV)$ the one parameter group of diffeomorphisms generated by V. We consider the case $L_t = \exp(tV)(L) = L$ and the family of diffeomorphisms $L \equiv L_t$ is $\exp(tV)$. We take $\mathcal{L}_t = \mathcal{L}$. We define $\mathcal{L} \cong i_t^*\mathcal{L}_t$ as follows. Let $p \in L$. $s \mapsto \exp(tsV)p$ is a path joining p to $\exp(tV)p$. The parallel transport along this path of the connection ∇ defines an isomorphism $\mathcal{L}_p \cong \mathcal{L}_{\exp(tV)p} = (i_t^*\mathcal{L}_t)_p$. We thus obtain an isomorphism $\mathcal{L} \cong i_t^*\mathcal{L}_t$.

We pull back the connection ∇ by this isomorphism and write it as $\nabla_t = \exp(tV)^*\nabla$. Then $F_\nabla = 2\pi\sqrt{-1}B$ implies

$$\frac{d}{dt}\exp(tV)^*\nabla\bigg|_{t=0} = i_V F_\nabla = \Im(2\pi\sqrt{-1}I_1).$$

Hence the family $(L_t, \mathcal{L}_t, \nabla_t)$ is mapped to zero by I. We thus verified that (1.7) is independent of the choice of the isomorphism $i_t : L \cong L_t$, $\mathcal{L} \cong i_t^*\mathcal{L}_t$ modulo exact form.

We next check that the left hand side of (1.7) is a closed form. Let V be as in (1.6). We may extend i_t to a symplectic embedding in a neighborhood of L by using Darboux-Weinstein theorem (see [AG] §4.1). By definition, we have

$$i_V\omega = \Re(I_1(V)).$$

We have

$$d\Re(I_1(V)) = d(i_V\omega) = -i_V d\omega + L_V\omega = 0.$$

(Here we use the fact that i_t preserves symplectic structure to show $L_V\omega = 0$.)

We next consider the imaginary part. We first observe that, by the same calculation as above, we have

$$d\Im(I_1(V)) = -2\pi L_V B. \tag{1.9}$$

By assumption, the curvature of ∇_t (which we regard as a connection on $\mathcal{L} \cong i_t^*\mathcal{L}_t$) is $2\pi\sqrt{-1}i_t^*B$. Hence

$$d\left(\frac{\partial}{\partial t}\nabla_t\bigg|_{t=0}\right) = 2\pi\sqrt{-1}L_V B \tag{1.10}$$

(1.9) and (1.10) cancel each other as required.

We next check that the differential of the family in an equivalence class of the relation $\sim$ will be send to an exact form. Let us use the notation of Definition 1.5. We put $i_t(x) = \Phi^t(x) = \Phi(x,t)$, $i_t(L_0) = L_t$. It is easy to show

$$\operatorname{Re} I_1(V) = i_V\omega = df_0$$

where f is the Hamiltonian generating the Hamiltonian isotopy i_t. Thus the real part is exact.

Let $\mathcal{L}_t$ be the restriction of $(\mathcal{L}, \nabla)$ to $L_0 \times \{t\} \cong L_t$. We define bundle isomorphism $i_t^* \mathcal{L}_t \cong \mathcal{L}_0$ by using parallel transport along the path $s \mapsto (x, st)$. (We use connection ∇ to define parallel transport.) Then, by (1.5.3), we have

$$\frac{\partial}{\partial t} i_t^* \nabla_t \Big|_{t=0} = 2\pi \sqrt{-1} i_V B.$$

Thus, for this choice of the isomorphism $i_t^* \mathcal{L}_t \sim \mathcal{L}_0$, the imaginary part of $I_1(V)$ vanishes. The proof of Lemma 1.8 is now complete. □

We remark that the right hand side of the isomorphism in Lemma 1.8 is a complex vector space. Hence we obtain an almost complex structure on our moduli space $\mathcal{LAG}(M, \Omega)$ on the subset where it is Hausdorff. It is easy to show that this almost complex structure is integrable and hence determine a complex structure. We call this complex structure the *classical complex structure* and write it $\mathcal{LAG}(M, \Omega)_{cl}$. (The construction above is a minor modification of one given in [Fu4] Chapter 1. There the case of symplectic torus is studied in more detail.)

Let us consider the direct product $\mathcal{LAG}(M, \Omega)_{cl} \times \mathcal{LAG}(M, \Omega)_{cl}$. Our purpose is to define an object of the derived category of coherent sheaves on it, which gives the family of Floer cohomologies. Namely let $((L_1, \mathcal{L}_1, \nabla_1), (L_2, \mathcal{L}_2, \nabla_2)) \in \mathcal{LAG}(M, \Omega)$. We consider Floer cohomology

$$HF((L_1, \mathcal{L}_1, \nabla_1), (L_2, \mathcal{L}_2, \nabla_2)) \tag{1.11}$$

which is a graded $\mathbb{C}$ vector space. (Actually there is a trouble related to point (D) below to define Floer cohomology with $\mathbb{C}$ coefficient.) We move the pair $((L_1, \mathcal{L}_1, \nabla_1), (L_2, \mathcal{L}_2, \nabla_2))$. Then we expect that (1.11) will be a holomorphic family of complex vector spaces, the family of Floer cohomologies, and expect that it will be a mirror of the construction of §0 in some case.

However one needs various modifications to define Floer cohomology for families in a rigorous way. The author is unable to complete the construction in a rigorous way in the general case at the time of writing this article. The main difficulties to be overcome are as follows.

(A) Floer cohomology is not always defined.

(A.1) We need a relative spin structure of Lagrangian submanifold for Floer cohomology (1.11) to be well defined over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ coefficient.

(A.2) The obstruction class defined in [FOOO] should vanish for Floer cohomology (1.11) to be well defined.

(B) Even in the case when Floer cohomology (1.11) is defined, it depends on various additional choices involved.

(C) We need additional data to define degree of the elements of Floer cohomology.

(D) Even in the case when Floer cohomology (1.11) is defined, the boundary operator is a formal power series whose convergence is not established yet.

(E) Even in the case when the boundary operator converges, the boundary operator does not depend continuously on Lagrangian submanifolds. Hence it does not give a holomorphic family of chain complexes in a naive sense.

(F) Even if we can overcome the problem (E) by taking "holomorphic structure including the quantum effect" the "holomorphic structure" on the family (1.11) define an object of derived category of coherent sheaves *not* on $\mathcal{LAG}(M,\Omega)_{cl} \times \mathcal{LAG}(M,\Omega)_{cl}$ but on $\mathcal{LAG}(M,\Omega)_{qm} \times \mathcal{LAG}(M,\Omega)_{qm}$. Here $\mathcal{LAG}(M,\Omega)_{qm}$ is an appropriate modification of $\mathcal{LAG}(M,\Omega)_{cl}$ by quantum effect.

Among the above difficulties (A)(B)(C) are basically settled in [FOOO]. The author is now working on (D)(E)(F). This article is a report on work in progress on it.

The author makes it clear which part is already settled and which are not. Especially the results stated as Theorem, Proposition, Lemma, are all rigorously established.

2. Infinitesimal family - infinitesimal (algebraic) deformation of Lagrangian submanifolds -

In this and the next sections, we review a part of the results in [FOOO] which are related to points (A2)(B) above. Roughly speaking we will construct a family of Floer homologies in an infinitesimal neighborhood of a point in $\mathcal{LAG}(M,\Omega)$.

We first define the universal Novikov ring Λ_{nov}. Let F be a commutative ring with unit. We put :

$$\Lambda_{nov,F} = \left\{ \sum a_i T^{\lambda_i} \,\middle|\, a_i \in F, \lambda_i \in \mathbb{R}, \lambda_i < \lambda_{i+1}, \lim_{i\to\infty} \lambda_i = \infty \right\}.$$

$\Lambda_{nov,F}$ is a complete valuation ring. Note the definition here is a bit different from the preprint version of [FOOO], where one extra parameter was added. The definition in this article seems to be more appropriate for applications to mirror symmetry. We define a subring $\Lambda_{0,nov,F}$ of $\Lambda_{nov,F}$ by

$$\Lambda_{0,nov,F} = \left\{ \sum a_i T^{\lambda_i} \in \Lambda_{nov,F} \,\middle|\, \lambda_i \geq 0 \right\},$$

and its ideal $\Lambda_{+,nov,F}$ by

$$\Lambda_{+,nov,F} = \left\{ \sum a_i T^{\lambda_i} \in \Lambda_{nov,F} \,\middle|\, \lambda_i > 0 \right\}.$$

We remark

$$\Lambda_{0,nov,F}/\Lambda_{+,nov,F} = F.$$

Hereafter we will write Λ_{nov}, $\Lambda_{0,nov}$, $\Lambda_{+,nov}$ in place of $\Lambda_{nov,\mathbb{C}}$, $\Lambda_{0,nov,\mathbb{C}}$, $\Lambda_{+,nov,\mathbb{C}}$ respectively.

Let L be a Lagrangian submanifold of a symplectic manifold (M,ω). We assume :

Assumption 2.1.
(2.1.1) $c^1(M) = 0$.
(2.1.2) The Maslov index $\eta : \pi_2(M,L) \to \mathbb{Z}$ (see [AG] Chapter 6) is zero.

Assumption 2.1 implies that the virtual dimension of the moduli space of the pseudoholomorphic maps $\varphi : (D^2, \partial D^2) \to (M,L)$ is independent of the homotopy class of φ and is equal to n, the dimension of L.

In the situation where Assumption 2.1 is not satisfied, the virtual dimension depends on the component of the moduli space (or homotopy class of φ) and we need to add one more formal parameter to $\Lambda_{nov,F}$ or alternatively our Floer cohomology has $\mathbb{Z}_{2\Sigma_L}$ grading instead of $\mathbb{Z}$ grading. (Here $2\Sigma_L$ is the generator of the image of Maslov index.) We do not discuss this point. See [Sei].

Assumption 2.1 is satisfied for special Lagrangian submanifold in Calabi-Yau manifolds.

Let $S(L)$ be the chain complex of smooth singular chains in L. An element of $S_k(L)$ defines a k-current and may also be regarded as a distribution valued $n-k$ form on L. If two elements of $S(L)$ determine the same current, then we say that they are equivalent to each other. Let $\overline{S}(L)$ be the quotient space by this equivalence relation. We put

$$\overline{S}^{n-k}(L) = \overline{S}_k(L)$$

and regard $\overline{S}^*(L)$ as a *cochain* complex. An element of $\overline{S}^*(L)$ is a distribution valued form. Hence the wedge product among them is not in general defined in the usual sense. However we have the following :

Proposition 2.2. *There exists a countably generated subcomplex $C(L;\mathbb{Q})$ of $\overline{S}^*(L)$ and a structure of A_∞ algebra on it, such that it is homotopy equivalent to the differential graded algebra $(\Gamma(L;\Lambda^*(L)), d, \wedge)$, the De-Rham complex.*

See [FOOO] Chapter 5, for the proof.

Let us review the definition of A_∞ algebra. Structure of A_∞ algebra on $C(L;\mathbb{Q})$ is by definition a series of operations

$$\overline{\mathfrak{m}}_k : \underbrace{C[1](L;\mathbb{Q}) \otimes \cdots \otimes C[1](L;\mathbb{Q})}_{k \text{ times}} \to C[1](L;\mathbb{Q})$$

$k = 1, 2, \cdots$ with the following properties. Here $C[1]^k(L;\mathbb{Q}) = C^{k+1}(L;\mathbb{Q})$.

(2.3.1) $\overline{\mathfrak{m}}_k$ is of degree $+1$.

(2.3.2)

$$\sum_{1 \le \ell < m \le k} \pm \overline{\mathfrak{m}}_{k-m+\ell}(x_1, \cdots, x_{\ell-1}, \overline{\mathfrak{m}}_{m-\ell+1}(x_\ell, \cdots, x_m), x_{m+1}, \cdots, x_k) = 0.$$

(See [FOOO] Chapter 6 for sign.) Hereafter we write

$$B_k C[1](L;\mathbb{Q}) = \underbrace{C[1](L;\mathbb{Q}) \otimes \cdots \otimes C[1](L;\mathbb{Q})}_{k \text{ times}}, \tag{2.4.1}$$

$$BC[1](L;\mathbb{Q}) = \sum_k B_k C[1](L;\mathbb{Q}). \tag{2.4.2}$$

We need also the notion of filtered A_∞ algebra. We take $C(L;\mathbb{Q})$ as in Proposition 2.2. Let $C(L;\Lambda_{0,nov})$ be the completion of the algebraic tensor product

$$C(L;\mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_{0,nov}.$$

Here we take the completion by using filtration induced by the valuation on $\Lambda_{0,nov}$. The structure of filtered A_∞ algebra on $C(L;\Lambda_{0,nov})$ is a series of operations

$$\mathfrak{m}_k : \underbrace{C[1](L;\Lambda_{0,nov}) \otimes \cdots \otimes C[1](L;\Lambda_{0,nov})}_{k \text{ times}} \to C[1](L;\Lambda_{0,nov})$$

for $k = 0, 1, \cdots$, such that (2.3.1),(2.3.2) hold and that $\mathfrak{m}_k$ preserves the filtration. Notes that $\mathfrak{m}_0$ is included for filtered A_∞ algebra but $\overline{\mathfrak{m}}_0 = 0$ for A_∞ algebra.

$BC[1](L;\Lambda_{0,nov})$ is defined in a similar way to (2.4). $\hat{B}C[1](L;\Lambda_{0,nov})$ is its completion.

Now the main result of [FOOO] is :

Theorem 2.5. *Let $(L, \mathcal{L}, \nabla) \in \mathcal{LAG}^\sim(M,\Omega)$. We assume that L has a relative spin structure*[2] *in the sense of [FOOO] Chapter 6. Then there exists a structure of filtered A_∞ algebra $\mathfrak{m}_k$ on $C(L;\Lambda_{0,nov})$ such that*

$$\mathfrak{m}_k \equiv \overline{\mathfrak{m}}_k \mod \Lambda_{+,nov}.$$

We remark that $C(L;\Lambda_{0,nov})$ is independent of $\mathcal{L}, \nabla$. However $\mathfrak{m}_k$ depends on $\mathcal{L}, \nabla$.

We here give a rough idea of the definition of $\mathfrak{m}_k$. The detail is in [FOOO]. Let $\tilde{\mathcal{M}}_{k+1}(L)$ be the set of all $(\varphi, \vec{z})$ where

$$\varphi : (D^2, \partial D^2) \to (M, L)$$

is a pseudoholomorphic map and

$$\vec{z} = (z_0, \cdots, z_{k+1}) \in (\partial D^2)^{k+1}.$$

We assume that $z_0, \cdots, z_{k+1}$ respects the counter clockwise order of ∂D^2.

The group $PSL(2,\mathbb{R}) = \mathrm{Aut}(D^2)$ acts on $\tilde{\mathcal{M}}_{k+1}(L)$ in an obvious way. Let $\mathcal{M}_{k+1}(L)$ be the quotient space.

