Floer homology for 3 manifold with boundary I

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Contents

§ 1 Introduction

The purpose of this series of papers is to define and study Floer homology of 3 manifold with boundary. In this paper, we announce the main analytic results and show the basic constructions based on them. The proof of the analytic results will be given in subsequent papers.
Our definition of Floer homology of 3 manifold with boundary gives an extension of topological field theory to manifolds with corners. (In usual topological field theory, manifolds with boundary are studied.)
So we start with briefly recalling Atiyah's axiom of topological field theory [2] and its generalization proposed by G. Segal [34]. The argument of this section is rather vague and heuristic since its main purpose is to describe naive ideas behind the constructions of later sections.
In topological field theory, we associate a number $Z(M)$ to each oriented closed manifold M of dimension n. Also, to each oriented manifold N of dimension $n-1$, we associate a vector space $H(N)$. For each compact oriented manifold M of dimension n, with boundary $\partial M = N$, we associate $Z(M) \in H(N)$. The axioms they are supposed to satisfy are the following.

(1.1) Each $H(N)$ has an inner product, such that $H(N)$ is a Hilbert space.
(1.2) $H(-N)$ is canonically isomorphic to the dual space of $H(N)$. Here $-N$ denotes the manifold N equipped with opposite orientation.
(1.3) $H(N_1 \cup N_2) = H(N_1) \otimes H(N_2)$. Here $N_1 \cup N_2$ denotes the disjoint union.
(1.4) Let M_1, M_2 be n-dimensional compact oriented manifolds. Suppose that N is a connected component of ∂M_1 and $-N$ is a connected component of ∂M_2. Let M be a manifold obtained by patching M_1 and M_2 along N. If $\partial M_1 = N \cup N'$, $\partial M_2 = -N \cup N''$, then

$$\Pi(Z(M_1) \otimes Z(M_2)) = Z(M)$$

where $\Pi : \big(H(N) \otimes H(N')\big) \otimes \big(H(-N) \otimes H(N'')\big) \to H(N') \otimes H(N'')$ is obtained by contracting $H(N) \otimes H(-N) \cong H(N) \otimes H(N)^*$.
(1.5) Let M be an n-dimensional compact oriented manifold. Suppose that N and $-N$ are connected components of ∂M. Let M^+ be the manifold obtained by gluing M with itself along N. Then

$$\Pi(Z(M)) = Z(M^+)$$

where $\Pi : \big(H(N) \otimes H(-N) \otimes H(N')\big) \to H(N')$. (Here $\partial M = N \cup -N \cup N'$.)

We do not intend to list up the complete set of axioms but only mention some of them.
The case we are mainly interested in is when $n = 4$, $Z(M)$ is the Donaldson's polynomial invariant [4] and $H(N)$ is Floer homology group [7].
In that case, we need to modify the axiom above a bit. Namely the Donaldson's polynomial invariant gives a map $Sym(H_*(M)) \to \boldsymbol{Q}$ instead of a number, also Floer homology group

is a graded abelian group instead of a vector space. We do not mention necessary modification here. The rigorous construction of relative version of Donaldson's polynomial invariant is known to experts for a long time. (The full detail of the proof is available in [19].)

Our main concern in this paper is to generalize the definition of topological field theory so that it includes also manifold Σ of dimension $n-2$ and extend Donaldson/Floer theory so that it satisfies this extended axiom.

A generalization of the axiom of topological field theory we discuss below is inspired by an idea of G. Segal. (The author heard of it during Donaldson's lecture [5].)

The generalization is as follows. For each oriented closed manifold Σ of dimension $n-2$, we associate a category $C(\Sigma)$, such that the morphism between two objects of $C(\Sigma)$ is a vector space. If N is a compact oriented $n-1$ dimensional manifold such that $\partial N = \Sigma$, then we associate an objects $H(N)$ of the category $C(\Sigma)$. Then the axiom is

(1.6) Let $\partial N_1 = \Sigma$, $\partial N_2 = -\Sigma$, and N be a closed oriented $n-1$ manifold, obtained by gluing N_1 and N_2 along Σ. Then

$$H(N) = Hom_{C(\Sigma)}(H(N_1), H(N_2)).$$

We might consider also the case that $\partial N_1 = \Sigma \cup \Sigma'$, $\partial N_2 = -\Sigma \cup \Sigma''$, but we do not try to do it. Also the case $\partial N = \Sigma \cup -\Sigma \cup \Sigma'$ might be discussed in a similar way to (1.5).

The axiom which corresponds to (1.2) is that $C(-\Sigma) = C(\Sigma)^o$, where C^o denote the opposite category, that is the category with the direction of arrows reversed.

It seems that there is an example of system satisfying similar axioms in the case when $n = 3$, based on conformal field theory and Witten invariant of 3 manifolds [36]. In that case, probably, $C(S^1)$ is a category of representation of an affine Lie algebra. It seems that this is the example Segal had in his mind. (Segal also mentioned a possibility to discuss Atiyah-Singer type index theory (of linear elliptic operator) for manifolds with corners under a similar frame work.) However there seems to be no reference so far constructing them rigorously.

Our main purpose is to find such a system (with some modification) based on gauge theory (or Donaldson/Floer theory) in the case $n = 4$.

A first candidate (more precisely the first approximation) of such a construction is suggested by Donaldson [5]. His suggestion is as follows. Let Σ be a surface. Then, the space

$$R(\Sigma) = \frac{\{\varphi : \pi_1(M) \to SU(2) \mid \text{homomorphism}\}}{SU(2)}$$

has a natural symplectic structure (Goldman [24]). (Here $SU(2)$ acts on the space of homomorphisms by conjugation.) The space $R(\Sigma)$ is singular. This fact causes a serious trouble for rigorous construction. So, in later sections, we replace it by the set of flat connections of a nontrivial $SO(3)$ bundle. In this section, we do not concern with technical point so we ignore the trouble caused by the singularity of $R(\Sigma)$.

We next consider the category $C_0(\Sigma)$ such that the objects of $C_0(\Sigma)$ is a Lagrangian submanifold (with some additional condition, which we specify later.) The morphism between two objects of $C_0(\Sigma)$ is, by definition, the Floer homology group of Lagrangian intersection.

Here we recall that Floer [8] defined a homology group $HF(L_1,L_2)$ for a pair of Lagrangian submanifolds L_1,L_2 (satisfying some additional assumptions which we do not mention here). Floer's construction is generalized by Oh [32] . Also it is pointed out in [32] , that we need several assumptions L_1,L_2 for $HF(L_1,L_2)$ to be well defined.

Oh did not discuss orientation problem and, as a consequence, his Floer homology group of Lagrangian intersection is with $\boldsymbol{Z}_2$ coefficient. The author is planning to discuss orientation problem in a forthcoming joint paper with H.Ohta and K.Ono . In this section, we do not mention it and in later sections we work with $\boldsymbol{Z}_2$ coefficient for preciseness.

Now suppose that there are three Lagrangian submanifolds L_1,L_2,L_3. Donaldson [5] suggested a map

(1.7) $$HF(L_1,L_2)\otimes HF(L_2,L_3)\to HF(L_1,L_3).$$

We will discuss it in § 2. (1.7) will be the composition of the morphisms in our category $C_0(\Sigma)$. (We write $C_0(\Sigma)=\mathcal{L}ag(R(\Sigma))$ in later sections.)

Donaldson proposed this category as the first approximation to the required one. Axiom (1.6) then is related to so called Atiyah-Floer conjecture, which we discuss here very briefly. Let N be a handle body. We consider $R(N)$, the set of conjugacy classes of homomorphisms from $\pi_1(N)$ to $SU(2)$. There is a map $res: R(N)\to R(\Sigma)$ defied by restriction. It is well known that this map is injective and the image $res(R(N))$ is a (singular) Lagrangian submanifold. Then one can try to put $H(N)=res(R(N))$. (We recall that $H(N)$ is supposed to be an object of the category $C_0(\partial N)$ and the object of $C_0(\Sigma)$ is a Lagrangian submanifold of $R(\Sigma)$.) Axiom (1.6) is then exactly Atiyah-Floer conjecture.

There are several papers (Yoshida 1992 [37], Lee-Lie 1995 [29]) which announce the proof of the Atiyah-Floer conjecture. The author does not concern with Atiyah-Floer conjecture directly in this paper. However this paper is closely related to it and its generalization.

As we mentioned before, the category $C_0(\Sigma)$ whose object is a Lagrangian submanifold is a first approximation to but is not itself the category we are looking for. The reason is as follows. For example, let us consider the case when $\Sigma=S^2$. Then, the set $R(S^2)$ is a one point. Hence the category $C_0(\Sigma)$ is trivial. On the other hand, to give a 3 manifold N with $\partial N=S^2$ is equivalent to give a closed 3 manifold. So in the case when $\Sigma=S^2$ the axiom (1.6) should give the description of Floer homology of connected sum of two closed 3 manifolds. Such a description is given in [20], [30]. From them, it is obvious that we need more information to describe Floer homology of $N_1\# N_2$ than "Lagrangian submanifold" of $R(S^2)=\text{point}$. There is a similar phenomenon in the case when $\Sigma=T^2$ with nontrivial $SO(3)$ bundle. See [3].

So the object of the category $C(\Sigma)$ we are looking for, should be a kind of mixture of Lagrangian submanifold of $R(\Sigma)$ and a chain complex. Our purpose in this paper is to define such an object and use it to define Floer homology of 3 manifolds with boundary.

Another trouble is that the restriction map $R(N)\to R(\Sigma)$ is in general an immersion and is not an embedding even after perturbation, and Floer homology is not well defined for immersed Lagrangian submanifolds.

The idea to define a category whose objects are a kind of mixtures of Lagrangian submanifold of $R(\Sigma)$ and a chain complex, is somewhat similar to the construction of K (Grotentick) group or group completion. Namely we consider the category $\mathcal{Ch}$ whose objects is a chain complex. We then consider the set of all functors : $C_0(\Sigma) \to \mathcal{Ch}$. This set can be regarded as the set of objects of some category $C(\Sigma) = \mathcal{Func}(\mathcal{Lag}(R(\Sigma), \mathcal{Ch})$.

For example since $R(S^2)$ is trivial, $C_0(S^2)$ is a trivial category. So $\mathcal{Func}(\mathcal{Lag}(R(\Sigma)), \mathcal{Ch}) \cong \mathcal{Ch}$. Therefore, relative invariant of 3-manifolds N with $\partial N = S^2$ is a chain complex (or its homology group). This is consistent with the fact that to give a 3 manifold N with $\partial N = S^2$ is equivalent to give a closed 3 manifold.

On the other hand, if L is an object of $C_0(\Sigma)$, (namely if L is a Lagrangian submanifold of $R(\Sigma)$), then we obtain an element of $\mathcal{Func}(C_0(\Sigma), \mathcal{Ch})$, which is a functor represented by L (§2). Thus, in the case of handle body N, the relative invariant is a functor represented by the Lagrangian submanifold $res(R(N)) \subseteq R(\partial N)$. An analogue of Yoneda's lemma (§ 12, 13) implies that the set of morphisms between two functors represented by Lagrangian submanifolds L_1 and L_2 is homotopy equivalent to $HF(L_1, L_2)$. Hence Atiyah-Floer conjecture will be Axiom (1.6).

To carry out this project, we need various kinds of results, one algebraic and geometric and two analytic.

One of the analytic results we need is one about ASD (Anti-Self-Adjoint) equation on 4 manifold with corners (or equivalently on 4 manifold with product ends and with boundaries which are diffeomorphic to the product of surfaces and $\boldsymbol{R}$.) The equation and boundary condition we study will be discussed in § 3. The construction and the study of the basic properties of the moduli space of the solutions of this equation is stated in this paper. Some part of it was written and distributed by the author in [13]. The complete proof of those results stated in this paper will appear in subsequence papers.

The second analytic result we need is one about the moduli space of pseudo holomorphic disks with Lagrangian boundary condition. In the case when there are two or less Lagrangian submanifolds, such a problem is studied by Gromov [25], Floer [8] and Oh [32]. In the case, when there are 3 or more Lagrangian submanifolds, we started its study in [22]. In fact one of the most basic problems which did not appear in the case of 1 or 2 Lagrangian submanifolds was already solved in [22] (see § 13 where they are applied). We announce in this paper, the results on the moduli space of pseudo holomorphic disks we need in this paper and prove it in subsequent papers.

We need also homological algebra to work out the story we mentioned above. The category $C_0(\Sigma)$ we are going to construct is in fact not a category in the usual sense, since the composition of the morphisms are not associative but is associative only modulo chain homotopy. (Also the identity map exists only in a modified sense.) We use the notion A^∞ category and topological A^∞ category introduced by the author [13], [15] for this purpose.

We need several algebraic arguments to establish basic properties we need to use A^∞category for our purpose. We present it in Part II of this paper.

It seems very interesting to the author that those formal and rather complicated formulas in homological algebra are directly related to the moduli space of ASD equation and/or pseudo holomorphic disks (and its compactification). One of the main purpose of this paper is to pursuit this analogy between algebra and analysis.

Donaldson [4] started to study this analogy by using cobordism argument to define an invariant. The cobordism he used is the moduli space of ASD equations.

Floer went one step further by defining homology group rather than a number. He used moduli space to define boundary operator and also to show various equalities basic for construction.

On the other hand, physicists (like Witten [35]) had a similar idea. They used degeneration of Riemann surface, or, in other words, compactification of the moduli space of pseudo holomorphic curves, to derive associativity law of quantum cohomology. (This idea is now a rigorous mathematics.)

In this paper, we go one more step further. In place of relating analysis to the algebraic construction which already exists, we develop homological algebra and analysis at the same time and finds analogy between them. (In [21] another example of such a study is presented.)

In order to develop topological field theory, it seems important to the author to pursuit this analogy of homological/homotopical algebra and the compactification of moduli space. The relation of homotopical algebra to (topological) field theory is discussed by various mathematicians, ([28] seems to be one of the first papers mentioning it explicitly.) They however are usually mixture of algebra, combinatorics and geometry. The point we want to emphasis in this paper is to relate algebra to geometry *and analysis* (the study of the moduli space of non linear partial differential equation).

The content of each section of this paper is as follows.

In § 2, we define A^∞category and A^∞functor and give two examples of A^∞ categories. One is based on Morse theory. This A^∞ category was constructed and discussed in detail in [22] so we discuss it only briefly. The second example is one we use mainly in this paper. This A^∞category is based on Lagrangian intersection theory. We consider here only simply connected Lagrangian submanifold to exclude the trouble pointed out by Oh [32]. This assumption seems too much restrictive for various applications. Especially to the applications proposed by M. Kontsevitch to Mirror symmetry and D-brane [27], [26]. (In this proposed application, our A^∞ category $\mathcal{Lag}(X)$ is supposed to play a role in symplectic geometry which the category of coherent sheaves plays in complex or algebraic geometry.)

Certainly we can relax this assumption somehow. (For example to monotone Lagrangian submanifold in the sense of [32].) However it seems yet unclear to the author what is the most natural assumption. So, in this paper, we work under this rather restrictive assumption and defer generalizations to the future.

In fact, we need more results and notions concerning homological algebra of A^∞category than those we gave in § 1. We present it in Part II. So the reader may need to see some part of Part II to read Part I. We arrange the material in this order since to read the formal arguments of Part II without motivation does not seem to be a happy job.

In § 3, we introduce a modified version of ASD (Anti-self-dual) equation which we mainly use to construct Floer homology for 3 manifold with boundary. This equation is a mixture of pseudo holomorphic curve equation to the representation space (of the surface group) and

ASD equation on 4 manifold. We describe in § 3 main analytic results on the moduli space of solutions of this equation.

In § 4, using the properties of the moduli space we described in § 3, we present a definition of Floer homology of 3 manifold with boundary.

§§ 5, 6 and 7 are devoted to the discussion of the well definedness of the relative Floer homology constructed in § 4. Namely we prove that the relative Floer homology is independent of the various choices involved. To state the well definedness, we need the definition of the homotopy equivalences between two A^∞ categories and A^∞ functors. They are given in Part II.

The well definedness is established in two stages. First, in § 5, we discuss the case when we do not change the metric of the boundary (surface) and change the metric (and perturbation) only at the interior of 3 manifold. In that case, the relative invariant is an object of the same A^∞ category $C(\Sigma) = \mathcal{Func}(\mathcal{Lag}(R(\Sigma),\mathcal{Ch}))$ (equivalently they are A^∞ functors from the same A^∞ category $\mathcal{Lag}(R(\Sigma))$ to $\mathcal{Ch}$.) Then well definedness in this case means that the two A^∞ functors are homotopic (in the sense we define in § 12).

In §§ 6,7, we consider the case when we change the metric etc. of the boundary (the Riemann surface.) If we change the metric of Σ, the symplectic manifold $R(\Sigma)$ will be the same but the Kähler metric (and the complex structure) on it changes. Hence we first prove that if we change the (compatible) almost complex structure of a symplectic manifold X, the resulting A^∞category $\mathcal{Lag}(X)$ does not change in the sense of homotopy equivalence. This point is proved in § 6 using analytic lemmata whose proof are deferred to the subsequent papers. Then finally, in § 7, we show that relative Floer homology up to homotopy is independent of the choice of the metric of the surface also.

In § 8, we discuss Axiom (1.6). Unfortunately the author does not know the proof of it in the general case. (So (1.6) is yet a conjecture.) However we are going to construct a canonical map $CF(N) \to Hom_{C(\Sigma)}(HF(N_1), HF(N_2))$ which the author believe to be an isomorphism. (Here $CF(N)$ is the chain complex defining Floer homology of closed 3 manifolds.) The construction of this map and its properties are in § 8.

In § 9, we consider the case $\Sigma \times [0,1]$. We can prove two results in this case. One is that the relative Floer homology $HF(\Sigma \times [0,1])$ is an A^∞functor represented by a Lagrangian submanifold of $R(\Sigma) \times R(\Sigma)$ and is the diagonal. Second is that the homomorphism $CF(N) \to Hom_{C(\Sigma)}(HF(\Sigma \times [0,1]), HF(N_2))$ in § 8 is a chain homotopy equivalence. These two facts are regarded as a generalization of a result by Dostglou and Salamon [6]. Their result may be regarded as the case when both of N_1 and N_2 are $\Sigma \times [0,1]$.

Part II is devoted to homological algebra of A^∞category.

In § 10, we define an A^∞category, $\mathcal{Func}(C^1,C^2)$, whose object is an A^∞ functor between two A^∞categories.

In § 11, we define homotopy equivalence between two A^∞categories which have an identity.

In § 12, we prove A^∞ version of Yoneda's lemma. Usual Yoneda's lemma asserts that the set of natural transformations between two functors represented by a and b is equal to the set of morphisms from b to a. Our A^∞ version asserts a similar conclusion but "equal to" are to be replaced by "homotopy equivalent to". This lemma is used in this paper for two purposes. First, in § 13, we use it to define weak homotopy equivalence of A^∞ categories

which do not have an identity but have an approximate identity. This point is important since our basic example $\mathcal{Lag}(X)$ does not have an identity. Second we use it in § 9. (There $HF(\Sigma \times [0,1])$ is an A^∞ functor which is represented by a diagonal.) It is also useful to simplify the study of A^∞ category since it implies that any A^∞ category with identity is homotopy equivalent to one for which the composition of morphisms are associative.
In § 13, we define an approximate identity and generalize A^∞ Yoneda's lemma to the case where there may not be an identity but there is only an approximate identity. Finally we discuss the existence of approximate identity of our A^∞category $\mathcal{Lag}(X)$.

Since we postpone the analytic detail to subsequent papers, we put * to the statements which will be proved in subsequent papers. The rule is that any results of Part I follows from statements with * which appeared before and during the proof of that statement. Assuming those statements with * the proof is given in this paper. There is one exception to this rule. Namely statements with * which are in § 13 is used in the proof of Theorem 9.3.
On the other hand, results in Part II are rigorously proved in this paper, except Theorem 13.13 and the statement appeared after that. To prove them we need statements with * in § 2,3,4 and § 13.

The basic idea of this paper was established around 1992 . Various parts of them had been announced by the author in various occasions [15] , [18], [17], [14], [23]. There are various reasons for the long delay of its publication.
First, the story is not yet complete since the key axiom (1.6) is in general a conjecture yet. We remark that the author proposed an idea to attack to this conjecture based on two last papers by Floer [10], [11], [3]. This idea is explained in [15]. We made a first step in this paper to realize it by proving (modulo analytic detail) the isomorphism in the case $\Sigma \times [0,1]$ in § 9. We do not discuss the next step in this paper since it is not yet complete.
The second reason is that to work out the technical detail of analysis announced in this paper is a quite heavy task The main part of the analysis was written and distributed in [13]. The referee of [13] requested the author to write more technical detail so he is going to write it in subsequence papers. Meanwhile to motivate those heavy analytic detail, it seems appropriate to present applications of it first. And it is the purpose of this paper.
Also, there used to be several points which had not been clear in symplectic geometry side of the story. By mainly the efforts of Oh [31] [32], (and also [22]) those points are now became much clearer. So we are now ready to write up the story up to the point we have attained.
A part of the idea in this paper is explained by D.Salamon in [33] § 3 following the idea of [15], [13]. The main idea in [15], [13], which was used by Salamon there, is to use Lagrangian submanifold of the representation space to set boundary condition. In [33] § 3, Salamon said that his Floer homology of 3-manifolds with boundary is different from ours. However it is different only because he proposed the different way to work out analytic detail to set the boundary condition, (based on a conjecture which is still open). It is almost certain that the resulting theory is the same. (Salamon described only a part of the structure constructed here, but if his conjecture, (Conjecture 3.2 in [33]) is correct it seems that all the

structures in this paper can be constructed also in his way, though more analytic detail than Conjecture 3.2 is to work out.)

PART I GEOMETRY

§ 2 A^∞ category

It seems natural for our purpose to involve informations more precise than homology group. Namely we work in the level of chain complex. For this purpose, we use the notion of A^∞ category defined in [18], [14]. In this section, we recall its definition and prove a few properties of it. More definitions and results are in Part II.

Definition 2.1 An A^∞category $\mathcal{C}$ consists of a set $\mathcal{Ob}(\mathcal{C})$ (of all objects) a chain complex $\mathcal{C}_*(a,b)$ (the set of morphisms) for each $a,b \in \mathcal{Ob}$ and homomorphisms $\eta_k : \mathcal{C}_*(a_0,a_1) \otimes \cdots \otimes \mathcal{C}_*(a_{k-1},a_k) \to \mathcal{C}_*(a_0,a_k)$ of degree $k-2$ (the k-th composition) for each $a_0,\cdots,a_k \in \mathcal{Ob}$ such that the following holds.

$$
\begin{aligned}
&(\partial\eta_k)(x_1 \otimes \cdots \otimes x_k) \\
&\qquad = \sum_{1\le i< j\le k} \pm\eta_{k-j+i}(x_1 \otimes \cdots \otimes \eta_{j-i+1}(x_i \otimes \cdots \otimes x_j) \otimes \cdots \otimes x_k) \cdot
\end{aligned}
\tag{2.1}
$$

Here we do not discuss the sign since in later sections we work over $\boldsymbol{Z}_2$ coefficient. Here and hereafter we write $(\partial\varphi)(u) = \partial(\varphi(u)) \pm \varphi(\partial u)$, $\partial(u \otimes v) = \partial u \otimes v \pm u \otimes \partial v$ unless otherwise defined.

Definition 2.2 Let $\mathcal{C}^1,\mathcal{C}^2$ be A^∞categories. An A^∞*functor* from $\mathcal{C}^1$ to $\mathcal{C}^2$ is a family of homomorphisms $F_0 : Ob(\mathcal{C}^1) \to Ob(\mathcal{C}^2)$, $F_1(a,b) : \mathcal{C}^1(a,b) \to \mathcal{C}^2(F_0(a),F_0(b))$, $\cdots$, $F_k(a_0,\cdots,a_k) : \mathcal{C}^1(a_0,a_1) \otimes \cdots \mathcal{C}^1(a_{k-1},a_k) \to \mathcal{C}^2(F_0(a_0),F_0(a_k))$ satisfying the following conditions :

(2.2.0) F_k is of degree $k-1$.

(2.2.1) F_1 is a chain map .

(2.2.2) $(\partial F_2)(x \otimes y) = \pm F_1(\eta_2(x \otimes y)) \pm \eta_2(F_1(x) \otimes F_1(y))$.

$$
\begin{aligned}
&\partial(F_3(x_1 \otimes x_2 \otimes x_3)) \pm F_3(\partial(x_1 \otimes x_2 \otimes x_3)) \\
&\qquad = \pm F_2(\eta_2(x_1 \otimes x_2) \otimes x_3) \pm F_2(x_1 \otimes \eta_2(x_2 \otimes x_3)) \\
&\qquad\quad \pm\eta_2(F_2(x_1 \otimes x_2) \otimes F_1(x_3)) \pm \eta_2(F_1(x_1) \otimes F_2(x_2 \otimes x_3)). \\
&\qquad\quad \pm\eta_3(F_1(x_1) \otimes F_1(x_2) \otimes F_1(x_3)) \\
&\qquad\quad \pm F_1(\eta_3(x_1 \otimes x_2 \otimes x_3))
\end{aligned}
\tag{2.2.3}
$$

$$
\begin{aligned}
&(\partial F_k)(x_1 \otimes \cdots \otimes x_k) \\
(2.2.k) \qquad &= \sum_{1 \le i < j \le k} \pm F_{k-j+i}(x_1 \otimes \cdots \otimes \eta_{j-i+1}(x_i \otimes \cdots \otimes x_j) \otimes \cdots \otimes x_k) \\
&\quad + \sum_{m_1 + \cdots + m_\ell = k} \pm \eta_\ell (F_{m_1}(x_1 \otimes \cdots \otimes x_{m_1}) \otimes \cdots \otimes F_{m_\ell}(x_{k-m_\ell-1} \otimes \cdots \otimes x_k))
\end{aligned}
.
$$

The following is the simplest example of A^∞ category.

Definition 2.3 $Ob(\mathcal{Ch})$=the set of all chain complexes. $\mathcal{Ch}(a,b)$=the graded abelian group of all homomorphisms, (not necessary to be a chain homomorphism.) η_2=the composition of two homomorphisms. η_3 and higher is 0.

Now, for an A^∞category C and $a \in Ob(C)$, we define an A^∞functor $F^a : C \to \mathcal{Ch}$ as follows.

Definition 2.4 $F_0^a(b) = C(a,b)$ $F_1^a(x)(y) = \eta_2(y \otimes x)$, $x \in C(b,c)$, $y \in C(a,b)$ $(F_1^a(x) \in Hom(C(a,b), C(a,c)))$, $\cdots$, $F_k^a(x_1 \otimes \cdots \otimes x_k)(y) = \eta_{k+1}(y \otimes x_1 \otimes \cdots \otimes x_k)$

Lemma 2.5 *$F^a : C \to \mathcal{Ch}$ is an A^∞-functor.*

Proof:

$$
\begin{aligned}
&(\partial F_k^a)(x_1 \otimes \cdots \otimes x_k)(y) \\
&\quad = \sum_{1 \le i < j \le k} \pm(\partial \eta_{k+1})(y \otimes x_1 \otimes \cdots \otimes x_k) \\
&\quad = \sum_{1 \le i < j \le k} \pm \eta_{k+1+j-i}(y \otimes x_1 \otimes \cdots \eta_{j-i}(x_i \otimes \cdots \otimes x_j) \otimes \cdots \otimes x_k) \\
&\qquad + \sum_{1 \le i \le k} \pm \eta_{i+1}(\eta_{k-i}(y \otimes x_1 \otimes \cdots x_i) \otimes \cdots \otimes x_k) \\
&\quad = \sum_{1 \le i < j \le k} \pm F_{k+j-i}^a(x_1 \otimes \cdots \eta_{j-i}(x_i \otimes \cdots \otimes x_j) \otimes \cdots \otimes x_k)(y) \\
&\qquad + \sum_{1 \le i \le k} \pm \eta_{k-i}(F_i^a(x_1 \otimes \cdots x_i)(y) \otimes x_{i+1} \otimes \cdots \otimes x_k) \\
&\quad = \sum_{1 \le i < j \le k} \pm F_{k+j-i}^a(x_1 \otimes \cdots \eta_{j-i}(x_i \otimes \cdots \otimes x_j) \otimes x_{i+1} \otimes \cdots \otimes x_k)(y) \\
&\qquad + \sum_{1 \le i \le k} \pm \eta_2(F_i^a(x_1 \otimes \cdots x_i) \otimes F_{k-i}^a(x_{i+1} \otimes \cdots \otimes x_k))(y)
\end{aligned}
.
$$

Using the fact that $\eta_3 = \cdots = 0$ in $\mathcal{Ch}$, we obtain the conclusion.

Definition 2.6 An A^∞functor from C to $\mathcal{Ch}$ is said to be *representable* if it is equal to the functor obtained above.

We define natural transformations between A^∞ functors etc. in a similar way. We then obtain an A^∞ category $\mathcal{F}unc(\mathcal{C}_1,\mathcal{C}_2)$ whose object is an A^∞ functor $F:\mathcal{C}_1 \to \mathcal{C}_2$ and whose morphisms are pre natural transformations. See Part II for the definitions of them.
In this paper we use rather a topological A^∞ category than an A^∞ category.