The evaluation map $ev : \mathcal{M}_{k+1}(L) \to L^{k+1}$ is defined by

$$ev([\varphi, \vec{z}]) = (\varphi(z_0), \cdots, \varphi(z_k)). \tag{2.6}$$

We define functions E, B, H on $\mathcal{M}_{k+1}(L)$ by

$$E(\varphi) = \int_{D^2} \varphi^*\omega \in \mathbb{R}, \qquad B(\varphi) = \int_{D^2} \varphi^* B \in \mathbb{R},$$
$$H(\varphi) = Hol_{\varphi(\partial D^2)}(\mathcal{L}, \nabla) \in U(1) = S^1.$$

Here $Hol_{\varphi(\partial D^2)}(\mathcal{L}, \nabla)$ denotes the holonomy of the connection ∇ along the loop $\varphi(\partial D^2)$.

[2] If L is spin then it has a relative spin structure.

Lemma 2.7. $\exp(2\pi\sqrt{-1}B(\varphi))H(\varphi)\otimes T^{E(\varphi)}$ *depends only on the homotopy class of* φ.

Proof. The independence of the absolute value of the coefficient and of $E(\varphi)$ is a consequence of the fact that L is a Lagrangian submanifold and the Stokes' theorem. The independence of the phase factor follows from $F_\nabla = 2\pi\sqrt{-1}B$. □

Now we are ready to explain the definition of the operation $\mathfrak{m}_k$ in Theorem 2.5. Let $\beta\in\pi_2(M,L)$. $\mathcal{M}_{k+1}(L;\beta)$ denotes the components of $\mathcal{M}_{k+1}(L)$ such that the homotopy class of φ is β.

By Lemma 2.7, we may write

$$\exp(2\pi\sqrt{-1}B(\beta))H(\beta)\otimes T^{E(\beta)}.$$

Let P_i be singular chains defining elements of $C(L;\mathbb{Q})$. We now put

$$\begin{aligned}\mathfrak{m}_k(P_1,\cdots,P_k) = \sum_\beta &\exp(2\pi\sqrt{-1}B(\beta))H(\beta)\\ &(\mathcal{M}_{k+1}(L;\beta)\times_{L^k}(P_1\times\cdots\times P_k))\otimes T^{E(\beta)}\end{aligned}\tag{2.8}$$

Here

$$\mathcal{M}_{k+1}(L;\beta)\times_{L^k}(P_1\times\cdots\times P_k)$$

is a fiber product taken by using $(ev_1,\cdots,ev_k)$, where

$$ev=(ev_0,\cdots,ev_k):\mathcal{M}_{k+1}(L;\beta)\to L^{k+1}.$$

We use ev_0 to regard $\mathcal{M}_{k+1}(L;\beta)\times_{L^k}(P_1\times\cdots\times P_k)$ as a current on L.

It is proved in [FOOO] Theorem 13.22 that (2.8) satisfies the required properties. (Actually the case when flat line bundle is included is not discussed in [FOOO]. However by using Lemma 2.7 we do not need any other argument than those given in [FOOO] to generalize it to include $\mathcal{L}$.)

Let us explain how Theorem 2.2 is related to the study of infinitesimal moduli space of Lagrangian submanifolds. (See [FOOO] Chapters 4 and 8 for more detail.) Let us consider the dual $C(L;\mathbb{Q})^*$ to $C(L;\mathbb{Q})$. We denote by $C(L;\Lambda_{0,nov})^*$ the completion of $C(L;\mathbb{Q})^*\otimes\Lambda_{0,nov,\mathbb{Q}}$. We shift its degree in the same way as before and take free tensor algebra $TC[1](L;\Lambda_{0,nov})^*$. Let $\hat{T}C[1](L;\Lambda_{0,nov})^*$ be the completion of $TC[1](L;\Lambda_{0,nov})^*$. Both $TC[1](L;\Lambda_{0,nov})^*$ and $\hat{T}C[1](L;\Lambda_{0,nov})^*$ are associative (but noncommutative) algebras over $\Lambda_{0,nov}$.

The dual of $\mathfrak{m}_k$ is a homomorphism

$$\mathfrak{m}_k^*: C[1](L;\Lambda_{0,nov})^*\to TC[1](L;\Lambda_{0,nov})^*.$$

We can extend $\mathfrak{m}_k^*$ uniquely to a derivation

$$\delta_k: TC[1](L;\Lambda_{0,nov})^*\to TC[1](L;\Lambda_{0,nov})^*.$$

The sum

$$\hat\delta=\sum_k\delta_k$$

converges as a $\Lambda_{0,nov}$ module homomorphism (derivation)

$$\hat{\delta} : \hat{T}C[1](L;\Lambda_{0,nov})^* \to \hat{T}C[1](L;\Lambda_{0,nov})^*.$$

We thus obtain a differential graded algebra $\left(\hat{T}C[1](L;\Lambda_{0,nov})^*, \cdot, \hat{\delta}\right)$. Its cohomology

$$\left(H(\hat{T}C[1](L;\Lambda_{0,nov})^*, \hat{\delta}), \cdot\right) \tag{2.9}$$

is an associative algebra. The result of [FOOO] Chapter 4 §15 immediately implies that the algebra (2.9) depends only on the $\sim$ equivalence class of $(L,\mathcal{L},\nabla)$.

Remark 2.10. Actually we proved in [FOOO] Chapter 4, that the weak homotopy type of filtered A_∞ algebra (the definition of weak homotopy equivalence of filtered A_∞ algebra is in [FOOO] Chapter 4) in Theorem 2.5 depends only on the $\sim$ equivalence class of $(L,\mathcal{L},\nabla)$. This statement is stronger than the independence of (2.9).

In [EGH], a similar construction is proposed in the case of contact homology. (L_∞ algebra appears in place of A_∞ algebra in that case. As a consequence the algebra corresponding to (2.9) in their situation is graded commutative.) Eliashberg-Givental-Hofer proposed to prove independence of the objects corresponding to (2.9) of various choices involved, in their situation. In their situation also, one can show the invariance of weak homotopy type of L_∞ algebra. (Compare also Chekanov [Ch].)

We now study a relation of our construction here to the infinitesimal structure of the moduli space $\mathcal{LAG}(M,\Omega)_{qm}$. We consider an infinitesimal deformation of an element $[L,\mathcal{L},\nabla]$ in $\mathcal{LAG}(M,\Omega)_{cl}$. By Lemma 1.7, the deformation is controlled by the cohomology group $H^1(L;\mathbb{C})$.

This is the first place where "quantum deformation" of the moduli space $\mathcal{LAG}(M,\Omega)_{cl}$ appears. Namely the deformation of $[L,\mathcal{L},\nabla]$ in $\mathcal{LAG}(M,\Omega)_{qm}$ is controlled by "*Spec*", $Spec(C(L;\Lambda_{0,nov}),\mathfrak{m})$ of the filtered A_∞ algebra in Theorem 2.5. The author does not know the precise definition of the "*Spec*" of filtered A_∞ algebra. (It should be some kind of super analogue of formal scheme.) (See however [FOOO] Chapter 8.) So, as its approximation, we consider the set of "$\Lambda_{0,nov}$ geometric points", which we can define rigorously as follows.

Definition 2.11. *We denote by* $\tilde{\mathcal{M}}'(M,\mathcal{L},\nabla)$ *the set of all continuous maps*

$$\phi : \hat{T}C[1](L;\Lambda_{0,nov})^* \to \Lambda_{0,nov}$$

of degree 0 *such that it is a ring homomorphism and* $\phi\circ\hat{\delta} = 0$.

The set $\tilde{\mathcal{M}}'(M,\mathcal{L},\nabla)$ above coincides to one in [FOOO] Chapter 4. To explain the later, let us consider the dual to ϕ.

The dual ϕ^* to ϕ is a map

$$\phi^* : \Lambda_{0,nov} \to \hat{B}C[1](L;\Lambda_{0,nov}). \tag{2.12}$$

Since ϕ is a ring homomorphism it follows that ϕ^* is a coalgebra homomorphism. Using the fact that $BC[1](L;\Lambda_{0,nov})$ is a free coalgebra, we can prove the following easily.

Lemma 2.13. *Coalgebra homomorphism (2.12) is given by*

$$\phi^*(1) = e^b, \quad e^b = 1 + b + b \otimes b + \cdots,$$

where $b \in C[1](L; \Lambda_{+,nov,\mathbb{C}})$. *Its dual* ϕ *is given by*

$$\phi(x^1 \otimes \cdots \otimes x^k) = x^1(b) \cdots x^k(b).$$

In order $e^b = 1 + b + b \otimes b + \cdots$ to converge in the (non Archimedean) topology induced by the valuation on Λ_{nov}, we assumed that

$$b \in C[1](L; \Lambda_{+,nov}). \tag{2.14}$$

(In case $b \in C[1](L; \Lambda_{0,nov})$, the infinite series e^b may not converge.) We assumed that ϕ^* is degree preserving. Therefore

$$b \in C[1]^0(L; \Lambda_{+,nov,\mathbb{C}}) = C^1(L; \Lambda_{+,nov,\mathbb{C}}).$$

We now put

$$\tilde{\mathcal{M}}(L, \mathcal{L}, \nabla) = \left\{ b \in C[1]^0(L; \Lambda_{+,nov,\mathbb{C}}) \middle| \hat{\delta}(e^b) = 0 \right\}.$$

The discussion so far implies that $\tilde{\mathcal{M}}(L, \mathcal{L}, \nabla)$ is identified to $\tilde{\mathcal{M}}'(M, \mathcal{L}, \nabla)$. We call an element of $\tilde{\mathcal{M}}(L, \mathcal{L}, \nabla)$ a *bounding chain.* $\tilde{\mathcal{M}}(L, \mathcal{L}, \nabla)$ is introduced in [FOOO] Chapter 4.

Let us write the equation $\hat{\delta}(e^b) = 0$ more explicitly. We first put

$$b = \sum b_{\lambda_i} T^{\lambda_i}, \quad b_{\lambda_i} \in C^1(L; \mathbb{C}), \quad \lambda_i \uparrow \infty.$$

We next define $\mathfrak{m}_{k,\lambda_i}$ by

$$\mathfrak{m}_k(P_1, \cdots, P_k) = \sum_i \mathfrak{m}_{k,\lambda_i}(P_1, \cdots, P_k) T^{\lambda_i}, \tag{2.15}$$

where $P_i \in C[1](L; \mathbb{C})$ and $\mathfrak{m}_{k,\lambda_i}(P_1, \cdots, P_k) \in C[1](L; \mathbb{C})$.

For a general filtered A_∞ algebra such discrete set $\{\lambda_1, \cdots, \lambda_k, \cdots\}$ may not exist. We say that our A_∞ algebra is *strongly gapped* if the operation $\mathfrak{m}_i$ can be written as in (2.15). (Note that we assumed that the set $\{\lambda_1, \cdots, \lambda_i, \cdots\}$ appeared in $\mathfrak{m}_k$ is independent of k and is discrete.)

In our case of the A_∞ structure in Theorem 2.5, the set $\{\lambda_1, \cdots, \lambda_k, \cdots\}$ is the set of all $E(\beta) \in \mathbb{R}_{\geq 0}$ where $\beta \in \pi_1(M, L)$ such that $\mathcal{M}(L, \beta)$ is nonempty. The set of such $E(\beta)$ is discrete by Gromov compactness.

Now the equation $\hat{\delta}(e^b) = 0$ can be written as

$$\begin{aligned} \mathfrak{m}_{0,\lambda}(1) + &\sum_{\lambda_{(0)}+\lambda_{(1)}=\lambda} \mathfrak{m}_{1,\lambda_{(0)}}(b_{\lambda_{(1)}}) \\ &+ \sum_{\lambda_{(0)}+\lambda_{(1)}+\lambda_{(2)}=\lambda} \mathfrak{m}_{2,\lambda_{(0)}}(b_{\lambda_{(1)}}, b_{\lambda_{(2)}}) \\ &+ \sum_{\lambda_{(0)}+\lambda_{(1)}+\lambda_{(2)}+\lambda_{(3)}=\lambda} \mathfrak{m}_{3,\lambda_{(0)}}(b_{\lambda_{(1)}}, b_{\lambda_{(2)}}, b_{\lambda_{(3)}}) + \cdots = 0. \end{aligned} \tag{2.16}$$

Let us consider the case when $\mathfrak{m}_0 = \mathfrak{m}_3 = \mathfrak{m}_4 = \cdots = 0$. In this case, our filtered A_∞ algebra is a differential graded algebra (DGA). (Actually we need to modify the sign of the operations to obtain DGA. See [FOOO] Chapter 4 Formula (13.4).) Then, the equation similar to (2.16) in this case is

$$db + b \wedge b = 0. \tag{2.17}$$

Equation (2.17) is similar to the equation of deformation of holomorphic structures on vector bundles

$$\overline{\partial}\mathfrak{B} + \mathfrak{B} \wedge \mathfrak{B} = 0. \tag{2.18}$$

(Here $\mathfrak{B} \in \Gamma(M; End(\mathcal{E}) \otimes \Lambda^{0,1})$ and $\mathcal{E}$ is a holomorphic vector bundle.) Equation (2.17) is also similar to the equation

$$F_A = dA + A \wedge A = 0 \tag{2.19}$$

of the deformation of flat bundles.

We recall that the equation (2.18) is equivalent to

$$(\overline{\partial} + \mathfrak{B}\wedge) \circ (\overline{\partial} + \mathfrak{B}\wedge) = 0. \tag{2.20}$$

Here

$$\overline{\partial}_{\mathfrak{B}} = \overline{\partial} + \mathfrak{B}\wedge : \Gamma(M; \mathcal{E} \otimes \Lambda^{0,k}) \to \Gamma(M; \mathcal{E} \otimes \Lambda^{0,k+1})$$

is the Dolbault operator with coefficient in the holomorphic vector bundle $(\mathcal{E}, \overline{\partial}_{\mathfrak{B}})$.

We can show a similar fact in our situation of A_∞ algebra. Let $b \in C[1]^0(L; \Lambda_{+,nov})$. We put

$$\mathfrak{m}_1^b(x) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \mathfrak{m}_{k+\ell+1}^b(\underbrace{b, \cdots, b}_{k \text{ times}}, x, \underbrace{b, \cdots, b}_{\ell \text{ times}})$$

for $x \in C[1](L; \Lambda_{nov,\mathbb{C}})$.

Lemma 2.21. $\mathfrak{m}_1^b(x) \circ \mathfrak{m}_1^b(x) = 0$ *if and only if* $\hat{\delta}(e^b) = 0$.

The proof is easy and is omitted. (See [FOOO] Chapter 4 Lemma 13.37.) We can define $\mathfrak{m}_k^b$ in a way similar to $\mathfrak{m}_1^b$. (See [FOOO] Chapter 4 Definition 13.39.) By Lemma 2.21, we can define Floer cohomology

$$HF((L, \mathcal{L}, \nabla, b); (L, \mathcal{L}, \nabla, b)) \cong \frac{\operatorname{Ker} \mathfrak{m}_1^b}{\operatorname{Im} \mathfrak{m}_1^b}, \tag{2.22}$$

for $b \in \tilde{\mathcal{M}}(L, \mathcal{L}, \nabla)$.

We remark that our space $\tilde{\mathcal{M}}(L, \mathcal{L}, \nabla)$ is too big to be an appropriate moduli space of Lagrangian submanifolds. For example, in the case of its analogy (2.18), we need to divide the set of its solutions by the action of gauge transformation group, to obtain a moduli space of holomorphic structures on the bundle $\mathcal{E}$.

In our case also, we need to define an equivalence relation and divide $\tilde{\mathcal{M}}(L, \mathcal{L}, \nabla)$ by it to obtain a moduli space $\mathcal{M}(L, \mathcal{L}, \nabla)$, which is an infinitesimal neighborhood of $[L, \mathcal{L}, \nabla]$ in $\mathcal{LAG}(M, \Omega)_{qm}$. Let us now define a gauge equivalence on $\tilde{\mathcal{M}}(L, \mathcal{L}, \nabla)$.

We take another formal parameter S. Let

$$C[1](L;\Lambda_{nov,\mathbb{C}})\langle S\rangle$$

be the set of all formal sums

$$X(S)=\sum_i X_{\lambda_i}(S)T^{\lambda_i}$$

where $X_{\lambda_i}(S)$ is a polynomial of S with coefficient in $C(L;\mathbb{C})$ and $\lambda_i\uparrow\infty$. We also remark that the degree of the polynomial $X_{\lambda_i}(S)$ may go to infinity as i goes to infinity. Hence $X(S)$ is a kind of formal power series with respect to S also. Note however, if $s\in\mathbb{R}$, the value $X(s)$ is well defined as an element of $C[1](L;\Lambda_{nov})$.

Now let

$$b(S)\in C[1]^0(L;\Lambda_{nov,\mathbb{C}})\langle S\rangle,\quad c(S)\in C[1]^{-1}(L;\Lambda_{nov,\mathbb{C}})\langle S\rangle.$$

We consider the equation

$$\frac{\partial b(S)}{\partial S}=\mathfrak{m}_1^{b(S)}(c(S)). \tag{2.23}$$

Note that the left and the right hand sides are well defined as elements of $C[1]^0(L;\Lambda_{nov,\mathbb{C}})\langle S\rangle$. As we remarked above, $b(0)$ and $b(1)$ are well defined.

Definition 2.24. Let $b,b'\in\tilde{\mathcal{M}}(L,\mathcal{L},\nabla)$. We say $b\sim b'$ if there exists a solution of (2.23) such that $b(0)=b$, $b(1)=b'$.

Proposition 2.25. *$\sim$ is an equivalence relation.*

The proof of transitivity is not trivial. Proposition 2.25 is proved in [FOOO] Chapter 8[3].

Lemma 2.26. *Let $b(S),c(S)$ be a solution of (2.23). We assume $b(0)\in\tilde{\mathcal{M}}(L,\mathcal{L},\nabla)$. Then $b(1)\in\tilde{\mathcal{M}}(L,\mathcal{L},\nabla)$.*

Proof.

$$\begin{aligned}\frac{\partial\mathfrak{m}(e^{b(S)})}{\partial S}&=\sum_{k=0}^{\infty}\frac{\partial}{\partial S}\mathfrak{m}_k(\underbrace{b(S),\cdots,b(S)}_{k\text{ times}})\\&=\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{\partial}{\partial S}\mathfrak{m}_{k+\ell+1}\left(\underbrace{b(S),\cdots,b(S)}_{k\text{ times}},\frac{\partial b(S)}{\partial S}\underbrace{b(S),\cdots,b(S)}_{\ell\text{ times}}\right)\\&=\mathfrak{m}_1^{b(S)}\left(\frac{\partial b(S)}{\partial S}\right)=(\mathfrak{m}_1^{b(S)}\circ\mathfrak{m}_1^{b(S)})(c(S))=0.\end{aligned}$$

More precisely, we proceed as follows : If $\mathfrak{m}(e^{b(S)})$ is zero modulo S^{k-1}, then $\mathfrak{m}_1^{b(S)}\circ\mathfrak{m}_1^{b(S)}$ is zero modulo S^{k-1}. Hence the above calculation shows that $\mathfrak{m}(e^{b(S)})$ is zero modulo S^k. □

[3]Chapter 8 is not included in the version of [FOOO] completed and distributed in December 2000. It will be included in the final version.

Definition 2.27. $\mathcal{M}(L,\mathcal{L},\nabla) = \tilde{\mathcal{M}}(L,\mathcal{L},\nabla)/\sim$.

We can prove the following. (See [FOOO] Chapters 4,8 for the proof.)

Lemma 2.28. *If $b \sim b'$ then*

$$HF((L,\mathcal{L},\nabla,b);(L,\mathcal{L},\nabla,b)) \cong HF((L,\mathcal{L},\nabla,b');(L,\mathcal{L},\nabla,b')).$$

Here we defined $\mathcal{M}(L,\mathcal{L},\nabla)$ only set theoretically. However we can define it as a formal scheme. Namely we have a formal map (the Kuranishi map) :

$$\text{Kura} : H^1(L;\mathbb{C}) \to H^2(L;\mathbb{C}) \otimes \Lambda_{0,nov} \tag{2.29}$$

such that its zero set is mapped surjectively to $\mathcal{M}(L,\mathcal{L},\nabla)$. (See [FOOO] Theorem D and [FOOO] Chapter 8 for detail.)

We now explain a relation of $\mathcal{M}(L,\mathcal{L},\nabla)$ to $\mathcal{LAG}(M,\Omega)$.

Let $[b] \in \mathcal{M}(L,\mathcal{L},\nabla)$. We consider linearized equation of the defining equation $\hat{\delta}(e^b) = 0$ of $\mathcal{M}(L,\mathcal{L},\nabla)$. Namely we differentiate

$$\hat{\delta}(e^{b(S)}) = 0,$$

with respect to S and obtain :

$$\mathfrak{m}_1^b(\Delta b) = 0. \tag{2.30}$$

Here

$$b(S) = b + S\Delta b + \cdots .$$

On the other hand, if $b(S) \sim b(0) = b$ then there exists $c(S)$ satisfying (2.23). Hence, in case $b(S) \sim b(0) = b$, we have

$$\Delta b = \mathfrak{m}_1^b(\Delta c).$$

Thus

$$T_{[b]}\mathcal{M}(L,\mathcal{L},\nabla) = HF^1((L,\mathcal{L},\nabla,b);(L,\mathcal{L},\nabla,b)). \tag{2.31}$$

(In case $\mathcal{M}(L,\mathcal{L},\nabla)$ is singular, the left hand side of (2.31) should be regarded as a Zariski tangent space.)

We next review a relation between Floer cohomology and usual cohomology. We remark that we may write

$$\mathfrak{m}_1^b = \sum_{i=0}^{\infty} \mathfrak{m}_{1,i}^b T^{\lambda_i}.$$

The first term $\mathfrak{m}_{1,0}^b$ is independent of b and coincides with the usual coboundary operator up to sign. Using this fact, we can construct a spectral sequence $E_r^{p,q}$ such that

$$E_1^{p,q} \cong gr_q(H^p(L;\Lambda_{0,nov}))$$
$$E_\infty^{p,q} \cong gr_q(HF^p(L,\mathcal{L},\nabla,b);(L,\mathcal{L},\nabla,b)).$$

(See [FOOO] Theorem E, for precise statement.)

Roughly speaking, it implies that Floer cohomology is a quantum deformation of the usual cohomology. The higher order terms $\mathfrak{m}_{1,i}^b$, $i \geq 1$ give the "quantum effect". Those terms come from the existence of the nonconstant pseudoholomorphic disks.

On the other hand, as we remarked before

$$T_{(L,\mathcal{L},\nabla)}\mathcal{LAG}(M,\Omega) \cong H^1(L;\mathbb{C}) \tag{2.32}$$

Comparing (2.31) and (2.32), we find that $\mathcal{M}(L,\mathcal{L},\nabla)$ is a quantum deformation of $\mathcal{LAG}(M,\Omega)$ at least at the level of tangent space.

3. Infinitesimal family - infinitesimal family of Floer homologies -

Now let us go back to our purpose, that is to construct infinitesimal family of Floer homologies on the infinitesimal neighborhood $\mathcal{M}(L,\mathcal{L},\nabla)$ in $\mathcal{LAG}(M,\Omega)_{qm}$ of $[L,\mathcal{L},\nabla]$. In other words, we are going to construct an object of derived category of coherent sheaves on the formal scheme $\mathcal{M}(L,\mathcal{L},\nabla)$. See [FOOO] Chapter 8 for detail on this section.

Let us recall the standard dictionary between geometry and ring theory. Let $[L,\mathcal{L},\nabla] \in \mathcal{LAG}(M,\Omega)$. Its infinitesimal neighborhood is (roughly speaking) a sub(formal)scheme of the affine formal scheme which is a spectrum of the complete valuational (free noncommutative) ring $\hat{T}C[1](L;\Lambda_{0,nov})^*$. The "ideal" defining our sub(formal)scheme is given by the boundary operator $\hat{d}$. Therefore the sheaf on the infinitesimal neighborhood of $([L_1,\mathcal{L}_1,\nabla_1],[L_2,\mathcal{L}_2,\nabla_2]) \in \mathcal{LAG}(M,\Omega) \times \mathcal{LAG}(M,\Omega)$ corresponds to "a module over a quotient ring of $\hat{T}C[1](L_1;\Lambda_{0,nov})^* \otimes \hat{T}C[1](L_2;\Lambda_{0,nov})^*$".

More precisely taking into account the noncommutativity of $\hat{T}C[1](L;\Lambda_{0,nov})^*$, a sheaf on the infinitesimal neighborhood of $([L_1,\mathcal{L}_1,\nabla_1],[L_2,\mathcal{L}_2,\nabla_2])$ corresponds to a $\hat{T}C[1](L_1;\Lambda_{0,nov})^*$, $\hat{T}C[1](L_2;\Lambda_{0,nov})^*$ differential graded bimodule. Actually we constructed in [FOOO] Chapter 4 §14.2 such differential graded bimodule. Namely the following theorem was proved there.

Theorem 3.1. *For each pair $([L_1,\mathcal{L}_1,\nabla_1],[L_2,\mathcal{L}_2,\nabla_2]) \in \mathcal{LAG}(M,\Omega)\times\mathcal{LAG}(M,\Omega)$, we can associate a left $\hat{T}C[1](L_1;\Lambda_{0,nov})^*$ and right $\hat{T}C[1](L_2;\Lambda_{0,nov})^*$ differential graded bimodule $\mathcal{D}$.*

In the rest of this section, we explain an outline of the construction of $\mathcal{D}$ and why it can be regarded as an infinitesimal family of Floer cohomologies.

We assume that L_1 is transversal to L_2. Then, as a Λ_{nov} module, we put :

$$\begin{aligned}\mathcal{D} = \hat{T}C[1](L_1;\Lambda_{nov})^* \otimes_{\Lambda_{0,nov}} \bigoplus_{p\in L_1\cap L_2} Hom(\mathcal{L}_{1,p},\mathcal{L}_{2,p}) \otimes \Lambda_{nov} \\ \otimes_{\Lambda_{0,nov}} \hat{T}C[1](L_2;\Lambda_{nov})^*.\end{aligned} \tag{3.2}$$

Bimodule structure is induced by the ring structures of $\hat{T}C[1](L_1;\Lambda_{nov})^*$ and of $\hat{T}C[1](L_2;\Lambda_{nov})^*$. The main point of the construction is the definition of the boundary operator. It is a combination of the definition of the boundary operator of Floer cohomology and the proof of Theorem 2.5.

Before explaining an outline of the construction of boundary operator, we continue our discussion based on the dictionary between ring theory and geometry. Let us consider the object $\mathfrak{D}$ of derived category corresponding to the differential graded bimodule $\mathcal{D}$. (3.2) implies that the fiber $\mathfrak{D}_{([b_1],[b_2])}$ of $\mathfrak{D}$ at $([b_1],[b_2]) \in \mathcal{M}(L_1,\mathcal{L}_1,\nabla_1) \times \mathcal{M}(L_2,\mathcal{L}_2,\nabla_2)$ is a cohomology of the graded differential Λ_{nov} module

$$\mathfrak{D}_{([b_1],[b_2])} \cong \Lambda_{0,nov} \otimes_{\phi^{b_1}} \mathcal{D} \otimes_{\phi^{b_2}} \Lambda_{0,nov}. \tag{3.3}$$

Here

$$\phi^{b_1} : \hat{T}C[1](L_1;\Lambda_{nov})^* \to \Lambda_{0,nov}$$

is defined by :

$$\phi^{b_1}(x^1 \otimes \cdots \otimes x^k) = x^1(b_1)\cdots x^k(b_k).$$

As Λ_{nov} module, we find easily that

$$\mathfrak{D}_{([b_1],[b_2])} \cong \bigoplus_{p\in L_1\cap L_2} Hom(\mathcal{L}_{1,p},\mathcal{L}_{2,p}) \otimes \Lambda_{nov,\mathbb{C}}.$$

Hence, as a Λ_{nov} module, $\mathfrak{D}_{([b_1],[b_2])}$ is independent of the choice of $([b_1],[b_2])$. However the boundary operator

$$\partial_{b_1,b_2} : \bigoplus_{p\in L_1\cap L_2} Hom(\mathcal{L}_{1,p},\mathcal{L}_{2,p}) \otimes \Lambda_{nov} \to \bigoplus_{p\in L_1\cap L_2} Hom(\mathcal{L}_{1,p},\mathcal{L}_{2,p}) \otimes \Lambda_{nov}$$

does depend on $([b_1],[b_2])$. Note that

$$\partial_{b_1,b_2} = 1 \otimes \delta \otimes 1,$$

and is

$$\partial_{b_1,b_2}(\alpha) = \sum_{k,\ell} \mathfrak{n}_{k,\ell}(\underbrace{b_1,\cdots,b_1}_{k \text{ times}},\alpha,\underbrace{b_2,\cdots,b_2}_{\ell \text{ times}}),$$

as we will define later in this section. We define :

Definition 3.4.

$$HF((L_1,\mathcal{L}_1,\nabla_1,b_1),(L_2,\mathcal{L}_2,\nabla_2,b_2)) \cong \frac{\operatorname{Ker}\partial_{b_1,b_2}}{\operatorname{Im}\partial_{b_1,b_2}}.$$

Let us explain how the discussion so far clarifies the troubles (A.2) and (B). The moduli space $\mathcal{M}(L,\mathcal{L},\nabla)$ may be empty in general. (Namely the equation (2.16) may not have a solution.) For such $(L,\mathcal{L},\nabla)$, the obstruction class mentioned in (A.2) is nonzero and Floer cohomology is not defined. If $\mathcal{M}(L,\mathcal{L},\nabla)$ is nonempty then Floer cohomology is defined. However it depends on the choice of the class $[b] \in \mathcal{M}(L,\mathcal{L},\nabla)$. So (A.2) and (B) are reduced to the problem to determine the moduli space $\mathcal{M}(L,\mathcal{L},\nabla)$. This problem is not easy.

We conjectured (in [FOOO]) that mirror object exists only in case when $\mathcal{M}(L,\mathcal{L},\nabla)$ is nonempty and the infinitesimal deformation theory of the mirror object (see [GM 1,2] for example) coincides with $\mathcal{M}(L,\mathcal{L},\nabla)$. Thus one possible approach to determine $\mathcal{M}(L,\mathcal{L},\nabla)$ is to prove homological mirror symmetry conjecture in the above sense and to use complex geometry.