Definition 2.7 A *topological* A^∞ *category* $\mathcal{C}$ consists of a topological space $\mathcal{O}b(\mathcal{C})$ (of all objects) a chain complex $\mathcal{C}_*(a,b)$ (the morphisms) for pairs $a,b \in \mathcal{O}b$ in a Bair subset of $\mathcal{O}b(\mathcal{C})\times\mathcal{O}b(\mathcal{C})$ and homomorphisms $\eta_k : \mathcal{C}_*(a_0,a_1)\otimes\cdots\otimes\mathcal{C}_*(a_{k-1},a_k)\to\mathcal{C}_*(a_0,a_k)$ (k-th composition) for $(a_0,\cdots,a_k)$ in a Bair subset of $\mathcal{O}b(\mathcal{C})^k$, such that the following holds.

$$(\partial\eta_k)(x_1\otimes\cdots\otimes x_k) = \sum_{1\le i<j\le k} \pm\eta_{k-j+i}(x_1\otimes\cdots\otimes\eta_{j-i+1}(x_i\otimes\cdots\otimes x_j)\otimes\cdots\otimes x_k).$$

We can modify the definition of A^∞ functor in a straight forward way and define a topological A^∞ functor.
There are two important examples $\mathcal{MS}(M)$ and $\mathcal{L}ag(X,\omega)$ of topological A^∞ categories. They are discussed in [22].
Let M be a Riemannian manifold. We have an A^∞ category $\mathcal{MS}(M)$ and call it the *Morse category*. Its object is a smooth function on a manifold M. For two objects $f_1, f_2 \in C^\infty(M)$, the set of morphisms $C_*(f_1,f_2)$ is the Morse-Witten complex of the function $f_2 - f_1$. (We remark that Morse-Witten complex is well defined if $\mathrm{grad}(f_2 - f_1)$ is Morse-Smale vector field. This condition is satisfied for $(f_1,f_2)\in C^\infty(M)^2$ in a Bair subset.) The k-th composition map $\eta_k : C^*(f_0,f_1)\otimes\cdots\otimes C^*(f_{k-1},f_k)\to C^*(f_0,f_k)$ is defined by using the moduli space of maps from a tree to M such that each edge is mapped to a gradient line of an appropriate $f_j - f_i$. We do not discuss the precise definition here, since we do not need it. See [22], [14], [18].
To define another basic example $\mathcal{L}ag(X,\omega)$, we start with a symplectic manifold (X,ω). To simplify various discussions, we restrict ourselves to the following cases.

Assumption 2.8 (X,ω) is monotone. Namely $c^1(X) = N\omega$ in $H^2(X;\boldsymbol{Z})$. Here $c^1(X)$ is the 1st Chern class of a compatible almost complex structure of (X,ω).

We put

Definition 2.9 $\mathcal{O}b(\mathcal{L}ag(X,\omega))$ is the set of all simply connected Lagrangian submanifolds of (X,ω).

Remark 2.10 One may replace the assumption simply connectivity by a weaker one.

For example by the monotonicity in the sense of Oh [32] .

We put C^∞ topology on the set of Lagrangian submanifolds.
Now, following [32], [31], we can define Floer's chain complex for generic pair of elements of $\mathcal{Ob}(\mathcal{Lag}(X,\omega))$. We recall the construction here since we need its generalization in later sections. We fix an almost complex structure J on X which is compatible with the symplectic structure ω. Let $\Lambda_1,\Lambda_2 \in \mathcal{Ob}(\mathcal{Lag}(X,\omega))$. We put $D = \{z \in \mathbf{C} \,|\, |z| < 1\}$.
For $p,q \in \Lambda_1 \cap \Lambda_2$, we consider the set of all maps $\varphi : D \to X$ satisfying the following conditions.

(2.11.1) φ is pseudo holomorphic. Namely $J\varphi_*(X) = \varphi_*(JX)$.
(2.11.2) We put $\partial_1 D = \{z \in \partial D \,|\, \operatorname{Im} z > 0\}$, $\partial_2 D = \{z \in \partial D \,|\, \operatorname{Im} z < 0\}$. Then φ is extended to a smooth map on $D \cup \partial D$ such that $\varphi(\partial_1 D) \subseteq \Lambda_1$, $\varphi(\partial_2 D) \subseteq \Lambda_2$.
(2.11.3) $\varphi(-1) = p$, $\varphi(+1) = q$.

Let $\mathcal{M}(X;\Lambda_1,\Lambda_2;p,q)$ be the set of all such maps. The following is proved in [32], [31]. We assume that Λ_1 is transversal to Λ_2.

Theorem 2.12 *There exists* $\mu : \Lambda_1 \cap \Lambda_2 \to \mathbf{Z}/2N$ *such that, for each generic pair* $\Lambda_1,\Lambda_2 \in \mathcal{Ob}(\mathcal{Lag}(X,\omega))$, *the space* $\mathcal{M}(X;\Lambda_1,\Lambda_2;p,q)$ *is a smooth manifold and its dimension is given by* :

$$\dim \mathcal{M}(X;\Lambda_1,\Lambda_2;p,q) \equiv \mu(p) - \mu(q) \mod 2N.$$

We define

Definition 2.13

$$C_k(X;\Lambda_1,\Lambda_2) = \bigoplus_{\substack{p \in \Lambda_1 \cap \Lambda_2 \\ \mu(p) = k}} \mathbf{Z}_2 \cdot [p]$$
$$\partial : C_k(X;\Lambda_1,\Lambda_2) \to C_{k-1}(X;\Lambda_1,\Lambda_2),$$
$$\partial[p] = \sum \#\overline{\mathcal{M}}(X;\Lambda_1,\Lambda_2;p,q)[q].$$

Following [32] we assume $N \ge 2$. We recall :

Theorem 2.14 (Floer [8], Oh [32]) $\partial\partial = 0$. *The homology group* $H_*(C_*(X;\Lambda_1,\Lambda_2),\partial)$ *is independent of the choice of almost complex structure and depends only on symplectic structure. It is also independent of the deformation of Lagrangian submanifolds.*

In order to motivate the construction we discuss later, we here quote some results from [8],

[32] which are basic in the proof of Theorem 2.14 .

Theorem 2.15 (Floer [8], Oh [32]) *Let* $p,q \in \Lambda_1 \cap \Lambda_2$ *be such that* $\mu(p) - \mu(q) = 2$. *Then, for generic choice of* Λ_1, Λ_2, *one dimensional component of* $\overline{\mathcal{M}}(X;\Lambda_1,\Lambda_2;p,q)$ *can be compactified to* $C\mathcal{M}(X;\Lambda_1,\Lambda_2;p,q)$ *such that it is a compact one dimensional manifold with boundary*

$$\bigcup_{r:\mu(r)=\mu(q)+1} \overline{\mathcal{M}}(X;\Lambda_1,\Lambda_2;p,r) \times \overline{\mathcal{M}}(X;\Lambda_1,\Lambda_2;r,q).$$

Here (and in a similar situation later) $\overline{\mathcal{M}}(X;\Lambda_1,\Lambda_2;p,r)$ *and* $\overline{\mathcal{M}}(X;\Lambda_1,\Lambda_2;r,q)$ *denote the union of its* 0 *dimensional components.*

We next define the k -th composition. We put

$$\mathcal{T}_{0,k+1} = \frac{\{(a_0,\cdots,a_k) \mid a_0 \in \partial D,\, a_0,\cdots,a_k \text{ respect the cyclic order of } \partial D\}}{\{h : D \to D \mid \text{biholomorphic}\}}.$$

We consider counter clockwise order of ∂D. The group $\{h : D \to D \mid \text{biholomorphic}\}$ acts on $\{(a_0,\cdots,a_k) \mid a_i \in \partial D,\, a_0,\cdots,a_k \text{ respect the cyclic order of } \partial D\}$ by

$$h(a_0,\cdots,a_k) = (h(a_0),\cdots,h(a_k)).$$

It is easy to see (and is proved in [22]) that $\mathcal{T}_{0,k+1}$ is diffeomorphic to $\boldsymbol{R}^{k-2}$.
Now let $(a_0,\cdots,a_k) \in \mathcal{T}_{0,k+1}$. We let $\partial_i D$ be the set of points of ∂D which lie between a_{i-1} and a_i. Let $p_i \in \Lambda_{i-1} \cap \Lambda_i$. ($p_0 \in \Lambda_k \cap \Lambda_0$.) We define $\mathcal{M}(X;(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k))$ to be the set of all maps $\varphi : D \to X$ satisfying the following conditions .

(2.16.1) φ is pseudo holomorphic. Namely $J\varphi_*(X) = \varphi_*(JX)$.
(2.16.2) φ is extended to a continuous map on $D \cup \partial D$ such that $\varphi(\partial_i D) \subseteq \Lambda_i$.
(2.16.3) $\varphi(a_i) = p_i$.

The proof of the following is similar to [32], [31] and will appear in a subsequent paper.

Theorem 2.17* *For generic* Λ_i, *the space* $\mathcal{M}(X;(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k))$ *is a smooth manifold and its dimension is given by :*

$$\dim \mathcal{M}\left(X;(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k)\right)=\sum_{i=1}^{k}\mu(p_i)-\mu(p_0)+k-2.$$

Theorem 2.18* *Let* $p_i\in\Lambda_{i-1}\cap\Lambda_i$ *be such that* $\sum\mu(p_i)-\mu(p_0)+k-2=1$. *Then, for generic* Λ_i, *the space* $\mathcal{M}\left(X;(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k)\right)$ *can be compactified to a compact one dimensional manifold* $C\mathcal{M}\left(X;(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k)\right)$ *whose boundary is a union of the following* 3 *kinds of sets :*

$$\bigcup_{\substack{p_i'\in\Lambda_{i-1}\cap\Lambda_i\\ \mu(q)=\mu(p_i)-1}}\mathcal{M}(X;(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_{i-1},p_i',p_{i+1},\cdots,p_k))\times\mathcal{M}(X;\Lambda_i,\Lambda_{i+1};p_i',p_i)$$

$$\bigcup_{\substack{p_0'\in\Lambda_k\cap\Lambda_0\\ \mu(p_0')=\mu(p_0)-1}}\mathcal{M}(X;(\Lambda_0,\cdots\Lambda_k);(p_0',p_1,\cdots,p_k))\times\mathcal{M}(X;\Lambda_0,\Lambda_k;p_0,p_0')$$

$$\bigcup_{\substack{p_{i,j}\in\Lambda_i\cap\Lambda_j\\ \sum_{\ell=i}^{j}\mu(p_\ell)-\mu(p_{i,j})+j-i-1=0}}\begin{array}{c}\mathcal{M}(X;(\Lambda_0,\cdots,\Lambda_i,\Lambda_j,\cdots\Lambda_k);(p_0,\cdots,p_{i-1},p_{i,j},p_{j+1},\cdots,p_k))\\ \times\mathcal{M}(X;(\Lambda_i,\cdots,\Lambda_j);(p_{i,j},p_i,\cdots,p_j))\end{array}$$

Figure 2.19

The reader may wonder why the sign of $\mu(p_0)$ is $-$ while the sign of other $\mu(p_i)$ is $+$. The reason is as follows : We regard $p_0\in\Lambda_0\cap\Lambda_k$ and the index (Floer degree) of p_0 regarded as an element of $\Lambda_0\cap\Lambda_k$ is n minus the degree of it regarded as an element of $\Lambda_k\cap\Lambda_0$.

Using Theorem 2.17 we define a homomorphism $\eta_k : C_*(M;\Lambda_0,\Lambda_1)\otimes\cdots\otimes C_*(M;\Lambda_{k-1},\Lambda_k)\to C_*(M;\Lambda_0,\Lambda_k)$ of degree $k-2$ by :

$$\eta_k([p_1]\otimes\cdots\otimes[p_k])=\sum\#\mathcal{M}(X;(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k))[p_0].$$

Using Theorem 2.18 in a similar way to the proof of Theorem 2.14, we can prove :

Theorem 2.20

$$(\partial\eta_k)(x_1\otimes\cdots\otimes x_k)=\sum_{1\le i<j\le k}\pm\eta_{k-j+i}(x_1\otimes\cdots\otimes\eta_{j-i+1}(x_i\otimes\cdots\otimes x_j)\otimes\cdots\otimes x_k).$$

We thus obtained a topological A^∞ category. We denote it by $\mathcal{Lag}(X,\omega)$.

The category $\mathcal{Lag}(X)$ is studied in the case when X is a cotangent bundle T^*M in [22], where it is proved that $\mathcal{Lag}(T^*M,\omega)$ contains $\mathcal{MS}(M)$ as a full subcategory.

Our main application of the category $\mathcal{Lag}(X)$ in this paper is the case when (X,ω) is the space of gauge equivalence classes of flat connections of an $SO(3)$ bundle on a Riemann surface.

Let Σ be an oriented closed 2 dimensional manifold. We choose a complex structure on it and hence a symplectic structure.

Let $E \to \Sigma$ be an $SO(3)$ bundle on Σ such that its second Stiefel Whiteney class is nontrivial on each connected component. (Such a bundle is unique.) Let $R(\Sigma;E)$ be the space of all gauge equivalence classes of flat $SO(3)$ connections on $E \to \Sigma$. (See § 3 for the choice of gauge transformation group.) $R(\Sigma;E)$ has a symplectic structure. Moreover the complex structure of Σ induces a Kähler metric of $R(\Sigma;E)$. We use this complex structure of $R(\Sigma;E)$. This symplectic manifold $R(\Sigma;E)$ is monotone and $N = 2$. Hence Floer homology has period 4 ([6]).

Remark 2.21 Since we did not perturb the complex structure of $R(\Sigma;E)$ the reader may wonder that there might be a trouble to compactify the moduli space $\mathcal{M}\big(X;(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k)\big)$. However, we know that there are no pseudo holomorphic sphere $\varphi : \boldsymbol{C}P^1 \to R(\Sigma;E)$ such that $\int_{\boldsymbol{C}P^1}\varphi^* c^1 < 0$ ([6]). Hence, in this case, we do not have to worry about "negative multiple cover problem".

The category we use to define Floer homology with boundary is $\mathcal{Func}(\mathcal{Lag}(R(\Sigma;E)),\mathcal{Ch})$. Namely for a three manifold N and an $SO(3)$ bundle $\tilde E$ on it such that $\partial N = \Sigma$, $\tilde E\big|_{\Sigma} = E$, we are going to define an A^∞functor $HF(N) : \mathcal{Lag}(R(\Sigma;E)) \to \mathcal{Ch}$, that is an object of $\mathcal{Func}(\mathcal{Lag}(R(\Sigma;E)),\mathcal{Ch})$ and regard it as a relative Floer homology.

At the end of this section, we briefly mention another example of A^∞ category $\mathcal{Sh}(M)$. Here M is a complex manifold. The object of $\mathcal{Sh}(M)$ is a coherent $O_M$module sheaf $\mathcal{F}$ together with its injective resolution $\mathcal{F} \to \mathcal{F}_*$. For two objects $\mathcal{F} \to \mathcal{F}_*$, $\mathcal{G} \to \mathcal{G}_*$, the morphism between them is

$$C_k(\mathcal{F},\mathcal{G}) \cong \oplus\Gamma\big(M; Hom(\mathcal{F}_i,\mathcal{G}_{i+k})\big).$$

This is clearly a chain complex. The composition is defined in an obvious way and higher compositions are all 0. It seems that it is expected that $\mathcal{Sh}(M)$ is somehow related to $\mathcal{Lag}(M^\vee)$ where $M^\vee$ is a Mirror of M.

§ 3 ASD equation on 4 manifold with corner

We use a kind of "moduli space of ASD (Anti-self-dual) connections" to define our relative Floer homology. In this section, we define and describe the properties of it, which we need for our construction.
We start with the following situation. Let M be an oriented 4 manifold with boundary and ends, and $\hat{E}$ be an $SO(3)$ bundle on it. We assume

Assumption 3.1
(3.1.1) There exist oriented compact 3 manifolds N_-, N_+ with boundaries such that $M-$(compact set) is diffeomorphic to $N_- \times (-\infty, 0] \cup N_+ \times [0, \infty)$.
(3.1.2) There exists an oriented closed 2 manifold Σ such that $\partial N_- \cong \partial N_+ \cong \Sigma$. The boundary of M is diffeomorphic to $\Sigma \times \boldsymbol{R}$.
(3.1.3) We assume that the restriction of $\hat{E}$ to each connected component of Σ is nontrivial.
(3.1.4) $w^2(\hat{E}) \in H^2(M; \boldsymbol{Z}_2)$ is in the image of $H^2(M; \boldsymbol{Z})$.

We put $\tilde{E} = \hat{E}\big|_{N_\pm}$, $E = \hat{E}\big|_{\Sigma}$. In case no confusion can occur, we simply write E in place of $\hat{E}$ or $\tilde{E}$. We take a Riemannian metric on $M, N_\pm, \Sigma$ such that :

(3.1.5) $M-$(compact set) is isometric to $N_- \times (-\infty, 0] \cup N_+ \times [0, \infty)$
(3.1.6) A neighborhood of the boundary of M is isometric to $\Sigma \times [-1,1] \times \boldsymbol{R}$ with $\Sigma \times \{1\} \times \boldsymbol{R}$ being the boundary.

Figure 3.2

Let $R(N_\pm; E)$ be the set of all gauge equivalence classes of flat $SO(3)$ connections of $\tilde{E}$ on $N_\pm$ and $R(\Sigma; E)$ be the set of all gauge equivalent classes of flat $SO(3)$ connections of E.

The definition of gauge transformation is the same as Floer used to define $SO(3)$ version of Floer homology. (It is described in [3], to which we refer for the detail.)

By (3.1.4), there exists a principal $U(2)$ bundle $P_{U(2)}$ such that

$$P_{U(2)} \times_{U(2)} \boldsymbol{R}^3 = \hat{E}.$$

Here $U(2)$ acts on $\boldsymbol{R}^3$ through $U(2) \to U(2)/U(1) = SO(3)$. On the other hand, $U(2)$ acts on $SU(2)$ by adjoint representation, and we put

$$Ad\hat{E} = P_{U(2)} \times_{U(2)} SU(2).$$

Our gauge transformation group is the set of all sections of $Ad\hat{E}$. If the first homology group of M, $N_\pm$ is trivial, it is the same as the set of $SO(3)$ gauge transformations. In general it is a subgroup of finite index of it.

In our situation, where Σ is disconnected, the space $R(\Sigma;E)$ is the direct product $\prod_i R(\Sigma_i;E)$ where Σ_i are connected components.

Let $res_\pm : R(N_\pm;E) \to R(\Sigma;E)$ be the map defined by restricting the flat connections.

The choice of Riemannian metric of Σ, determines a complex structure on Σ. It induces a Kähler structure on $R(\Sigma;E)$. Let ω be the Kähler form. It is well known that ω is independent of the metric of Σ. For a technical reason we use the symplectic structure $-\omega$ and compatible complex structure $-J$ on $R(\Sigma;E)$.

We remark that the real dimension of $R(\Sigma;E)$ is $\sum(6g_i - 6)$, where g_i is a genus of a connected component of Σ. The following lemma is well known.

Lemma 3.2 $res_\pm^*\omega = 0$.

In a "generic" case, $res_\pm : R(N_\pm;E) \to R(\Sigma;E)$ is know to be a Lagrangian immersion. However the term "generic" may need to be precise. We will concern with this point later in this section.

Let Λ be a simply connected Lagrangian submanifold of $R(\Sigma;E)$. In other word, $\Lambda \in \mathcal{Ob}(\mathcal{Lag}(R(\Sigma;E)))$. Let $a_\pm \in R(N_\pm;E)$ such that $res_\pm(a_\pm) \in \Lambda$.

We, for a moment, make the following assumption for simplicity.

Assumption 3.3

(3.3.1) A neighborhood of $a_\pm$ in $R(N_\pm;E)$ is a smooth manifold of dimension $\sum(3g_i - 3)$.

(3.3.2) $res_\pm$ are transversal to Λ at $a_\pm$.

As we mentioned in the introduction, the basic idea is to use Λ and $a_\pm$ as the boundary condition of the ASD equation. But if we do it directly, the resulting moduli space is of "$-\infty$ dimensional". The reason is that requiring the connection to be flat on $\Sigma \times \mathbf{R}$ are too much. Therefore, we modify ASD equation as follows.

Let us recall that a neighborhood of the boundary of M is diffeomorphic to $\Sigma\times[-1,1]\times\boldsymbol{R}$ and the metric there is the product metric $g_0 = g_\Sigma \oplus ds^2 \oplus dt^2$. We change this metric as follows. Let $\chi:[-1,1]\to[0,1]$ be a smooth function such that $\chi(s)=1$ in a neighborhood of -1, $\chi(s)>0$ if s is negative and $\chi(s)=0$ if s is positive. Then we consider the "metric" $g=\chi(s)^2 g_\Sigma \oplus ds^2 \oplus dt^2$. Of course this "metric" is degenerate in the domain $s\ge 0$. However we can still define the ASD equation as follows. Let $\mathcal{A}$ be a connection of $\hat{E}$. Then $\mathcal{A} = A + \Phi ds + \Psi dt$. Here $A = A(s,t)$ is a two parameter family of connections of E and $\Phi,\Psi \in \Gamma(\Sigma\times[-1,1]\times\boldsymbol{R}, ad\hat{E})$. (Here $ad\hat{E} = P_{U(2)}\times_{U(2)} su(2)$.) Then the ASD equation is

$$\frac{\partial A}{\partial t} - d_A\Psi + *(\frac{\partial A}{\partial s} - d_A\Phi) = 0 \tag{3.4.1}$$

$$\chi(s)^2\left(\frac{\partial\Phi}{\partial t} - \frac{\partial\Psi}{\partial s} - [\Phi,\Psi]\right) + *F_A = 0 \tag{3.4.2}$$

More precisely (3.4) is ASD equation in the case when $\chi>0$ and hence we just regard (3.4) as ASD equation also in our case where the metric is degenerate. Therefore Equation (3.4) can be extended smoothly to M (as ASD equation).

Now, we consider Equation (3.4) in the domain $s>0$. Equation (3.4.2) in this case is $F_A = 0$. Thus, we have two parameter family of flat connections, namely the map $[0,1]\times\boldsymbol{R}\to R(\Sigma;E)$. Equation (3.4.1) then means that this map $[0,1]\times\boldsymbol{R}\to R(\Sigma;E)$ is holomorphic. (See [6] for the discussion about it.) (We remark that Hodge $*$ gives the usual complex structure on $R(\Sigma;E)$ and we are using minus of it.) Thus, for this equation, it is natural to assume that its value at $\Sigma\times\{1\}\times\boldsymbol{R}$ is contained in Λ.

Keeping the above observation in mind, we define the moduli space $\hat{\mathcal{M}}(M;\Lambda;a_-,a_+)$ as follows. We fix a connection $\mathcal{A}_0$ on M such that $\mathcal{A}_0$ is flat on $N_-\times(-\infty,R]$ $\cup N_+\times[R,\infty)$ and coincides with $a_\mp$ there. We consider smooth connections $\mathcal{A}$ on M such that $\mathcal{A}-\mathcal{A}_0$ is of L^2 class. (Under Assumption 3.3 we do not have to worry so much about the norm we take.) Let us write $\boldsymbol{\mathcal{A}}(M,E;a_-,a_+)$ the set of all such $\mathcal{A}$.

Now we put

$$\hat{\mathcal{M}}(M;\Lambda;a_-,a_+) = \left\{ \mathcal{A} \in \mathcal{A}(M,E;a_-,a_+) \;\middle|\; \begin{array}{l} \mathcal{A} \text{ solves (3.4.1), (3.4.2), and is an ASD} \\ \text{connection on } M - [-1,1]\times \boldsymbol{R} \times \Sigma. \\ \text{The gauge equivalence class of the restriction} \\ \text{of } \mathcal{A} \text{ to } \{(t,1)\}\times\Sigma \text{ belongs to } \Lambda \text{ for each } t. \end{array} \right\}$$

We next divide it by gauge transformation group as follows. We remark that $a_\pm$ is irreducible since the restriction of E to each connected component of Σ is nontrivial as $SO(3)$ bundle. Hence the natural boundary condition at $t \to \pm\infty$ for the gauge transformation is that it will converges to identity there. More precisely, since we assume that $\mathcal{A} - \mathcal{A}_0$ is of L^2 class, it is natural to assume that gauge transformation minus identity is of L^2_1 class (namely the L^2 norm of its first derivative is finite.) Namely we put

$$\mathcal{G}(M,E) = \left\{ g \in \Gamma(M, Ad\ E) \;\middle|\; \begin{array}{l} \|g - id\|_{L^2_1} < \infty \\ g \text{ is smooth} \end{array} \right\}.$$

Here $Ad\ E = P_{U(2)} \times_{U(2)} SU(2)$.

To work out analytic detail, we need to study connections and gauge transformations which is not necessary smooth but is in an appropriate Sobolev space. In the paper, we do not concern with analytic point.

It is easy to see that $\mathcal{G}(M,E)$ acts on $\hat{\mathcal{M}}(M;\Lambda;a_-,a_+)$ as gauge transformations. We use L^2_k topology on $\hat{\mathcal{M}}(M;\Lambda;a_-,a_+)$, and L^2_{k+1} topology on $\mathcal{G}(M,E)$ for k large. Then the action is continuous. Let $\mathcal{M}(M;\Lambda;a_-,a_+)$ be the quotient space.

Remark 3.5 We remark that if g is a smooth gauge transformation, $\mathcal{A} \in \hat{\mathcal{M}}(M;\Lambda;a_-,a_+)$, and if $g^*\mathcal{A} \in \hat{\mathcal{M}}(M;\Lambda;a_-,a_+)$ then $g \in \mathcal{G}(M,E)$. This is a consequence of the fact that $a_\pm$ is irreducible.

We next discuss the transversality of our moduli spaces. We choose and fix a compact subset of M which is disjoint to $\boldsymbol{R}\times[-1,1]\times\Sigma$. We consider the set of all smooth Riemannian metrics on M which coincides with g outside this compact set and put C^∞ topology on it. We say, "for a generic metric, $\cdots$ holds", if $\cdots$ is satisfied for a metric in a Bair subset of this set of metrics.

We assume that Assumption 3.3 is satisfied for any element $a_\pm \in R(N_\pm, E)$ with $res_\pm a_\pm \in \Lambda$. Now the first result we need on our moduli space $\mathcal{M}(M;\Lambda;a_-,a_+)$ is as follows.

Theorem 3.6* *There exists a map* $\mu_\pm : \{a_\pm \in R(N_\pm, E) \mid res_\pm a_\pm \in \Lambda\} \to \mathbf{Z}/4\mathbf{Z}$ *such that the following holds for a generic metric.*

$\mathcal{M}(M;\Lambda; a_-, a_+)$ is a smooth manifold of dimension $\mu_-(a_-) - \mu_+(a_+) + c(M,\hat{E})$ modulo 4. Here $c(M,\hat{E})$ is an integer depending only on $(M,\hat{E})$. (We remark that $\mu_\pm$ is independent of M and depends only on $(N_\pm, \tilde{E})$.)

The proof will be given in a subsequence paper. Theorem 3.6 is used in later section to define and study various maps and operations of relative Floer homology.

In order to study Floer homology for 3 manifold with boundary, we consider the case when $N_- \cong N_+ \cong N$ and $M = N \times \mathbf{R}$. In that case, as was the case when N is closed [7], [16], we need to take our perturbation invariant of $\mathbf{R}$ action by translation. For this purpose, we use perturbation using holonomy rather than metric. We recall it here. First, let V be a closed 3 manifold and $\tilde{E}$ be an $SO(3)$ bundle on it. Let $\mathcal{A}(V,\tilde{E})$ be the set of all connections on V. Let $\ell : S^1 \to M$ be a loop and $\varphi : SO(3) \to \mathbf{R}$ be a smooth function invariant of the conjugation. It defines a map $\Phi'_{\ell,\varphi} : \mathcal{A}(V,\tilde{E}) \to \mathbf{R}$ by

$$\Phi'_{\ell,\varphi}(A) = \varphi(Hol_A(\ell)). \tag{3.7}$$

Here $Hol_\ell(A)$ is a holonomy of the connection A along the loop ℓ. We can modify $\Phi'_{\ell,\varphi} : \mathcal{A}(V,\tilde{E}) \to \mathbf{R}$ by taking a tubular neighborhood of ℓ and taking an average so that it is smooth. (See [16] § 2.) Let $\Phi_{\ell,\varphi} : \mathcal{A}(V,\tilde{E}) \to \mathbf{R}$ be the map obtained in this way. We consider the gradient vector field $\mathrm{gra}_{\ell,\varphi}(A) = \mathrm{grad}_A \Phi_{\ell,\varphi}$ of it. (We use L^2 norm to define a metric on $\mathcal{A}(V,\tilde{E})$. We can check easily that gradient vector field is well defined.)

Lemma 3.8 $\mathrm{gra}_{\ell,\varphi}(A)$ *depends only on the restriction of A to a small neighborhood of $\ell(S^1)$. The support of $\mathrm{gra}_{\ell,\varphi}(A) \in T_A\mathcal{A}(V,\tilde{E}) = \mathcal{A}(V,\tilde{E})$ is contained in a small neighborhood of $\ell(S^1)$.*

The lemma is immediate from definition. Because of the lemma, we can define $\mathrm{gra}_{\ell,\varphi}(A)$ for a loop $\ell : S^1 \to N_- \cong N_+ \cong N$ and a connection A on it.

In case we have several (finitely many) loops $\ell_i : S^1 \to N_- \cong N_+ \cong N$ and $\varphi : \dfrac{SO(3) \times \cdots \times SO(3)}{SO(3)} \to \mathbf{R}$, we obtain $\mathrm{gra}_{\vec{\ell},\varphi}(A)$ in a similar way. (See [16] § 2.)

Now we perturb our moduli space of flat connections $R(N;E;\vec{\ell},\varphi)$ as follows.

$$\tilde{R}(N;E;\vec{\ell},\varphi) = \left\{ A \,\middle|\, \begin{array}{l} \text{a connection on } N \\ * F_A = \mathrm{gra}_{\vec{\ell},\varphi}(A) \end{array} \right\}.$$

Since $\mathrm{gra}_{\vec{\ell},\varphi}(A)$ is gauge invariant, $\tilde{R}(N;E;\vec{\ell},\varphi)$ is invariant of the gauge transformations. Let $R(N;E;\vec{\ell},\varphi)$ be the quotient space.

Lemma 3.9* *We can find a finitely many loops $\ell_i : S^1 \to N$ such that $R(N;E;\vec{\ell},\varphi)$ is a smooth manifold of dimension $\sum(3g_i - 3)$ for generic φ. And $res : R(N;E;\vec{\ell},\varphi) \to R(\Sigma;E)$ is a Lagrangian immersion.*

The proof can be done by a method which is now standard. But, since the author does not want to include analytic argument in this paper, we postpone the proof to subsequent papers.