Now we present an outline of the construction of the boundary operator : $\mathcal{D} \to \mathcal{D}$ in [FOOO] §14. Actually we construct its dual. Namely we put

$$\begin{aligned}\mathcal{D}^* = \hat{B}C[1](L_1;\Lambda_{nov}) \otimes_{\Lambda_{0,nov}} &\bigoplus_{p\in L_1\cap L_2} Hom(\mathcal{L}_{1,p},\mathcal{L}_{2,p}) \otimes \Lambda_{nov} \\ &\otimes_{\Lambda_{0,nov}} \hat{B}C[1](L_2;\Lambda_{nov}),\end{aligned}$$

and will define $\hat{d} : \mathcal{D}^* \to \mathcal{D}^*$. We remark that $\hat{B}C[1](L_1;\Lambda_{nov})$ is a coalgebra. Namely :

$$\Delta(x_1\otimes\cdots\otimes x_n) = \sum_{k=0}^{n}(x_1\otimes\cdots\otimes x_k)\otimes(x_{k+1}\otimes\cdots\otimes x_n). \tag{3.5}$$

(3.5) induces a bicomodule structure on $\mathcal{D}^*$. We remark that coalgebra structure (3.5) is a dual to the algebra structure on $\hat{T}C[1](L_1;\Lambda_{nov})^*$, and the bicomodule structure on $\mathcal{D}^*$ is dual to the bimodule structure on $\mathcal{D}$. Therefore, in order to construct a derivation on $\mathcal{D}$, it suffices to construct a coderivation $\hat{d}$ on $\mathcal{D}^*$.

We remark that $\mathcal{D}^*$ is a free bicomodule. Hence if we define

$$\mathfrak{n} : \mathcal{D}^* \to \bigoplus_{p\in L_1\cap L_2} Hom(\mathcal{L}_{1,p},\mathcal{L}_{2,p}) \otimes \Lambda_{nov}$$

then bicomodule homomorphism $\hat{d} : \mathcal{D}^* \to \mathcal{D}^*$ will be induced by

$$\begin{aligned}\hat{d}(\mathbf{x}\otimes\alpha\otimes\mathbf{y}) = (\hat{\delta}(\mathbf{x})\otimes\alpha\otimes\mathbf{y}) + (1\otimes\mathfrak{n}\otimes 1)(\Delta\mathbf{x}\otimes\alpha\otimes\Delta\mathbf{y}) \\ + (-1)^{\deg\mathbf{x}+\deg p+2}(\mathbf{x}\otimes\alpha\otimes\hat{\delta}(\mathbf{y})),\end{aligned} \tag{3.6}$$

where $\alpha \in Hom(\mathcal{L}_{1,p},\mathcal{L}_{2,p})$. ((3.6) is a bicomodule homomorphism and is a coderivation automatically. We need to check that it is a boundary operator, namely $\hat{d}\hat{d} = 0$.)

To define $\mathfrak{n}$, we use again the moduli space of pseudoholomorphic disks. Let $p,q \in L_1\cap L_2$. We consider the following moduli space.

$$\tilde{\mathcal{M}}(L_1,L_2;p,q) = \{\varphi : [0,1]\times\mathbb{R}\to M \mid \text{Condition 3.7 below.}\}.$$

Condition 3.7.
(3.7.1) φ is pseudoholomorphic.
(3.7.2) $\varphi(0,\tau)\in L_1$, $\varphi(1,\tau)\in L_2$.
(3.7.3) $\lim_{\tau\to-\infty}\varphi(t,\tau) = p$, $\lim_{\tau\to+\infty}\varphi(t,\tau) = q$.

There is an obvious $\mathbb{R}$ action on $\tilde{\mathcal{M}}(L_1,L_2;p,q)$ induced by the shift of the second factor of $[0,1]\times\mathbb{R}$. Let $\mathcal{M}(L_1,L_2;p,q)$ be the quotient space by this action.

We divide $\mathcal{M}(L_1,L_2;p,q)$ according to the homotopy class as follows. Let β be a homotopy class of a map satisfying (3.7.2),(3.7.3). We denote by $\mathcal{M}(L_1,L_2;p,q;\beta)$ the subset consisting of $[\varphi]\in\mathcal{M}(L_1,L_2;p,q)$ such that $\varphi\in\beta$

We next define a weight function on $\mathcal{M}(L_1,L_2;p,q)$. Let $[\varphi]\in\mathcal{M}(L_1,L_2;p,q)$. We put

$$E(\varphi) = \int_{[0,1]\times\mathbb{R}}\varphi^*\omega\in\mathbb{R}, \quad B(\varphi) = \int_{[0,1]\times\mathbb{R}}\varphi^*B\in\mathbb{R}.$$

We also define

$$H(\varphi) : Hom(\mathcal{L}_{1,p},\mathcal{L}_{2,p}) \to Hom(\mathcal{L}_{1,q},\mathcal{L}_{2,q}),$$

by

$$H(\varphi)(\alpha) = h_{\varphi(\{1\}\times\mathbb{R})}(\mathcal{L}_2)\circ\alpha\circ h_{\varphi(\{0\}\times\mathbb{R})}(\mathcal{L}_1)^{-1}.$$

Here $h_{\varphi(\{1\}\times\mathbb{R})}(\mathcal{L}_2) : \mathcal{L}_{2,p}\to\mathcal{L}_{2,q}$ is a parallel transport of $(\mathcal{L}_2,\nabla_2)$ along the path $\varphi(\{1\}\times\mathbb{R})$. $h_{\varphi(\{0\}\times\mathbb{R})}(\mathcal{L}_1) : \mathcal{L}_{1,p}\to\mathcal{L}_{1,q}$ is defined in a similar way.

Lemma 3.8. $\exp(2\pi\sqrt{-1}B(\varphi))H(\varphi)(\alpha)\otimes T^{E(\varphi)}$ *depends only on homotopy class of* φ.

The proof is the same as the proof of Lemma 2.7.

We next recall that we assumed Assumption 2.1. It implies that there exists $\eta : L_1 \cap L_2 \to \mathbb{Z}$ such that

$$\dim \mathcal{M}(L_1, L_2; p, q) = \eta(p) - \eta(q) - 1.$$

Here dim is a virtual dimension (in the sense of Kuranishi structure [FOn 1,2]). (Note that Assumption 2.1 implies that the dimension is independent of the components.)

We put $\deg \alpha = n - \eta(p)$ is $\alpha \in Hom(\mathcal{L}_{1,p}, \mathcal{L}_{2,p})$. Now we define

$$\mathfrak{n}_{0,0} : \bigoplus_{p \in L_1 \cap L_2} Hom(\mathcal{L}_{1,p}, \mathcal{L}_{2,p}) \otimes \Lambda_{nov} \to \bigoplus_{q \in L_1 \cap L_2} Hom(\mathcal{L}_{1,q}, \mathcal{L}_{2,q}) \otimes \Lambda_{nov}$$

by

$$\begin{aligned} \mathfrak{n}_{0,0}(\alpha) = & \sum_{q, \deg q = \deg p + 1} \sum_{\beta} \sum_{\varphi \in \mathcal{M}(L_1, L_2; p, q; \beta)} \\ & \pm \exp(2\pi\sqrt{-1}B(\varphi)) H(\varphi)(\alpha) \otimes T^{E(\varphi)}. \end{aligned} \tag{3.9}$$

Here $\alpha \in Hom(\mathcal{L}_{1,p}, \mathcal{L}_{2,p})$. (We omit the definition of the sign. See [FOOO] Chapter 6.) Gromov's compactness theorem implies that the right hand side is contained in

$$\bigoplus_{q \in L_1 \cap L_2} Hom(\mathcal{L}_{1,q}, \mathcal{L}_{2,q}) \otimes \Lambda_{nov}.$$

(We need to work out transversality and perturb so that $\mathcal{M}(L_1, L_2; p, q; \beta)$ is actually zero dimensional to define (3.9) rigorously. ($\mathcal{M}(L_1, L_2; p, q; \beta)$ is of virtual dimension 0 by $\deg q = \deg p + 1$.))

We will define $\mathfrak{n}_{k,\ell}$ by combining the definition (3.9) and the construction in §2. Here

$$\begin{aligned} \mathfrak{n}_{k,\ell} : B_k C[1](L_1; \Lambda_{0,nov}) \otimes & \left(\bigoplus_{p \in L_1 \cap L_2} Hom(\mathcal{L}_{1,p}, \mathcal{L}_{2,p}) \otimes \Lambda_{nov} \right) \\ & \otimes B_\ell C[1](L_1; \Lambda_{0,nov}) \to \bigoplus_{q \in L_1 \cap L_2} Hom(\mathcal{L}_{1,q}, \mathcal{L}_{2,q}) \otimes \Lambda_{nov}. \end{aligned}$$

We consider the set of $(\varphi, \vec{z}, \vec{w})$ such that

(3.10.1) $\varphi \in \tilde{\mathcal{M}}(L_1, L_2; p, q; \beta)$.
(3.10.2) $\vec{z} = ((0, z_1), \cdots, (0, z_k))$ where $z_i \in \mathbb{R}$.
(3.10.3) $\vec{w} = ((1, w_1), \cdots, (1, w_\ell))$ where $w_i \in \mathbb{R}$.

We divide the totality of such triples $(\varphi, \vec{z}, \vec{w})$ by an obvious $\mathbb{R}$ action. Let $\mathcal{M}(L_1, L_2; p, q; k, \ell; \beta)$ be the quotient space. We define an evaluation map :

$$ev : (ev_{1,1}, \cdots, ev_{1,k}, ev_{2,1}, \cdots, ev_{2,\ell}; \beta) : \mathcal{M}(L_1, L_2; p, q; k, \ell) \to L_1^k \times L_2^\ell$$

by

$$ev([\varphi, \vec{z}, \vec{w}]) = (\varphi(z_1), \cdots, \varphi(w_\ell)).$$

Let $P_{1,i}$ be singular chains of L_1 and $P_{2,i}$ be singular chains of L_2. We put

$$\begin{aligned}&\mathcal{M}(L_1,L_2;p,q;k,\ell;P_{1,1},\cdots\cdots,P_{1,k};P_{2,1},\cdots,P_{2,\ell};\beta)\\&\quad=\mathcal{M}(L_1,L_2;p,q;k,\ell;\beta)_{ev}\times_{L_1^k\times L_2^\ell}P_{1,1}\times\cdots P_{1,k}\times P_{2,1}\times\cdots P_{2,\ell}.\end{aligned}$$

Now we define :

$$\begin{aligned}&\mathfrak{n}_{k,\ell}(P_{1,1}\otimes\cdots P_{1,k}\otimes\alpha\otimes P_{2,1}\otimes\cdots P_{2,\ell})\\&\quad=\sum_{q\in L_1\cap L_2}\sum_{\beta}\sum_{[\varphi,\vec z,\vec w]\in\mathcal{M}(L_1,L_2;p,q;k,\ell;P_{1,1},\cdots\cdots,P_{1,k};P_{2,1},\cdots,P_{2,\ell};\beta)}\\&\qquad\qquad\pm\exp(2\pi\sqrt{-1}B(\varphi))H(\varphi)(\alpha)\otimes T^{E(\varphi)}.\end{aligned}$$

The first sum is taken over all q such that the moduli space

$$\mathcal{M}(L_1,L_2;p,q;k,\ell;P_{1,1},\cdots\cdots,P_{1,k};P_{2,1},\cdots,P_{2,\ell};\beta)$$

is of zero dimensional. (We omit the definition of the sign. See [FOOO] Chapter 6.) We finally define our operator $\mathfrak{n}$ by

$$\mathfrak{n}=\sum_{k,\ell}\mathfrak{n}_{k,\ell}.$$

Then we define $\hat d$ by (3.6). It is proved in [FOOO] that $\hat d$ is a boundary operator. (Namely $\hat d\circ\hat d=0$.)

4. Local family

In the last section, we discussed infinitesimal family of Floer homologies. However, we did not actually move Lagrangian submanifolds. The construction in §3,4 was carried out by using a fixed pair of Lagrangian submanifolds. In this section, we begin to move our pair of Lagrangian submanifolds. The results up to the last section is rigorously established in [FOOO]. Several parts of this and later sections are not yet rigorously established. But those which is called Theorem, Proposition, Lemma, are proved rigorously at this stage. Also many parts of the construction of this article was rigorously established in the case of affine Lagrangian submanifolds in simplex torus in [Fu4]. Actually the study of the case of torus was one of the major sources of ideas described in this article.

Let us begin with describing the situation we work with. We need two Lagrangian submanifolds to define Floer cohomology. To simplify the notation, we fix one of them L_0. Also we fix a complex line bundle $\mathcal{L}_0$ together with a connection ∇_0 on it satisfying (1.1.3). We move another Lagrangian submanifold $L(v)$ and a line bundle $\mathcal{L}(\sigma)$ together with its connection satisfying (1.1.3). To simplify notation, we denote by $\mathcal{L}(\sigma)$ a pair of a line bundle and a connection. Here (v,σ) is a parameter in $\mathbb{R}^{2b}\cong H^1(L(0);\mathbb{R})\oplus H^1(L(0);\mathbb{R})$.

We choose representatives $(L(v),\mathcal{L}(\sigma))\in\mathcal{LAG}(M,\Omega)$ as follows. By Darboux-Weinstein theorem, a neighborhood of $L(0)=L$ in (M,ω) can be identified to a neighborhood U of the zero section in the cotangent bundle $T^*L(0)$. We fix closed 1 forms $e_1,\cdots,e_b$ representing a basis of $H^1_{Dr}(L(0);\mathbb{R})$. We define

$$L(v)=\text{The graph of }v_1e_1+\cdots+v_be_b\subseteq U\subseteq M,$$

for small v_i.

Next we define $\mathcal{L}(\sigma)$ on $L(v)$. We remark that the restriction of the projection $T^*L(0) \to L(0)$ to $L(v)$ is a diffeomorphism. Hence there exists a canonical diffeomorphism $L(v) \to L(0)$. Using this diffeomorphism, we pull back the complex line bundle $\mathcal{L} = \mathcal{L}(0)$ and write it $\mathcal{L}(\sigma) \to L(v)$. As a complex line bundle, it is independent of σ. We will define a connection $\nabla(\sigma)$ on it.

We first define $\varphi : L(0) \times [0,1] \to M$ by

$$\varphi(x,t) = t(v_1e_1(x) + \cdots + v_be_b(x)) \in U \subseteq M.$$

We then put

$$\nabla(\sigma) = \nabla(0) + 2\pi\sqrt{-1}\left(\sigma_1e_1 + \cdots + \sigma_be_b + \int_0^1 i_{\frac{\partial}{\partial t}}\varphi^*B\right). \tag{4.1}$$

It is easy to check (1.1.3) by a direct calculation. Thus we obtain a map

$$V \to \mathcal{LAG}(M,\Omega).$$

Here V is a neighborhood of zero in $H^1(L(0);\mathbb{R}) \oplus H^1(L(0);\mathbb{R})$. We remark that the complex structure on $\mathcal{LAG}(M,\Omega)_{cl}$ introduced in section 1, induce one on $H^1(L(0);\mathbb{R}) \oplus H^1(L(0);\mathbb{R})$. In case when $B = 0$, the induced complex structure is usual one where $v \in H^1(L(0);\mathbb{R})$ is a real part and $\sigma \in H^1(L(0);\mathbb{R})$ is an imaginary part. In case when $B \neq 0$ the induced complex structure is still linear but v, σ are no longer real and imaginary parts. Hereafter we use induced complex structure on V.

We assume that $L(0)$ is relatively spin. Then relative spin structure of $L(0)$ induces one on $L(v)$.

We also assume that $L(0)$ is transversal to L_0. Then, by shrinking V if necessary, we may assume that $L(v)$ is transversal to L_0. Note that we are studying local family here. In case we are studying global family, we can no longer assume that $L(v)$ is transversal to L_0. We will discuss this point in §6.

Now we want to construct a chain complex of holomorphic vector bundles $(CF((L_0,\mathcal{L}_0),(L(v),\mathcal{L}(\sigma)$ $(v,\sigma) \in V$. We define the complex (but not yet holomorphic) bundles by

$$\begin{cases} CF((L_0,\mathcal{L}_0),(L(v),\mathcal{L}(\sigma)))_{\mathbb{C}} = \displaystyle\bigoplus_{p\in L_0\cap L(v)} Hom\left(\mathcal{L}_0|_{p(v)}, \mathcal{L}(\sigma)|_{p(v)}\right), \\ CF((L_0,\mathcal{L}_0),(L(v),\mathcal{L}(\sigma))) = CF((L_0,\mathcal{L}_0),(L(v),\mathcal{L}(\sigma)))_{\mathbb{C}} \otimes \Lambda_{nov}. \end{cases} \tag{4.2}$$

Note for each individual (v,σ), the Λ_{nov} modules $CF((L_0,\mathcal{L}_0),(L(v),\mathcal{L}(\sigma)))$ are members of the chain complex to define Floer homology $HF((L_0,\mathcal{L}_0),(L(v),\mathcal{L}(\sigma)))$.

We will define a structure of holomorphic vector bundle on the members of the complex (4.2). We need to assume the following for this purpose.

Assumption 4.3. There exists a Lagrangian submanifold L_{st} (which may have a boundary), and a complex line bundle $\mathcal{L}_{st}$ equipped with a connection, with the following properties.

(4.4.1) The curvature of $\mathcal{L}_{st}$ is $2\pi\sqrt{-1}B$.
(4.4.2) For each v, $L(v)$ intersects with L_{st} at one point. They intersect transversally there.

Example 4.5. Let $M \to N$ be a Lagrangian fibration. It is well known that the fiber is a torus. We consider $\operatorname{Re} V \subseteq N$ and let $L(v)$ be the fiber of $v \in \operatorname{Re} V \subseteq N$.

Let $s : \operatorname{Re} V \to M$ be a section of $M \to N$ whose image is a Lagrangian submanifold. Then $L_{st} = s(\operatorname{Re} V)$ satisfies Assumption 4.3. (We remark that we may allow $M \to N$ to have a singular fiber. However we have more to work out in case there is a singular fiber. Namely in case when there is a singular fiber, it is inevitable to include quantum correction to the complex structure of moduli space, (that is the point (F) mentioned in §0). We will not discuss it here and leave them for future research.) This situation is one appeared in Strominger-Yau-Zaslow's idea [SYZ] to construct a mirror as a moduli space of special Lagrangian submanifolds plus line bundles on it. (However in their situation the fibration has a singular fiber expect the case of symplectic torus.)

We are going to define a holomorphic structure by defining a (holomorphic) chart of the bundle (4.2).

To be a bit more precise, we are going to find a T dependent family of holomorphic structures. (Here T is a formal parameter in the Novikov ring Λ_{nov}.) Let us explain this point first. We remark that the T appeared in the formula in the form $T^{E(\varphi)}$. Here $E(\varphi)$ is the integration of the symplectic form on an appropriate disks. So replacing the symplectic form ω by $C\omega$ is equivalent to replacing T by T^C. In mirror symmetry, we study the limit $\omega \to +\infty$ and compare it to the calculation in the mirror. (The limit as $\omega \to +\infty$ is called the *large volume limit* and its mirror is called the *large complex structure limit.*) In case $T < 1$ the limit $\omega \to +\infty$ corresponds to the limit $T \to 0$.

Note that B also appeared as $\exp(2\pi\sqrt{-1}B(\varphi))$ in the formula. Hence the factor

$$\exp(2\pi\sqrt{-1}B(\varphi))T^{E(\varphi)}$$

will become e^{Ω} if we put $T = e^{-1}$.

More precisely we proceed as follows. We study the following family of complexified simplex structure. We first fix ω, B_0. Let $z \in \mathfrak{h} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. We put

$$\Omega_z = 2\pi\sqrt{-1}B_0 - 2\pi\sqrt{-1}z\omega. \tag{4.6}$$

Namely

$$B_z = B_0 - \frac{1}{2\pi}\operatorname{Re} z\omega$$

is the our B field and $2\pi \operatorname{Im} z\ \omega$ is our symplectic form. We remark that in case when L is a Lagrangian submanifold with respect to ω, the restriction of B_z to L coincides with B_0. Hence Condition 1.1 for $B = B_z$ is independent of z. The functions E, B on $\mathcal{M}_{k+1}(L)$ depend on z. We write them E_z, B_z respectively.

We now first put $T = e^{-1}$ then

$$\exp(2\pi\sqrt{-1}B_z(\varphi))T^{E_z(\varphi)} = \exp(2\pi\sqrt{-1}B_0(\varphi) + 2\pi\sqrt{-1}zE(\varphi)). \tag{4.7}$$

where $E(\varphi)$ is the symplectic area with respect to φ. We then change our definition of T and put

$$T = \exp(2\pi\sqrt{-1}z) \tag{4.8}$$

then the right hand side of (4.6) will be

$$\exp(2\pi\sqrt{-1}B_0(\varphi))T^{E(\varphi)}. \tag{4.9}$$

This is actually the same weight function as we used before (by putting $B_0 = B$)). In this sense, we may regard the parameter T as one parametrizing the family (4.6) by (4.8). (4.8) sends the neighborhood of $+\infty\sqrt{-1}$ in $\mathfrak{h}$ to $D(\epsilon)^*$ an ϵ neighborhood of the origin in $\mathbb{C}$ minus origin.

Note that the map (4.8) $z \mapsto T$ is not one to one. Actually if we regard (4.9) as a function of complex parameter T then it is multivalued. We discuss this point later.

We suppose that there exists a family of complex manifolds $(M,\Omega_z)^\wedge$ which are mirror to (M,Ω_z).

Let us assume for a moment that $[\omega] \in H^2(M;\mathbb{Z})$. Then it is known in physics literature that the mirror of (M,Ω_z) and of (M,Ω_{z+1}) is the same. (See for example [As].) Thus we can take T as a parameter in place of z. Hence there exists a family

$$(M,\Omega_{\log T/2\pi\sqrt{-1}})^\wedge \to \mathfrak{M} \to D(\epsilon)^*. \tag{4.10}$$

One may try to extend it to 0 but the fiber of 0 will necessary be singular.

Suppose that one is given a Lagrangian submanifold L with some extra data on M. Then we suppose that its mirror gives a family of sheaves on the fibers of (4.10). In case ϵ is infinitesimally small $D(\epsilon)^*$ may be regarded as a formal scheme :

$$spec(\Lambda_{nov}).$$

(Note $D(\epsilon)$ may be regarded as $spec(\Lambda_{0,nov})$ in the same sense.) Hence $\mathfrak{M}$ is supposed to be a scheme over these formal schemes. The infinitesimal family we constructed is an infinitesimal family of sheaves (objects of derived category of sheaves) over ring Λ_{nov}. This is consistent with the discussion above. In (4.2) we introduced a family of vector bundles whose fiber is vector space (module over $\mathbb{C}$). So it is not consistent with above and hence we need to work with Λ_{nov} module.

Here we add a remark about the multivaluedness of (4.9). We recall that (4.9) is the weight appeared in the definition of $\mathfrak{m}_k$. The A_∞ structure $\mathfrak{m}_k$ will turn out to give the defining equation of the moduli space $\mathcal{M}(L,\mathcal{L},\nabla)$. $\mathcal{M}(L,\mathcal{L},\nabla)$ is expected to coincides with the (infinitesimal) moduli space of a derived category of objects on the mirror. As we discussed above, those moduli spaces are z dependent family. Since we have a T dependent family of complex manifolds as in (4.10), it seems natural to have T dependent family of moduli spaces $\mathcal{M}(L,\mathcal{L},\nabla)$ in place of z dependent family. This is a problem related to the multivaluedness of (4.9). We may explain this point in the following way. Let us consider the case when our Lagrangian submanifold L is (strongly) rational. Here we say that L is (strongly) rational if there exists $\lambda \in \mathbb{Q}$ such that for each pseudoholomorphic disk $\varphi : (D^2,\partial D^2) \to (M,L)$ the energy $E(\varphi)$ is in the submonoid $\lambda\mathbb{Z}_{\ge 0}$. This means that if we replace T by T^λ then $\mathfrak{m}_k$ is single valued. In other words, our moduli space $\mathcal{M}(L,\mathcal{L},\nabla)$ is well defined as a family on $\tilde{D}(\epsilon)^*$, where $\tilde{D}(\epsilon)^*$ is a finite covering of $D(\epsilon)^*$. We remark that this kind of situation occurs naturally in complex or algebraic geometry.

If we do not assume L to be rational, then we need to take infinite covering of $D(\epsilon)^*$ to have a T dependent objects. This seems to suggests that our mirror object is transcendental. Since the author does not have an explicit example to work out what happens in such a case, he does not try to go further on this line of ideas.

Remark 4.11. We need to consider two different kinds of parameters of infinitesimal deformations. Namely T parametrize the deformation of the complex structures of the mirror as we explained above. On the other hand, we consider also the formal neighborhood of the moduli space of vector bundles. Namely the element b and the defining equation $\hat{\delta}(e^b) = 0$ of $\mathcal{M}(L, \mathcal{L}, \nabla)$ are formal power series not only of T but also of the parameters in $H^1(L;\mathbb{C})$. In this section, we are trying to construct a moduli space which is local (but not infinitesimally small) in $H^1(L;\mathbb{C})$ direction but is infinitesimally small in T direction.

As we explained above, we will construct holomorphic structure of (4.2) depending on T. We are going to construct a T dependent frame for this purpose. The idea to find such a frame is to choose one so that boundary operators will be as close to be a holomorphic family as possible.

Our assumption that $L(v)$ is transversal to L_0 implies that the number of intersection points $L(v) \cap L_0$ is independent of v. Therefore, by shrinking V if necessary, we may assume that there exists $p_1(v), \cdots, p_N(v)$ depending smoothly on v such that

$$L(v) \cap L_0 = \{p_1(v), \cdots, p_N(v)\}. \tag{4.12}$$

(Thus (4.2) is topologically a trivial bundle.) Let us denote by $p_0(v)$ the unique intersection point of L_{st} and $L(v)$. We assume hereafter that our Lagrangian submanifolds are all connected. We first choose and fix a frame

$$s_i(0,0) \in Hom\left(\mathcal{L}_0|_{p_i(v)}, \mathcal{L}(\sigma)|_{p_i(v)}\right) \cong \mathbb{C}.$$

We choose a path $\ell_{i,0}$ in $L(0)$ joining $p_0(0)$ ($\in L(0) \cap L_0$) to $p_i(0)$. We can then find a smooth family of paths $\ell_{i,v}$ in $L(v)$ joining $p_0(v)$ to $p_i(v)$.

On the other hand, there is a smooth family of paths $\gamma_{i,v}$ in L_0 joining $p_i(v)$ to $p_i(0)$. Moreover there is a smooth family of paths $\gamma'_{i,v}$ in L_{st} joining $p_0(v)$ to $p_0(0)$.

We may choose those families of paths so that there exists a smooth family of disks $\psi_v : D^2 \to M$ such that $\partial\psi_v(D^2) = \ell_{i,0} \cup \ell_{i,v} \cup \gamma_{i,v} \cup \gamma'_{i,v}$. See Figure 1.

Figure 1

We next consider the family of vector spaces

$$Hom\left(\mathcal{L}_{st}|_{p_0(v)}, \mathcal{L}(\sigma)|_{p_0(v)}\right)$$

parametrized by $(v,\sigma) \in V$ and T. This vector bundle on $V \times D(\epsilon)^*$ is trivial. We fix a trivialization of it. Let

$$\mathbf{v}_{p_0(v)} \in Hom\left(\mathcal{L}_{st}|_{p_0(v)}, \mathcal{L}(\sigma)|_{p_0(v)}\right)$$

be a frame. Namely it depends smoothly on v and is nonzero. We may choose it so that it is independent of σ. (Note our bundle $\mathcal{L}_\sigma$ as a complex vector bundle is independent of σ.) We now put

$$\begin{aligned} s_i(v,\sigma,T) = \exp\left(2\pi\sqrt{-1}B(\psi_v)\right) \otimes T^{E(\psi_v)} \Big(& h_{\ell_{i,v}}(\mathcal{L}(\sigma)) \circ \mathbf{v}^{-1}_{p_0(v)} \\ & \circ h_{\gamma'_{i,0}}(\mathcal{L}_{st}) \circ \mathbf{v}_{p_0(0)} \circ h^{-1}_{\ell_{i,0}}(\mathcal{L}(0)) \circ s_i(0,0) \circ h^{-1}_{\gamma_{i,v}}(\mathcal{L}_0)\Big). \end{aligned} \tag{4.13}$$

Here h in the right hand side is the parallel transport of the connection, that is one of $\mathcal{L}(0)$, $\mathcal{L}(v)$, $\mathcal{L}_0$, $\mathcal{L}_{st}$. For example $h_{\gamma'_{i,0}}(\mathcal{L}_{st})$ is a parallel transport along the path $\gamma'_{i,0}$ of the connection $\mathcal{L}_{st}$. We can easily see that the right hand side is an element of

$$\bigoplus_{p(v)\in L_0\cap L(v)} Hom\left(\mathcal{L}_0|_{p(v)}, \mathcal{L}(\sigma)|_{p(v)}\right). \tag{4.14}$$

We define a T dependent family of holomorphic structures of the bundle (4.14) so that $s_i(v,\sigma,T)$ is a holomorphic frame.

Lemma 4.15. *The T dependent family of holomorphic structures defined by using the holomorphic frame $s_i(v,\sigma,T)$ is independent of the choices of the paths $\ell_{i,0}, \ell_{i,v}, \gamma_{i,v}, \gamma'_{i,v}$ and the map ψ_v.*

This lemma is proved in [Fu4] §5 in the case of affine Lagrangian submanifold of a symplectic torus. The proof of the general case is similar and is omitted.

Now we defined holomorphic structures on vector bundles in (4.2). We defined a boundary operator in the last section. We next discuss holomorphicity of the boundary operator with respect to (v,σ). Actually various holomorphic structures are designed so that Lemma 4.18 below holds. To state the lemma we need some notations.