Now for $\ell_i : S^1 \to N$ and $\varphi : \dfrac{SO(3)\times\cdots\times SO(3)}{SO(3)} \to \boldsymbol{R}$, we modify equation (3.4.1) and ASD equation as follows. We first remark that we may take $\ell_i : S^1 \to N$ so that its image is outside $\Sigma\times[-1,1]\subseteq N$. Therefore, we need no perturbation of the equation on $\Sigma\times[-1,1]\times\boldsymbol{R}\subseteq N\times\boldsymbol{R}$. Next we recall that the perturbed equation is

(3.10) $$F_{\mathcal{A}} + \tilde{*}F_{\mathcal{A}} - \mathrm{gra}_{\vec{\ell},\varphi}(a(t))\wedge dt - *\mathrm{gra}_{\vec{\ell},\varphi}(a(t)) = 0\,.$$

(See [16].) Here $\mathcal{A} = A + a\wedge dt$, $\tilde{*}$ and $*$ are Hodge star operator of 4 and 3 manifolds respectively. We remark that the support of $\mathrm{gra}_{\vec{\ell},\varphi}(A(t))$ is contained in a small neighborhood of the union of $\ell_i : S^1 \to N$ and depends only on a restriction of $A(t)$ to a small neighborhood of the union of $\ell_i : S^1 \to N$ and use (3.4) at $\Sigma\times[-1,1]$. So we can use Equation (3.10) outside $\Sigma\times[-1,1]\subseteq N$. Using Equation (3.10), we modify the definition of $\hat{\mathcal{M}}(N\times\boldsymbol{R};\Lambda;a_-,a_+)$ as follows. Let $a_\pm \in \tilde{R}(N;E;\vec{\ell},\varphi)$ be connections such that $res_\pm(a_\pm)\in\Lambda$. We assume that $res_\pm$ is transversal to Λ at $a_\pm$. (This assumption is satisfied for generic Λ.) We define a connection $\mathcal{A}_0$ of $\hat{E}$ on $M = N\times\boldsymbol{R}$ such that it coincides with a_+ (resp. a_-) on $N\times[R,\infty)$ (resp. $N\times(-\infty,-R]$). Let $\mathcal{A}(N\times\boldsymbol{R},E;a_+,a_-)$ be the set of all smooth connections $\mathcal{A}$ of $\hat{E}$ on $N\times\boldsymbol{R}$ such that $\mathcal{A}-\mathcal{A}_0$ is of L^2-class. We put

$$\hat{\mathcal{M}}(N\times \boldsymbol{R};\Lambda;a_-,a_+;\ell,\varphi)$$
$$=\left\{\mathcal{A}\in\mathcal{A}(N\times\boldsymbol{R};a_-,a_+)\ \middle|\ \begin{array}{l}\mathcal{A}\text{ solves (3.4.1), (3.4.2), on }\Sigma\times[-1,1]\times\boldsymbol{R}\\ \text{and (3.10) on }N\times\boldsymbol{R}-\Sigma\times[-1,1]\times\boldsymbol{R}.\\ \text{The gauge equivalence class of the restriction}\\ \text{of }\mathcal{A}\text{ to }\Sigma\times\{(t,1)\}\text{ belongs to }\Lambda\text{ for each }t.\end{array}\right\}$$

It is again invariant of $\mathcal{G}(N\times\boldsymbol{R},E)$. Let $\mathcal{M}(N\times\boldsymbol{R};\Lambda;a_-,a_+;\ell,\varphi)$ be the quotient space by this action.

Now an analogue of Theorem 3.6 for this perturbation is as follows :

Theorem 3.11* *There exist a finitely many loops* $\ell_i : S^1\to N_-\cong N_+\cong N$ *such that, for generic choice of the triple* (g_N,φ,Λ), *the following holds.*

Let g_N *is a metric on* N *which coincides with the given one outside a fixed compact subset of* N, $\varphi:\dfrac{SO(3)\times\cdots\times SO(3)}{SO(3)}\to\boldsymbol{R}$, *and* Λ *is a simply connected Lagrangian submanifold of* $R(\Sigma;E)$.

(3.11.1) $R(N;E;\vec{\ell},\varphi)$ *is a smooth manifold of dimension* $\sum(3g_i-3)$.
(3.11.2) $res: R(N;E;\vec{\ell},\varphi)\to R(\Sigma;E)$ *is transversal to* Λ.
(3.11.3) $\mathcal{M}(N\times\boldsymbol{R};\Lambda;a_-,a_+;\ell,\varphi)$ *is a smooth manifold whose dimension is* $\mu(a_-)-\mu(a_+)$ *modulo* 4.

The proof is given in a subsequent paper. If we take generic (g,φ) then the set of Λ satisfying (3.11.1) $\cdots$ (3.11.3) is a Bair subspace of $\mathcal{O}b(\mathcal{L}ag(R(\Sigma))$. The moduli space $\mathcal{M}(N\times\boldsymbol{R};\Lambda;a_-,a_+;\ell,\varphi)$ has a free action of $\boldsymbol{R}$ induced by the translation along $\boldsymbol{R}$ direction. Let $\overline{\mathcal{M}}(N\times\boldsymbol{R};\Lambda;a_-,a_+;\ell,\varphi)$ be the quotient space.

When we consider a 3 dimensional manifold M such that it is not a direct product and that $N_\pm$ does not satisfy Assumption 3.3, we need to combine two perturbations we introduced. (This argument is parallel to one we need to define Donaldson invariant for 4 manifold with boundary [19].)

We start with the situation of Assumption (3.1). We take $(\vec{\ell}_\pm,g_\pm,\varphi_\pm)$ on $N_\pm$ so that (3.11) holds for generic Λ. Let us take a generic Λ and $a_\pm\in R(N_\pm;E;\vec{\ell}_\pm,\varphi_\pm)$ such that $res_\pm(a_\pm)\in\Lambda$. Choose sufficiently large R and a smooth function $\chi_\pm:\boldsymbol{R}\to[0,1]$ such that

$$\chi_{\pm}(t)=\begin{cases}1 & \text{if} \quad \pm t > \pm 2R \\ 0 & \text{if} \quad \pm t < \pm(2R-1)\end{cases}.$$

We then modify the ASD equation as follows. We recall $M \supseteq N_- \times(-\infty,-R], N_+ \times [R,\infty)$. We perturb ASD equation on $\left(N_- - \Sigma\times[-1,1]\right)\times(-\infty,-R]\cup\left(N_+ - \Sigma\times[-1,1]\right)\times[R,\infty)$ only. The perturbed equation is

(3.12) $F_{\mathcal{A}} + \tilde{*}F_{\mathcal{A}} - \chi_{\pm}(t)\ \mathrm{gra}_{\bar{\ell},\varphi}(a(t)) \wedge dt - \chi_{\pm}(t) * \mathrm{gra}_{\bar{\ell},\varphi}(a(t)) = 0$ if $\pm t \ge \pm R$.

We remark Equation (3.12) coincides with (3.10) if $\pm t \ge \pm 2R$. We take a compact subset of M disjoint from $(\Sigma\times[-1,1])\times \boldsymbol{R} \cup N_- \times(-\infty,-R]\cup N_+ \times[R,\infty)$ and consider the perturbation of the metric of M supported on this set.

We then put

$$\hat{\mathcal{M}}\left(M,\Lambda;a_-,a_+;(\ell_-,\varphi_-),(\ell_+,\varphi_+)\right)$$
$$=\left\{\mathcal{A}\in\mathcal{A}(M,E;a_-,a_+)\ \middle|\ \begin{array}{l}\mathcal{A}\text{ solves (3.4.1), (3.4.2), on } [-1,1]\times\Sigma\times\boldsymbol{R}\text{ and (3.12) on}\\ \left(N_- - \Sigma\times[-1,1]\right)\times(-\infty,-R]\cup\left(N_+ - \Sigma\times[-1,1]\right)\times[R,\infty)\\ \text{and is an ASD connection elesewhere.}\\ \quad\text{The gauge equivalence class of the restriction of } \mathcal{A} \text{ to}\\ \Sigma\times\{(t,1)\} \text{ belongs to } \Lambda \text{ for each } t.\end{array}\right\}$$

which is invariant of the action of $\mathcal{G}(M,E)$. Let $\mathcal{M}\left(M,\Lambda;a_-,a_+;(\ell_-,\varphi_-),(\ell_+,\varphi_+)\right)$ be the quotient space. Then an analogy of Theorem 3.6 is as follows.

Theorem 3.13* *There exists a map* $\mu_{\pm}:\left\{a_{\pm}\in R(N_{\pm},E)\mid res_{\pm}a_{\pm}\in\Lambda\right\}\to \boldsymbol{Z}/4\boldsymbol{Z}$ *such that the following holds for a generic metric on* M.

$\mathcal{M}\left(M,\Lambda;a_-,a_+;(\ell_-,\varphi_-),(\ell_+,\varphi_+)\right)$ *is a smooth manifold whose dimension is* $\mu_-(a_-)-\mu_+(a_+)+c(M,\hat{E})$ *modulo* 4. *Here* $c(M,\hat{E})$ *is an integer depending only on* $(M,\hat{E})$.

We next discuss compactification of our moduli spaces. We discuss only the case when moduli space is 0 or 1 dimensional. More precisely we need to consider the component whose dimension is 0 or 1, since the dimension depends on the component. To simplify the notation we write $\mathcal{M}\left(M,\Lambda;a_-,a_+;(\ell_-,\varphi_-),(\ell_+,\varphi_+)\right)$ for the union of components of the minimal dimension.

Theorem 3.14* *If* $\mu_-(a_-)-\mu_+(a_+)=1$ *then* $\overline{\mathcal{M}}(N\times \boldsymbol{R};\Lambda;a_-,a_+;\ell,\varphi)$ *consists of finitely many points for generic* (g_N,φ,Λ). *If* $\mu_-(a_-)-\mu_+(a_+)+c(M,\hat{E})=0$, *then* $\mathcal{M}\big(M,\Lambda;a_-,a_+;(\ell_-,\varphi_-),(\ell_+,\varphi_+)\big)$ *consists of finitely many points for generic choice of* $(g_M,\Lambda;(\ell_-,\varphi_-),(\ell_+,\varphi_+))$.

Theorem 3.15* *If* $\mu_-(a_-)-\mu_+(a_+)=2$ *then* $\overline{\mathcal{M}}(N\times \boldsymbol{R};\Lambda;a_-,a_+;\ell,\varphi)$ *has a compactification* $C\overline{\mathcal{M}}(N\times \boldsymbol{R};\Lambda;a_-,a_+;\ell,\varphi)$ *whose boundary is identified with*

$$\bigcup_{b:\mu(b)=\mu(a_-)-1}\overline{\mathcal{M}}(N\times \boldsymbol{R};\Lambda;a_-,b;\ell,\varphi)\times\overline{\mathcal{M}}(N\times \boldsymbol{R};\Lambda;b,a_+;\ell,\varphi).$$

Theorem 3.16* *If* $\mu_-(a_-)-\mu_+(a_+)+c(M,\hat{E})=1$ *then* $\mathcal{M}\big(M,\Lambda;a_-,a_+;(\ell_-,\varphi_-),(\ell_+,\varphi_+)\big)$ *has a compactification* $C\mathcal{M}\big(M,\Lambda;a_-,a_+;(\ell_-,\varphi_-),(\ell_+,\varphi_+)\big)$ *whose boundary is identified with the union of the following* 2 *kinds of spaces :*

$$\bigcup_{b:\mu(b)=\mu(a_+)+1}\mathcal{M}\big(M,\Lambda;a_-,b;(\ell_-,\varphi_-),(\ell_+,\varphi_+)\big)\times\overline{\mathcal{M}}(N\times \boldsymbol{R};\Lambda;b,a_+;\ell,\varphi)$$

$$\bigcup_{b:\mu(b)=\mu(a_-)-1}\overline{\mathcal{M}}(N\times \boldsymbol{R};\Lambda;a_+,b;\ell,\varphi)\times\mathcal{M}\big(M,\Lambda;b,a_+;(\ell_-,\varphi_-),(\ell_+,\varphi_+)\big).$$

The proof is postponed again. The proof, in fact, follows the basic strategy established in the definition of relative Donaldson invariant.

For the application in later sections, we need also to use the moduli space similar to and more general than those discussed in this section. Namely the case when there are two or more Lagrangian submanifolds. They are mixture of the moduli spaces discussed in this section and the last section. We will introduce them at the stage when we need it.

§ 4 Relative Floer homology

In this section, we use the moduli spaces discussed in §§ 2,3 to define Floer homology of 3 manifold with boundary. Let us recall what we want to construct.

Let N be a compact 3 manifold with boundary $\partial N = \Sigma$, and E be an $SO(3)$ bundle on N whose restriction to each connected component of Σ is nontrivial. We obtain $R(\Sigma;E)$, the moduli space of the flat $SO(3)$ connections on Σ. We fix a Riemannian metric on Σ and hence a complex structure of Σ. Then $R(\Sigma;E)$ is a Kähler manifold. Hence, by § 2, we obtain a topological A^∞ category $\mathcal{Lag}(R(\Sigma;E))$. What we want to construct is a topological A^∞ functor $HF(N,E) : \mathcal{Lag}(R(\Sigma;E)) \to \mathcal{Ch}$ associated to N. For this purpose, we fix a Riemannian metric g_N on N which coincides with the product metric on $\Sigma \times [-1,1]$, a neighborhood of $\partial N = \Sigma$ in N. Also we choose $\ell_i : S^1 \to N$, $\varphi : \dfrac{SO(3)\times\cdots\times SO(3)}{SO(3)} \to \boldsymbol{R}$ so that the conclusion of Theorem 3.11 holds for generic element Λ of $\mathcal{Ob}\big(\mathcal{Lag}(R(\Sigma;E))\big)$.

We then have $\mu : \{a \in R(N,E) \mid res\, a \in \Lambda\} \to \boldsymbol{Z}/4\boldsymbol{Z}$ such that $\mathcal{M}(N\times\boldsymbol{R};\Lambda;a_-,a_+;\ell,\varphi)$ is a smooth manifold whose dimension is $\mu(a_-)-\mu(a_+)$ modulo 4 for $a_\pm \in \{a \in R(N,E) \mid res\, a \in \Lambda\}$. We now put

$$(4.1) \qquad \big((HF(N;g_N,\vec{\ell},\varphi))_0(\Lambda)\big)_k = \bigoplus_{a\in\{a\in R(N,E)\mid res\,a\in\Lambda\}:\mu(a)=k} \boldsymbol{Z}_2[a].$$

Here we take $\boldsymbol{Z}_2$ coefficient since we do not discuss the orientation in this paper.

(4.1) is a $\boldsymbol{Z}_4$ graded abelian group. We are going to define a boundary operator $\partial : \big((HF_2(N;g_N,\vec{\ell},\varphi))_0(\Lambda)\big)_k \to \big((HF_2(N;g_N,\vec{\ell},\varphi))_0(\Lambda)\big)_{k-1}$. Let $a_\pm \in \{a \in R(N,E) \mid res\, a\in\Lambda\}$ such that $\mu(a_+) = k-1$, $\mu(a_-)=k$. Then, by Theorem 3.11, $\overline{\mathcal{M}}(N\times\boldsymbol{R};\Lambda;a_-,a_+;\ell,\varphi)$ is a space of dimension 0. By Theorem 3.14, $\overline{\mathcal{M}}(N\times\boldsymbol{R};\Lambda;a_-,a_+;\ell,\varphi)$ consists of finitely many points. We put

$$\langle\partial a_-,a_+\rangle = \#\overline{\mathcal{M}}(N\times\boldsymbol{R};\Lambda;a_-,a_+;\ell,\varphi)$$

$$\partial[a_-] = \sum_{a_+:\mu(a_+)=\mu(a_-)-1}\langle\partial a_-,a_+\rangle[a_+].$$

Theorem 4.2

$$\partial\partial = 0.$$

Proof:

This follows from Theorem 3.15 in exactly the same way as [7].

We thus obtain a chain complex $(HF(N;g_N,\vec{\ell},\varphi))_0(\Lambda)$. It gives a part of the structure we need to define our topological A^∞ functor $HF(N,E): \mathcal{Lag}(R(\Sigma;E)) \to \mathcal{Ch}$.

According to Definition 2.2, other part of the structures we need to define a topological A^∞ functor is a map

$$
(4.3)\qquad \begin{aligned}(HF(N;g_N,\vec{\ell},\varphi))_k(\Lambda_0,\cdots,\Lambda_k): C(\Lambda_0,\Lambda_1)\otimes\cdots\otimes C(\Lambda_{k-1},\Lambda_k)\\ \to Hom\Big((HF(N;g_N,\vec{\ell},\varphi))_0(\Lambda_0),(HF(N;g_N,\vec{\ell},\varphi))_0(\Lambda_k)\Big)\end{aligned}
$$

for generic Λ_i. To construct (4.3), we use a moduli space similar to but a bit different from one in § 3. Let :

(4.4.1) $$a_0 \in \{a \in R(N,E) \mid res\, a \in \Lambda_0\},$$
(4.4.2) $$a_i \in \Lambda_{i-1}\cap\Lambda_i,\ \ i=1,\cdots,k,$$
(4.4.3) $$a_{k+1} \in \{a \in R(N,E) \mid res\, a \in \Lambda_k\}.$$

We consider the set of multiples $(\mathcal{A},(t_1,\cdots,t_k))$ such that :

(4.5.1) $\mathcal{A}$ is a smooth connection of E on $N\times\boldsymbol{R}$.
(4.5.2) $t_1<\cdots<t_k$.
(4.5.3) $\mathcal{A}-\mathcal{A}_0$ is of L^2-class, where $\mathcal{A}_0$ is a connection of E on $N\times\boldsymbol{R}$ which is equal to a_0 at $N\times(-\infty,-R]$ and to a_{k+1} at $N\times[R,\infty)$.
(4.5.4) $\mathcal{A}$ satisfies Equation (3.4.1), (3.4.2) at $\Sigma\times[-1,1]\times\boldsymbol{R}$.
(4.5.5) $\mathcal{A}$ satisfies Equation (3.10) at $N\times\boldsymbol{R}-(N-\Sigma\times[-1,1])\times\boldsymbol{R}$.
(4.5.6) $[A(1,t_i)]=a_i$.
(4.5.7) If $t_i<t<t_{i+1}$, then $[A(1,t)]\in\Lambda_i$. Here we put $t_0=-\infty,\ t_{k+1}=\infty$.

We denote $\hat{\mathcal{M}}\big(N\times\boldsymbol{R};(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});\ell,\varphi\big)$ the set of all $(\mathcal{A},(t_1,\cdots,t_k))$ satisfying (4.5.1), $\cdots$, (4.5.7). Let $\mathcal{M}\big(N\times\boldsymbol{R};(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});\ell,\varphi\big)$ be its quotient by the gauge transformation group. There is an $\boldsymbol{R}$ action on $\mathcal{M}\big(N\times\boldsymbol{R};(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});\ell,\varphi\big)$ induced by the translation along $\boldsymbol{R}$ direction. Let $\overline{\mathcal{M}}\big(N\times\boldsymbol{R};(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});\ell,\varphi\big)$ be the quotient space by this action.

Now an analogy of the results in §§ 2,3 is as follows.

Theorem 4.6* *For generic* $\vec{\ell}$, g_N, φ *and* Λ_i, *the following holds.*

(4.6.1)

$\overline{\mathcal{M}}(N\times\boldsymbol{R};(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});\ell,\varphi)$ *is a smooth manifold of dimension* $\sum_{i=0}^{k}\mu(a_i)-\mu(a_{k+1})+k-1$ *modulo* 4.

(4.6.2)

If $\sum\mu(a_i)-\mu(a_{k+1})+k-1=0$, *then* $\overline{\mathcal{M}}(N\times\boldsymbol{R};(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});\ell,\varphi)$ *consists of finitely many points.*

(4.6.3)

If $\sum\mu(a_i)-\mu(a_{k+1})+k-1=1$, *then* $\overline{\mathcal{M}}(N\times\boldsymbol{R};(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});\ell,\varphi)$ *has a compactification* $C\overline{\mathcal{M}}(N\times\boldsymbol{R};(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});\ell,\varphi)$ *such that its boundary is a union of the following* 5 *kinds of spaces.*

(a) $$\bigcup_{a'_0\in R(N,E):\,res\,a'_0\in\Lambda_0,\ \mu(a'_0)=\mu(a_0)+1}\overline{\mathcal{M}}(N\times\boldsymbol{R};\Lambda_1;a_0,a'_0;\ell,\varphi)\times\overline{\mathcal{M}}(N\times\boldsymbol{R};(\Lambda_0,\cdots,\Lambda_k);(a'_0,a_1,\cdots,a_{k+1});\ell,\varphi)$$

(b) $$\bigcup_{a'_{k+1}\in R(N,E):\,res\,a'_{k+1}\in\Lambda_k,\ \mu(a'_{k+1})=\mu(a_{k+1})-1}\overline{\mathcal{M}}(N\times\boldsymbol{R};(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_k,a'_{k+1});\ell,\varphi)\times\overline{\mathcal{M}}(N\times\boldsymbol{R};\Lambda_k;a'_{k+1},a_{k+1};\ell,\varphi)$$

(c) $$\bigcup_{\substack{a'_i\in\Lambda_{i-1}\cap\Lambda_i,\\ \mu(a'_i)=\mu(a_i)+1}}\overline{\mathcal{M}}\Big(N\times\boldsymbol{R};(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{i-1},a'_i,a_{j+1}\cdots,a_{k+1});\ell,\varphi\Big)\times\overline{\mathcal{M}}(R(\Sigma);(\Lambda_{i-1},\Lambda_i);a'_i,a_i)$$

(d) $$\bigcup_{a_{i,j}\in\Lambda_i\cap\Lambda_j,}\overline{\mathcal{M}}\Big(N\times\boldsymbol{R};(\Lambda_0,\cdots,\Lambda_i,\Lambda_j,\cdots,\Lambda_k);(a_0,\cdots,a_{i-1},a_{i,j},a_{i+1}\cdots,a_{k+1});\ell,\varphi\Big)\times\overline{\mathcal{M}}(R(\Sigma);(\Lambda_i,\cdots,\Lambda_j);(a_{i,j},a_i\cdots,a_j);\ell,\varphi)$$

(e) $$\bigcup_{b\in R(N,E):\,res\,b\in\Lambda_i}\overline{\mathcal{M}}(N\times\boldsymbol{R};(\Lambda_0,\cdots,\Lambda_i);(a_0,\cdots,a_i,b);\ell,\varphi)\times\overline{\mathcal{M}}(N\times\boldsymbol{R};(\Lambda_i,\cdots,\Lambda_k);(b,a_{i+1},\cdots,a_{k+1});\ell,\varphi).$$

Here $\overline{\mathcal{M}}(R(\Sigma);(\Lambda_{i-1},\Lambda_i);a'_i,a_i)$ *and* $\overline{\mathcal{M}}(R(\Sigma);(\Lambda_i,\cdots,\Lambda_j);(a_{i,j},a_i\cdots,a_j);\ell,\varphi)$ *are the moduli space introduced in §* 2.

Figure 4.7.

Theorem 4.6 may look quite complicated. But it is a natural generalization of the results in §§ 2 and 3 and the proof is similar. In fact, in Theorem 4.6, the 3 manifold N plays the same role as Lagrangian submanifolds play in Theorem 2.18. See also Figure 4.7. The proof is given in a subsequent paper.

Now we are going to use this moduli space to define $(HF(N;g_N,\vec{\ell},\varphi))_k(\Lambda_0,\cdots,\Lambda_k)$. Let a_i be as in (4.4.1),(4.4.2),(4.4.3) and $\sum\mu(a_i)-\mu(a_{k+1})+k-1=0$. We put

$$\begin{aligned}&\left\langle\left((HF(N;g_N,\vec{\ell},\varphi))_k(\Lambda)([a_1]\otimes\cdots\otimes[a_k]\right)([a_0]),[a_{k+1}]\right\rangle\\&\qquad=\#\overline{\mathcal{M}}(N\times\boldsymbol{R};(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});\ell,\varphi)\end{aligned}$$

and then

$$\begin{aligned}&\left((HF(N;g_N,\vec{\ell},\varphi))_k(\Lambda)([a_1]\otimes\cdots\otimes[a_k])\right)([a_0])\\&=\sum_{a_{k+1}}\left\langle\left((HF(N;g_N,\vec{\ell},\varphi))_k(\Lambda)([a_1]\otimes\cdots\otimes[a_k]\right)([a_0]),[a_{k+1}]\right\rangle[a_{k+1}]\cdot\end{aligned}$$

Theorem 4.8

$(HF(N;g_N,\vec{\ell},\varphi))_k$, $k=0,1,2,\cdots$ is a topological A^∞ functor.

Proof:

We need to verify Formulas (2.2). But it is an immediate consequence of (4.6.3) and the fact that the order of the boundary of closed one dimensional manifold is even. In fact, the spaces (a), $\cdots$, (e) in (4.6.3) corresponds to the terms

$$\partial\left((HF(N;g_N,\vec{\ell},\varphi))_k(\Lambda)([a_1]\otimes\cdots\otimes[a_k])([a_0])\right)$$

$$(HF(N;g_N,\vec{\ell},\varphi))_k(\Lambda)([a_1]\otimes\cdots\otimes[a_k])(\partial[a_0])$$

$$(HF(N;g_N,\vec{\ell},\varphi))_k(\Lambda)([a_1]\otimes\cdots\otimes\partial[a_i]\otimes\cdots\otimes[a_k])([a_0])$$

$$(HF(N;g_N,\vec{\ell},\varphi))_{k-j+i}(\Lambda)([a_1]\otimes\cdots\otimes\eta([a_i]\otimes\cdots\otimes[a_j])\otimes\cdots\otimes[a_k])([a_0])$$

$$\left((HF(N;g_N,\vec{\ell},\varphi))_{k-i}(\Lambda)([a_{i+1}]\otimes\cdots\otimes[a_k])\right)\left((HF(N;g_N,\vec{\ell},\varphi))_i(\Lambda)([a_1]\otimes\cdots\otimes[a_i])([a_0]\right),$$

respectively.

Thus we defined a relative Floer homology as an A^∞ functor $HF(N,E):\mathcal{Lag}(R(\Sigma;E))\to\mathcal{Ch}$.

Let us mention some of the invariants we obtain from Theorem 4.8.

First we fix a simply connected Lagrangian submanifold Λ. We then get a homology group of the chain complex $HF(N,g_N,\vec{\ell},\varphi)_0(\Lambda)$ which we write $HF(N,g_N,\vec{\ell},\varphi;\Lambda)$.

Second, for two simply connected Lagrangian submanifolds Λ_1, Λ_2, we obtain a map

$$\cup : HF(N, g_N, \vec{\ell}, \varphi; \Lambda_1) \otimes HF(R(\Sigma); \Lambda_1, \Lambda_2) \to HF(N, g_N, \vec{\ell}, \varphi; \Lambda_2).$$

Here $HF(R(\Sigma); \Lambda_1, \Lambda_2)$ is the Floer homology group of Lagrangian intersection.

Third, we have Massey type operation. Namely let $x \in HF(N, g_N, \vec{\ell}, \varphi)_0(\Lambda)$, $y \in CF(\Lambda_1, \Lambda_2)$, $y \in CF(\Lambda_2, \Lambda_3)$ such that

$$\begin{aligned} \partial x = \partial y = \partial z = 0 \\ \eta_2(x \otimes y) = \partial\alpha \\ \eta_2(y \otimes x) = \partial\beta . \end{aligned}$$

Here $CF(\Lambda_1, \Lambda_2)$ is the Chain complex giving $HF(R(\Sigma); \Lambda_1, \Lambda_2)$ and η is the product operator introduced in §§ 2 and 4. We then put

$$w = \eta_2(\alpha \otimes z) \pm \eta_2(x \otimes \beta) \pm \eta_3(x \otimes y \otimes z).$$

We have $\partial w = 0$ and its homology class modulo elements of $[x] \cup * + * \cup [z]$ is independent of x, y, z in their homology classes.

We can define higher Massey type products in a similar way. The results of § 5,6,7 imply that these structures are independent of various choices involved.

§ 5 Well definedness I

Let $N, E, \ell_{i,\pm} : S^1 \to N$, $\varphi_\pm : \dfrac{SO(3)\times\cdots\times SO(3)}{SO(3)} \to \boldsymbol{R}$ be as in § 3. We remark that we took metrics $g_N^\pm$ on N and g_Σ on $\Sigma = \partial N$ so that a neighborhood of ∂N is isometric to the direct product $\Sigma\times[-1,1]$. We then defined topological A^∞ functors :

$$HF(N; g_{N,+}, \vec{\ell}_+, \varphi_+) : \mathcal{Lag}(R(\Sigma)) \to \mathcal{Ch}$$
$$HF(N; g_{N,-}, \vec{\ell}_-, \varphi_-) : \mathcal{Lag}(R(\Sigma)) \to \mathcal{Ch}.$$

We write $N_\pm$ in place of $(N, g_{N,\pm})$. Since the A^∞ category $\mathcal{Lag}(R(\Sigma))$ depends only on the Kähler manifold $R(\Sigma)$, it is the same for these two A^∞ functors. The main result discussed in this section is as follows :

Theorem 5.1

The topological A^∞ functor $HF(N_-, \vec{\ell}_-, \varphi_-)$ is homotopic to $HF(N_+, \vec{\ell}_+, \varphi_+)$
.

The definition that two A^∞ functors to be homotopic is given in § 12. By Definition 11.8, Theorem 5.1 is equivalent to say that two objects $HF(N_-, \vec{\ell}_-, \varphi_-), HF(N_+, \vec{\ell}_+, \varphi_+)$ of $\mathcal{Func}(\mathcal{Lag}(R(\Sigma)), \mathcal{Ch})$ are homotoy equivalent.
In this section, we give the proof of Theorem 5.1 modulo analytic detail.
In order to prove Theorem 5.1, we first define a natural transformation $T(g)$ from $HF(N_-, \vec{\ell}_-, \varphi_-)$ to $HF(N_+, \vec{\ell}_+, \varphi_+)$. For this purpose, we take a metric g on $N\times\boldsymbol{R}$ such that

(5.2.1) g is a product metric on $\Sigma\times[-1,1]\times\boldsymbol{R}$.
(5.2.2) $g = g_{N,-}\oplus dt^2$ on $N\times(-\infty,-R]$ for a sufficiently large R.
(5.2.3) $g = g_{N,+}\oplus dt^2$ on $N\times[R,\infty)$ for a sufficiently large R.

We put $M = N\times\boldsymbol{R}$, and use the moduli space $\mathcal{M}\big(M,\Lambda; a_-, a_+; (\ell_-, \varphi_-), (\ell_+, \varphi_+)\big)$ introduced in § 3. Here Λ is a generic simply connected Lagrangian submanifold of $R(\Sigma)$ and $a_\pm \in R(N)$ such that $res_\pm(a_\pm)\in\Lambda$. By perturbing the metric outside $\Sigma\times[-1,1]\times\boldsymbol{R}$, $N\times(-\infty,-R]$, $N\times[R,\infty)$, we may assume that transversality is satisfied for $\mathcal{M}\big(M,\Lambda; a_-, a_+; (\ell_-, \varphi_-), (\ell_+, \varphi_+)\big)$.
We define

$$T(g)_0(\Lambda) : HF(N_-, \vec{\ell}_-, \varphi_-)(\Lambda) \to HF(N_+, \vec{\ell}_+, \varphi_+)(\Lambda)$$

by

$$\left(T(g)_0(\Lambda)\right)[a_-] = \sum_{a_-} \#\mathcal{M}\left(M,\Lambda;a_-,a_+;(\ell_-,\varphi_-),(\ell_+,\varphi_+)\right)[a_+].$$

Lemma 5.3

$T(g)_0(\Lambda)$ is a chain map.