Let $\varphi_0 : D^2 \to M$ be a map satisfying Condition 3.7 (3.7.2),(3.7.3) for $L_0, L(0)$, $p(0), q(0)$. We can then find a family $\varphi_v : D^2 \to M$ of maps satisfying Condition 3.7 (3.7.2),(3.7.3) for $L_0, L(v), p(v), q(v)$. We consider

$$\begin{aligned}\exp(2\pi\sqrt{-1}B(\varphi_v))H(\varphi_v;\sigma)\otimes T^{E(\varphi_v)} &: Hom(\mathcal{L}_{0,p(v)}, \mathcal{L}_{1,p(v)}(\sigma)) \\ &\to Hom(\mathcal{L}_{0,q(v)}, \mathcal{L}_{1,q(v)}(\sigma)) \otimes \Lambda_{nov}.\end{aligned} \tag{4.16}$$

(We write $H(\varphi_v;\sigma)$ in place of $H(\varphi_v)$ to emphasize its σ dependence. The other part of (4.16) is independent of σ.) Lemma 4.18 asserts the holomorphicity of this map with respect to (v,σ). We remark that T is a formal parameter. On the other hand, $E(\varphi_v)$ does depend on v. So to state the holomorphicity, we need to clarify what we mean by the differential of $T^{E(\varphi_v)}$ with respect to v. Also we need to be careful about the multivaluedness of $T^{E(\varphi_v)}$. For this purpose, it is simplest to proceed as follows. We consider the complexified simples structure Ω_z in (4.6). Here T and z is related by (4.8). Instead of (4.16), we consider

$$\begin{aligned}&\exp(2\pi\sqrt{-1}B_0(\varphi_v) + 2\pi z\sqrt{-1}E(\varphi_v))H(\varphi_v;\sigma) \\ &\qquad : Hom(\mathcal{L}_{0,p(v)}, \mathcal{L}_{1,p(v)}(\sigma)) \to Hom(\mathcal{L}_{0,q(v)}, \mathcal{L}_{1,q(v)}(\sigma))\end{aligned} \tag{4.17}$$

Lemma 4.18. *(4.17) is holomorphic with respect to the complex structure on $\mathcal{LAG}(M,\Omega_z)$, defined in section 1 and holomorphic structure on $Hom(\mathcal{L}_{0,p(v)}, \mathcal{L}_{1,p(v)}(\sigma))$ and on $Hom(\mathcal{L}_{0,q(v)}, \mathcal{L}_{1,q(v)}(\sigma))$ by Lemma 4.15.*

The proof is similar to [Fu4] Theorem γ and is omitted.

In Lemma 4.18 we used (4.17) in place of (4.16). This was possible because the map (4.16) obviously converges when we replace the formal parameter T by some explicit complex number. But when we go to the next step it becomes impossible to do so and we need to work on formal power series such as (4.16). We do not try to rewrite here Lemma 4.18 in terms of formal power series. It is not so difficult but cumbersome.

Now we remark that (4.16) is the weight we put on the boundary operator. So Lemma 4.18 is related to the holomorphicity of the boundary operator with respect to the parameters (v,σ). But an important point here is that Lemma 4.18 does *not* imply the holomorphicity of boundary operators with respect to the parameters (v,σ). This is the main point of this article and will be discussed in more detail in the next section.

We here mention what will follow in case when the boundary operator $\partial_{(v,\sigma)}$ is holomorphic with respect to (v,σ). More precisely the boundary operator also depends on the bounding chain b. So we *assume* that there exists a family of bounding chains $b(v,\sigma)$ depending smoothly on (v,σ) such that the family of boundary operators

$$(4.19)\quad \partial_{(v,\sigma),b(v,\sigma),z} : CF((L_0,\mathcal{L}_0),(L(v),\mathcal{L}(\sigma)))_{\mathbb{C}} \to CF((L_0,\mathcal{L}_0),(L(v),\mathcal{L}(\sigma)))_{\mathbb{C}}$$

converges (after replacing (4.16) by (4.17)) and is holomorphic with respect to (v,σ), for $z \in \mathfrak{h}_C = \{z \in \mathfrak{h} \,|\, \operatorname{Im} z > C\}$, and sufficiently large C. Then we will have a complex of holomorphic vector bundles on $V \times \mathfrak{h}_C$. This complex will then define a (z dependent family of) objects of coherent sheaves on a subset V of $(M,\Omega_z)^\wedge$, which we expect to be a restriction of mirror object of our Lagrangian submanifold L_0. (Note according to Strominger-Yau-Zaslow [SYZ], the mirror of $(L(v),\mathcal{L}(\sigma))$ are skyscraper sheaves.)

As we mentioned above, our assumption (the holomorphicity of (4.19)) does not hold in general. The reason is "quantum effect" which we will discuss in the next section.

5. Wall crossing of Floer homology

We now explain why (4.19) is not holomorphic with respect to (v,σ) in general. Actually there are two points to be discussed.

(5.1.1) Can we take $b(v,\sigma)$ which is holomorphic with respect to (v,σ) in an appropriate sense ?
(5.1.2) Is (4.19) holomorphic for such $b(v,\sigma)$?

Note $b(v,\sigma) \in \mathcal{M}(L(v),\mathcal{L}(\sigma))$, which is a quantized version of the infinitesimal neighborhood of the moduli space of the pair $(L(v),\mathcal{L}(\sigma))$. (5.1.1) is related to the problem how this infinitesimal deformation space changes when we move $(L(v),\mathcal{L}(\sigma))$. Various parts of this question is common with similar questions in the study of the moduli spaces appeared in various related context. (For example in the situation we mentioned in §0.) However there is one new point here. That is the operators $\mathfrak{m}_k$ which depends on (v,σ) and which is basic in the definition of $\mathcal{M}(L(v),\mathcal{L}(\sigma))$ *may jump*. Namely $\mathfrak{m}_k$ is not continuous with respect to (v,σ) in general. The other points on (5.1.1) can be analyzed in the frame work of established general theory of deformations. Moreover this new point, that is the discontinuity of $\mathfrak{m}_k$, is similar to other problems which appears in (5.1.2). So to simplify the situation, we do not discuss (5.1.1) and concentrate on (5.1.2). By this reason, we put the following rather restrictive assumption here.

Assumption 5.2. For each v, there exists no pseudoholomorphic disk bounding $L(v)$ other than trivial ones. (Here by trivial pseudoholomorphic disk we mean a constant map.)

As mentioned above, we may remove this assumption. But to do so we need to work out a lot on general deformation theory.

Now Assumption 5.2 implies that the operation $\mathfrak{m}_k$ coincides to the operation $\overline{\mathfrak{m}}_k$ in Proposition 2.2. In particular we may put $b(v,\sigma) \equiv 0$. From now on in this section, we write $\partial_{(v,\sigma)}$ in place of $\partial_{(v,\sigma),b(v,\sigma)}$.

Now we discuss holomorphicity of (4.16) under Assumption 5.2. Let us recall the definition of $\partial_{(v,\sigma)}$ given in §3. In our case, we need only $\mathfrak{n}_{0,0}$ since $b(v,\sigma) = 0$. Namely we have

$$
\begin{aligned}
\partial_{(v,\sigma)}(\alpha(v,\sigma)) = \sum_j \sum_\beta & \sum_{\varphi \in \mathcal{M}(L_0,L(v);p_i(v),p_j(v);\beta)} \\
& \pm \exp(2\pi\sqrt{-1}B(\varphi))H(\varphi;\sigma)(\alpha(v,\sigma)) \otimes T^{E(\varphi)}.
\end{aligned}
\tag{5.3}
$$

where

$$\alpha(v,\sigma) \in Hom(\mathcal{L}_{0,p_i(v)}, \mathcal{L}_{1,p_i(v)}(\sigma)).$$

The notations are as in (3.9).

Now Lemma 4.18 asserts that each term of (5.3) is holomorphic with respect to (v,σ). However :

Observation 5.4. The order $\sharp\mathcal{M}(L_0,L(v);p_i(v),p_j(v);\beta)$ jumps as we move v.

We call this phenomenon *wall crossing of Floer homology.*

Remark 5.5. Wall crossing was discovered by Donaldson [D] in the case of Donaldson invariant (gauge theory) of 4 manifolds with $b_2^+ = 1$, where wall crossing is caused by the reducible connections.

The wall crossing we are discussing here has various similarities to the case of gauge theory. Also in both cases, wall crossing is closely related to indefinite theta function. (See [GZ] for gauge theory case and [Fu4] for symplectic Floer homology case.) The author does not know conceptional explanation of this similarity.

Observation 5.4 means that the family of operators (5.3) is discontinuous and hence is not holomorphic.

An examples where wall crossing of Floer homology actually occurs is given in [FOOO] Chapter 7 §28.

We discuss here an idea how to overcome this trouble and "define" holomorphic object by putting some quantum effect. For this purpose, we study parametrized version of the moduli space of pseudoholomorhic disks. We put

$$\mathcal{M}(L_0,L;i,j;para) = \bigcup_v \mathcal{M}(L_0,L(v);p_i(v),p_j(v)) \times \{v\} \tag{5.5}$$

and

$$\pi(\varphi,v) = v, \quad \pi : \mathcal{M}(L_0,L;i,j;para) \to \mathbb{R}^{\mathrm{rank} H^1(L;\mathbb{R})}.$$

Each component of the moduli space $\mathcal{M}(L_0,L;i,j;para)$ is finite dimensional. By using the general theory of Kuranishi structure developed in [FOn 1,2] we can handle this moduli space as if it is a manifold with corners, (as long as we are interested in its fundamental class over $\mathbb{Q}$.) The moduli space is independent of σ parameter. So we add that direction as a direct summand. Namely we put

$$\mathcal{M}(L_0, L; i, j; para)^{++} = \mathcal{M}(L_0, L; i, j; para) \times H^1(L; \sqrt{-1}\mathbb{R}).$$

We define

$$\mathcal{M}(L_0, L; i, j; para)^{++} \to \mathbb{R}^{\mathrm{rank} H^1(L;\mathbb{R})} \times H^1(L; \sqrt{-1}\mathbb{R}) \supseteq V$$

using π and let $\mathcal{M}(L_0, L; i, j; para)^{+}$ be the inverse image of V. We then have

$$\pi : \mathcal{M}(L_0, L; i, j; para)^{+} \to V. \tag{5.6}$$

We next divide it into components as before. Namely we take a homotopy class β of maps φ satisfying (3.7.2),(3.7.3). (We replace L_1, L_2, p, q in (3.7.2),(3.7.3) by $L_0, L(v), p_i(v), p_j(v)$.) And let $\mathcal{M}(L_0, L; i, j; para; \beta)^{+}$ be the subset of $\mathcal{M}(L_0, L; i, j; para)^{+}$ consisting of elements of homotopy class β.

We consider the vector bundle on V whose fiber at (v, σ) is $Hom(\mathcal{L}_0|_{p_i(v)}, \mathcal{L}(\sigma)|_{p_i(v)})$. We denote it by $\mathcal{E}_i \to V$. We define $\mathcal{E}_j \to V$ in a similar way. We consider its pull back $\pi^*\mathcal{E}_i \to \mathcal{M}(L_0, L; i, j; para)^{+}$, and $\pi^* Hom(\mathcal{E}_i, \mathcal{E}_j)$. Then

$$(\varphi, v, \sigma) \mapsto \exp(2\pi\sqrt{-1}B(\varphi))H(\varphi; \sigma) \otimes T^{E(\varphi)} \tag{5.7}$$

is a section of $\pi^* Hom(\mathcal{E}_i, \mathcal{E}_j) \otimes \Lambda_{nov}$. We use this weight function and the fundamental chain of $\mathcal{M}(L_0, L; i, j; para)^{+}$ to "define"

$$\mathfrak{B}^k_{i,j} \in W^{-\infty}\left(V; Hom(\mathcal{E}_i, \mathcal{E}_j) \otimes \Lambda^k(V) \otimes \Lambda_{nov}\right). \tag{5.8}$$

Here $\Lambda^k(V)$ is a vector bundle of degree k forms, and $W^{-\infty}$ is the space of distribution valued sections. Let us explain the definition of $\mathfrak{B}^k_{i,j}$.

Let Ψ be a smooth $n-k$ form of compact support with value in the dual bundle $Hom(\mathcal{E}_i, \mathcal{E}_j)^*$ to $Hom(\mathcal{E}_i, \mathcal{E}_j)$. ($\Psi$ is a test function.) We then define

$$\begin{aligned}\int_V \langle \mathfrak{B}_{i,j;\beta} \wedge \Psi \rangle = \int_{\mathcal{M}(L_0,L;i,j;para;\beta)^+} \\ \langle \exp(2\pi\sqrt{-1}B(\varphi))H(\varphi;\sigma), \pi^*\Psi \rangle\end{aligned} \tag{5.9}$$

Note $E(\varphi)$ depends only of β and v where v parametrize Lagrangian submanifold $L(v)$. So we write $T^{E(\beta,v)}$ in place of $L(\varphi)$. Then

$$\mathfrak{B}^k_{i,j} \text{``}=\text{''} \sum_\beta \mathfrak{B}_{i,j;\beta} \otimes T^{E(\beta,v)}. \tag{5.10}$$

Here the sum is taken over all β such that the virtual dimension of $\mathcal{M}(L_0, L; i, j; para; \beta)^{+}$ is $\dim V - k$.

Note however that the function space such as $W^{-\infty}\left(V; Hom(\mathcal{E}_i, \mathcal{E}_j) \otimes \Lambda^k(V) \otimes \Lambda_{nov}\right)$ is not easy to define, since Λ_{nov} is an infinite dimensional vector space. So we need some work to make sense (5.10). We do not discuss this point here. This is the (only) reason we put "define" in (5.8) and = in (5.10), in the quote.

Roughly speaking, $\mathfrak{B}^k_{i,j}$ is dual to the current $\pi_*(\mathcal{M}(L_0,L;i,j;para)^+)$ times the weight (5.7). We here remark that the moduli space $\mathcal{M}(L_0,L;i,j;para)^+$ may be noncompact, hence it does not have a fundamental class. (For example there is a case where the moduli space is codimension one and the image $\pi(\mathcal{M}(L_0,L;i,j;para;\beta)^+)$ is dense in V. Such an example is observed in [Fu4].) However, for each C, the sum of moduli spaces $\mathcal{M}(L_0,L;i,j;para;\beta)^+$ for $E(\beta,v) < C$ is compact by Gromov compactness. Hence we may expect that (5.10) makes sense as an asymptotic expansion.

Now we divide "k form" $\mathfrak{B}^k_{i,j}$ into the sum of $i,k-i$ forms using complex structure of V in an obvious way. Let $\mathfrak{B}^{0,k}_{i,j}$ be the $0,k$ part of $\mathfrak{B}^k_{i,j}$.

We then "obtain" a distribution coefficient multiplication operator

$$\mathfrak{B}^{0,k}_{i,j}\wedge : \bigoplus_{\ell}\Gamma(V;\mathcal{E}_i\otimes\Lambda^{0,\ell}) \to \bigoplus_{\ell} W^{-\infty}(V;\mathcal{E}_j\otimes\Lambda^{0,\ell+k}\otimes\Lambda_{nov}).$$

By summing up $\mathfrak{B}^{0,k}_{i,j}\wedge$, we "obtain"

$$\begin{aligned}\mathfrak{B}\wedge : &\bigoplus_{a+b=c}\Gamma(V;CF^a((L_0,\mathcal{L}_0),(L(v),\mathcal{L}(\sigma)))_{\mathbb{C}}\otimes\Lambda^{0,b})\\ &\to \bigoplus_{a+b=c+1} W^{-\infty}(V;CF^a((L_0,\mathcal{L}_0),(L(v),\mathcal{L}(\sigma)))_{\mathbb{C}}\otimes\Lambda^{0,b})\otimes\Lambda_{nov}.\end{aligned}$$

"Theorem 5.9".

$$\left(\overline{\partial}+\mathfrak{B}\wedge\right)\circ\left(\overline{\partial}+\mathfrak{B}\wedge\right)=0.$$

We remark that since $\mathfrak{B}$ is distribution valued (and is a kind of asymptotic series) we need to be very careful to iterate the operation $\overline{\partial}+\mathfrak{B}\wedge$ twice. This is related to the problem of transversality among the current realized by $\mathcal{M}(L_0,L;i,j;para;\beta)^+$. We can work on energy filtration on $E(\beta)$ and use the theory of Kuranishi structure in a way similar to [FOOO] Chapter 5 to settle this trouble. However we do not provide the detail here. This is one of the reasons Theorem 5.9 is in the quote.

We remark that in [Fu4] a version of "Theorem 5.9" is proved and used to show a part of homological mirror symmetry of torus.

The wall crossing mentioned in this section is related to the fact that $\mathfrak{B}^{0,k}_{i,j}$ for $k>0$ can be nonzero.

Now we come to the following convergence question.

Open problem 5.10. *B converges as a current for small T. $\overline{\partial}+\mathfrak{B}\wedge$ defines an operator*

$$\begin{aligned}\overline{\partial}+\mathfrak{B}\wedge : &\bigoplus_{a+b=c}\Gamma(V;CF^a((L_0,\mathcal{L}_0),(L(v),\mathcal{L}(\sigma)))_{\mathbb{C}}\otimes\Lambda^{0,b})\\ &\to \bigoplus_{a+b=c+1} W^{-\infty}(V;CF^a((L_0,\mathcal{L}_0),(L(v),\mathcal{L}(\sigma)))_{\mathbb{C}}\otimes\Lambda^{0,b}).\end{aligned}\tag{5.11}$$

To attach this problem directly, we need to estimate the volume of the current $\pi_*(\mathcal{M}(L_0,L;i,j;para;\beta)^+)$ when $E(\beta)$ goes to infinity. We remark that Gromov compactness implies that

$$\sum_{E(\beta)\le E}\pi_*(\mathcal{M}(L_0,L;i,j;para;\beta)^+)$$

is compact. If we can prove that

$$\sum_{E(\beta)\le E} \mathrm{Vol}(\pi_*(\mathcal{M}(L_0,L;i,j;para;\beta)^+)) < C\exp(CE) \tag{5.12}$$

for some constants C, then $\mathfrak{B}$ will converge as a current for small T. Proving (5.12) is an "open string version" of the problem of the convergence of Gromov-Witten potential. This problem is very hard to solve directly.

On the other hand, in the case of affine Lagrangian submanifold in symplectic torus, we know that the left hand side of (5.12) is of polynomial order in E. (See [Fu 4] §10.) Hence $\mathfrak{B}$ converges for arbitrary T. We checked it in [Fu 4] based on the explicit description of the moduli space. In the case of Gromov-Witten potential, it converges for all the examples where Gromov-Witten invariants are calculated. However so far no proof of convergence, which is not based on explicit calculation of Gromov-Witten invariant, is known.

In [KS], Kontsevitch-Soibelman proposed to use rigit analytic geometry [Ber], [B-GR] and homological mirror symmetry itself to prove convergence. In our context, their proposal may be reinterpreted as follows. First, we construct an infinitesimal family of Floer homologies as a formal power series of T. We next show that it will be a local family, where local means that in $\mathcal{LAG}(M,\Omega)_{qm}$ coordinate it is well defined on an open set which is small but is of nonzero size, (it may only be a formal power series with respect to T). We next might glue them in the $\mathcal{LAG}(M,\Omega)_{qm}$ direction. (See next section for some idea how to glue them.) We next may compactify $\mathcal{LAG}(M,\Omega)_{qm}$. We then would have an object of derived category of coherent sheaves of a proper scheme on formal power series ring, Λ_{nov}. Then one might expect the convergence with respect to T would be implied by the GAGA of formal scheme and/or Artin type approximation theorem. (Here rigid analytic geometry might be applied.) Since the author is unable to realize this part of project at the time of writing this article, he stops talking on Open problem 5.10 here.

We finally explain how we will have a local family of Floer homologies as an object of derived category, provided Open problem 5.10 is settled.

One further trouble to do so is the fact that $\mathfrak{B}$ is singular (and is not a smooth form). However we can settle this point as follows. We can construct a family of *smooth* forms $\mathfrak{B}_{i,j;\beta;smooth}$ such that if we define $\mathfrak{B}_{smooth}$ in the same way as $\mathfrak{B}$, by using $\mathfrak{B}_{i,j;\beta;smooth}$ in place of $\mathfrak{B}_{i,j}$, then we have

$$(\overline{\partial} + \mathfrak{B}_{smooth}) \circ (\overline{\partial} + \mathfrak{B}_{smooth}) = 0$$

and that there is a (distribution valued) chain homotpy equivalence from $\overline{\partial} + \mathfrak{B}$ to $\overline{\partial} + \mathfrak{B}_{smooth}$. We can construct such smoothing of $\mathfrak{B}_{i,j;\beta}$ and chain homotopy equivalence, by working on induction on $E(\beta)$. (See [Fu4] Chapter 4 where a similar construction is used.) Then we have a complex of sheaves

$$\begin{aligned} \overline{\partial} + \mathfrak{B}_{smooth}\wedge : &\bigoplus_{a+b=c} \Gamma(V; CF^a((L_0,\mathcal{L}_0),(L(v),\mathcal{L}(\sigma)))_{\mathbb{C}} \otimes \Lambda^{0,b}) \\ &\to \bigoplus_{a+b=c+1} \Gamma(V; CF^a((L_0,\mathcal{L}_0),(L(v),\mathcal{L}(\sigma)))_{\mathbb{C}} \otimes \Lambda^{0,b}). \end{aligned} \tag{5.13}$$

($c = 0, 1, \cdots$.) Note that (5.13) is a complex of $\mathcal{O}_V$ module sheaves. We also remark that the complex (5.13) is elliptic and the symbol is the same as Dolbaut complex with coefficient in vector bundles $CF((L_0,\mathcal{L}_0),(L(v),\mathcal{L}(\sigma)))_{\mathbb{C}}$. Thus, by a standard elliptic theory, the cohomology sheaf is coherent. Hence (5.13) will define an object of derived category of coherent sheaves.

6. Gluing objects of derived category

The discussion of sections 4,5 are local. There, modulo very serious convergence problem, we gave an outline of the construction of the local family of Floer cohomologies. In this section, we discuss a way to glue them. To be specific, we consider the following situation. Let $\pi : M \to N$ be a fiber bundle. We assume that M is a symplectic manifold and the fibers are Lagrangian submanifolds. For simplicity, we put $B = 0$. For $v \in N$ we put $L_v = \pi^{-1}(v)$. According to Strominger-Yau-Zaslow [SYZ], the mirror of M then is

$$M^\wedge = \{(v,\sigma) \mid v \in N, \sigma \in R(L_v)\} \tag{6.1}$$

where $R(L_v)$ is a moduli space of gauge equivalence class of flat $U(1)$ connections on the trivial bundle on L_v. In case there is a singular fiber L_v, the discussion will be harder. Namely we need to consider quantum effect which modify the complex structure of $M^\wedge \subseteq \mathcal{LAG}(M,\omega)_{cl}$. This is the point (F) mentioned in section 1. We do not discuss it here. So it seems that the only case we can apply our argument directly is $M = T^{2n}$. However the argument we present here may be a big part of the building blocks to handle the general case.

We next assume that there is a global section $s : N \to M$ whose image is a Lagrangian submanifold, L_{st}. Let us take another Lagrangian submanifold L_0. We consider $(L_0,\mathcal{L}_0) \in \mathcal{LAG}(M,\omega)$. We fix also $b_0 \in \tilde{\mathcal{M}}(L_0,\mathcal{L}_0)$. Our purpose here is to consider the family of Floer homology

$$\mathcal{E}(L_0,\mathcal{L}_0,b_0)_{(v,\sigma)} = HF((L_0,\mathcal{L}_0,b_0),(L(v),\mathcal{L}(\sigma))) \tag{6.2}$$

where $(v,\sigma) \in M^\wedge$. As in the last section, we are assuming that there is no nontrivial pseudoholomorphic disks bounding $L(v)$.

The main reason the author is interested in constructing (6.2) is

Conjecture 6.3. The mirror object of $(L_0,\mathcal{L}_0,b_0)$ on $M^\wedge$ is an object of the derived category of coherent sheaves on $M^\wedge$ given as the family of Floer cohomologies (6.2).

This conjecture for example implies

$$\mathrm{Ext}(\mathcal{E}(L_0,\mathcal{L}_0,b_0),\mathcal{E}(L_1,\mathcal{L}_1,b_1)) \cong HF((L_0,\mathcal{L}_0,b_0),(L_1,\mathcal{L}_1,b_1)). \tag{6.4}$$

In the case of affine Lagrangian submanifold in symplectic torus, the isomorphism, (6.4) which is functorial, is established in [Fu4], generalizing [Ko2],[PZ] which discussed the case of elliptic curve.

In sections 4,5, we discussed a construction of $\mathcal{E}(L_0,\mathcal{L}_0,b_0)_{(v,\sigma)}$ on a small open set of $\mathcal{LAG}(M,\omega)$. In this section, we discuss how to glue them (in case we can solve the trouble on convergence.) We remark that, we assumed that L_0 is transversal to $L(v)$ in §4,5. This assumption is not so restrictive in case our construction is local on v. However, for the purpose of this section, this assumption is too restrictive. Namely the assumption that $L(v)$ is always transversal to L_0 means that L_0 is a finite cover of N and $\pi : L_0 \to N$ is a covering map. There are many other Lagrangian submanifolds which do not satisfy this assumption even in the case of symplectic torus. (On the other hand in the case of symplectic torus the author does not know any example of L_0 transversal to the fibers other than affine Lagrangian submanifolds.)

Example 6.5[4][5]. Let us consider symplectic (T^4, ω). Note that T^4 is hyper Kähler. So complex submanifold with respect to a complex structure I will become a Lagrangian submanifold of (T^4, ω), where ω is a Kähler form of another complex structure J. We assume that (T^4, I) is an Abelian surface. Then there are lots of complex submanifolds Σ of higher genus in (T^4, I). We may take Lagrangian fibration $(T^4, \omega) \to T^2$. Then $\Sigma \to T^2$ is a branched covering. At the branch point $v \in T^2$, the fiber L_v is not transversal to Σ.

The following lemma is easy to prove.

Lemma 6.6. *There are finitely many* $(L_{0,i}, \mathcal{L}_{0,i}) \in \mathcal{LAG}(M, \omega)$ *and there exists a finite open covering* $\cup V_i = N$, *such that* $(L_{0,i}, \mathcal{L}_{0,i}) \sim (L_0, \mathcal{L}_0)$ *and that* $L_{0,i}$ *is transversal to* $L(v)$ *for* $v \in V_i$.

We put $M_i^\wedge = \pi^{-1}(V_i) \subset M^\wedge$. If the construction of sections 4,5 works we will have a family of Floer homologies

$$HF((L_{0,i}, \mathcal{L}_{0,i}, b_i), (L(v), \mathcal{L}(\sigma))) \to M_i^\wedge. \tag{6.7}$$

Thus our problem is as follows.

Problem 6.8. Let X be a complex manifold and $X = \cup_i X_i$ is an open covering. Let $\mathfrak{E}_i$ be an object of derived category of coherent sheaves on X_i. Find a sufficient condition for the existence of $\mathfrak{E}$ such that $\mathfrak{E}$ is an object of derived category of coherent sheaves on X and that the restriction of $\mathfrak{E}$ to X_i is chain homotopy equivalent to $\mathfrak{E}_i$.

In the rest of this section, we give a sufficient condition and offer an argument that (6.7) satisfies this condition. (The part that our condition is enough to give $\mathfrak{E}$ is rigorous. But since the construction of (6.7) still contains a gap (the convergence question) the part to show (6.7) satisfies our sufficient condition is not rigorous.)

Let us first consider the case when $X = X_1 \cup X_2$. In this case, our sufficient condition is that $\mathfrak{E}_1|_{X_1 \cap X_2}$ is chain homotopy equivalent to $\mathfrak{E}_2|_{X_1 \cap X_2}$. In fact, let

$$\varphi_{12} : \mathfrak{E}_1|_{X_1 \cap X_2} \to \mathfrak{E}_2|_{X_1 \cap X_2}$$

be the chain homotopy equivalence. We construct its mapping cylinder $Cyl(\mathfrak{E}_1, \mathfrak{E}_2; \varphi_{12})$ as follows.

$$Cyl(\mathfrak{E}_1, \mathfrak{E}_2; \varphi_{12}) = \mathfrak{E}_1|_{X_1 \cap X_2} \oplus \mathfrak{E}_2|_{X_1 \cap X_2} \oplus \mathfrak{E}_1|_{X_1 \cap X_2}[1]$$

[4]The author thanks N.C. Leung who pointed out this example to the author during the conference "Algebraic geometry and Integrable system related to String theory".

[5]In the conference (of which this volume is a proceeding), the author discussed this example and mention how pseudoholomorphic disks will bifurcate at the singular fiber. The detail of the discussion there will appear elsewhere. The approach taken in this section is different from the approach discussed in the conference.

as graded $\mathcal{O}_{X_1\cap X_2}$ module. We twist the differential and define

$$\delta(x_1,x_2,x_{12}) = \left(\delta x_1 + (-1)^{\deg x_{12}}x_{12}, \delta x_2 + (-1)^{\deg x_{12}}\varphi_{12}(x_{12}), \delta(x_{12})\right).$$

Here δ in the right hand side are boundary operators of $\mathfrak{E}_1, \mathfrak{E}_2$ respectively.

Now we define $\mathfrak{E}$ so that its germ $\mathfrak{E}_p$ is obtained by

$$\mathfrak{E}_p = \begin{cases} \mathfrak{E}_{1,p} & \text{if } p \in X_1 - X_1\cap X_2 \\ \mathfrak{E}_{2,p} & \text{if } p \in X_2 - X_1\cap X_2 \\ Cyl(\mathfrak{E}_{1,p},\mathfrak{E}_{2,p};\varphi_{12}) & \text{if } p \in X_1\cap X_2. \end{cases}$$

It is easy to see that $\mathfrak{E}$ has required properties.

In case when $X = X_1\cup X_2\cup X_3$, we need first to assume the existence of φ_{12},φ_{23},φ_{13}. (Here $\varphi_{ij} : \mathfrak{E}_i\vert_{X_i\cap X_j} \to \mathfrak{E}_j\vert_{X_i\cap X_j}$ is a chain homotoy equivalence.) We need moreover that $\varphi_{23}\circ\varphi_{12}$ is chain homotopic to φ_{13} on $X_1\cap X_2\cap X_3$.