Proof: This follows from Theorem 3.16 in the same way, the proof of well definedness of Floer homology of closed 3 manifolds [7] . (We remark $c(N\times\boldsymbol{R},E)=0$.)

Thus we constructed $T(g)_0(\Lambda)$. We are going to construct $T(g)_k(\Lambda_0,\cdots,\Lambda_k)$. Let $a_i\in\Lambda_{i-1}\cap\Lambda_i$, $a_0\in R(N_-)$, $a_{k+1}\in R(N_+)$ such that $res(a_0)\in\Lambda_0$, $res(a_{k+1})\in\Lambda_k$. (Here and hereafter we write $R(N_-)$ in place of $R(N_-,E;\vec{\ell}_-,\varphi_-)$.) Imitating the construction of § 4, we define the moduli space $\mathcal{M}\left(M,(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});(\ell_-,\varphi_-),(\ell_+,\varphi_+)\right)$ as follows. (Here and hereafter we write M in place of $N\times\boldsymbol{R}$ in case when the metric we consider is not direct product metric but the metric g.) We consider the set of multiples $(\mathcal{A},(t_1,\cdots,t_k))$ such that :

(5.4.1) $\mathcal{A}$ is a smooth connection of E on M.

(5.4.2) $t_1<\cdots<t_k$.

(5.4.3) $\mathcal{A}-\mathcal{A}_0$ is of L^2-class, where $\mathcal{A}_0$ is a connection of E on M which is equal to a_0 at $N\times(-\infty,-R]$ and to a_{k+1} at $N\times[R,\infty)$.

(5.4.4) $\mathcal{A}$ satisfies the equation (3.4.1), (3.4.2) at $\Sigma\times[-1,1]\times\boldsymbol{R}$.

(5.4.5) $\mathcal{A}$ satisfies the equation (3.12) at $N\times(-\infty,-R]\cup N\times[R,\infty)$.

(5.4.6) $\mathcal{A}$ is ASD at $M-\left((N-\Sigma\times[-1,1])\times(-\infty,-R]\cup(N-\Sigma\times[-1,1])\times[R,\infty)\right)$.

(5.4.7) $[A(1,t_i)]=a_i$.

(5.4.8) If $t_i<t<t_{i+1}$, then $[A(1,t)]\in\Lambda_i$. Here we put $t_0=-\infty$, $t_{k+1}=\infty$.

We denote $\hat{\mathcal{M}}\left(M,(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});(\ell_-,\varphi_-),(\ell_+,\varphi_+)\right)$ the set of all $(\mathcal{A},(t_1,\cdots,t_k))$ satisfying (5.4.1), $\cdots$, (5.4.8). Let $\mathcal{M}\left(M,(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});(\ell_-,\varphi_-),(\ell_+,\varphi_+)\right)$ be its quotient by the gauge transformation group. (We remark that we can not and do not divide by $\boldsymbol{R}$ action.)

Lemma 5.5*

For generic g, the following holds.

(5.5.1)

$\mathcal{M}\big(M,(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});(\ell_-,\varphi_-),(\ell_+,\varphi_+)\big)$ *is a smooth manifold of dimension* $\sum\mu(a_i)-\mu(a_{k+1})+k$.

(5.5.2)

If $\sum\mu(a_i)-\mu(a_{k+1})+k=0$, *then* $\mathcal{M}\big(M,(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});(\ell_-,\varphi_-),(\ell_+,\varphi_+)\big)$ *consists of finitely many points.*

(5.5.3)

If $\sum\mu(a_i)-\mu(a_{k+1})+k=1$, *then* $\mathcal{M}\big(M,(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});(\ell_-,\varphi_-),(\ell_+,\varphi_+)\big)$ *has a compactification* $C\mathcal{M}\big(M,(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});(\ell_-,\varphi_-),(\ell_+,\varphi_+)\big)$ *such that its boundary is a union of the following* 6 *kinds of spaces.*

$$\bigcup_{\substack{a'_{k+1}\in R(N,E):\, res_+ b\in\Lambda_k,\\ \mu(a'_{k+1})=\mu(a_{k+1})-1}} \begin{array}{l}\mathcal{M}\big(M,(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a'_{k+1});(\ell_-,\varphi_-),(\ell_+,\varphi_+)\big)\\ \times\overline{\mathcal{M}}(N_+\times\boldsymbol{R};\Lambda_k;a'_{k+1},a_{k+1})\end{array}$$

$$\bigcup_{\substack{a'_0\in R(N,E):\, res_- a'_0\in\Lambda_0,\\ \mu(a'_0)=\mu(a_0)+1}} \begin{array}{l}\overline{\mathcal{M}}(N_-\times\boldsymbol{R};\Lambda_0;a_0,a'_0;\ell_-,\varphi_-)\\ \mathcal{M}\big(M,(\Lambda_0,\cdots,\Lambda_k);(a'_0,\cdots,a_{k+1});(\ell_-,\varphi_-),(\ell_+,\varphi_+)\big)\end{array}$$

$$\bigcup_{\substack{a'_i\in\Lambda_{i-1}\cap\Lambda_i\\ \mu(a'_i)=\mu(a_i)-1}} \begin{array}{l}\mathcal{M}\big(M,(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a'_i,\cdots,a_{k+1});(\ell_-,\varphi_-),(\ell_+,\varphi_+)\big)\\ \times\overline{\mathcal{M}}\big(R(\Sigma);\Lambda_{i-1},\Lambda_i;a'_i,a_i\big)\end{array}$$

$$\bigcup_{a_{i,j}\in\Lambda_i\cap\Lambda_j} \begin{array}{l}\mathcal{M}\Big(M,\big(\Lambda_0,\cdots,\Lambda_i,\Lambda_j,\cdots,\Lambda_k\big);\big(a_0,\cdots,a_{i-1},a_{i,j},a_{j+1}\cdots,a_{k+1}\big);(\ell_-,\varphi_-),(\ell_+,\varphi_+)\Big)\\ \times\overline{\mathcal{M}}\Big(R(\Sigma);\big(\Lambda_i,\cdots,\Lambda_j\big);a_{i,j},a_i,\cdots,a_j\Big)\end{array}$$

$$\bigcup_{b\in R(N,E):\, res_+ b\in\Lambda_i} \begin{array}{l}\mathcal{M}\big(M,(\Lambda_0,\cdots,\Lambda_i);(a_0,\cdots,a_i,b);(\ell_-,\varphi_-),(\ell_+,\varphi_+)\big)\\ \overline{\mathcal{M}}\big(N_+;(b,a_{i+1},\cdots,a_{k+1});(\Lambda_i,\cdots,\Lambda_k),(\ell_+,\varphi_+)\big)\end{array}$$

$$\bigcup_{b\in R(N,E):\, res_- b\in\Lambda_i} \begin{array}{l}\mathcal{M}\big(N_-;(a_0,\cdots,a_i,b);(\ell_-,\varphi_-)\big)\times\\ \overline{\mathcal{M}}\big(M,(\Lambda_0,\cdots,\Lambda_i);(b,a_{i+1},\cdots,a_{k+1});(\ell_-,\varphi_-),(\ell_+,\varphi_+)\big)\end{array}.$$

Proof of Lemma 5.5 is again a straight forward analogue of the proof of Theorem 4.6 and is given in subsequent papers.

Using $\mathcal{M}\big(M,(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});(\ell_-,\varphi_-),(\ell_+,\varphi_+)\big)$ we define

$$\begin{aligned}T(g)_k(\Lambda_0,\cdots,\Lambda_k):C(\Lambda_0,\Lambda_1)\otimes\cdots\otimes C(\Lambda_{k-1},\Lambda_k)\\ \to Hom(HF(N_-)(\Lambda_0),HF(N_+)(\Lambda_k))\end{aligned}$$

as follows.

$$T(g)_k(\Lambda_0,\cdots,\Lambda_k)([a_1]\otimes\cdots\otimes[a_k])([a_0])$$
$$=\sum_{a_{k+1}}\#\mathcal{M}\big(M,(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});(\ell_-,\varphi_-),(\ell_+,\varphi_+)\big)\,[a_{k+1}]$$

The consequence of Lemma 5.5 is the following. (We write $T(g)_k$ in place of $T(g)_k(\Lambda_0,\cdots,\Lambda_k)$ in the lemma.)

Lemma 5.6

$$\begin{aligned}
&\partial\big((T(g)_k([a_1]\otimes\cdots\otimes[a_k]))([a_0])\big)\\
&\quad+\big(T(g)_k([a_1]\otimes\cdots\otimes[a_k])\big)(\partial[a_0])\\
&\quad+\sum T(g)_{k-i+1}([a_{i+1}]\otimes\cdots\otimes[a_k])\Big(HF\big(N_-,\vec{\ell}_-,\varphi_-\big)_i([a_1]\otimes\cdots\otimes[a_i])([a_0])\Big)\\
&\quad+\sum\big(T(g)_k([a_1]\otimes\cdots\otimes\eta([a_i]\otimes\cdots\otimes[a_j])\otimes\cdots\otimes[a_k])\big)([a_0])\\
&\quad+\sum HF\big(N_+,\vec{\ell}_+,\varphi_+\big)_k([a_{i+1}]\otimes\cdots\otimes[a_k])\big(T(g)_i([a_1]\otimes\cdots\otimes[a_i])([a_0])\big)\\
&\quad+\sum\big(T(g)_k([a_1]\otimes\cdots\otimes\partial[a_i]\otimes\cdots\otimes[a_k])\big)([a_0])\\
&=0
\end{aligned}$$

Lemma 5.6 means that pre natural transformations $T(g)$ is a natural transformation from $HF(N_-,\vec{\ell}_-,\varphi_-)$ to $HF(N_+,\vec{\ell}_+,\varphi_+)$. (Definition 10.3).

Lemma 5.6 is immediate from Lemma 5.5. In fact the terms in the equality correspond to the 6 kinds of spaces appeared in (5.5.3).

We next consider two metrics g, g' on $N\times\boldsymbol{R}=M$, which satisfy Condition (5.2).

Lemma 5.7 *There exists a pre natural transformation* $S:HF(N_-,\vec{\ell}_-,\varphi_-)\to HF(N_+,\vec{\ell}_+,\varphi_+)$ *such that* $\partial S=T(g)-T(g')$.

Proof: Let g_u, $u\in[0,1]$ be a family of metrics on $N\times\boldsymbol{R}=M$ such that g_s satisfies Condition (5.2) and that $g_0=g$, $g_1=g'$. We put

$$\mathcal{M}_{para}\big(M,(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});(\ell_-,\varphi_-),(\ell_+,\varphi_+)\big)$$
$$=\bigcup_{u\in[0,1]}\mathcal{M}\big((M,g_u),(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});(\ell_-,\varphi_-),(\ell_+,\varphi_+)\big).$$

and define :

$$\left(S_0(\Lambda)\right)[a_-] = \sum_{a_+} \# \mathcal{M}_{para}(M,\Lambda;a_-,a_+)\,[a_+]$$

$$\begin{aligned} & S_k(\Lambda_0,\cdots,\Lambda_k)([a_1]\otimes\cdots\otimes[a_k])\left([a_0]\right) \\ & \quad = \mathcal{M}_{para}\left(M,(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});(\ell_-,\varphi_-),(\ell_+,\varphi_+)\right)[a_{k+1}] \end{aligned}.$$

Then using a lemma* similar to Lemma 5.5, (which we prove in subsequent papers), we obtain Lemma 5.7.

Now the rest of the proof of Theorem 5.1 is quite similar to the proof of the well definedness of Floer homology of closed 3 manifold.

Let h be the metric on $M = N\times\boldsymbol{R}$ obtained by pulling back g by the diffeomorphism $M\to M$, $(x,t)\mapsto(x,-t)$. It induces a natural transformation $T(h): HF(N_+,\vec{\ell}_+,\varphi_+)\to HF(N_-,\vec{\ell}_-,\varphi_-)$.

We next take a sufficiently large R and define $g\#_R h$ by

$$\left(g\#_R h\right)(x,t) = \begin{cases} g(x,t+2R) & t\le -R \\ g_{N_+}\oplus dt^2 & -R\le t\le R. \\ h(x,t-2R) & R\le t \end{cases}$$

Lemma 5.8 *For sufficiently large* R, $T(g\#_R h) = \eta\left(T(h),T(g)\right)$. *Here the right hand side is the composition of pre natural transformation, defined in* § 11.

Proof: We can use Taubes' type gluing result to show the following equality*

$$\begin{aligned} & \mathcal{M}\left((M,g\#_R h),(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1});(\ell_-,\varphi_-),(\ell_+,\varphi_+)\right) \\ & \quad = \bigcup_i \mathcal{M}\left((M,g),(\Lambda_0,\cdots,\Lambda_i);(a_0,\cdots,a_i);(\ell_-,\varphi_-),(\ell_+,\varphi_+)\right) \qquad , \\ & \qquad\qquad \times\mathcal{M}\left((M,h),(\Lambda_i,\cdots,\Lambda_k);(a_i,\cdots,a_{k+1});(\ell_+,\varphi_+).(\ell_-,\varphi_-)\right) \end{aligned}$$

for sufficiently large R. Hence the lemma follows from definition.

We can prove that $T(g_{N\pm}\oplus dt^2)$ is the identity functor, (defined in § 11), by using free $\boldsymbol{R}$ action on moduli spaces appeared in the definition of $T(g_N\oplus dt^2)$. Hence, by Lemmata 5.6 and 5.8, $\eta\left(T(h),T(g)\right)$ is homotopic to identity. In a similar way $\eta\left(T(g),T(h)\right)$ is homotopic to identity. The proof of Theorem 5.1 (modulo analytic detail) is now complete.

§ 6 Well definedness II

In this section, we discuss how the A^∞ category $\mathcal{Lag}(X,\omega,J)$ changes when we change the compatible almost complex structure J. It might be possible to study more general problem namely to include the deformation of symplectic structure. It seems interesting to study the functor { symplectic manifolds } $\to$ { A^∞ categories }, $X \to \mathcal{Lag}(X)$ and its relation to the deformation of symplectic structure more systematically. (Since the language of A^∞ category is suited to study the deformations.) However to discuss it is out of the scope of this paper. We will discuss it in [12]. We study here only the change of almost complex structure. We consider symplectic manifold (X,ω) such that $c^1(X) = N[\omega]$, $N \ge 2$. Let J_1, J_2 be two almost complex structures compatible with ω. Our main result is :

Theorem 6.1 *$\mathcal{Lag}(X,\omega,J_1)$ is weakly homotopy equivalent to $\mathcal{Lag}(X,\omega,J_2)$. The weak homotopy equivalence is canonical up to homotopy.*

The definition of homotoy equivalence of A^∞ category is in § 11, and the definition of weak homotopy equivalence of topological A^∞ category is in § 13.

To prove Theorem 6.1, we use parametrized versions of the moduli spaces introduced in § 3. Let I be a manifold with boundary (we use the case $I = [0,1]$ and $I = [0,1]^2$ mainly later). Let J_u, $u \in I$ be a smooth family of almost complex structures of X compatible with symplectic structure ω.

Let Λ_i be Lagrangian submanifolds of (X,ω), $p,q \in \Lambda_1 \cap \Lambda_2$, and $\mathcal{M}(X;\Lambda_1,\Lambda_2;p,q)$ be the moduli space of pseudo holomorphic disks introduced in § 2. To specify the almost complex structure we use, we write $\mathcal{M}((X,J_u);\Lambda_1,\Lambda_2;p,q)$. Let $\overline{\mathcal{M}}((X,J_u);\Lambda_1,\Lambda_2;p,q)$ be its quotient by the action of $\boldsymbol{R}$. We put

$$\overline{\mathcal{M}}_I(X;\Lambda_1,\Lambda_2;p,q) = \bigcup_{u\in I}\overline{\mathcal{M}}((X,J_u);\Lambda_1,\Lambda_2;p,q).$$

For $p_i \in \Lambda_{i-1} \cap \Lambda_i$, $p_0 \in \Lambda_k \cap \Lambda_0$, let $\overline{\mathcal{M}}((X,J_u);(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k))$ be as in § 2 and we put

$$\mathcal{M}_I(X;(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k)) = \bigcup_{u\in I}\overline{\mathcal{M}}((X,J_u);(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k)).$$

The following lemmata are straight forward analogue of Theorems 2.12 , 2,18 etc. Their proofs will be in subsequent papers.

Lemma 6.2* *For each generic pair* $\Lambda_1, \Lambda_2 \in \mathcal{Ob}(\mathcal{Lag}(X,\omega))$, *and generic family* J_u, *the space* $\overline{\mathcal{M}}_I(X;\Lambda_1,\Lambda_2;p,q)$ *is a smooth manifold and its dimension is given by* :

$$\dim \mathcal{M}(X;\Lambda_1,\Lambda_2;p,q) \equiv \mu(p) - \mu(q) + \dim I \mod 2N.$$

Lemma 6.3* *If* $\mu(p) - \mu(q) + \dim I = 1$, *then* $\overline{\mathcal{M}}_I(X;\Lambda_1,\Lambda_2;p,q)$ *consists of finitely many points.*

Lemma 6.4* *If* $\mu(p) - \mu(q) + \dim I = 2$, *then* $\overline{\mathcal{M}}_I(X;\Lambda_1,\Lambda_2;p,q)$ *can be compactified to* $C\mathcal{M}_I(X;\Lambda_1,\Lambda_2;p,q)$ *such that it is a compact one dimensional manifold with boundary and its boundary is a union of*

$$\bigcup_{\substack{r:\mu(p)>\mu(r)>\mu(q)\\ u\in I}} \overline{\mathcal{M}}((X,J_u);\Lambda_1,\Lambda_2;p,r) \times \overline{\mathcal{M}}((X,J_u);\Lambda_1,\Lambda_2;r,q)$$

$$\overline{\mathcal{M}}_{\partial I}(X;\Lambda_1,\Lambda_2;p,q).$$

Lemma 6.5* *For generic* Λ_i, *and generic family* J_u, *the space* $\mathcal{M}_I(X;(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_{k+1}))$ *is a smooth manifold and its dimension is given by :*

$$\dim \mathcal{M}_I(X;(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k)) \equiv \sum_{i=1}^{k} \mu(p_i) - \mu(p_0) + k - 2 + \dim I \mod 2N.$$

Lemma 6.6* *If* $\sum_{i=1}^{k} \mu(p_i) - \mu(p_0) + k - 2 + \dim I = 0$, *then* $\mathcal{M}_I(X;(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k))$ *consists of finitely many points.*

Lemma 6.7* *If* $\sum_{i=1}^{k} \mu(p_i) - \mu(p_0) + k - 2 + \dim I = 1$, *then* $\mathcal{M}_I(X;(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k))$ *can be compactified to a compact one dimensional manifold* $C\mathcal{M}_I(X;(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k))$ *whose boundary is a union of the following* 3 *kinds of spaces* :

(a) $$\bigcup_{u\in I}\bigcup_{p_i'\in\Lambda_{i-1}\cap\Lambda_i} \mathcal{M}((X,J_u);(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_{i-1},p_i',p_{i+1},\cdots p_k)) \times \mathcal{M}((X,J_u);\Lambda_{i-1},\Lambda_i;p_i',p_i)$$

(b) $$\bigcup_{p_{i,j}\in\Lambda_i\cap\Lambda_j} \mathcal{M}((X,J_u);(\Lambda_0,\cdots,\Lambda_i,\Lambda_j,\cdots\Lambda_k);(p_0,\cdots,p_{i-1},p_{i,j},p_{j+1},\cdots,p_k)) \times \mathcal{M}((X,J_u);(\Lambda_i,\cdots,\Lambda_j);(p_{i,j},p_i,\cdots,p_j))$$

(c) $$\mathcal{M}_{\partial I}(X;(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k)).$$

We are going to use these lemmata to show Theorem 6.1. Let J_1, J_2 be two almost complex structures compatible with ω. We take a path of almost complex structures J_u, $u \in [1,2]$ which are compatible with ω. Our first task is to construct a topological A^∞ functor $F(J_u) : \mathcal{L}ag(X,\omega,J_1) \to \mathcal{L}ag(X,\omega,J_2)$.

Let Ξ be a finite set of simply connected Lagrangian submanifolds of (X,ω). Let $I_0 \in [1,2]$ be the finite subset with the following properties.

(6.8.1) If $\Lambda_1,\Lambda_2 \in \Xi$, $p,q \in \Lambda_1 \cap \Lambda_2$, $p \neq q$, $\mu(p) - \mu(q) = 0$, and if $\overline{\mathcal{M}}((X,J_u);\Lambda_1,\Lambda_2;p,q)$ is nonempty then $u \in I_0$.
(6.8.2) If $\Lambda_0,\cdots,\Lambda_k \in \Xi$, $\sum \mu(p_i) - \mu(p_0) + k - 1 = 0$ and if $\mathcal{M}\big((X,J_u);(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k)\big)$ is nonempty, then $u \in I_0$.
(6.8.3) For each $u \in I_0$, only one of the moduli spaces in (6.8.1), (6.8.2) is non empty. The order of that moduli space is one.

Finiteness of such a subset I_0 is a consequence of Lemmata 6.3 and 6.6. By perturbing the family J_u, we can achieve (6.8.3). Let $u_1 < u_2 < \cdots < u_{N-1} < u_N$ be all the elements of I_0. We choose u'_i such that

$$1 = u'_0 < u_1 < u'_1 < u_2 < \cdots < u'_{N-2} < u_{N-1} < u'_{N-1} < u_N < u'_N = 2 .$$

We consider the full subcategory $C^i(\Xi)$ of $\mathcal{L}ag((X,\omega,J_{u'_i}))$ such that $\mathcal{O}b(C^i(\Xi)) = \Xi$. We are going to construct an A^∞functor $F^i(\Xi) : C^i(\Xi) \to C^{i+1}(\Xi)$.

These A^∞ functors are identity map on the set of objects. Namely we put $F^i_0(\Lambda) = \Lambda$. We next define $F^i_1(\Xi)(\Lambda_1,\Lambda_2) : \mathcal{L}ag((X,\omega,J_{u'_i}))(\Lambda_1,\Lambda_2) \to \mathcal{L}ag((X,\omega,J_{u'_{i+1}}))(\Lambda_1,\Lambda_2)$ by :

$$F^i_1(\Xi)(\Lambda_1,\Lambda_2)([p]) = \sum \#\overline{\mathcal{M}}_{[u'_i,u'_{i+1}]}(X;\Lambda_1,\Lambda_2;p,q)\,[q] .$$

We also define $F^i_k(\Xi)(\Lambda_0,\cdots,\Lambda_k) : \mathcal{L}ag((X,\omega,J_{u'_i}))(\Lambda_0,\Lambda_1) \otimes \cdots \otimes \mathcal{L}ag((X,\omega,J_{u'_i}))(\Lambda_{k-1},\Lambda_k) \to \mathcal{L}ag((X,\omega,J_{u'_{i+1}}))(\Lambda_0,\Lambda_k)$ by :

$$F^i_k(\Xi)(\Lambda_0,\cdots,\Lambda_k)\big([p_1]\otimes\cdots\otimes[p_k]\big) = \sum \#\mathcal{M}_{[u'_i,u'_{i+1}]}\big(X;(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k)\big)[p_0] .$$

We remark that, by (6.8.3), one of the following holds.

(6.9.1)

$\#\mathcal{M}_{[u'_i,u'_{i+1}]}\big(X;(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k)\big)$ is nonzero for exactly one choice of $\Lambda_0,\cdots\Lambda_k;p_0,\cdots,p_k$. (Here we consider only $\mathcal{M}_{[u'_i,u'_{i+1}]}\big(X;(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k)\big)$ which is of zero dimensional.) $\#\overline{\mathcal{M}}_{[u'_i,u'_{i+1}]}(X;\Lambda_1,\Lambda_2;p,q)$ are all nonempty for $\mu(q)=\mu(p)$, $p\neq q$.

(6.9.2)

$F^i_k(\Xi)(\Lambda_0,\cdots,\Lambda_k)$ are all zero for $k\geq 2$. There exist unique Λ_1,Λ_2, p,q, $p\neq q$, $\mu(q)=\mu(p)$, such that $\#\overline{\mathcal{M}}_{[u'_i,u'_{i+1}]}(X;\Lambda_1,\Lambda_2;p,q)$ is nonzero.

We then have the following :

Lemma 6.10 *$F^i : C^i \to C^{i+1}$ is an A^∞ functor*

Proof: We use the symbol ∂_u for the boundary operators of $\mathcal{L}ag(X,J_u)$.
Suppose (6.9.1) holds. Then by Lemma 6.4, we find that $F^i_1(\Xi)(\Lambda_1,\Lambda_2):\mathcal{L}ag((X,\omega,J_{u'_i}))(\Lambda_1,\Lambda_2)\to\mathcal{L}ag((X,\omega,J_{u'_{i+1}}))(\Lambda_1,\Lambda_2)$ is identity and also that $\#\overline{\mathcal{M}}_u(X;\Lambda_1,\Lambda_2;p,q)$ for $\mu(p)-\mu(q)=1$ is dependent of $u\in[u'_i,u'_{i+1}]$. Namely ∂_u is independent of $u\in[u'_i,u'_{i+1}]$. We write it as ∂. Then Lemma 6.7 implies that

$$
\begin{aligned}
&\partial\Big(F^i_k(\Xi)(\Lambda_0,\cdots,\Lambda_k)\big([p_1]\otimes\cdots\otimes[p_k]\big)\Big)\\
&\qquad+\sum\pm F^i_k(\Xi)(\Lambda_0,\cdots,\Lambda_k)\big([p_1]\otimes\cdots\otimes\partial[p_\ell]\otimes\cdots\otimes[p_k]\big)\\
&\qquad+\sum\pm F^i_{k-\ell+m}(\Xi)(\Lambda_0,\cdots,\Lambda_k)\Big([p_1]\otimes\cdots\otimes\eta_{\ell-m+1}\big([p_\ell]\otimes\cdots\otimes[p_m]\big)\otimes\cdots\otimes[p_k]\Big).\\
&\qquad+\eta^{i+1}_k\big([p_1]\otimes\cdots\otimes[p_k]\big)-\eta^i_k\big([p_1]\otimes\cdots\otimes[p_k]\big)\\
&=0
\end{aligned}
\tag{6.11}
$$

Here we write η^i_k for the k-th composition in $\mathcal{L}ag(X,J_{u'_i})$. We remark that $\eta^i_{\ell-m+1}\big([p_\ell]\otimes\cdots\otimes[p_m]\big)$ is equal to $\eta^{i+1}_{\ell-m+1}\big([p_\ell]\otimes\cdots\otimes[p_m]\big)$, in the case the term $F^i_{k-\ell+m}(\Xi)(\Lambda_0,\cdots,\Lambda_k)\Big([p_1]\otimes\cdots\otimes\eta^*_{\ell-m+1}\big([p_\ell]\otimes\cdots\otimes[p_m]\big)\otimes\cdots\otimes[p_k]\Big)$ is nonzero, because of (6.9.1). Hence we simply wrote $\eta_{\ell-m+1}\big([p_\ell]\otimes\cdots\otimes[p_m]\big)$ in Formula (6.11). Formula (6.11) means that $F^i(\Xi):C^i\to C^{i+1}$ is an A^∞ functor.
The proof in the case when (6.9.2) holds is similar.

Remark 6.12 In Lemma 6.10, we consider only the case when $\Lambda_i\neq\Lambda_j$ for $i\neq j$. So it is a bit imprecise to say that $F^i(\Xi)$ is an A^∞ functor. However, to define a topological A^∞ functor $F(J_u):\mathcal{L}ag(X,\omega,J_1)\to\mathcal{L}ag(X,\omega,J_2)$, it is enough to consider this case only.

We now define an A^∞functor $F(\Xi): C^0(\Xi) \to C^N(\Xi)$ by $F(\Xi) = F^{N-1}(\Xi) \circ \cdots \circ F^0(\Xi)$. (We remark that the composition of A^∞ functors are associative by Lemma 11.12.)

By definition, it is easy to see that, for $\Xi \subseteq \Xi'$, we have a commutative diagram of A^∞ functors

$$\begin{array}{ccc} C^0(\Xi) & \xrightarrow{F(\Xi)} & C^N(\Xi) \\ \downarrow & & \downarrow \\ C^0(\Xi') & \xrightarrow{F(\Xi')} & C^{N'}(\Xi') \end{array}$$

Diagram 6.12

here the vertical arrows are natural inclusions. We remark that Diagram 6.12 not only commutes up to homotopy but commutes exactly. It follows immediately that we obtain a topological A^∞functor $F(J_u): \mathcal{Lag}(X,\omega,J_1) \to \mathcal{Lag}(X,\omega,J_2)$.

We construct the converse. We consider J_{3-u}, which is a homotopy from J_2 to J_1. We have :

Lemma 6.13 *$F(J_{3-u}) \circ F(J_u)$ and $F(J_u) \circ F(J_{3-u})$ are identity functors.*

Remark 6.14 Lemma 6.13 asserts more that we need. Namely we show that $F(J_{3-u}) \circ F(J_u)$ and $F(J_u) \circ F(J_{3-u})$ are identity functors in place of showing them to be homotopic to identity. One reason we can prove it is that the morphisms of $\mathcal{Lag}(X,\omega,J_1)$ and $\mathcal{Lag}(X,\omega,J_2)$ are isomorphic as abelian group. The other reason is that the chain map induced by $F(J_u)$ to the set of morphisms is filtered. (Compare [3] where a similar facts in the situation of gauge theory Floer homology is proved and applied. As a consequence the chain complex of gauge theory Floer homology is well defined up to isomorphism (not only up to chain homotopy), in the case when the set of flat connections are discrete and when $H^1(M^3, ad\ a) = 0$ for all flat connections.) (We do no uses this fact in the proof.) However the topological A^∞ functor $F(J_u): \mathcal{Lag}(X,\omega,J_1) \to \mathcal{Lag}(X,\omega,J_2)$ itself does depend on the choice of J_u and only its homotopy class is well defined, as we will see below. It seems that this additional fact that $F(J_u)$ is an isomorphism is not so useful. However thanks to it, we do not have to worry on the potential trouble caused by the fact that $\mathcal{Lag}(X,\omega,J)$ has no identity but has only an approximate identity.