Let us do it more systematically. We need to use A_∞ structure for this purpose.

To describe our sufficient condition, we need some notations. Let (C_i,δ_i), $i = 1,\cdots,m$ be chain complex. We assume that we are given degree zero homomorphisms

$$\varphi_{i_1,\cdots,i_k} : C_{i_1} \to C_{i_k}[2-k]$$

for each $1 \le i_1 < \cdots < i_k \le m$.

Condition 6.9.

(6.9.1) $\varphi_i = \delta_i$, the boundary operators of C_i.

(6.9.2) $\varphi_{ij} : C_i \to C_j$ is a chain homotopy equivalence.

(6.9.3) For each $1 \le i_1 < \cdots < i_k \le m$, we have :

$$\sum_{\ell=1}^{k} \varphi_{i_\ell,\cdots,i_k}\circ\varphi_{i_1,\cdots,i_{\ell-1}} + \sum_{\ell=2}^{k-1} \pm\varphi_{i_1,\cdots,\hat{i}_\ell,\cdots,i_k} = 0.$$

We specify the sign in (6.9.3) later.

For example, in case $k = 3$, Condition 6.9 implies

$$\delta_3\circ\varphi_{123} \pm \varphi_{123}\circ\delta_3 \pm \varphi_{23}\circ\varphi_{12} \pm \varphi_{13} = 0.$$

Namely $\varphi_{23}\circ\varphi_{12}$ is chain homotopic to φ_{13} with chain homotopy φ_{123}.

We are going to construct a *multi-mapping cylinder* $Cyl(C_1,\cdots,C_m;\varphi_*)$ under assumption (6.9.1) (6.9.3). (We do not assume (6.9.2) yet.)

We put

$$C_{i_1,\cdots,i_k} = C_{i_1}[k-1], \quad Cyl(C_1,\cdots,C_m;\varphi_*) = \bigoplus_{1\le i_1<\cdots<i_k\le m} C_{i_1,\cdots,i_k},$$

as an Abelian group. We define its boundary operator δ as follows. Let $x \in C_{i_1,\cdots,i_k}$ then

$$\delta(x) = \sum_\ell \delta_{i_1,\cdots,i_\ell}(x) + \sum_\ell \delta'_{i_1,\cdots,\hat{i}_\ell,\cdots,i_k}(x),$$

(6.10.1) $$\delta_{i_1,\cdots,i_\ell}(x) = \pm\varphi_{i_1,\cdots,i_\ell}(x) \in C_{i_\ell,\cdots,i_k}.$$

(6.10.2) $$\delta'_{i_1,\cdots,\hat{i}_\ell,\cdots,i_k}(x) = \pm x \in C_{i_1,\cdots,\hat{i}_\ell,\cdots,i_k}.$$

We specify the signs in (6.10.1) and (6.10.2) later.

It is easy to see that the degree of δ is 1. One can easily see

Lemma 6.11. *Condition (6.9.3) is equivalent to* $\delta\delta = 0$.

Actually multi-mapping cylinder is obtained by repeating the construction of mapping cylinder.

Lemma 6.12. *There exists a chain map*

$$\Psi : Cyl(C_1, \cdots, C_{m-1}; \varphi_*) \to C_m$$

such that $Cyl(C_1, \cdots, C_m; \varphi_*)$ *is isomorphic to a mapping cylinder of* Ψ.

Proof. Let $x \in C_{i_1,\cdots,i_k} \subset Cyl(C_1, \cdots, C_{m-1}; \varphi_*)$. We put

$$\Psi(x) = \varphi_{i_1,\cdots,i_k,m}(x).$$

We can check that (6.9.3) implies that Ψ is a chain map. It is easy to see from definition that $Cyl(C_1, \cdots, C_m; \varphi_*)$ is isomorphic to a mapping cylinder of Ψ. □

Here we fix sign in (6.9.3), (6.10.1), (6.10.2). We have proved Lemma 6.11 and Lemma 6.12 up to sign. It is easy to see that there is a unique sign convention so that Lemma 6.12 holds together with sign. (Note the sign of the boundary operator in the mapping cylinder is well established.) We thus fixed the sign. We do not try to calculate the sign explicitly since we do not need it in this article.

Now Lemma 6.12 implies the following. We define $I_i : C_i \to Cyl(C_1, \cdots, C_m; \varphi_*)$ in an obvious way. (C_i is a component of $Cyl(C_1, \cdots, C_m; \varphi_*)$.)

Lemma 6.13. $I_m : C_m \to Cyl(C_1, \cdots, C_m; \varphi_*)$ *is a chain homotopy equivalence. The composition* $I_j \circ \varphi_{ij}$ *is chain homotopic to* I_i. I_i *is a chain homotopy equivalence if (6.9.2) is satisfied.*

Proof. The first assertion is immediate from Lemma 6.12 and well known fact on mapping cylinder. We put $H(x) = x \in C_{ij} \subset Cyl(C_1, \cdots, C_m; \varphi_*)$ for $x \in C_i$. Then we find easily that

$$\delta \circ H \pm H \circ \delta = I_i \pm I_j \circ \varphi_{ij}.$$

The second statement follows. The last statement is immediate from first two. □

We can generalize Lemma 6.13 as follows. Let $1 \le i_1 < \cdots < i_k \le m$. We construct multi-mappingcylinder $Cyl(C_{i_1}, \cdots, C_{i_k}; \varphi_*)$. Since $Cyl(C_{i_1}, \cdots, C_{i_k}; \varphi_*)$ is a sum of direct summands of $Cyl(C_1, \cdots, C_m; \varphi_*)$ there exists a chain map

$$I_{i_1,\cdots,i_k} : Cyl(C_{i_1}, \cdots, C_{i_k}; \varphi_*) \to Cyl(C_1, \cdots, C_m; \varphi_*). \tag{6.14}$$

Lemma 6.13 implies that $I_{i_1,\cdots,i_k}$ is a chain homotopy equivalence if (6.9.2) is satisfied. Moreover the compositions of these chain maps are compatible.

Now the discussion so far implies the following :

Theorem 6.15. *Let* $X = \cup X_i$ *is a open covering of a complex manifold. We put* $X_{i_1,\cdots,i_k} = \cap_{j=1}^k X_{i_j}$. *Let* $\mathfrak{E}_i$ *is a complexes of coherent sheaves on* X_i. *We assume that, for each* $1 \le i_1 < \cdots < i_k \le m$, *there exists a homomorphism*

$$\varphi_{i_1,\cdots,i_k} : \mathfrak{E}_{i_1}|_{X_{i_1,\cdots,i_k}} \to \mathfrak{E}_{i_k}|_{X_{i_1,\cdots,i_k}}[2-k]$$

of coherent sheaves. We assume that they satisfy Condition 6.9. Then there exists a complex of coherent sheaves $\mathfrak{E}$ on X and a chain homotopy equivalence $\mathfrak{I}_i : \mathfrak{E}_i \to \mathfrak{E}|_{X_i}$ such that $\mathfrak{I}_j \circ \varphi_{ij}$ is chain homotopic to $\mathfrak{I}_i$ on $X_i \cap X_j$.

Proof. We define

$$\mathfrak{E}_p = Cyl(\mathfrak{E}_{i_1}, \cdots, \mathfrak{E}_{i_k}; \varphi_*)_p \tag{6.16}$$

if $p \in X_{i_1,\cdots,i_k} - \cup_{\ell \neq i_1,\cdots,i_k} X_{i_1,\cdots,i_k} \cap X_\ell$. We can show easily that (6.15) satisfy the required properties by using Lemmas 6.11, 6.12, 6.13 and the compatibility of (6.14). □

Thus we obtained an answer to Problem 6.8.

We next explain why we expect the family of Floer cohomologies to satisfy the assumption of Theorem 6.15. Let us go back to the situation we mentioned at the beginning of this section. We are going to construct a degree preserving homomorphism of $\mathcal{O}_{M^\wedge}$ module sheaves

$$\begin{aligned} \varphi_{i_1,\cdots,i_k} :& HF((L_{0,i_1}, \mathcal{L}_{0,i_1}, b_{i_1}), (L(v), \mathcal{L}(\sigma))) \\ &\to HF((L_{0,i_k}, \mathcal{L}_{0,i_k}, b_{i_k}), (L(v), \mathcal{L}(\sigma)))[2-k]. \end{aligned} \tag{6.17}$$

The idea of the construction is as follows. It is proved in [FOOO] Chapter 4 that Floer cohomology is invariant of Hamiltonian isotopy. This gives φ_{ij}. In a way similar to the argument there, we can show that homotopy between Hamiltonian isotopies defines a homotopy between chain homotopies. One can do a similar construction for higher homotopy between homotopies, which proves that $\varphi_{i_1,\cdots,i_k}$ satisfying Condition 6.9 exists.

Let us carry out this construction below. We first remark that we may take our Lagrangian submanifolds $L_{0,i}$ so that they are close to each other. We also remark that the set of Lagrangian submanifold (plus other parameters, for example perturbation, which we do not specify in the article) is locally contractible. Hence we have a family of Lagrangian submanifolds which contains $L_{0,i}$. More precisely we have the following Lemma 6.16. Let Δ^{m-1} be the $m-1$ dimensional simplex and let $\mathfrak{v}_1, \cdots, \mathfrak{v}_m$ be its vertices.

Lemma 6.18. *There exists a smooth map $\Phi : \Delta^{m-1} \times L_0 \to M$ and a complex line bundle $\mathcal{L}_0 \to \Delta^{m-1} \times L_0$ equipped with a connection, with the following properties.*

(6.19.1) *For each $x \in \Delta^{m-1}$, the restriction of Φ to $\{x\} \times L_0$ is a Lagrangian embedding.*

(6.19.2) *For each path $x(t)$ in Δ^{m-1}, the family of Lagrangian embeddings $\Phi|_{\{x(t)\}\times L_0}$ is a Hamiltonian isotopy.*

(6.19.3) *The curvature of $\mathcal{L}_0$ is $2\pi\sqrt{-1}\Phi^* B$.*

(6.19.4) *The restriction of Φ and $\mathcal{L}_0$ to $\{\mathfrak{v}_i\} \times L_0$ coincides with $(L_{0,i_1}, \mathcal{L}_{0,i_1})$.*

The proof is easy and is omitted. We use this family to construct (6.17). We need one more lemma for this purpose. Let $\Omega(\mathfrak{v}_i, \mathfrak{v}_j; \Delta^{m-1})$ be the set of all smooth paths in Δ^{m-1} joining $\mathfrak{v}_i$ to $\mathfrak{v}_j$. Let $\Delta^{k-1}_{i_1,\cdots,i_k}$ be the $k-1$ simplex whose vertices are $\mathfrak{v}_{i_1}, \cdots, \mathfrak{v}_{i_k}$. By composing the path we obtain

$$\Omega(\mathfrak{v}_1, \mathfrak{v}_\ell; \Delta^{\ell-1}_{1,\cdots,\ell}) \times \Omega(\mathfrak{v}_\ell, \mathfrak{v}_m; \Delta^{m-\ell}_{\ell,\cdots,m}) \to \Omega(\mathfrak{v}_1, \mathfrak{v}_m; \Delta^{m-1}). \tag{6.20}$$

(To be precise we need to discuss the way to put parameter for the path joining two paths. We omit it since the way to handle it is classical in the context of A_∞ structure. (See for example [Ad].))

Lemma 6.21. *There exists $m-1$ dimensional submanifold $W(\mathfrak{v}_1,\mathfrak{v}_m;\Delta^{m-1})$ with corners in $\Omega(\mathfrak{v}_1,\mathfrak{v}_m;\Delta^{m-1})$ such that the following equality holds (using inclusion (6.20)).*

$$\begin{aligned}\partial W(\mathfrak{v}_1,\mathfrak{v}_m;\Delta^{m-1}) &= \bigcup_{\ell=1}^{k-1} W(\mathfrak{v}_1,\mathfrak{v}_\ell;\Delta^{\ell-1}_{1,\cdots,\ell})\times W(\mathfrak{v}_\ell,\mathfrak{v}_m;\Delta^{m-\ell}_{\ell,\cdots,m})\\ &\cup\bigcup_{\ell=1}^{k-1} W(\mathfrak{v}_1,\mathfrak{v}_m;\Delta^{m-2}_{1,\cdots,\hat{\ell},\cdots,m}).\end{aligned} \tag{6.22}$$

Proof. We can find a Morse function g_m on Δ^{m-1} with the following properties.

(6.23.1) The restriction of g_m to $\Delta^{k-1}_{i_1,\cdots,i_k}$ coincides with g_k.
(6.23.2) The critical point set of g_m is the set of vertices $\{\mathfrak{v}_1,\cdots,\mathfrak{v}_m\}$. The critical points are all nondegenerate.
(6.23.3) The Morse index of $\mathfrak{v}_i$ is $i-1$.

Let $W(\mathfrak{v}_1,\mathfrak{v}_m;\Delta^{m-1})$ be the set of all gradient lines of g_m joining $\mathfrak{v}_1$ to $\mathfrak{v}_m$. It is easy to check (6.22) by using (6.23). □

Now using the Lemma 6.21, we will "construct" the homomorphism (6.17). (The reason we put "construct" in the quote is that we can perform the construction only in the level of formal power series by the same convergence trouble as in the last section.)

Let $p(v)\in\Phi(\{\mathfrak{v}_1\}\times L_0)\cap L(v) = L_{0,1}\cap L(v)$, $q(v)\in\Phi(\{\mathfrak{v}_m\}\times L_0)\cap L(v) = L_{0,m}\cap L(v)$, $\ell\in W(\mathfrak{v}_1,\mathfrak{v}_m;\Delta^{m-1})$ ($\ell:\mathbb{R}\to\Delta^{m-1}$). We consider the maps $u:[0,1]\times\mathbb{R}\to M$ with the following properties.

Condition 6.24.
(6.25.1) u is pseudoholomorphic.
(6.25.2) $u(0,\tau)\in\Phi(\{\ell(\tau)\}\times L_0)$. $u(1,\tau)\in L_1(v)$.
(6.25.3) $\lim_{\tau\to-\infty}u(t,\tau)=p(v)$, $\lim_{\tau\to+\infty}u(t,\tau)=q(v)$.

We let $\mathcal{M}(L_0,L_1(v);p(v),q(v);\Phi;\ell)$ be the isomorphism class of such u. Let $\alpha\in Hom(\mathcal{L}_{0,p(v)},\mathcal{L}(\sigma)_{p(v)})$. We put the weight

$$\exp(2\pi\sqrt{-1}B(u))H(u;\sigma)(\alpha)\otimes T^{E(u)}. \tag{6.26}$$

Using this weight function and the moduli space

$$\bigcup_{v,\ell}\mathcal{M}(L_0,L_1(v);p(v),q(v);\Phi;\ell)\times\{(v,\ell)\}$$

in the same was as the last section, we define (6.17). By comparing Formulas (6.22) and (6.9.3) we find that $\varphi_{i_1,\cdots,i_k}$ defined in this way satisfies Condition 6.9. Hence we can apply Theorem 6.15, if we can prove the convergence of the operators.

Thus the biggest problem left to construct family of Floer cohomologies is the convergence, Question 5.10.

To be continued somewhere someday.

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