Remark 6.15 By the same reason as Lemma 6.12, Lemma 6.13 is in a bit imprecisely stated in the sense that $F(J_{3-u}) \circ F(J_u)$ is only a topological A^∞ functor. Especially

$(F(J_{3-u})\circ F(J_u))(\Lambda_0,\cdots,\Lambda_k)$ is well defined only when $\Lambda_0,\cdots,\Lambda_k$ are mutually distinct. The precise statement of Lemma 6.13 is that $F(J_{3-u})\circ F(J_u)$ *and* $F(J_u)\circ F(J_{3-u})$ coincides with the identity functor where it is defined. This statement however is enough to show that $F(J_u)^*\circ F(J_{3-u})^*:\mathcal{Func}(\mathcal{Lag}(X,J_1),\mathcal{Ch})\to\mathcal{Func}(\mathcal{Lag}(X,J_1),\mathcal{Ch})$ is the identity functor. (We remark that $F(J_u)^*$ is not only an topological A^∞ functor but also an A^∞ functor.) This fact, by definition, is enough to show that $\mathcal{Lag}(X,J_1)$ is weakly homotopy equivalent to $\mathcal{Lag}(X,J_2)$. Our A^∞ Yoneda's lemma (Proposition 13.7) implies that weak homotopy equivalence implies that the operations (products and (higher) Massey products) we obtain from A^∞ structures are preserved by weak homotopy equivalence.

Lemma 6.13, in fact, immediately follows from definition of $F^i(\Xi)$ and Definition 12.8, once we remark that it suffices to work only on the small domain $[u_i',u_{i+1}']$. In fact, $F(J_u)$ and $F(J_{3-u})$ restricted on this small domain coincides up to sign, (which we do not consider in this paper).

To complete the proof of Theorem 6.1, we are going to show that the A^∞ functor $F(J_u)$ is independent of the choice of J_u up to homotopy. (The definition of two A^∞ functors homotopic to each other is given in § 12.) We next prove the following Lemma 6.16. Let $J_{u,1}$ and $J_{u,2}$ be two paths of almost complex structures which are compatible with ω and such that $J_{1,i}=J_1$, $J_{2,i}=J_2$.

We need to state Lemma a bit carefully since our A^∞ category $\mathcal{Lag}(X,\omega,J)$ do not have an identity. The way to handle this case is discussed in § 13. Fortunately our functor are identity on objects. Using this fact we can simplify the discussion.

Lemma 6.16 *There exists a natural trnasformations* $T_{12}:F(J_{s,1})\to F(J_{s,2})$, $T_{21}:F(J_{s,2})\to F(J_{s,1})$ *such that the composition* $T_{12}\circ T_{21}$ *and.*$T_{21}\circ T_{12}$ *coincides identity transformation where it is defined.*

We remark that $1_{F(J_{s,1})}:F(J_{s,1})\to F(J_{s,1})$ is not everlywhere defined. Lemma 6.16 means that it coincides to $T_{12}\circ T_{21}$ when both are well defined. This is enouth for example to show the composition

$$\mathcal{Rep}(\mathcal{Lag}(X,\omega,J_1))^o\to\mathcal{Lag}(X,\omega,J_1)\xrightarrow{F(J_{s,1})}\mathcal{Lag}(X,\omega,J_2)\to\mathcal{Rep}(\mathcal{Lag}(X,\omega,J_2))^o$$

is homotopic to

$$\mathcal{R}ep(\mathcal{L}ag(X,\omega,J_1))^o \to \mathcal{L}ag(X,\omega,J_1) \xrightarrow{F(J_{s2})} \mathcal{L}ag(X,\omega,J_2) \to \mathcal{R}ep(\mathcal{L}ag(X,\omega,J_2))^o.$$

Proof: We first remark that the set of all almost complex structures compatible to a given symplectic structure ω is contractible. (Gromov [25].) Hence we have a family $J_{u,v}$, $(u,v) \in [1,2]\times[1,2]$ extending $J_{u,1}$, $J_{u,2}$ and such that $J_{1,v} = J_1$, $J_{2,v} = J_2$

Let Ξ be a finite set of objects of (X,ω). Let $\mathcal{L}ag(X,\omega,J)(\Xi)$ be a full subcategory of $\mathcal{L}ag(X,\omega,J)$ the set of whose objects is Ξ. Let $C^{(u,v)}(\Xi)$ be the full subcategory of $\mathcal{L}ag(X,\omega,J_{u,v})$ such that its objects are elements of Ξ. It suffices to show that the restriction of $F(J_{u,1})$ to $C^{(1,1)}(\Xi)$ is homotopic to the restriction of $F(J_{u,2})$ to $C^{(1,2)}(\Xi) = C^{(1,1)}(\Xi)$ by the homotopy compatible with the inclusion $C^{(u,v)}(\Xi) \to C^{(u,v)}(\Xi')$. (We need to be careful to say that two functors are homotopic on $C^{(1,1)}(\Xi)$ since $C^{(1,1)}(\Xi)$ does not have the identity. We mean that they are homotopic in a similar sense as Lemma 6.16.)

We first find finite subsets I_0, I_0', I_1, I_1' of $[1,2]$ with the following Properties 6.19.

Let L_0 be the set of all (u,v) such that the moduli space of virtual dimension -2 of the pseudo holomorphic disks in $(X,J_{u,v})$ is nonempty. Precisely $(u,v)\in L_0$ if one of the following holds :

(6.17.1) There exist $\Lambda_1,\Lambda_2 \in \Xi$, $p,q \in \Lambda_1 \cap \Lambda_2$, $\mu(p)-\mu(q) = -1$, such that $\overline{\mathcal{M}}((X,J_{u,v});\Lambda_1,\Lambda_2;p,q)$ is nonempty

(6.17.2) There exist $\Lambda_0,\cdots,\Lambda_k \in \Xi$, $\sum \mu(p_i) - \mu(p_0) + k = 0$, such that $\mathcal{M}\big((X,J_{u,v});(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k)\big)$ is nonempty.

Let L_1 be the set of all (u,v) such that the moduli space of virtual dimension -1 of the pseudo holomorphic disks in $(X,J_{u,v})$ is nonempty. Precisely $(u,v)\in L_0$ if one of the following holds :

(6.18.1) There exist $\Lambda_1,\Lambda_2 \in \Xi$, $p,q \in \Lambda_1 \cap \Lambda_2$, $p \ne q$, $\mu(p)-\mu(q) = 0$, such that $\overline{\mathcal{M}}((X,J_{u,v});\Lambda_1,\Lambda_2;p,q)$ is nonempty.

(6.18.2) There exist $\Lambda_0,\cdots,\Lambda_k \in \Xi$, $\sum \mu(p_i) - \mu(p_0) + k = 0$ such that $\mathcal{M}\big((X,J_{u,v};(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k)\big)$ is nonempty.

Property 6.19

(6.19.1) $L_0 \subseteq I_0 \times I_1$.

(6.19.2) For each $(u,v)\in L_0$ at most only one of the moduli spaces in (6.17.1) or (6.17.2) is nonempty. And the order of that moduli space is one.
(6.19.3) For each $(u,v)\in L_1$ at most only one of the moduli spaces in (6.18.1) or (6.18.2) is nonempty. And the order of that moduli space is one.
(6.19.4) We put $I_0=\{u_1,\cdots,u_N\}$, $I_0'=\{u_0',\cdots,u_N'\}$ such that

$$1=u_0'<u_1<u_1'<u_2<\cdots<u_{N-2}'<u_{N-1}<u_{N-1}'<u_N<u_N'=2,$$

We put $I_0=\{u_1,\cdots,u_N\}$, $I_0'=\{u_0',\cdots,u_N'\}$ such that

$$1=u_0'<u_1<u_1'<u_2<\cdots<u_{N-2}'<u_{N-1}<u_{N-1}'<u_N<u_N'=2.$$

(6.19.5) We put

$$\ulcorner=\{u_i'\}\times[v_i',v_{i+1}']\cup[u_i',u_{i+1}']\times\{v_{i+1}'\},\quad \lrcorner=[u_i',u_{i+1}']\times\{v_i'\}\cup\{u_{i+1}'\}\times[v_i',v_{i+1}'].$$

(6.19.6) We then have,

$$\#(L_1\cap\ulcorner)+\#L_0\cap[u_i',u_{i+1}']\times[v_j',v_{j+1}']\le 2$$

$$\#(L_1\cap\lrcorner)+\#L_0\cap[u_i',u_{i+1}']\times[v_j',v_{j+1}']\le 2$$

Figure 6.20

The existence of such I_0,I_0', I_1,I_1' after perturbing $J_{u,v}$ is a consequence of Lemmata 6.3 and 6.6.

We put $C^{i,j}(\Xi)^*=C^{u_i',v_j'}(\Xi)^*$. We remark $C^{i,j}(\Xi)^*=\mathcal{R}ep(C^{i,j}(\Xi),\mathcal{C}h)$ and $C^{i,j}(\Xi)$ is a full subcategory of $\mathcal{L}ag((X,\omega,J_{u_i',v_j'}))$. Using families $J_{u,v}$: $(u,v)\in\{u_i'\}\times[v_j',v_{j+1}']$ in a similar way, we obtain an A^∞functor $G^{i,j}:C^{i,j}(\Xi)\to C^{i,j+1}(\Xi)$. We also use the family $J_{s,t}$: $(u,v)\in\{u_i',u_{i+1}'\}\times\{v_j'\}$ we obtain an A^∞functor $F^{i,j}:C^{i,j}(\Xi)\to C^{i+1,j}(\Xi)$. We remark that

$$F(J_{u,1})=F^{N,0}\circ\cdots\circ F^{1,0},$$
$$F(J_{u,2})=F^{N,M}\circ\cdots\circ F^{1,M}.$$

Hence to prove Lemma 6.16, it suffices to show that $\left(F^{i,j+1}\circ G^{i,j}\right)^*$ is homotopic to $\left(G^{i+1,j}\circ F^{i,j}\right)^*$.

We define $T_1^{i,j}(\Lambda_1,\Lambda_2):C^{i,j}(\Lambda_1,\Lambda_2)\to C^{i+1,j+1}(\Lambda_1,\Lambda_2)$ by

$$T_1^{i,j}(\Xi)(\Lambda_1,\Lambda_2)([p])=\sum\#\overline{\mathcal{M}}_{[u'_i,u'_{i+1}]\times[v'_j,v'_{j+1}]}(X;\Lambda_1,\Lambda_2;p,q)\,[q]$$

and $T_k^{i,j}(\Lambda_0,\cdots,\Lambda_k):C^{i,j}(\Lambda_0,\Lambda_1)\otimes\cdots\otimes C^{i,j}(\Lambda_{k-1},\Lambda_k)\to C^{i+1,j+1}(\Lambda_0,\Lambda_k)$ by

$$T_k^{i,j}(\Lambda_0,\cdots,\Lambda_k)\big([p_1]\otimes\cdots\otimes[p_k]\big)=\sum\#\mathcal{M}_{[u'_i,u'_{i+1}]\times[v'_j,v'_{j+1}]}\big(X;(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k)\big)[p_0].$$

Using Lemmata 6.4 and 6.7 we can prove that T^{ij} gives a natural transformation from $G^{i+1,j}\circ F^{i,j}$ to $G^{i+1,j}\circ F^{i,j}$ as follows.

In case Figure 6.20 (d) or (e), we find that

$$\begin{gathered}F^{i,j+1}\circ G^{i,j}=G^{i+1,j}\circ F^{i,j}=identity\\ T_1^{i,j}=identity\\ T_k^{i,j}=0\quad k\ge 2.\end{gathered}$$

In case Figure 6.20 (c) , we find that

$$\begin{gathered}F^{i,j+1}\circ G^{i,j}=G^{i+1,j}\circ F^{i,j}\ne identity,\\ T_1^{i,j}=identity\\ T_k^{i,j}=0\quad k\ge 2.\end{gathered}$$

In case Figure 6.20 (a), we find that

$$F^{i,j+1}\circ G^{i,j}\ne G^{i+1,j}\circ F^{i,j}=identity$$

and $T^{i,j}$ gives a natural transformation.

In case Figure 7.20 (b), we find that

$$F^{i,j+1}\circ G^{i,j}=identity\ne G^{i+1,j}\circ F^{i,j}$$

and $T^{i,j}$ gives a natural transformation.

We also find that (up to sign) the same map is a natural transformation from $G^{i+1,j}\circ F^{i,j}$ to $G^{i+1,j}\circ F^{i,j}$, and the composition induces an identity functors from $\left(F^{i,j+1}\circ G^{i,j}\right)^*$ and

$\left(G^{i+1,j} \circ F^{i,j}\right)^*$ to itself. (To show that the composition induces identity functors, we remark that we only need to consider $\Lambda_0,\cdots,\Lambda_k$ which are mutually distinct.) The proof of Lemma 6.16 is now complete modulo analytic detail.

§ 7 Well definedness III

Now we combine the arguments of §§ 5 and 6 to prove the independence of the relative Floer homology of the choices of the metrics and perturbation in general

Let N, E, $\ell_{i,\pm}: S^1 \to N$, $\varphi_{\pm}: \dfrac{SO(3)\times\cdots\times SO(3)}{SO(3)} \to \boldsymbol{R}$ be as in § 3. We remark that we took metric $g_{N\pm}$ on N and $g_{\Sigma,\pm}$ on $\Sigma = \partial N$ so that a neighborhood of ∂N is isometric to the direct product $\Sigma \times [-1,1]$ with metric $g_{\Sigma,\pm} \otimes ds^2$. The metrics $g_{\Sigma,\pm}$ on Σ determines a complex structures $J_{\pm}$ on $R(\Sigma)$. We put $J_1 = J_-$, $J_2 = J_+$. We then have A^∞ functors :

$$HF(N; g_{N,+}, \vec{\ell}_+, \varphi_+) : \mathcal{L}ag(R(\Sigma), J_2) \to \mathcal{C}h,$$
$$HF(N; g_{N,-}, \vec{\ell}_-, \varphi_-) : \mathcal{L}ag(R(\Sigma), J_1) \to \mathcal{C}h.$$

Let $g_{\Sigma,t}$ be a one parameter family of metrics on Σ joining $g_{\Sigma,-}$ and $g_{\Sigma,+}$. The metric $g_{\Sigma,t}$ induces an almost complex structure J_t on $R(\Sigma)$.

By Theorem 6.1 it induces an A^∞ functor $F(J_t) : \mathcal{L}ag(R(\Sigma), J_1) \to \mathcal{L}ag(R(\Sigma), J_2)$ Our result is then :

Theorem 7.1 *The composition* $HF(N; g_{N,+}, \vec{\ell}_+, \varphi_+) \circ F(J_t)$: $\mathcal{L}ag(R(\Sigma), J_1) \to \mathcal{L}ag(R(\Sigma), J_2) \to \mathcal{C}h$ *is homotopic to* $HF(N; g_{N,-}, \vec{\ell}_-, \varphi_-)$.

Remark 7.2 We remark that we use the variable t for the parameter of the family of almost complex structures J_t. This is because we will identity this parameter to the coordinate of $\boldsymbol{R}$, the second factor of $N \times \boldsymbol{R}$. This coordinate will turn out to be identified to one of the coordinate of the domain of the holomorphic disk $\varphi : [-1,1] \times \boldsymbol{R} \to R(\Sigma)$. In § 6, the parameter u is independent of the coordinate of the domain of holomorphic disk.

To explain the origin of this difference we recall that, to the well definedness of Floer homology in symplectic geometry, there are two kinds of proofs.

One [8] uses the parametrized version of moduli space of pseudo holomorphic curves and the parameter is independent of the coordinate of the domain. The other (for example [9]) identifies the parameter of the family of the almost complex structures (or another perturbation) to one of the coordinates of the domain.

In our situation, where we need also to show the well definedness of higher composition operator, the author does not know how to work out the second method, (since it breaks the symmetry of the action of $\mathcal{Hol}(D^2, J) = SL(2,\mathbf{R})$ we need).

On the other hand, to study the well definedness in gauge theory side, it seems much more natural to identify the parameter to one of the coordinate of 4 manifold than just using one parameter family of metrics etc. on 4 manifold. (If we do it in the second way, we need more difficult analysis including center manifold theory, though it is possible to do so.) The reason is that since the order of the intersection $res(R(N,\vec{\ell},\varphi)) \cap \Lambda$ depends on the perturbation $\vec{\ell},\varphi$. (In the situation of § 6, the intersection $\Lambda \cap \Lambda'$ is (of course) independent of the almost complex structure.)

So we used both of them. (The first one in § 6 and the second one in § 5.) This mixture causes a small technical trouble but it can be handled in the way we are going to explain during the proof of Theorem 7.1.

The proof of Theorem 7.1 is a combination of ones of Theorems 5.1 and 6.1.

We first remark that we can change the parameter of the holonomy perturbation $\ell_{i,\pm} : S^1 \to N$, $\varphi_\pm : \dfrac{SO(3)\times\cdots\times SO(3)}{SO(3)} \to \mathbf{R}$ without changing the metric on the surface, using Theorem 5.1. Hence we may assume that $\vec{\ell}_+ = \vec{\ell}_-$ and $\varphi_+ = \varphi_-$. So, for the rest of the proof, we omit these parameters.

We next divide the interval $[1,2]$ in a similar way to § 6 as follows. We consider the moduli space

$$\overline{\mathcal{M}}_I(R(\Sigma);\Lambda_1,\Lambda_2;p,q) = \bigcup_{t\in I}\overline{\mathcal{M}}((R(\Sigma),J_t);\Lambda_1,\Lambda_2;p,q).$$

$$\mathcal{M}_I\big(R(\Sigma);(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k)\big) = \bigcup_{t\in I}\mathcal{M}\big((R(\Sigma),J_t);(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k)\big).$$

for $I \subseteq [1,2]$. Let Ξ be a finite set of simply connected Lagrangian submanifolds of $R(\Sigma)$. Then Lemmata 6.3 and 6.6 again imply that we have a finite subset $I_0 \subseteq [1,2]$ with the following properties.

(7.3.1) If $\Lambda_1,\Lambda_2 \in \Xi$, $\mu(p) - \mu(q) = 0$, $p \ne q$ and if $\overline{\mathcal{M}}((R(\Sigma),J_t);\Lambda_1,\Lambda_2;p,q)$ is nonempty then $t \in I_0$.

(7.3.2) If $\Lambda_0,\cdots,\Lambda_k \in \Xi$, $\mu(p_i) - \mu(p_0) + k - 1 = 0$ and if $\mathcal{M}\big((R(\Sigma),J_t);(\Lambda_0,\cdots\Lambda_k);(p_0,\cdots,p_k)\big)$ is nonempty, then $t \in I_0$.

(7.3.3) For each $s \in I_0$ only one of the moduli spaces in (7.3.1), (7.3.2) is nonempty.

Such a finite set I_0 exists for generic Ξ. Let $t_1 < t_2 < \cdots < t_{N-1} < t_N$ be all the elements of I_0. We choose t_i' such that

$$1 = t_0' < t_1 < t_1' < t_2 < \cdots t_{N-2}' < t_{N-1} < t_{N-1}' < t_N < t_N' = 2 .$$

We consider the full subcategory $\mathcal{C}^i(\Xi)$ of $\mathcal{L}ag((X,\omega,J_{t_i'}))$ such that $\mathcal{O}b(\mathcal{C}^i(\Xi)) = \Xi$.

In § 6 an A^∞ functor $F^i(\Xi) : \mathcal{C}^i(\Xi) \to \mathcal{C}^{i+1}(\Xi)$ was constructed and $F(J_t) : \mathcal{L}ag(R(\Sigma), J_1) \to \mathcal{L}ag(R(\Sigma), J_2)$ is a composition of them on $\mathcal{C}^0(\Xi)$.

We remark that for each $t_i < t < t_{i+1}$, the full subcategory $\mathcal{C}(\Xi,t)$ of $\mathcal{L}ag((X,\omega,J_t))$ is canonically isomorphic to $\mathcal{C}^i(\Xi)$.

In fact, the sets of objects are both Ξ and the set of morphisms as an abelian group are clearly isomorphic to each other. Moreover (7.2.1) , (7.2.2) and Lemmata 6.4 and 6.7, imply that the boundary operators and (higher) composition operators (which give the structure of A^∞ category) exactly coincide.

For each i we choose $t_i^\pm$ such that $t_i^- < t_i < t_i^+$ and that $t_i^+ - t_i^-$ is smaller than a number we specify later. By the above remark $\mathcal{C}(\Xi,t_i^+)$ and $\mathcal{C}(\Xi,t_{i+1}^-)$ are canonically isomorphic to each other and to $\mathcal{C}^i(\Xi)$. Hence we identify them.

Let $g_{N,t}$ be a family of metrics on N such that the restriction of $g_{N,t}$ to $\Sigma \times [-1,1]$ is isometric to $g_{\Sigma,t} \oplus ds^2$. We have an A^∞ functor $HF(N;g_{N,t}) : \mathcal{C}(\Xi,t) \to \mathcal{C}h$.

We first show :

Lemma 7.4 *$HF(N;g_{N,t_i^+}) : \mathcal{C}^i(\Xi) \to \mathcal{C}h$ is homotopic to $HF(N;g_{N,t_{i+1}^-}) : \mathcal{C}^i(\Xi) \to \mathcal{C}h$. (Here we identify $\mathcal{C}(\Xi,t_i^+) = \mathcal{C}(\Xi,t_{i+1}^-) = \mathcal{C}^i(\Xi)$.)*

The proof of Lemma 7.3 is a straight forward generalization of the proof of Theorem 5.1. We first extend family $g_{N,t}$, $t \in [t_i^+, t_{i+1}^-]$ so that it is constant outside $[t_i^+, t_{i+1}^-]$. We then get a metric on $N \times \boldsymbol{R}$. We modify this metric so that it is degenerate at $\Sigma \times [0,1] \times \boldsymbol{R}$. Then using this degenerate metric we construct the “moduli space of ASD connections” in the same way as § 5. Then the required homotopy is constructed by the same formula as § 5. (7.3.1) and (7.3.2) can be used to show that boundary of this “moduli space of ASD connections” behaves in the same way as the case when the metric at the boundary is constant.

To complete the proof, we need to compare the A^∞ functors $HF(N;g_{N,t_i^-}):C^{i-1}(\Xi)\to \mathcal{C}h$ and $HF(N;g_{N,t_i^+}):C^{i}(\Xi)\to \mathcal{C}h$. In fact we prove

Proposition 7.5 *If $t_i^+ - t_i^-$ is sufficiently small then $HF(N;g_{N,t_i^-}):C^{i-1}(\Xi)\to \mathcal{C}h$ is equal to the composition $HF(N;g_{N,t_i^+})\circ F^{i-1}:C^{i-1}(\Xi)\to \mathcal{C}h$.*

To prove Proposition 7.5, we again construct the moduli space of ASD connections as follows. For each fixed $\hat t\in[t_i^-,t_i^+]$ we consider the direct product metric $g_{N,\hat t}\oplus dt^2$ on $N\times \mathbf{R}$. (Here $\hat t$ is a fixed number and is independent of t, the coordinate of $\mathbf{R}$. This is confusing but is inevitable. We recall that t is identified to the $\mathbf{R}$ coordinate in Lemma 7.4.)

We modify it so that it is degenerate at $\Sigma\times[0,1]\times\mathbf{R}$. Then, using this degenerate metric, we construct the "moduli space of ASD-connections" in the same way as § 5 and obtain : $\mathcal{M}\big((N\times\mathbf{R},g_{N,\hat t});\Lambda;a_-,a_+\big)$ and $\mathcal{M}\big((N\times\mathbf{R},g_{N,\hat t});(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1})\big)$ as in §3 and §4.

We divide them by $\mathbf{R}$ action. (We remark that we are using direct product metric hence the moduli spaces are in variant of $\mathbf{R}$ action.) We obtain $\overline{\mathcal{M}}\big((N\times\mathbf{R},g_{N,\hat t});\Lambda;a_-,a_+\big)$, $\overline{\mathcal{M}}\big((N\times\mathbf{R},g_{N,\hat t});(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1})\big)$.

We then put

$$\mathcal{M}_{para}(N\times\mathbf{R};\Lambda;a_-,a_+)=\bigcup_{\hat t\in[t_i^-,t_i^+]}\overline{\mathcal{M}}\big((N\times\mathbf{R},g_{N,\hat t});\Lambda;a_-,a_+\big)$$

$$\mathcal{M}_{para}\big((N\times\mathbf{R},g_{N,t});(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1})\big)$$
$$=\bigcup_{\hat t\in[t_i^-,t_i^+]}\overline{\mathcal{M}}\big((N\times\mathbf{R},g_{N,\hat t});(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1})\big)$$

We then have the following :

Lemma 7.6* *For generic family of metrics $g_{N,t}$ the space $\mathcal{M}_{para}(N\times\mathbf{R};\Lambda;a_-,a_+)$ is a manifold of dimension is $\mu(a_-)-\mu(a_+)$ modulo* 4.

For generic family of metrics $g_{N,t}$ the space

$\mathcal{M}_{para}\big((N\times\boldsymbol{R},g_{N,t});(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1})\big)$ *is a manifold of dimension* $\sum\mu(a_i)-\mu(a_{k+1})+k$ *modulo* 4.

This lemma is a straight forward analogue of Theorems 3.13 and 4.6. We again use transversality to show the following :

Lemma 7.7* *We can choose the family* $g_{N,t}$ *generic so that the following holds.*

If $\mu(a_-)-\mu(a_+)=0$ *and* $a_+\neq a_-$, *then* $\overline{\mathcal{M}}\big((N\times\boldsymbol{R},g_{N,t_i});\Lambda;a_-,a_+\big)$ *is empty.*

If $\sum\mu(a_i)-\mu(a_{k+1})+k$ *then* $\overline{\mathcal{M}}\big((N\times\boldsymbol{R},g_{N,t_i});(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1})\big)$ *is empty.*

We remark that the virtual dimension of the moduli space in Lemma 7.5 is -1 since we fix $\hat{t}=t_i$. Therefore Lemma 7.7 is a consequence of the usual dimension counting argument. Now we use Lemma 7.7 and obtain :

Lemma 7.8 *We can choose* $t_i^+-t_i^-$ *sufficiently small so that the following holds. If* $\mu(a_-)-\mu(a_+)=0$ *and* $a_+\neq a_-$ *then* $\mathcal{M}_{para}(N\times\boldsymbol{R};\Lambda;a_-,a_+)$ *is empty.*

If $\sum\mu(a_i)-\mu(a_{k+1})+k=0$ *then* $\overline{\mathcal{M}}\big((N\times\boldsymbol{R},g_{N,t_i});(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1})\big)$ *is empty.*

We next consider the boundary of the moduli spaces in Lemma 7.6 in the case when the dimension is 1. Most of the candidate of the boundary components do not appear by virtue of Lemma 7.8. It will turn out that we find Proposition 7.5. To be precise we have the following :

Lemma 7.9* *If* $\mu(a_-)-\mu(a_+)=0$ *then one dimensional manifold* $\mathcal{M}_{para}(N\times\boldsymbol{R};\Lambda;a_-,a_+)$ *has a compactification* $C\overline{\mathcal{M}}_{para}(N\times\boldsymbol{R};\Lambda;a_-,a_+)$ *such that its boundary is identified with the union of*

(a) $$\bigcup_{\hat{t}\in[t_i^-,t_i^+]}\bigcup_{b}\Big(\overline{\mathcal{M}}\big((N\times\boldsymbol{R},g_{N,\hat{t}});\Lambda;a_-,b\big)\Big)\times\Big(\overline{\mathcal{M}}\big((N\times\boldsymbol{R},g_{N,\hat{t}});\Lambda;b,a_+\big)\Big)$$

(b) $$\overline{\mathcal{M}}\Big(\big(N\times\boldsymbol{R},g_{N,t_i^-}\big);\Lambda;a_-,a_+\Big)$$

(c) $$\overline{\mathcal{M}}\left(\left(N\times \boldsymbol{R},g_{N,t_i^+}\right);\Lambda;a_-,a_+\right).$$

We remark that (b) and (c) gives the boundary operators of the chain complexes $HF(N;g_{N,t_i^-})_0(\Lambda)$, $HF(N;g_{N,t_i^+})_0(\Lambda)$ respectively. On the other hand Lemma 7.8 implies that (a) is in fact empty. Hence $HF(N;g_{N,t_i^-})_0(\Lambda)=HF(N;g_{N,t_i^+})_0(\Lambda)$.

We next have :

Lemma 7.10* *If* $\sum\mu(a_i)-\mu(a_{k+1})+k=0$ *then one dimensional manifold* $\mathcal{M}_{para}\left(\left(N\times\boldsymbol{R},g_{N,t}\right);(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1})\right)$ *has a compactification* $C\mathcal{M}_{para}\left(\left(N\times\boldsymbol{R},g_{N,t}\right);(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1})\right)$ *whose boundary is identified with the union of the following* 7 *kinds of spaces :*

(a) $$\bigcup_{\hat t\in[t_i^-,t_i^+]}\ \bigcup_{\substack{a_0'\in R(N):\\ res\,a_0'\in\Lambda_0}}\overline{\mathcal{M}}\left(\left(N\times\boldsymbol{R},g_{N,\hat t}\right);\Lambda_0;(a_0,a_0')\right)\times\overline{\mathcal{M}}\left(\left(N\times\boldsymbol{R},g_{N,\hat t}\right);(\Lambda_0,\cdots,\Lambda_k);(a_0',a_1,\cdots,a_{k+1})\right)$$

(b) $$\bigcup_{\hat t\in[t_i^-,t_i^+]}\ \bigcup_{\substack{a_{k+1}'\in R(N):\\ res\,a_{k+1}'\in\Lambda_k}}\overline{\mathcal{M}}\left(\left(N\times\boldsymbol{R},g_{N,\hat t}\right);(\Lambda_0,\cdots,\Lambda_k);(a_0\cdots,a_k,a_{k+1}')\right)\times\overline{\mathcal{M}}\left(\left(N\times\boldsymbol{R},g_{N,\hat t}\right);(a_{k+1}',a_{k+1});\Lambda_k\right)$$

(c) $$\bigcup_{\hat t\in[t_i^-,t_i^+]}\ \bigcup_{\substack{b\in R(N)\\ res\,b\in\Lambda_\ell}}\overline{\mathcal{M}}\left(\left(N\times\boldsymbol{R},g_{N,\hat t}\right);(a_0,\cdots,a_\ell,b);(\Lambda_0,\cdots,\Lambda_\ell)\right)\times\overline{\mathcal{M}}\left(\left(N\times\boldsymbol{R},g_{N,\hat t}\right);(b,a_{\ell+1},\cdots,a_{k+1});(\Lambda_\ell,\cdots,\Lambda_k)\right)$$

(d) $$\bigcup_{\hat t\in[t_i^-,t_i^+]}\ \bigcup_{\substack{a_\ell'\in\Lambda_{\ell-1}\cap\Lambda_\ell,\\ \mu(a_\ell')=\mu(a_\ell)-1}}\overline{\mathcal{M}}\left(\left(N\times\boldsymbol{R},g_{N,\hat t}\right);(a_0,\cdots,a_{\ell-1},a_\ell',a_{\ell+1}\cdots,a_{k+1});(\Lambda_0,\cdots,\Lambda_k)\right)\times\overline{\mathcal{M}}((R(\Sigma),J_{\hat t});\Lambda_{\ell-1},\Lambda_\ell;a_\ell',a_\ell)$$

(e) $$\bigcup_{\hat t\in[t_i^-,t_i^+]}\ \bigcup_{a_{\ell,m}\in\Lambda_\ell\cap\Lambda_m}\overline{\mathcal{M}}\left(\left(N\times\boldsymbol{R},g_{N,\hat t}\right);(\Lambda_0,\cdots,\Lambda_\ell,\Lambda_m,\cdots\Lambda_k);(a_0,\cdots,a_{\ell-1},a_{\ell,m},a_{m+1}\cdots,a_{k+1})\right)\times\overline{\mathcal{M}}\left((R(\Sigma),J_{\hat t});(\Lambda_\ell\cdots,\Lambda_m);(a_{\ell,m},a_\ell,\cdots,a_m)\right)$$

(f) $$\overline{\mathcal{M}}\left(\left(N\times\boldsymbol{R},g_{N,t_i^-}\right);(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1})\right)$$

(g) $$\overline{\mathcal{M}}\left(\left(N\times\boldsymbol{R},g_{N,t_i^+}\right);(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1})\right).$$

The proof is again a straight forward analogue of the proof of Theorem 4.6. By Lemma

7.7, (a),(b),(c) are empty. The order of the spaces (f) and (g) give $HF(N;g_{N,t_i^-})_k(\Lambda_0,\cdots,\Lambda_k)$ and $HF(N;g_{N,t_i^+})_k(\Lambda_0,\cdots,\Lambda_k)$, respectively. On the other hand, the space $\overline{\mathcal{M}}((R(\Sigma),J_{\hat t}),\Lambda_{\ell-1};\Lambda_\ell,a'_\ell,a_\ell)$ is nonempty only for $\hat t = t_i$ and the order of this set at $\hat t = t_i$ gives F_0^i of our A^∞ functor $F^i(\Xi):C^i(\Xi)\to C^{i+1}(\Xi)$. Finally $\overline{\mathcal{M}}((R(\Sigma),J_t),\Lambda_\ell\cdots,\Lambda_m,a_{\ell,m},a_\ell,\cdots,a_m)$ gives $F^i_{m-\ell}$. Thus Lemma 7.10 implies Proposition 7.5

The proof of Theorem 7.1 is now complete. (Modulo analytic detail which we will present in subsequent papers.)

§ 8 Gluing homomorphism

Let (Σ, E) as in §4. Let (N_-, E_-), (N_+, E_+) be such that $\partial(N_\pm, E_\pm) = (\Sigma, E)$. We take metrics on them as in § 4. To save the notation, we assume that transversality is satisfied without introducing perturbation based on holonomy, which we discussed in § 3. (The modification we need to include perturbation by holonomy is an obvious analogue of the arguments in § 3.)

Let $N = -N_- \cup_\Sigma N_+$. Let $CF(N,E)$ be the Floer's chain complex which defines the Floer homology $HF(N)$ [7]. (The $SO(3)$ version we are discussing here is in [3].) The purpose of this section is to construct a chain map $Glue : CF(N) \to \mathcal{F}unc(HF(N_-), HF(N_+))$.

For this purpose, we construct a 4 manifold M as follows. We take $N_- \times (-\infty, 0]$ and $N_+ \times [0,\infty)$. $N_\pm$ has a color diffeomorphic to $\Sigma \times [0,1]$. We glue $\Sigma \times [0,1] \times \{0\} \subseteq N_- \times \{0\}$ and $\Sigma \times [-1,1] \times \{0\} \subseteq N_+ \times \{0\}$. We then obtain a 4 manifold with corner $\Sigma \times \{0\} \times \{0\}$. We smooth this corner and obtain a 4 manifold. It has a boundary component diffeomorphic to N. We remove it and obtain an oriented 4 manifold M such that

(8.1.1)
$$\partial M = \Sigma \times \mathbf{R}.$$
(8.1.2)
$$M - (\text{compact}) = \left(N_- \times (-\infty, 0)\right) \cup \left(N_+ \times (0,\infty)\right) \times (-N \times (-\infty, 0)).$$

Figure 8.2

The bundles E_i and E are extended to M. We take a metric on M such that

(8.3.1)

A neighborhood of the boundary of M is isometric to $\Sigma \times [-1,1] \times \mathbf{R}$.

(8.3.2)

The diffeomorphism (8.1.2) is an isometry.

We now define a moduli space similar to one in § 3. Let a be a flat connection on N, and $a_\pm$ be a flat connection on $N_\pm$. Let Λ be a generic element of $\mathcal{O}b(\mathcal{L}ag(R(\Sigma)))$. We assume that $res_\pm(a_\pm) \in \Lambda$.

We assumed that the set of flat connections on N is discrete and $H^1(N, ad\, a) = 0$, (since we assumed that the transversality holds without taking perturbation by holonomy.) Let $\mathcal{A}_0$ be a connection on (M,E) which coincides to a, $a_\pm$ respectively outside a compact sets. Let $\mathcal{A}(M,E;a;a_-,a_+)$ be the set of all smooth connections $\mathcal{A}$ on (M,E), such that $\mathcal{A}-\mathcal{A}_0$ is of L^2-class. Let us consider the space

$$\hat{\mathcal{M}}(M;\Lambda;a;a_-,a_+)=\left\{\mathcal{A}\in\mathcal{A}(M;a;a_-,a_+)\;\middle|\;\begin{array}{l}\mathcal{A}\text{ solves (3.4.1), (3.4.2), and is an ASD}\\ \text{connection on } M-\Sigma\times[-1,1]\times\boldsymbol{R}.\\ \text{The gauge equivalence class of the restriction}\\ \text{of } \mathcal{A} \text{ to } \Sigma\times\{(t,1)\} \text{ belongs to } \Lambda \text{ for each } t.\end{array}\right\}.$$

We consider the set of gauge transformations g of (M,E) such that $g-\text{identity}$ is of L^2_1 class. This group acts on $\hat{\mathcal{M}}(M;\Lambda;a;a_-,a_+)$. Let $\mathcal{M}(M;\Lambda;a;a_-,a_+)$ be the quotient space.

In a way similar to the other analytic results announced in earlier sections, we can prove the following theorem. Let $\mu(a)$ be the Floer degree of the flat connection a. And let $\overline{\mathcal{M}}(N;a,b)$ be the moduli space used by Floer [7] to define boundary operator of Floer homology of 3 manifold N. Namely it is the moduli space of the solution of ASD equation which is asymptotic to a and b as $t\to\mp\infty$. Here we remark that our convention of the Floer degree is

$$\overline{\mathcal{M}}(N,a;b)=\mu(a)-\mu(b)-1$$

and the definition of the boundary operator in $CF(N,E)$ is

$$\partial[a]=\sum\#\overline{\mathcal{M}}(N;a,b)\,[b].$$

Theorem 8.4* *If we take the metric on M and the simply connected Lagrangian submanifold Λ generic, then the following holds.*

(8.4.1) $\mathcal{M}(M;\Lambda;a;a_-,a_+)$ *is a smooth manifold of dimension* $\mu(a_-)-\mu(a_+)+\mu(a)$ *modulo* 4.

(8.4.2) *If* $\mu(a_-)-\mu(a_+)+\mu(a)=0$, *then* $\mathcal{M}(M;\Lambda;a;a_-,a_+)$ *consists of finitely many points.*

(8.4.3) *If* $\mu(a_-)-\mu(a_+)+\mu(a)=1$, *then* $\mathcal{M}(M;\Lambda;a;a_-,a_+)$ *has a compactification whose boundary is identified to the union of*

$$\overline{\mathcal{M}}(N;a,b)\times\mathcal{M}(M;\Lambda;b;a_-,a_+).$$

$$\overline{\mathcal{M}}(N_-\times\boldsymbol{R};\Lambda;a_-,a_-')\times\mathcal{M}(M;\Lambda;a;a_-',a_+).$$

$$\mathcal{M}(M;\Lambda;a;a_-,a'_+)\times\overline{\mathcal{M}}(N_+\times\boldsymbol{R};\Lambda;a'_+,a_+).$$

The moduli spaces $\overline{\mathcal{M}}(N_-\times\boldsymbol{R};\Lambda;a_-,a'_-)$, $\overline{\mathcal{M}}(N_+\times\boldsymbol{R};\Lambda;a'_+,a_+)$ was introduced in § 3. Using Theorem 8.4, we define

$$Glue(a)_0 : HF(N_-)(\Lambda)\to HF(N_+)(\Lambda)$$

by

$$Glue(a)_0([a_-])=\sum_{a_+}\#\mathcal{M}(M;\Lambda;a;a_-,a_+)\,[a_+].$$

In order to define $Glue(a)_k$, we use a similar moduli space as § 4. Let $\Lambda_i\in \mathcal{Ob}(\mathcal{Lag}(R(\Sigma)))$ are generic elements, $a_i\in\Lambda_{i-1}\cap\Lambda_i$, $i=1,\cdots,k$, $a_0\in\{a\in R(N_-,E)\mid res\ a\in\Lambda_0\}$, $a_{k+1}\in\{a\in R(N_+,E)\mid res_+\ a\in\Lambda_k\}$. Let $\mathcal{A}(M,E;a;a_+,a_-)$ be as above. We consider $(\mathcal{A},(t_1,\cdots,t_k))$ such that

(8.5.1) $\mathcal{A}$ is a smooth connection of E on M.
(8.5.2) $t_1<\cdots<t_k$.
(8.5.3) $\mathcal{A}-\mathcal{A}_0$ is of L^2-class. Here $\mathcal{A}_0$ is a connection which coincides with $a,a_\pm$ outside a compact set.
(8.5.4) $\mathcal{A}$ satisfies Equation (3.4.1), (3.4.2) at $\Sigma\times[-1,1]\times\boldsymbol{R}$.
(8.5.5) $\mathcal{A}$ satisfies Equation (3.12) at $N\times(-\infty,-R]\cup N\times[R,\infty)$.
(8.5.6) $\mathcal{A}$ is ASD at $N\times\boldsymbol{R}-\big((N-\Sigma\times[-1,1])\times(-\infty,-R]\cup(N-\Sigma\times[-1,1])\times[R,\infty)\big)$.
(8.5.7) $[A(1,t_i)]=a_i$.
(8.5.8) If $t_i<t<t_{i+1}$, then $[A(1,t)]\in\Lambda_i$. Here we put $t_0=-\infty$, $t_{k+1}=\infty$.

Let $\hat{\mathcal{M}}\big(M;(\Lambda_0,\cdots,\Lambda_k);a;(a_0,\cdots,a_{k+1})\big)$ be the set of such $(\mathcal{A},(t_1,\cdots,t_k))$. We divide it by gauge transformation group to obtain $\mathcal{M}\big(M;(\Lambda_0,\cdots,\Lambda_k);a;(a_0,\cdots,a_{k+1})\big)$. Then, in a similar way to Theorem 4.6, we have the following :

Theorem 8.6* *For a generic metric on* M *and* Λ_i, *the following holds.*

(8.6.1)
$\mathcal{M}\big(M;(\Lambda_0,\cdots,\Lambda_k);a;(a_0,\cdots,a_{k+1})\big)$ *is a smooth manifold of dimension* $\sum\mu(a_i)-\mu(a_{k+1})+\mu(a)+k$.

(8.6.2)
If $\sum\mu(a_i)-\mu(a_{k+1})+\mu(a)+k=0$, *then*

$\mathcal{M}\left(M,E;a;(a_-,a_1,\cdots,a_k,a_+);(\Lambda_0,\cdots,\Lambda_k)\right)$ *consists of finitely many points.*

(8.6.3)

If $\sum \mu(a_i) - \mu(a_{k+1}) + \mu(a) + k = 1,$ *then* $\mathcal{M}\left(M;(\Lambda_0,\cdots,\Lambda_k);a;(a_0,a_1,\cdots,a_k,a_{k+1})\right)$ *has a compactification* $C\mathcal{M}\left(M;(\Lambda_0,\cdots,\Lambda_k);a;(a_0,a_1,\cdots,a_k,a_{k+1})\right)$ *such that its boundary is a union of the following 7 types of sets.*

(a) $$\overline{\mathcal{M}}(N;a,b)\times\mathcal{M}\left(M;(\Lambda_0,\cdots,\Lambda_k);b;(a_0,\cdots,a_{k+1})\right)$$

(b) $$\overline{\mathcal{M}}(N_-\times\boldsymbol{R};\Lambda_0;a_0,a_0')\times\mathcal{M}\left(M;(\Lambda_0,\cdots,\Lambda_k);a;(a_0',a_1,\cdots,a_k,a_{k+1})\right)$$

(c) $$\mathcal{M}\left(M;(\Lambda_0,\cdots,\Lambda_k);a;(a_0,\cdots,a_{k+1}')\right)\times\overline{\mathcal{M}}(N_+\times\boldsymbol{R};\Lambda_k;a_{k+1}',a_{k+1})$$

(d) $$\bigcup_{\substack{b\in R(N_+)\\ res_+(b)\in\Lambda_i}}\mathcal{M}\left(M;(\Lambda_0,\cdots,\Lambda_i);a;(a_0,\cdots a_i,b)\right)\times \mathcal{M}\left(N_+\times\boldsymbol{R};(\Lambda_i,\cdots,\Lambda_k);(b,a_{i+1},\cdots,a_{k+1})\right)$$

(e) $$\bigcup_{\substack{b\in R(N_-)\\ res_+(b)\in\Lambda_i}}\mathcal{M}\left(N_-\times\boldsymbol{R};(\Lambda_0,\cdots,\Lambda_i);(a_0,\cdots a_i,b)\right)\times \mathcal{M}\left(M;(\Lambda_i,\cdots,\Lambda_k);a;(b,a_{i+1},\cdots,a_{k+1})\right)$$

(f) $$\bigcup_{a_i'\in\Lambda_{i-1}\cap\Lambda_i}\mathcal{M}\left(M;(\Lambda_0,\cdots,\Lambda_k);a;(a_0,\cdots a_i',\cdots,a_{k+1})\right)\times\overline{\mathcal{M}}\left(R(\Sigma);\Lambda_{i-1},\Lambda_i;a_i',a_i\right)$$

(g) $$\bigcup_{a_{i,j}\in\Lambda_i\cap\Lambda_j}\mathcal{M}\left(M;(\Lambda_0,\cdots,\Lambda_i,\Lambda_j,\cdots,\Lambda_k);a;(a_0,\cdots a_{i-1},a_{i,j},a_{j+1}\cdots a_{k+1})\right) \times\overline{\mathcal{M}}\left(R(\Sigma),(\Lambda_i,\cdots,\Lambda_j);(a_{i,j},a_i\cdots,a_j)\right).$$

Now we put

$$Glue(a)_k([a_1]\otimes\cdots\otimes[a_k])([a_0]) = \sum_{a_{k+1}}\#\mathcal{M}\left(M;(\Lambda_0,\cdots,\Lambda_k);a;(a_0,\cdots,a_{k+1})\right)[a_{k+1}].$$

The main theorem of this section is :

Theorem 8.7 *Glue* $:CF(N)\to \mathcal{F}unc(HF(N_-),HF(N_+))$ *is a chain map.*

Proof: Let us verify $Glue(\partial a)_k = \left(\partial Glue(a)\right)_k$. This follows from (8.6.3). In fact (a), (b), (c), (d), (e), (f), (g) corresponds to

(a) $Glue(\partial a)_k([a_1]\otimes\cdots\otimes[a_k])([a_0])$,

(b) $Glue(a)_k([a_1]\otimes\cdots\otimes[a_k])(\partial[a_0])$,

(c) $\partial\left(Glue(a)_k([a_1]\otimes\cdots\otimes[a_k])([a_0])\right)$,

(d) $\left(Glue(a)_{k-i}([a_{i+1}]\otimes\cdots\otimes[a_k])\right)\left(HF_i(N_-)([a_1]\otimes\cdots\otimes[a_i])([a_0])\right)$,

(e) $\left(HF_{k-i}(N_+)([a_{i+1}]\otimes\cdots\otimes[a_k])\right)\left(Glue(a)_i([a_1]\otimes\cdots\otimes[a_i])([a_0])\right)$,

(f) $Glue(a)_k([a_1]\otimes\cdots\otimes\partial[a_i]\otimes\cdots\otimes[a_k])([a_0])$,

(g) $Glue(a)_{k-j+i}([a_1]\otimes\cdots\otimes\eta_{j-i+1}([a_i]\otimes\cdots\otimes[a_j])\otimes\cdots[a_k])([a_0])$,

respectively. Theorem 8.7 then follows from definition.

Theorem 8.8 *The chain homotopy types of* $Glue : CF(N)\to \mathcal{Func}(HF(N_-),HF(N_+))$ *is independent of the metric on* M.

This theorem and similar well definedness statements of the map $Glue$ follows in a way similar to §§5,6,7.

Unfortunately the following is yet a conjecture.

Conjecture 8.9 $Glue : CF(N)\to \mathcal{Func}(HF(N_-),HF(N_+))$ *is a chain homotopy equivalence.*

We next prove the following functoriality of our homomorphism. This functoriality is suggested by Donaldson [5].

Let (N_i,E_i) be compact oriented 3 manifolds such that $(\partial N_i,E_i)=(\Sigma,E)$ for $i=1,2,3$. We assume that the restriction of E_i to each connected component of Σ is nontrivial. Let N_{ij} be closed 3 manifolds obtained by gluing $-N_i$ and N_j along Σ. There exists a 4 manifold M_{123} with boundary such that $\partial M_{123}=-N_{12}\cup -N_{23}\cup N_{13}$.

Figure 8.10

Relative version of Donaldson's polynomial invariant defines a map

(8.11) $$Q(M_{123}) : CF(N_{12}) \otimes CF(N_{23}) \to CF(N_{13}),$$

which is well defined up to chain homotopy. On the other hand, we constructed maps

(8.12) $$Glue_{ij} : CF(N_{ij}) \to \mathcal{F}unc(HF(N_i), HF(N_j)).$$

We then have :

Theorem 8.13 *The following diagram commutes up to chain homotopy.*

$$\begin{array}{ccc}
CF(N_{12}) \otimes CF(N_{23}) & \xrightarrow{Q(M_{123})} & CF(N_{13}) \\
\downarrow Glue_{12} \otimes Glue_{23} & & \downarrow Glue_{13} \\
\mathcal{F}unc(HF(N_1), HF(N_2)) \otimes \mathcal{F}unc(HF(N_2), HF(N_3)) & \xrightarrow{\Phi_2} & \mathcal{F}unc(HF(N_1), HF(N_3))
\end{array}$$

Diagram 8.14

Here Φ_2 is the composition in A^∞ category $\mathcal{F}unc(\mathcal{L}ag(\Sigma), \mathcal{C}h)$. Let us prove Theorem 8.13 modulo analytic detail which will appear in a subsequence paper.

Let M_{ij} be 4 manifolds with boundaries and ends which we used to define $Glue_{ij}$. Namely :

(8.14.1) A neighborhood of the boundary of M_{ij} is isometric to $\Sigma \times [-1,1] \times \boldsymbol{R}$.
(8.14.2) M_{ij} minus a compact set is isometric to $\left(N_i \times (-\infty, 0)\right) \cup \left(N_j \times (0, \infty)\right) \times \left(-N_{ij} \times (-\infty, 0)\right)$.

We remove boundaries from M_{123} and write it by the same symbol. We next take and fix a metric on M_{123} such that M_{123} minus compact set is isometric to $\left(N_{12} \times (-\infty, 0)\right) \cup \left(N_{23} \times (-\infty, 0)\right) \cup \left(N_{13} \times (\infty, 0)\right)$.

We remark the following

Lemma 8.15 *There exists a diffeomorphism*

$$M_{12} \cup_{N_2} M_{23} \cong M_{123} \cup_{N_{13}} M_{13}.$$

The proof is obvious from the following figure.

Figure 8.16

We put $M = M_{12} \cup_{N_2} M_{23} \cong M_{123} \cup_{N_{13}} M_{13}$ and take a family of metrics g_u on M with the following properties.

(8.17.1) For $u < -R$, (M,g_u) contains a subset isometric to $M_{13} \times (-u,u)$. The complement $(M,g_u) - (M_{13} \times (u,-u))$ together with its metric is independent of $u < -R$ and is isometric to $(M_{123} - N_{13} \times (0,\infty)) \cup (-M_{13} \times (-\infty,0))$.

(8.17.2) For $u > R$, (M,g_u) contains a subset isometric to $N_2 \times (-u,u)$. The complement $(M,g_u) - (N_2 \times (u,-u))$ together with its metric is independent of $u > R$ and is isometric to $(N_{12} - N_2 \times (0,\infty)) \cup (N_{23} - N_2 \times (-\infty,0))$.

Let $c_{12} \in R(N_{12})$ and $c_{23} \in R(N_{23})$. We take also $\Lambda_i \in \mathcal{Ob}(\mathcal{Lag}(R(\Sigma))$ $i = 0,\cdots,k$, $a_i \in \Lambda_{i-1} \cap \Lambda_i$ for $i = 1,\cdots,k$. We choose furthermore $a_0 \in R(N_1)$ with $res(a_0) \in \Lambda_0$ and $a_{k+1} \in R(N_3)$ with $res(a_{k+1}) \in \Lambda_k$.

For each u we construct a moduli space $\mathcal{M}((M,g_u);(\Lambda_0,\cdots,\Lambda_k);(c_{12},c_{23});(a_0,\cdots,a_{k+1}))$ as follows. Let $\mathcal{A}_0(M,c_{12},c_{13},a_0,a_{k+1})$ be a connection on M which coincides to $c_{12},c_{13},a_0,a_{k+1}$ outside a compact set. We consider $(\mathcal{A},(t_1,\cdots,t_k))$ such that

(8.18.1) $\mathcal{A}$ is a smooth connection of E on M.
(8.18.2) $t_1 < \cdots < t_k$.
(8.18.3) $\mathcal{A} - \mathcal{A}_0$ is of L^2-class.
(8.18.4) $\mathcal{A}$ satisfies the equation (3.4.1), (3.4.2) at $\Sigma \times [-1,1] \times \boldsymbol{R}$.
(8.18.5) $\mathcal{A}$ satisfies the equation (3.12) at $N_1 \times (-\infty,-R] \cup N_3 \times [R,\infty)$.
(8.18.6) $\mathcal{A}$ is ASD with respect to the metric g_u at other part of M.
(8.18.7) $[A(1,t_i)] = a_i$.
(8.18.8) If $t_i < t < t_{i+1}$, then $[A(1,t)] \in \Lambda_i$. Here we put $t_0 = -\infty$, $t_{k+1} = \infty$.

Let $\mathcal{M}((M,g_u);(\Lambda_0,\cdots,\Lambda_k);(c_{12},c_{23});(a_0,\cdots,a_{k+1}))$ be the space of Gauge equivalence class of such elements $(\mathcal{A},(t_1,\cdots,t_k))$.

Using $\mathcal{M}((M,g_u);(\Lambda_0,\cdots,\Lambda_k);(c_{12},c_{23});(a_0,\cdots,a_{k+1}))$ in exactly the same way as before, we obtain a chain map

$$Q(M,g_u) : CF(N_{12}) \otimes CF(N_{23}) \to \mathcal{Func}(HF(N_1), HF(N_3)) .$$

Using an argument similar to §§ 5,7, we find that $Q(M,g_u)$ up to chain homotopy is independent of u.

We then use Taubes' type gluing argument based on (8.17) and prove the following :

Lemma 8.19* *For sufficiently large* u, *we have* $Q(M,g_u) = \Phi_2 \circ \left(Glue_{12} \otimes Glue_{23}\right)$. *For sufficiently small* u, *we have* $Q(M,g_u) = Glue_{13} \circ Q(M_{123})$.

This complete the proof of Theorem 8.13 modulo analytic detail.

§ 9 The case $N = \Sigma \times [0,1]$

Let $\Delta \subseteq R(\Sigma)^- \times R(\Sigma)$ be the diagonal.

Theorem 9.1

$HF(\Sigma \times [0,1]) \in \mathcal{F}un\!c(\mathcal{L}ag(R(\Sigma)^- \times R(\Sigma)), \mathcal{C}h)$ is homotopy equivalent to an A^∞ functor represented by $\Delta \in \mathcal{O}b(\mathcal{L}ag(R(\Sigma)^- \times R(\Sigma))$.

Theorem 9.2

Let $\partial N_0 = \Sigma \cup -\Sigma$. We glue N_0 with $\Sigma \times [0,1]$ to obtain a closed 3 *manifold N. Then $CF(N)$ is homotopy equivalent to $HF(\Sigma \times [0,1])(\Delta)$.*

Theorem 9.3

Let N_0, N be an in Theorem 9.2. *Then the chain map, $Glue : CF(N) \to \mathcal{F}unc(HF(\Sigma \times [0,1]), HF(N_0))$ defined in* §8 *is a chain homotopy equivalence.*

We give a proof of them modulo analytic detail. We consider $\Sigma \times [-5,5]$ rather than $\Sigma \times [0,1]$ for the convenience of the notation. We choose a cut function $\chi : [-5,5] \to [0,1]$ such that :

$$\chi(s) = \begin{cases} 0 & s < -4 \\ 1 & s \in [-3,3] \\ 0 & s > 4 \\ > 0 & s \in [-4,4] \end{cases}.$$

We consider the degenerate metric $g_\varepsilon = \varepsilon^2 \chi(s)^2 g_\Sigma + ds^2$ on $\Sigma \times [-5,5]$. We use this degenerate metric and define a moduli space of ASD connections in the same way as § 3 as follows.

We consider simply connected Lagrangian submanifolds $\Lambda_i \subseteq R(\Sigma)^- \times R(\Sigma)$. Let $(a_{i,-}, a_{i,+}) \in \Lambda_{i-1} \cap \Lambda_i$, $(a_0, a_0) \in \Delta \cap \Lambda_0$, and $(a_{k+1}, a_{k+1}) \in \Lambda_k \cap \Delta$. (Here $k = 0,1,2,\cdots$.) We remark that $\Delta = R(\Sigma \times [-5,5])$. We use these date to fix a boundary condition. On the other hand the equation we use is

$$\frac{\partial A}{\partial t} - d_A \Psi + *(\frac{\partial A}{\partial s} - d_A \Phi) = 0 \tag{9.4.1}$$

(9.4.2) $$\varepsilon^2\chi(s)^2\left(\frac{\partial\Phi}{\partial t}-\frac{\partial\Psi}{\partial s}-[\Phi,\Psi]\right)+*F_A=0.$$

We fix a connection $\mathcal{A}_0$ on $\Sigma\times[-5,5]\times\boldsymbol{R}$ such that $\mathcal{A}_0$ is flat on $\Sigma\times[-5,5]\times(-\infty,R]\cup\Sigma\times[-5,5]\times[R,\infty)$ and coincides with a_0, a_{k+1} there. Let $\mathcal{A}(M,E;a_0,a_{k+1})$ be the set of all smooth connections $\mathcal{A}$ on $\Sigma\times[-5,5]\times\boldsymbol{R}$ such that $\mathcal{A}-\mathcal{A}_0$ is of L^2 class. Then the moduli space we study is

$$\hat{\mathcal{M}}(\Sigma\times[-5,5],\varepsilon;\Lambda_0;a_0,a_1)=\left\{\mathcal{A}\in\mathcal{A}(M,E;a_0,a_1)\,\middle|\,\begin{array}{l}\mathcal{A}\text{ solves (9.4.1), (9.4.2).}\\ ([A(-5,t)],[A(5,t)])\in\Lambda_0\end{array}\right\}$$

We divide it by the set of smooth gauge transformations g such that $g-id\in L^2_1$. Let $\mathcal{M}(\Sigma\times[-5,5];\varepsilon;\Lambda_1;a_0,a_1)$ be the quotient. We furthermore divide it by the $\boldsymbol{R}$ action and let $\overline{\mathcal{M}}(\Sigma\times[-5,5],\varepsilon;\Lambda_1;a_0,a_1)$ be the quotient.

We next consider $(\mathcal{A},(t_1,\cdots,t_k))$ such that

(9.5.1) $\mathcal{A}$ is a smooth connection on $\Sigma\times[-5,5]\times\boldsymbol{R}$.
(9.5.2) $t_1<\cdots<t_k$.
(9.5.3) $\mathcal{A}-\mathcal{A}_0$ is of L^2-class.
(9.5.4) $\mathcal{A}$ satisfies the equation (9.4.1), (9.4.2).
(9.5.5) $[A(\pm 5,t_i)]=a_{i,\pm}$.
(9.5.6) If $t_{i-1}<t<t_i$, then $[A(-5,t),(A(5,t)]\in\Lambda_i$. Here we put $t_0=-\infty$, $t_{k+1}=\infty$.

Let $\mathcal{M}\left(\Sigma\times[-5,5],\varepsilon;(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1})\right)$ be the set of all gauge equivalence classes of such $(\mathcal{A},(t_1,\cdots,t_k))$. We again divide it by $\boldsymbol{R}$ action to obtain $\overline{\mathcal{M}}\left(\Sigma\times[-5,5],\varepsilon;(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1})\right)$. We use these moduli spaces in the same way as §4 to get an A^∞functor : $\mathcal{Lag}(R(\Sigma)^-\times R(\Sigma))\to\mathcal{Ch}$. This A^∞functor is $HF(\Sigma\times[-5,5])$ and its homotopy type is independent of $\varepsilon>0$.

For $\varepsilon=0$, our moduli space is one of holomorphic disks in $R(\Sigma)^-\times R(\Sigma)$. We first assert :

Theorem 9.6* *For sufficiently small* ε *the moduli spaces* $\mathcal{M}(\Sigma\times[-5,5];\varepsilon;\Lambda_1;a_0,a_1)$ *and* $\overline{\mathcal{M}}\left(\Sigma\times[-5,5],\varepsilon;(\Lambda_1,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1})\right)$ *are diffeomorphic to* $\mathcal{M}(\Sigma\times[-5,5];0;\Lambda_1;a_0,a_1)$, $\overline{\mathcal{M}}\left(\Sigma\times[-5,5],0;(\Lambda_1,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1})\right)$ *respectively in case the dimension is* 0.

The proof is a minor modification of the argument by Dostglou-Salamon in [6] combined with the proof of theorems in § 3. The detail will be given in a subsequent paper.

We next study $\overline{\mathcal{M}}\left(\Sigma\times[-5,5],0;(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1})\right)$. We use a kind of reflection principle. Let $(\mathcal{A},(t_1,\cdots,t_k))\in\overline{\mathcal{M}}\left(\Sigma\times[-5,5],0;(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1})\right)$. We define $h:[0,5]\times\boldsymbol{R}\to R(\Sigma)^-\times R(\Sigma)$ by

$$h(s,t)=([A(-s,t)],[A(s,t)])\ .$$

We remark that Equation (9.4.2) for $\varepsilon=0$ implies that $A(s,t)$ is a flat connection. Hence h defines a map $:[0,5]\times\boldsymbol{R}\to R(\Sigma)^-\times R(\Sigma)$. Then (9.4.1) implies that h is holomorphic. We next remark that $h(\{0\}\times\boldsymbol{R})\subseteq\Delta$ by the definition and $h(\{5\}\times\boldsymbol{R})\subseteq\Lambda_0\cup\cdots\cup\Lambda_k$. Hence using the notation of § 2, we have

$$h\in\mathcal{M}\left(R(\Sigma)^-\times R(\Sigma);(\Delta,\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1})\right).$$

(We use a biholomorphic map $(0,5)\times\boldsymbol{R}\cong D$ for this identification.) Thus we proved

Lemma 9.7 *$\mathcal{M}\left(\Sigma\times[-5,5],0;(\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1})\right)$ is homeomorphic to $\mathcal{M}\left(R(\Sigma)^-\times R(\Sigma);(\Delta,\Lambda_0,\cdots,\Lambda_k);(a_0,\cdots,a_{k+1})\right)$.*

Combining Lemma 9.7 and Theorem 9.6, we obtain Theorem 9.1.

We next turn to the proof of Theorem 9.2. Let N_0 be as in Theorem 9.2. We define a metric on it so that it is isometric to $(\Sigma\cup-\Sigma)\times[-1,1]\times\boldsymbol{R}$ near the boundary. We glue N_0 with $\Sigma\times[-5,5]$ equipped with metric $g_{N,u}=(1-u^2\chi(s)^2)g_\Sigma+dt^2$. We then obtain a manifold N with metric $g_{N,u}$. For $u<1$ this metric is smooth and hence we can use it to define Floer homology of N. For $u=1$, the metric $g_{N,1}$ is singular on $\Sigma\times[-3,3]\subseteq\Sigma\times[-5,5]$. However we can use the same method as § 3 to construct moduli space of ASD connections on $(N,g_{N,1})$. For $a_-,a_+\in R(N)$, let $\overline{\mathcal{M}}((N,g_{N,u}),a_-,a_+)$ be the moduli space of ASD connections on $N\times\boldsymbol{R}$ with metric $g_{N,u}\oplus dt^2$, which is asymptotic to a_-,a_+ as $t\to\mp\infty$. (We divide it by the gauge transformation group and $\boldsymbol{R}$ action.) (We again assume that transversality is satisfied without perturbation, for simplicity of notation. The modification to include perturbation by holonomy is an obvious analogue of the arguments in § 3.)

Theorem 9.8 $\overline{\mathcal{M}}((N,g_{N,u}),a_-,a_+)$ *is diffeomorphic to* $\overline{\mathcal{M}}((N,g_{N,1}),a_-,a_+)$ *if* $1-u$ *is sufficiently small and if* $\mu(a)-\mu(b)=1$.

The proof is again a minor modification of the argument of [6] combined with one we use to prove theorems in § 3, and will be given in subsequence papers.

We next prove the following lemma.

Lemma 9.9 $\overline{\mathcal{M}}((N,g_{N,1}),a_-,a_+)$ *is diffeomorphic to* $\overline{\mathcal{M}}(N_0;\Delta;a_-,a_+)$.

We recall notations in Lemma 9.9. We recall $a_-,a_+\in R(N)$. We remark also that $R(N)=\left\{a\in R(N_0)\,\middle|\, resa\in\Delta\subseteq R(\Sigma)^-\times R(\Sigma)\right\}$. Hence $a_-,a_+\in \left\{a\in R(N_0)\,\middle|\, resa\in\Delta\subseteq R(\Sigma)^-\times R(\Sigma)\right\}$. Thus $\overline{\mathcal{M}}((N,g_{N,0}),a_-,a_+)$ is one defined in §3.

The proof of Lemma 9.9 is in fact easy. Let $\overline{\mathcal{M}}((N,g_{N,1}),a_-,a_+)$. We cut N at $\Sigma\times\{0\}\subseteq\Sigma\times[-3,3]$ to obtain N_0. On $N_0\times\boldsymbol{R}$, the connection $\mathcal{A}$ gives a solution of (3.4) whose boundary value at $(-\Sigma\cup\Sigma)\times\boldsymbol{R}$ is contained in Δ. Hence we obtain an element of $\overline{\mathcal{M}}(N_0;\Delta;a_-,a_+)$. It is easy to see that this map gives a diffeomorphism required in Lemma 9.9.

Theorem 9.8 and Lemma 9.9 imply that $HF(N_0)_0(\Delta)$ is isomorphic to $CF(N)$. Thus we established an isomorphism $CF(N)\approx HF(N_0)_0(\Delta)$ as a chain complex in the case we use the metric $g_{N,u}$ with $1-u$ small. Theorem 9.2 then follows.

We next are going to prove Theorem 9.3 (modulo analytic detail). We remark that Theorems 9.1, 9.2 and Lemma 13.28 imply that there exists a homotopy equivalence : $CF(N)\approx \mathcal{F}unc(HF(\Sigma\times[-5,5]),HF(N_0))$.

We however still need to show that this homotopy equivalence is realized by the map $Glue:CF(N)\to\mathcal{F}unc(HF(\Sigma\times[-5,5]),HF(N_0))$ constructed in § 8. (We want to prove it since the map $Glue$ in §8 is defined in other cases also and enjoys various functoriality. So proving Theorem 9.3 can be expected to be a first step toward the proof of Conjecture 8.9.) To prove it we proceed as follows. (The analytic detail again will appear in subsequent papers.)

We take the 4 manifold M we used to construct $Glue: CF(N) \to \mathcal{F}unc(HF(\Sigma \times [-5,5]), HF(N_0))$. In this case, we have :

(9.111) A neighborhood of the boundary of M is isometric to $(-\Sigma \cup \Sigma) \times [-1,1] \times \boldsymbol{R}$.
(9.11.2) $M - (\text{compact}) = (\Sigma \times [-5,5] \times (-\infty, 0)) \cup (N_0 \times (0,\infty)) \times (-N \times (-\infty, 0))$.

We take a flat 2 manifold W as in Figure 9.12 such that $\Sigma \times W$ is embedded to M by isometry.

Figure 9.12

We remark that $\Sigma \times W$ contains $\Sigma \times [-2,2] \times \boldsymbol{R}$, where one end of it is identified with $\Sigma \times [-2,2] \times (-\infty, -100]$ and the other end is identified with $\Sigma \times [-2,2] \times [100,\infty) \subseteq -N \times (-\infty, -100]$.

We consider a subset $W_1 \subseteq W$ with flat metric such that

$$W_1 = [-5,5] \times (-\infty, -100] \cup [-2,2] \times \boldsymbol{R} \cup [-5,-3] \times \boldsymbol{R} \cup [3,5] \times \boldsymbol{R} .$$

We choose $W_1 \supseteq W_2 \supseteq [-5,5] \times (-\infty, -200] \cup [-1,1] \times \boldsymbol{R} \cup [-5,-4] \times \boldsymbol{R} \cup [4,5] \times \boldsymbol{R}$, such that W_2 has a smooth boundary.

Figure 9.13

Let $\chi' : W \to [0,1]$ be a smooth function such that $\chi' \equiv 0$ outside W_1 and $\chi'^{-1}(1) = W_2$. We use also the function χ introduced at the beginning of this section. We consider a family of degenerate metrics $g_{M,\lambda}$ on M by

$$g_{M,\lambda}(x,(s,t)) = \chi(s,t)^2 \left(1 - \lambda^2 \chi'(s,t)^2\right) g_\Sigma \oplus ds^2 \oplus dt^2$$

on $(s,t) \in ([-5,-3] \cup [3,5]) \times \boldsymbol{R} \subseteq W$ and

$$g_{M,\lambda}(x,p) = \left(1 - \lambda^2 \chi'(p)^2\right) g_\Sigma \oplus g_W$$

on $p \in W - ([-5,-3] \cup [3,5]) \times \boldsymbol{R}$. (We do not change the metric outside $\Sigma \times W$.)

We remark that $g_{M,0}$ is the metric we used to define

$Glue : CF(N) \to \mathcal{F}un(HF(\Sigma \times [-5,5]), HF(N_0))$. So by the same argument as §§ 5,6,7, we find that we can use $g_{M,\lambda}$, $\lambda < 1$ also to define $Glue : CF(N) \to \mathcal{F}un(HF(\Sigma \times [-5,5]), HF(N_0))$. To be more precise, we define a moduli space $\mathcal{M}\big((M, g_{M,\lambda}); (\Lambda_0, \cdots, \Lambda_k); a; (a_0, \cdots, a_{k+1})\big)$ as follows. (Here $\Lambda_i \in \mathcal{O}b(\mathcal{L}ag(R(\Sigma)^- \times R(\Sigma)))$, $a \in R(N)$, $(a_0, a_0) \in \Delta \cap L_0$, $a_i = (a_i^-, a_i^+) \in \Lambda_{i-1} \cap \Lambda_i$, $a_{k+1} \in R(N_0)$, $res\, a_{k+1} \in \Lambda_k$.) We consider $(\mathcal{A}, (t_1, \cdots, t_k))$ such that

(9.14.1) $\mathcal{A}$ is a smooth connection on M.
(9.14.2) $t_1 < \cdots < t_k$.
(9.14.3) $\mathcal{A} - \mathcal{A}_0$ is of L^2 class. Here $\mathcal{A}_0$ is a connection which coincides with a, a_0, a_{k+1} outside a compact set.
(9.14.4) $\mathcal{A}$ is an ASD connection with respect to the degenerate metric $g_{M,\lambda}$.
(9.14.5) $[A(\pm 5, t_i)] = a_{i,\pm}$.
(9.14.7) If $t_{i-1} < t < t_i$, then $([A(-5,t)],[A(5,t)]) \in \Lambda_i$. Here we put $t_0 = -\infty$, $t_{k+1} = \infty$.

We consider the set of all such $(\mathcal{A}, (t_1, \cdots, t_k))$ and divide it by the gauge transformation group. We then obtain $\mathcal{M}\big((M, g_{M,\lambda}); (\Lambda_0, \cdots, \Lambda_k); a; (a_0, \cdots, a_{k+1})\big)$. We can use $\mathcal{M}\big((M, g_{M,\lambda}); (\Lambda_0, \cdots, \Lambda_k); a; (a_0, \cdots, a_{k+1})\big)$ to define $Glue : CF(N) \to \mathcal{F}un(HF(\Sigma \times [-5,5], HF(N_0)))$ for $\lambda < 1$.

In case $\lambda = 1$, our metric is degenerate on $\Sigma \times W_2$. We can handle the new degeneration in a similar way and define $\mathcal{M}\big((M, g_{M,1}); (\Lambda_0, \cdots, \Lambda_k); a; (a_0, \cdots, a_{k+1})\big)$. We need a bit care to construct $\mathcal{M}\big((M, g_{M,1}); (\Lambda_0, \cdots, \Lambda_k); a; (a_0, \cdots, a_{k+1})\big)$ as a smooth manifold of finite dimension since the boundary of the domain where the metric is begin to degenerate is now Σ times a curve and the curve is not straight. However the basic idea to handle it is the same as one in § 3 and is given in a subsequent paper.

We then have :

Lemma 9.15* *If* $1 - \lambda$ *is small, then* $\mathcal{M}\big((M, g_{M,\lambda}); (\Lambda_0, \cdots, \Lambda_k); a; (a_0, \cdots, a_{k+1})\big)$ *is diffeomorphic to* $\mathcal{M}\big((M, g_{M,1}); (\Lambda_0, \cdots, \Lambda_k); a; (a_0, \cdots, a_{k+1})\big)$ *in case dimension is* 0.

The proof is again by a combination of [6] and the proof of the result of § 3 and is in a subsequence paper.

We thus may assume that

$$Glue(a)_k([a_1]\otimes\cdots\otimes[a_k])([a_0]) = \sum_{a_{k+1}} \#\mathcal{M}\big((M,g_{M,1});(\Lambda_0,\cdots,\Lambda_k);a;(a_0,\cdots,a_{k+1})\big)\,[a_{k+1}].$$

Now we study the moduli space $\mathcal{M}\big((M,g_{M,1});(\Lambda_0,\cdots,\Lambda_k);a;(a_0,\cdots,a_{k+1})\big)$. We remark that there exists a subspace $\Sigma\times[-1,1]\times\boldsymbol{R}\subseteq(M,g_{M,1})$ with degenerate metric. We cut it at $\Sigma\times\{0\}\times\boldsymbol{R}$ to obtain $(M_0',g_{M_0'})$. We find that $\partial M_0' = \partial M\cup(-\Sigma\times\Sigma)\times\boldsymbol{R}$. To distinguish $(-\Sigma\times\Sigma)\times\boldsymbol{R} = \partial M_0' - \partial M$ from $(-\Sigma\times\Sigma)\times\boldsymbol{R} = \partial M$, we write it as $(-\Sigma'\times\Sigma')\times\boldsymbol{R}$.

We remark that M_0' is diffeomorphic to $(N_0\times\boldsymbol{R}) - ((\Sigma\cup-\Sigma)\times\{0\})$. (Here $\Sigma\cup-\Sigma\subseteq\partial N$.)

We then find that $\mathcal{M}\big((M,g_{M,1});(\Lambda_0,\cdots,\Lambda_k);a;(a_0,\cdots,a_{k+1})\big)$ is identified to the following moduli space. We consider $(\mathcal{A},(t_1,\cdots,t_k))$ such that

(9.16.1) $\mathcal{A}$ is a smooth connection on M_0'.
(9.16.2) $t_1<\cdots<t_k$.
(9.16.3) $\mathcal{A}-\mathcal{A}_0'$ is of L^2-class. Here $\mathcal{A}_0'$ is a pull back of the connection $\mathcal{A}_0$ to M_0'.
(9.16.4) $\mathcal{A}$ is an ASD connection with respect to the degenerate metric $g_{M_0'}$.
(9.16.5) $[A(\pm 5,t_i)] = a_{i,\pm}$.
(9.16.6) If $t_{i-1}<t<t_i$, then $([A(-5,t)],[A(5,t)])\in\Lambda_i$. Here we put [redacted], $t_{k+1}=\infty$.
(9.16.7) Let us consider the restriction of $\mathcal{A}$ $(-\Sigma'\times\Sigma')\times\boldsymbol{R}$. It gives a map $2\,\mathrm{points}\times\boldsymbol{R}\to R(\Sigma)$, or equivalently the map $\boldsymbol{R}\to R(\Sigma)^-\times R(\Sigma)$. We assume that its image is in the diagonal.

Let $\mathcal{M}\big((M_0',g_{M_0'});(\Delta,\Lambda_0,\cdots,\Lambda_k);a;(a_0,\cdots,a_{k+1})\big)$ be the space of gauge equivalent classes of such $(\mathcal{A},(t_1,\cdots,t_k))$.

We find that

$$\mathcal{M}\big((M,g_{M,1});(\Lambda_0,\cdots,\Lambda_k);a;(a_0,\cdots,a_{k+1})\big) = \mathcal{M}\big((M_0',g_{M_0'});(\Delta,\Lambda_0,\cdots,\Lambda_k);a;(a_0,\cdots,a_{k+1})\big).$$

We next compare $\mathcal{M}\big((M_0',g_{M_0'});(\Delta,\Lambda_0,\cdots,\Lambda_k);a;(a_0,\cdots,a_{k+1})\big)$ with $\overline{\mathcal{M}}\big(N_0\times\boldsymbol{R};(\Delta,\Lambda_0,\cdots,\Lambda_k);(a,a_0,\cdots,a_{k+1})\big)$. We recall that to define $\overline{\mathcal{M}}\big(N_0\times\boldsymbol{R};(\Delta,\Lambda_0,\cdots,\Lambda_k);(a,a_0,\cdots,a_{k+1})\big)$, we used product "metric" which is degenerate at the neighborhood of the boundary of $N\times\boldsymbol{R}$. We are going to construct a family of metrics joining $(M_0', g_{M_0'})$ and $N \times \boldsymbol{R}$ with product metric.

We put $W_2 = W_2^+ \cup W_2^-$, where $W_2^\pm$ is the intersection of W_2 with the part where $\pm s > 0$. We may assume that the reflection of W_2^+ by the t axis is W_2^-.

Let $W_3 = W_3^+ \cup W_3^-$ is a small open neighborhood of the closure of W_2. There exists a unique biconformal map $\varphi : Int(W_3^+) \to (4.5, 5) \times \boldsymbol{R}$ such that

$$\begin{aligned} &\varphi(\{0\} \times \boldsymbol{R}) = \{5\} \times \boldsymbol{R}_- \\ &\varphi(\{5\} \times \boldsymbol{R}) \subseteq \{5\} \times \boldsymbol{R}_+ \\ &\lim_{t \to -\infty} \varphi(s,t) = (5,0). \end{aligned} \tag{9.17}$$

Figure 9.18.

φ induces $\varphi : Int(W_3^-) \to (-4.5, -5) \times \boldsymbol{R}$ by reflection. We then obtain a diffeomorphism

$$\Phi : \Sigma \times Int(W_3) - \{0\} \times \boldsymbol{R} \cong \Sigma \times (-5, -4.5) \times \boldsymbol{R} \cup \Sigma \times (4.5, 5) \times \boldsymbol{R}$$

Let us define $f : W_3 \to \boldsymbol{R}_{\ge 0}$ by

$$\varphi^*(ds^2 \oplus dt^2) = f^{-2} g_W . \tag{9.19}$$

here g_W is the flat metric on W. We then find that

$$\Phi_*(g_{M_0}) = \left(\chi' \circ \Phi^{-1}\right)^2 g_\Sigma \oplus \left(f \circ \Phi^{-1}\right)^2 \left(ds^2 \oplus dt^2\right).$$

We remark that f is positive in a neighborhood of the closure of W_2. So we find open sets U_1, U_2 with $\overline{W}_2 \subseteq U_1 \subseteq \overline{U}_1 \subseteq U_2$ and a metric $g'_{M_0'}$ on M_0' such that $g_{M_0'} = g'_{M_0'}$ in U_1 and

$$\Phi_*(g'_{M_0'}) = \left(\chi' \circ \Phi^{-1}\right)^2 g_\Sigma \oplus \left(ds^2 \oplus dt^2\right) \text{ outside } \Phi(U_2).$$

We can use this metric in place of $g_{M_0'}$ to define $Glue(a)$. Namely we may assume

$$\begin{aligned} &Glue(a)_k([a_1] \otimes \cdots \otimes [a_k])([a_0]) \\ &\quad = \sum_{a_{k+1}} \# \mathcal{M}\Big((M_0', g'_{M_0'}); \big(\Delta, \Lambda_0, \cdots, \Lambda_k\big); a; \big(a_0, \cdots, a_{k+1}\big)\Big) [a_{k+1}] \cdot \end{aligned} \tag{9.20}$$

We may find f' such that

$$\Phi_*(g'_{M'_0}) = \left(\chi' \circ \Phi^{-1}\right)^2 g_\Sigma \oplus \left(f' \circ \Phi^{-1}\right)^2 \left(ds^2 \oplus dt^2\right).$$

We then define a one parameter family of metrics $g_{N_0 \times \boldsymbol{R},\kappa}$ on $N_0 \times \boldsymbol{R}$ as follows.

$$\begin{aligned} g_{N_0\times \boldsymbol{R},\kappa} &= \left(\chi + \kappa\left(\chi' \circ \Phi^{-1}\right)\left(f' \circ \Phi^{-1}\right)\right)^2 g_\Sigma \\ &\oplus \left(\kappa\left(f' \circ \Phi^{-1}\right) + (1-\kappa)\right)^2 \left(ds^2 \oplus dt^2\right) \end{aligned}. \tag{9.21}$$

Using this family of metrics, we define a family of moduli spaces as follows.

We consider $(\mathcal{A},(t_1,\cdots,t_k))$ such that

(9.22.1) $\mathcal{A}$ is a smooth connection on $N_0 \times \boldsymbol{R}$.
(9.22.2) $0 < t_1 < \cdots < t_k$.
(9.22.3) $\mathcal{A} - \mathcal{A}_0$ is of L^2-class. Here $\mathcal{A}_0$ is a connection on $N_0 \times \boldsymbol{R}$ which coincides with a , a_{k+1} outside a compact set.
(9.22.4) $\mathcal{A}$ is an ASD connection with respect to the metric $g_{N \times \boldsymbol{R},\kappa}$.
(9.22.5) $[A|_{\partial N \times t_i}] = a_i$.
(9.22.5) $[A|_{\partial N \times 0}] = a$.
(9.22.6) If $t_{i-1} < t < t_i$, then $[A|_{\partial N \times t_i}] \in \Lambda_i$. Here we put $t_{-1} = -\infty$, $t_0 = 0$, $t_{k+1} = \infty$, $\Lambda_{-1} = \Delta$.

Let $\mathcal{M}\Big((N \times \boldsymbol{R}, g_{N_0 \times \boldsymbol{R},\kappa});(\Delta,\Lambda_0,\cdots,\Lambda_k);(a,a_0,\cdots,a_{k+1})\Big)$ be the moduli space of gauge equivalence classes of such$(\mathcal{A},(t_1,\cdots,t_k))$.

We then obtain the following :

Lemma 9.23
$$\begin{aligned} &\mathcal{M}\Big((N_0 \times \boldsymbol{R}, g_{N_0 \times \boldsymbol{R},1});(\Delta,\Lambda_0,\cdots,\Lambda_k);(a,a_0,\cdots,a_{k+1})\Big) \\ &= \mathcal{M}\Big((M'_0, g'_{M'_0});(\Delta,\Lambda_0,\cdots,\Lambda_k);a;(a_0,\cdots,a_{k+1})\Big)\,[a_{k+1}] \end{aligned}.$$

This lemma is in fact clear from definition. Therefore by an argument similar to §§ 5,6,7, we may assume that

$$\begin{aligned} &Glue(a)_k([a_1]\otimes\cdots\otimes[a_k])([a_0]) \\ &= \sum_{a_{k+1}} \#\mathcal{M}\Big((N_0 \times \boldsymbol{R}, g_{N_0 \times \boldsymbol{R},0});(\Delta,\Lambda_0,\cdots,\Lambda_k);(a,a_0,\cdots,a_{k+1})\Big)\,[a_{k+1}]. \end{aligned}$$

On the other hand, we have :

Lemma 9.24
$$\begin{aligned}&\mathcal{M}\big((N_0\times \boldsymbol{R}, g_{N\times\boldsymbol{R},0});(\Delta,\Lambda_0,\cdots,\Lambda_k);(a,a_0,\cdots,a_{k+1})\big)\\ &\qquad = \overline{\mathcal{M}}\big((N_0\times \boldsymbol{R}, g_{N_0}\oplus dt^2);(\Delta,\Lambda_0,\cdots,\Lambda_k);(a,a_0,\cdots,a_{k+1})\big).\end{aligned}$$

Here the right hand side is the moduli space we use to define $HF(N_0)(\Delta,\Lambda_0,\cdots,\Lambda_k)$. Lemma 9.24 is obvious from definition. Therefore, up to homotopy, we have

(9.25)*
$$\begin{aligned}&Glue(a)_k([a_1]\otimes\cdots\otimes[a_k])([a_0])\\ &\qquad = HF(N_0)(\Delta,\Lambda_0,\cdots,\Lambda_k)\big([a]\otimes[a_0]\otimes[a_1]\otimes\cdots\otimes[a_k]\big).\end{aligned}$$

By the proof of Lemma 12.28, we find that the right hand side is the map obtained by Theorems 9.1 9.2 and Lemma 12.28. The proof of Theorem 9.3 is now completed modulo analytic detail.

PART II HOMOLOGICAL ALGEBRA

§10 $\mathcal{F}unc(\mathcal{C}^1,\mathcal{C}^2)$

In part II, we discuss basic properties of A^∞ category. First we define a natural transformation between A^∞ functors. Let $\mathcal{C}^1,\mathcal{C}^2$ be A^∞ categories and $F^1:\mathcal{C}^1\to\mathcal{C}^2$, $F^2:\mathcal{C}^1\to\mathcal{C}^2$ be A^∞ functors.

Definition 10.1 A *pre natural transformation* $T:F^1\to F^2$ of degree d, consists of $T_0(a)\in\mathcal{C}^2_d(F^1_0(a),F^2_0(a))$ for each $a\in\mathcal{O}b(\mathcal{C}_1)$, and $T_k(x_1\otimes\cdots\otimes x_k)\in\mathcal{C}^2_{d+k+\sum e_i}(F^1_0(a_0),F^2_0(a_k))$ for each $x_i\in\mathcal{C}^1_{e_i}(a_{i-1},a_i)$, $i=1,\cdots,k$, such that T_k are homomorphisms.

We write T_0 in place of $T_0(a)$ when no confusion can occur.

For each pre natural transformation $T:F^1\to F^2$ of degree d, we define its *boundary* ∂T as follows. ∂T is a pre natural transformation of degree $d-1$: $F^1\to F^2$ defined by :

(10.2.0) $(\partial T)_0(a)=\partial\big(T_0(a)\big)$.

(10.2.1) $(\partial T)_1(x)=\pm\eta_2(F^1_1(x)\otimes T_0(a_1))\pm\eta_2(T_0(a_0)\otimes F^2_1(x))\pm\partial\big(T_1(x)\big)\pm T_1(\partial x)$ for each $x\in\mathcal{C}_1(a_0,a_1)$.

(10.2.k)

$$
\begin{aligned}
(\partial T)_k(x_1\otimes\cdots\otimes x_k)
&= \pm\partial\left(T_k(x_1\otimes\cdots\otimes x_k)\right)\pm\left(T_k(\partial(x_1\otimes\cdots\otimes x_k))\right)\\
&+\sum_{1\le i<j\le k}\pm T_{k-i+j}\left(x_1\otimes\cdots\otimes\eta_{j-i+1}(x_i\otimes\cdots\otimes x_j)\otimes\cdots\otimes x_k\right)\\
&+\sum_{k_1,k_2}\sum\pm\eta_{k_1+k_2+1}\Big(F^1_{\ell_1}(x_1\otimes\cdots\otimes x_{\ell_1})\otimes\cdots\otimes F^1_{\ell_{k_1}}(x_{\ell_1+\cdots+\ell_{k_1-1}+1}\otimes\cdots\otimes x_{\ell_1+\cdots+\ell_{k_1}})\\
&\qquad\otimes T_m(x_{\ell_1+\cdots+\ell_{k_1}+1}\otimes\cdots\otimes x_{\ell_1+\cdots+\ell_{k_1}+m})\otimes\\
&\qquad F^2_{n_1}(x_{\ell_1+\cdots+\ell_{k_1}+m+1}\otimes\cdots\otimes x_{\ell_1+\cdots+\ell_{k_1}+m+n_1})\otimes\cdots\otimes F^2_{n_{k_2}}(\cdots\otimes x_k)\Big)
\end{aligned}
$$

Here $\sum_{k_1,k_2}\sum$ means the summations over all k_1,k_2, $\ell_1,\cdots,\ell_{k_1},m,n_1,\cdots,n_{k_2}$, such that $\ell_1+\cdots+\ell_{k_1}+m+n_1+\cdots+n_{k_2}=k$. ($\ell_i>0$, $m_i>0$. But $m=0$ is allowed. In that case $T_0(x_i\otimes\cdots\otimes x_{i-1})$ means $T_0(a_i)$.)

Definition 10.3 A pre natural transformation $T:F^1\to F^2$ is said to be a *natural transformation* if $\partial T=0$.

Remark 10.4

Let $F^i:C\to \mathcal{C}h$, $i=1,2$ be A^∞ functors and $T:F^1\to F^2$ be a natural transformation. Then $(\partial T)_0=0$ means that $T_0(a):F_0^1(a)\to F_0^2(a)$ is a chain map. $(\partial T)_1=0$ means that if $x\in C(a_0,a_1)$ and $\partial x=0$ then the following diagram commutes up to chain homotopy $T_1(x)$.

$$
\begin{array}{ccc}
F_0^1(a_0) & \xrightarrow{T_0(a_0)} & F_0^2(a_0)\\
\downarrow F_1^1(x) & & \downarrow F_1^2(x)\\
F_0^2(a_1) & \xrightarrow{T_0(a_1)} & F_0^2(a_1)
\end{array}
$$

Diagram 10.5

Formula (10.2.k) looks rather complicated. Lemma 10.7 gives a motivation of this definition.

Definition 10.6 Let $b,c\in \mathcal{O}b(C)$ and $y\in C_d(c,b)$. b,c determine A^∞ functors $F^b,F^c:C\to\mathcal{C}h$ by $F_0^b(a)=C(b,a)$ etc. (§ 2.) Using y we define

$T^y(a) \in \mathcal{C}h(F^b(a), F^c(a))$, $T_k^y(x_1 \otimes \cdots \otimes x_k) \in \mathcal{C}h(F_0^b(a_0), F_0^c(a_k))$ as follows.

(10.6.0) $T_0^y(a)(z) = \eta_2(y \otimes z)$, where $z \in F_0^b(a) = C(b,a)$.
(10.6.k) $T_k^y(x_1 \otimes \cdots \otimes x_k)(z) = \eta_{k+2}(y \otimes z \otimes x_1 \otimes \cdots \otimes x_k)$, where $z \in F_0^b(a_0) = C(b,a_0)$, $x_i \in C(a_{i-1}, a_i)$, $i = 1, \cdots, k$.

T^y is a pre natural transform of degree d. We have :

Lemma 10.7

$$\partial T^y = T^{\partial y} .$$

Proof:

We first verify (10.2.1). We use $\partial \circ \eta_2 = \pm \eta_2 \circ \partial$ and obtain :

$$\begin{aligned} \left(\partial T^y\right)_0(z) &= \pm\partial\left(T_0^y(z)\right) \pm T_0^y(\partial z) \\ &= \pm\partial\left(\eta_2(y \otimes z)\right) \pm \eta_2(y \otimes \partial z) \\ &= \pm(\partial\eta_2)(y \otimes z) \pm \eta_2(\partial y \otimes z) \\ &= T_0^{\partial y}(z) \end{aligned}.$$

Let us verify (10.2.2) . We calculate

$$\begin{aligned} T_1^{\partial y}(x)(z) &= \pm\eta_3(\partial y \otimes z \otimes x) \\ &= \pm(\partial\eta_3)(y \otimes z \otimes x) \pm \partial\left(\eta_3(y \otimes z \otimes x)\right) \\ &\quad \pm\eta_3(y \otimes \partial z \otimes x) \pm \eta_3(y \otimes z \otimes \partial x) \\ &= \pm\eta_2\left(\eta_2(y \otimes z) \otimes x\right) \pm \eta_2\left(y \otimes \eta_2(z \otimes x)\right) , \\ &\quad \pm\partial\left(T_1^y(x)\right)(z) \pm T_1^y(\partial x)(z) \\ &= \pm\eta_2\left(T_0^y \otimes F_1^c(x)\right)(z) \pm \eta_2\left(F_1^b(x) \otimes T_0^y\right)(z) \\ &\quad \pm\partial\left(T_1^y(x)\right)(z) \pm T_1^y(\partial x)(z) \end{aligned}$$

as required. (10.2.k) for general k follows from the following calculation.

$$
\begin{aligned}
T_k^{\partial y}(x_1\otimes\cdots\otimes x_k)(z) &= \pm\eta_{k+2}(\partial y\otimes z\otimes x_1\otimes\cdots\otimes x_k)\\
&= \pm\partial\left(\eta_{k+2}(y\otimes z\otimes x_1\otimes\cdots\otimes x_k)\right)\\
&\quad\pm\eta_{k+2}(y\otimes\partial z\otimes x_1\otimes\cdots\otimes x_k)\\
&\quad\pm\eta_{k+2}(y\otimes z\otimes\partial\left(x_1\otimes\cdots\otimes x_k\right))\\
&\quad\pm(\partial\eta_{k+2})(y\otimes z\otimes x_1\otimes\cdots\otimes x_k)\\
&= \partial\left(T_k^y(x_1\otimes\cdots\otimes x_k)\right)(z)\pm\left(T_k^y(\partial\left(x_1\otimes\cdots\otimes x_k\right))\right)(z)\\
&\quad\pm\sum_{1\le i<j\le k}\eta_{k+2-j+i}(y\otimes z\otimes x_1\otimes\cdots\otimes\eta_{j-i+1}(x_i\otimes\cdots\otimes x_j)\otimes\cdots\otimes x_k).\\
&\quad\pm\sum_{1\le i\le k}\eta_{k+2-i}(y\otimes\eta_{i+1}(z\otimes\cdots\otimes x_i)\otimes\cdots\otimes x_k)\\
&\quad\pm\sum_{0\le i\le k-1}\eta_{k+1-i}(\eta_{i+2}(y\otimes z\otimes\cdots\otimes x_i)\otimes\cdots\otimes x_k)\\
&= \partial\left(T_k^y(x_1\otimes\cdots\otimes x_k)\right)(z)\pm\left(T_k^y(\partial\left(x_1\otimes\cdots\otimes x_k\right))\right)(z)\\
&\quad\pm\sum_{1\le i<j\le k}T_{k-i+j}^y\left(x_1\otimes\cdots\otimes\eta_{j-i+1}(x_i\otimes\cdots\otimes x_j)\otimes\cdots\otimes x_k\right)(z)\\
&\quad\pm\sum_{1\le i\le k}\eta_2\left(F_i^b(x_1\otimes\cdots\otimes x_i)\otimes T_{k-i}^y(x_{i+1}\otimes\cdots\otimes x_k)\right)(z)\\
&\quad\pm\sum_{1\le i\le k}\eta_2\left(T_{k-i}^y(x_1\otimes\cdots\otimes x_{i-1})\otimes F_{k-i+1}^c(x_i\otimes\cdots\otimes x_k)\right)(z).
\end{aligned}
$$

We next prove the following :

Proposition 10.8

$\partial(\partial T)=0$ *for any pre natural functor* T.

Proof

$(\partial\partial T)_0=0$ is immediate from. (10.2.0).

$$
\begin{aligned}
(\partial\partial T)_1(x) &= \pm\eta_2\left(F_1^1(x)\otimes(\partial T)_0\right)\pm\eta_2\left((\partial T)_0\otimes F_1^2(x)\right)\pm\partial\left((\partial T)_1(x)\right)\pm(\partial T)_1(\partial x)\\
&= \pm\eta_2\left(F_1^1(x)\otimes\partial(T_0)\right)\pm\eta_2\left(\partial(T_0)\otimes F_1^2(x)\right)\\
&\quad\pm\partial\left(\eta_2\left(F_1^1(x)\otimes T_0\right)\right)\pm\partial\left(\eta_2\left(T_0\otimes F_1^2(x)\right)\right)\pm\partial\partial\left(T_1(x)\right)\pm\partial\left(T_1(\partial x)\right)\quad,\\
&\quad\pm\eta_2\left(F_1^1(\partial x)\otimes T_0\right)\pm\eta_2\left(T_0\otimes F_1^2(\partial x)\right)\pm\partial\left(T_1(\partial x)\right)\pm T_1(\partial\partial x)\\
&= 0
\end{aligned}
$$

since η_2 is a chain map. Let us prove $(\partial\partial T)_k=0$. We calculate

(10.9)

$$
\begin{aligned}
&(\partial\partial T)_k(x_1\otimes\cdots\otimes x_k)\\
&\quad= \pm\partial\big((\partial T)_k(x_1\otimes\cdots\otimes x_k)\big)\pm(\partial T)_k\big(\partial(x_1\otimes\cdots\otimes x_k)\big)\\
&\qquad+\sum_{1\le i<j\le k}\pm(\partial T)_{k-i+j}\big(x_1\otimes\cdots\otimes\eta_{j-i+1}(x_i\otimes\cdots\otimes x_j)\otimes\cdots\otimes x_k\big)\\
&\qquad+\sum_{k_1,k_2}\sum\pm\eta_{k_1+k_2+1}\Big(F^1_{\ell_1}(x_1\otimes\cdots\otimes x_{\ell_1})\otimes\cdots\otimes F^1_{\ell_{k_1}}(x_{\ell_1+\cdots+\ell_{k_1-1}+1}\otimes\cdots\otimes x_{\ell_1+\cdots+\ell_{k_1}})\\
&\qquad\qquad\otimes(\partial T)_m(x_{\ell_1+\cdots+\ell_{k_1}+1}\otimes\cdots\otimes x_{\ell_1+\cdots+\ell_{k_1}+m})\otimes\\
&\qquad\qquad F^2_{n_1}(x_{\ell_1+\cdots+\ell_{k_1}+m+1}\otimes\cdots)\otimes\cdots\otimes F^2_{n_{k_2}}(\cdots\otimes x_k)\Big)
\end{aligned}
$$

To calculate Formula (10.9) in this way, seems to much complicated. So we use a symbolic notation and write (10.9) as :

$$
\begin{aligned}
&\pm\partial\big((\partial T)(\cdots)\big)+\sum\pm(\partial T)(\cdots\partial x\cdots))+\sum\pm(\partial T)(\cdots\eta(\cdots)\cdots)\\
&\qquad+\sum\pm\eta(F^1(\cdots)\cdots F^1(\cdots)\otimes(\partial T)(\cdots)\otimes F^2(\cdots)\cdots\otimes F^2(\cdots))
\end{aligned}
\tag{10.10}
$$

Namely we omit index $*$ in T_*, F^i_*, η_*, etc. and we omit x_i if it is not of the form ∂x. We also omit $\otimes$ when no confusion can occur. We use this notation frequently in the rest of this paper. Using this notation , Formulae (2.1), (2.2.k), (10.2.k) are

$$
\sum\pm\eta(\cdots\eta(\cdots)\cdots)=\partial\eta(\cdots)+\sum\pm\eta(\cdots\partial x\cdots)),
\tag{2.1}
$$

$$
\partial\big(F(\cdots)\big)+\sum\pm F(\cdots\partial x\cdots)=\sum\pm F(\cdots\eta(\cdots)\cdots)+\sum\pm\eta\big(F(\cdots)\cdots F(\cdots)\big),
\tag{2.2.k}
$$

$$
\begin{aligned}
(\partial T)(\cdots)=\pm\partial(T(\cdots))+\sum\pm T(\cdots\partial x\cdots)+\sum\pm T(\cdots\eta(\cdots)\cdots)&\\
+\sum\pm\eta\big(F^1(\cdots)\cdots F^1(\cdots)T(\cdots)F^2(\cdots)\cdots F^2(\cdots)\big)&,
\end{aligned}
\tag{10.2.k}
$$

respectively. Using (2.1),(2.2.k), (10.2.k), we calculate (10.10) and obtain

$$
\begin{aligned}
&\sum \pm\partial\big(T(\cdots\partial x\cdots)\big)\\
&\qquad+\sum \pm\partial\big(T(\cdots\eta(\cdots)\cdots)\big)\\
&\qquad+\sum \pm\partial\Big(\eta\Big(F^1\cdots F^1TF^2\cdots F^2\Big)\Big)\\
&\quad+\sum \pm\partial\big(T(\cdots\partial x\cdots)\big)\\
&\qquad+\sum \pm T(\cdots\partial x\cdots\eta(\cdots)\cdots)\\
&\qquad+\sum \pm T(\cdots\eta(\cdots\partial x\cdots)\cdots)\\
&\qquad+\sum \pm T(\cdots\eta(\cdots)\cdots\partial x\cdots)\\
&\qquad+\sum \pm\eta\Big(F^1\cdots F^1(\cdots\partial x\cdots)\cdots F^1TF^2\cdots F^2\Big)\\
&\qquad+\sum \pm\eta\Big(F^1\cdots F^1T(\cdots\partial x\cdots)F^2\cdots F^2\Big)\\
&\qquad+\sum \pm\eta\Big(F^1\cdots F^1TF^2\cdots F^2(\cdots\partial x\cdots)\cdots F^2\Big)\\
&\quad+\sum \pm\partial\big(T(\cdots\eta(\cdots)\cdots)\big)\\
&\qquad+\sum \pm T(\cdots\partial x\cdots\eta(\cdots)\cdots)\\
&\qquad+\sum \pm T(\cdots\partial(\eta(\cdots))\cdots)\cdots)\\
&\qquad+\sum \pm T(\cdots\eta(\cdots)\cdots\partial x\cdots)\\
&\qquad+\sum \pm T(\cdots\eta(\cdots\eta(\cdots)\cdots)\cdots)\\
&\qquad+\sum \pm\eta\Big(F^1\cdots F^1(\cdots\eta(\cdots)\cdots)\cdots F^1TF^2\cdots F^2\Big)\\
&\qquad+\sum \pm\eta\Big(F^1\cdots F^1T(\cdots\eta(\cdots)\cdots)F^2\cdots F^2\Big)\\
&\qquad+\sum \pm\eta\Big(F^1\cdots F^1TF^2\cdots F^2(\cdots\eta(\cdots)\cdots)\cdots F^2\Big)\\
&\quad+\sum \pm\eta(F^1\cdots F^1\partial(T(\cdots))F^2\cdots F^2)\\
&\qquad+\sum \pm\eta(F^1\cdots F^1T(\cdots\partial x\cdots)F^2\cdots F^2)\\
&\qquad+\sum \pm\eta(F^1\cdots F^1T(\cdots\eta(\cdots)\cdots)F^2\cdots F^2)\\
&\qquad+\sum \pm\eta(F^1\cdots F^1\eta\Big(F^1\cdots F^1TF^2\cdots F^2\Big)F^2\cdots F^2)
\end{aligned}
$$

here we write F^1 etc. in place of $F^1(\cdots)$ etc. By applying obvious cancellation and applying (2.2.1) once, this is equal to

$$
\begin{aligned}
&\sum \pm\partial\left(\eta\left(F^1\cdots F^1TF^2\cdots F^2\right)\right)\\
&\quad+\sum \pm\eta\left(F^1\cdots F^1(\cdots\partial x\cdots)\cdots F^1TF^2\cdots F^2\right)\\
&\quad+\sum \pm\eta\left(F^1\cdots F^1TF^2\cdots F^2(\cdots\partial x\cdots)\cdots F^2\right)\\
(10.11)\qquad &\quad+\sum \pm\eta\left(F^1\cdots F^1(\cdots\eta(\cdots)\cdots)\cdots F^1TF^2\cdots F^2\right)\\
&\quad+\sum \pm\eta\left(F^1\cdots F^1TF^2\cdots F^2(\cdots\eta(\cdots)\cdots)\cdots F^2\right)\\
&\quad+\sum \pm\eta(F^1\cdots F^1\partial(T(\cdots))F^2\cdots F^2)\\
&\quad+\sum \pm\eta(F^1\cdots F\eta\left(F^1\cdots F^1TF^2\cdots F^2\right)F^2\cdots F^2)
\end{aligned}
$$

We use (2.2.k) to find that (10.11) is equal to

$$
\begin{aligned}
&\sum \pm\partial\left(\eta\left(F^1\cdots F^1TF^2\cdots F^2\right)\right)\\
&\quad+\sum \pm\eta\left(F^1\cdots\eta(F^1\cdots F^1)\cdots F^1TF^2\cdots F^2\right)\\
&\quad+\sum \pm\eta\left(F^1\cdots\partial(F^1(\cdots))\cdots F^1TF^2\cdots F^2\right)\\
(10.12)\qquad &\quad+\sum \pm\eta\left(F^1\cdots F^1TF^2\cdots\eta(F^2\cdots F^2)\cdots F^2\right)\\
&\quad+\sum \pm\eta\left(F^1\cdots F^1TF^2\cdots\partial(F^2(\cdots))\cdots F^2\right)\\
&\quad+\sum \pm\eta(F^1\cdots F^1\partial(T(\cdots))F^2\cdots F^2)\\
&\quad+\sum \pm\eta(F^1\cdots F^1\eta\left(F^1\cdots F^1TF^2\cdots F^2\right)F^2\cdots F^2)
\end{aligned}
$$

In view of (2.1), we find that (10.12) is zero. The proof of Proposition 10.8 is now complete.

Definition 10.13

Let C^1, C^2 be A^∞ categories and $F^1 : C^1 \to C^2$, $F^2 : C^1 \to C^2$ be A^∞ functors. We write $C(F^1,F^2)$ the graded abelian group of all pre natural transformations between them. $C(F^1,F^2)$ is a chain complex by Proposition 10.8.

A natural transformation T is said to be *exact* if $T = \partial T'$ for some pre natural transformation T'.

Two natural transformations T^1, T^2 are said to be *homotopic* if $T^1 - T^2$ is exact.

By Lemma 10.7 we have a chain map

$$(10.14)\qquad C(c,b) \to \mathcal{F}unc(F^b, F^c)$$

which plays an important role in our application. We discuss in §§ 12, 13 a sufficient condition for (10.14) to be a chain homotopy equivalence (together with higher compositions) .

We next define (higher) compositions of pre natural transformations.

Definition 10.15

Let $F^i : C^1 \to C^2$, $i = 0,\cdots,h$ be A^∞ functors and $T^i : F^{i-1} \to F^i$ be pre natural transformations. Then their h-th composition $\eta_h(T^1 \otimes \cdots \otimes T^h)$ is defined as follows.

(1016.0) $$\left(\eta_h(T^1 \otimes \cdots \otimes T^h)\right)_0(a) = \eta_h(T_0^1(a) \otimes \cdots \otimes T_0^h(a)).$$

(10.16.k) $$\begin{aligned}&\left(\eta_h(T^1 \otimes \cdots \otimes T^h)\right)_k(x_1 \otimes \cdots \otimes x_k)\\ &\quad = \sum \pm \eta(F^0 \cdots F^0 T^1 F^1 \cdots F^1 T^2 \cdots F^{h-1} \cdots F^{h-1} T^h F^h \cdots F^h)\end{aligned}.$$

More precisely (10.16.k) is :

$$\left(\eta_h(T^1\otimes\cdots\otimes T^h)\right)_k(x_1,\cdots,x_k)$$

$$\begin{aligned}
=\sum_{k_0,\cdots,k_h}\sum\pm\eta_{k_0+\cdots+k_h+h}\Bigg[&F^0_{\ell^0_1}\left(x_1\otimes\cdots\otimes x_{\ell^0_1}\right)\otimes\cdots\otimes F^0_{\ell^0_{k_1}}\left(x_{\ell^0_1+\cdots+\ell^0_{k_0-1}+1}\otimes\cdots\otimes x_{\ell^0_1+\cdots+\ell^0_{k_0}}\right)\\
&\otimes T^1_{m_1}\left(x_{\ell^0_1+\cdots+\ell^0_{k_0}+1}\otimes\cdots\otimes x_{\ell^0_1+\cdots+\ell^0_{k_0}+m_1}\right)\otimes\\
&F^1_{\ell^1_1}\left(x_{\ell^0_1+\cdots+\ell^0_{k_0}+m_1+1}\otimes\cdots\otimes x_{\ell^0_1+\cdots+\ell^0_{k_0}+m_1+\ell^1_1}\right)\otimes\\
&\cdots\otimes F^1_{\ell^1_{k_2}}\left(x_{\ell^0_1+\cdots+\ell^0_{k_0}+m_1+\ell^1_1+\cdots+\ell^1_{k_1-1}+1}\otimes\cdots\otimes x_{\ell^0_1+\cdots+\ell^0_{k_0}+m_1+\ell^1_1+\cdots+\ell^1_{k_1}}\right)\\
&\otimes T^2_{m_2}\left(x_{\ell^0_1+\cdots+\ell^0_{k_0}+m_1+\ell^1_1+\cdots+\ell^1_{k_1}+1}\otimes\cdots\otimes x_{\ell^0_1+\cdots+\ell^0_{k_0}+m_1+\ell^1_1+\cdots+\ell^1_{k_1}+m_2}\right)\otimes\\
&\bullet\\
&\bullet\\
&\bullet\\
&\otimes T^h_{m_h}\left(x_{\sum_{i=0}^{h-1}\sum_{j=1}^{k_i}\ell^i_j+\sum_{i=1}^{h-1}m_i+1}\otimes\cdots\otimes x_{\sum_{i=0}^{h-1}\sum_{j=1}^{k_i}\ell^i_j+\sum_{i=1}^{h}m_i}\right)\otimes\\
&F^h_{\ell^h_1}\left(x_{\sum_{i=0}^{h-1}\sum_{j=1}^{k_i}\ell^i_j+\sum_{i=1}^{h}m_i+1}\otimes\cdots\otimes x_{\sum_{i=0}^{h-1}\sum_{j=1}^{k_i}\ell^i_j+\sum_{i=1}^{h}m_i+\ell^h_1}\right)\otimes\\
&\cdots\otimes F^h_{\ell^h_{k_h}}\left(x_{\sum_{i=0}^{h-1}\sum_{j=1}^{k_i}\ell^i_j+\sum_{i=1}^{h}m_i+\sum_{j=1}^{k_h-1}\ell^h_j+1}\otimes\cdots\otimes x_{\sum_{i=0}^{h}\sum_{j=1}^{k_i}\ell^i_j+\sum_{i=1}^{h}m_i}\right)\Bigg].
\end{aligned}$$

(We remark that $m_i=0$ is allowed in Formula (10.16.k) .)

The main result of this section is :

Theorem 10.17 *$\mathcal{F}unc(C^1,C^2)$ is an A^∞ category. Here the object of $\mathcal{F}unc(C^1,C^2)$ is an A^∞ functor, morphisms of it is a pre natural transformation, and the (higher) composition map is as in Definition* 10.15.

Proof:

We are going to verify Formula (2.1). We use Φ_h to denote the h-th composition in $\mathcal{F}unc(C^1,C^2)$ and η_h for h-th composition in C^1,C^2, in order to avoid the confusion.

We are going to verify

$$\sum \pm\Phi(T\cdots T\Phi(T\cdots T)T\cdots T) = \partial\big(\Phi(T\cdots T)\big) + \sum \pm\Phi(T\cdots T(\partial T)T\cdots T)).$$

Here and hereafter we omit the index i in F^i, T^i. The formula

$$\sum \pm\big(\Phi(T\cdots T\Phi(T\cdots T)T\cdots T)\big)_0(a) = \pm\big(\partial\,(\Phi(T\cdots T))\big)_0(a) + \sum \pm\big(\Phi(T\cdots T(\partial T)\,T\cdots T))\big)_0\,(a)$$

is immediate from (10.16.0) and (2.1) for C^1, C^2. Hence it suffices to show

(10.18)
$$\begin{aligned}&\sum \pm\big(\Phi(T\cdots T\Phi(T\cdots T)T\cdots T)\big)_k(\cdots)\\ &\quad \pm\big(\partial(\Phi(T\cdots T))\big)_k(\cdots) + \sum \pm\big(\Phi(T\cdots T(\partial T)T\cdots T))\big)_k(\cdots) = 0\end{aligned}.$$

We calculate the first term of (10.18) according to the definition and obtain

(10.19)
$$\begin{aligned}&\sum \pm\big(\Phi(T\cdots T\Phi(T\cdots T)T\cdots T)\big)_k(\cdots)\\ &\quad = \sum \pm\eta(F\cdots FTF\cdots F\Phi(T\cdots T)F\cdots FTF\cdots F)\\ &\quad = \sum \pm\eta(F\cdots FTF\cdots F\eta(F\cdots FTF\cdots FTF\cdots F)F\cdots FTF\cdots F)\end{aligned}.$$

We remark that in the right hand side of (10.19) the summation is taken over all choices so that the number of T's in $\eta(F\cdots FTF\cdots FTF\cdots F)$ is not smaller than 2 and smaller than k. We next calculate the second term of (10.18) and obtain

$$
\begin{aligned}
&\left(\partial(\Phi(T\cdots T))\right)_k(\cdots)\\
&\quad=\partial\left(\Phi(T\cdots T)(\cdots)\right)+\sum\pm\Phi(T\cdots T)(\cdots\partial x\cdots)\\
&\qquad\quad+\sum\pm\Phi(T\cdots T)(\cdots\eta(\cdots)\cdots)+\sum\pm\eta(F\cdots F\Phi(T\cdots T)F\cdots F)\\
&\quad=\sum\pm\partial\left(\eta(F\cdots FTF\cdots FTF\cdots F)\right)\\
&\qquad\quad+\sum\pm\eta(F\cdots FTF\cdots F(\cdots\partial x\cdots)\cdots FTF\cdots F)\\
&\qquad\quad+\sum\pm\eta(F\cdots FTF\cdots T(\cdots\partial x\cdots)\cdots FTF\cdots F)\\
&\qquad\quad+\sum\pm\eta(F\cdots FTF\cdots F(\cdots\eta(\cdots)\cdots)\cdots FTF\cdots F)\\
&\qquad\quad+\sum\pm\eta(F\cdots FTF\cdots T(\cdots\eta(\cdots)\cdots)\cdots FTF\cdots F)\\
&\qquad\quad+\sum\pm\eta(F\cdots F\eta(F\cdots FTF\cdots FTF\cdots F)F\cdots F)\\
&\quad=\sum\pm\partial\left(\eta(F\cdots FTF\cdots FTF\cdots F)\right)\\
&\qquad\quad+\sum\pm\eta(F\cdots FTF\cdots\eta(F\cdots F)\cdots FTF\cdots F)\\
&\qquad\quad+\sum\pm\eta(F\cdots FTF\cdots\partial(F(\cdots))\cdots FTF\cdots F)\\
&\qquad\quad+\sum\pm\eta(F\cdots FTF\cdots T(\cdots\partial x\cdots)\cdots FTF\cdots F)\\
&\qquad\quad+\sum\pm\eta(F\cdots FTF\cdots T(\cdots\eta(\cdots)\cdots)\cdots FTF\cdots F)\\
&\qquad\quad+\sum\pm\eta(F\cdots F\eta(F\cdots FTF\cdots FTF\cdots F)F\cdots F)
\end{aligned}
\tag{10.20}
$$

We next calculate the 3rd term of (10.18) and obtain

$$
\begin{aligned}
&\sum\pm\left(\Phi(T\cdots T(\partial T)T\cdots T)\right)_k(\cdots)\\
&\quad=\sum\pm\eta(F\cdots FTF\cdots((\partial T)(\cdots))\cdots FTF\cdots F)\\
&\quad=\sum\pm\eta(F\cdots FTF\cdots\partial(T(\cdots))\cdots FTF\cdots F)\\
&\qquad\quad+\sum\pm\eta(F\cdots FTF\cdots T(\cdots\partial x\cdots)\cdots FTF\cdots F)\\
&\qquad\quad+\sum\pm\eta(F\cdots FTF\cdots T(\cdots\eta(\cdots)\cdots)\cdots FTF\cdots F)\\
&\qquad\quad+\sum\pm\eta(F\cdots FTF\cdots\eta(F\cdots FTF\cdots F)\cdots FTF\cdots F)
\end{aligned}
\tag{10.21}
$$

Therefore (10.19)+(10.20)+(10.21) is equal to

$$
\begin{aligned}
&\sum \pm\eta(F\cdots FTF\cdots F\eta(F\cdots FTF\cdots FTF\cdots F)F\cdots FTF\cdots F)\\
&\quad+\sum \pm\partial(\eta(F\cdots FTF\cdots FTF\cdots F))\\
&\quad+\sum \pm\eta(F\cdots FTF\cdots\eta(F\cdots F)\cdots FTF\cdots F)\\
&\quad+\sum \pm\eta(F\cdots FTF\cdots\partial(F(\cdots))\cdots FTF\cdots F)\\
&\quad+\sum \pm\eta(F\cdots F\eta(F\cdots FTF\cdots FTF\cdots F)F\cdots F)\\
&\quad+\sum \pm\eta(F\cdots FTF\cdots\partial(T(\cdots))\cdots FTF\cdots F)\\
&\quad+\sum \pm\eta(F\cdots FTF\cdots\eta(F\cdots FTF\cdots F)\cdots FTF\cdots F)
\end{aligned}
\tag{10.22}
$$

(10.22) vanishes by (2.1). The proof of Theorem 10.17 is now complete.

For an A^∞ category $\mathcal{C}$, we define its opposite category $\mathcal{C}^o$ by $\mathcal{O}b(\mathcal{C}^o)=\mathcal{O}b(\mathcal{C})$, $\mathcal{C}^o(a,b)=\mathcal{C}(b,a)$. Hence we can define and prove results similar to those we showed in this section on contravariant functors in place of covariant functors.

Our final task in this section is to extend Definitions 2.4, 10.3, 10.6 and Lemma 10.7 and to construct an A^∞ functor $\mathcal{F}:\mathcal{C}^o \to \mathcal{F}unc(\mathcal{C},\mathcal{C}h)$. The definition is as follows.

For $a\in\mathcal{O}b(\mathcal{C})$, $\mathcal{F}_0(a)\in\mathcal{O}b(\mathcal{F}unc(\mathcal{C},\mathcal{C}h))$ is an A^∞ functor $:\mathcal{C}\to\mathcal{C}h$ such that for $b\in\mathcal{O}b(\mathcal{C})$

$$
\left(\mathcal{F}_0(a)\right)_0(b)=\mathcal{C}(a,b)\in\mathcal{O}b(\mathcal{C}h)
\tag{10.23}
$$

For $x_i\in\mathcal{C}(b_{i-1},b_i)$, $z\in\mathcal{C}(a,b_0)\in\left(\mathcal{F}_0(a)\right)_0(b_0)$ we put

$$
\begin{gathered}
\left(\mathcal{F}_0(a)\right)_h(x_1\cdots x_h)\in Hom(\mathcal{C}(a,b_0),\mathcal{C}(a,b_k))\\
\left(\left(\mathcal{F}_0(a)\right)_h(x_1\cdots x_h)\right)(z)=\eta_{h+2}(zx_1\cdots x_h).
\end{gathered}
\tag{10.24}
$$

We wrote $\mathcal{F}_0(a)$ as F^a in Definition 2.4. Lemma 2.5 implies that it is an A^∞ functor.

We next define $\mathcal{F}_\ell(y_1\cdots y_\ell)\in\mathcal{F}unc(\mathcal{F}_0(a_0),\mathcal{F}_0(a_\ell))$ for each $y\in\mathcal{C}(a_i,a_{i-1})=\mathcal{C}^o(a_{i-1},a_i)$, $a_i\in\mathcal{O}b(\mathcal{C})$. Namely $\mathcal{F}_\ell(y_1\cdots y_\ell)$ is a pre natural transformation : $\mathcal{F}_0(a_0)\to\mathcal{F}_0(a_\ell)$. For $b\in\mathcal{O}b(\mathcal{C})$, we define

$$
\begin{gathered}
\left(\mathcal{F}_\ell(y_1\cdots y_\ell)\right)_0(b)\in Hom(\mathcal{C}(a_0,b),\mathcal{C}(a_\ell,b))=\mathcal{C}h\left(\left(\mathcal{F}_0(a_0)\right)(b),\left(\mathcal{F}_\ell(a_0)\right)(b)\right)\\
\left(\left(\mathcal{F}_\ell(y_1\cdots y_\ell)\right)_0(b)\right)(z)=\eta_{\ell+1}(y_1\cdots y_\ell z).\ \ (z\in\mathcal{C}(a_0,b)).
\end{gathered}
\tag{10.25}
$$

Next for $x_i\in\mathcal{C}(b_{i-1},b_i)$, $z\in\mathcal{C}(a,b_0)\in\left(\mathcal{F}_0(a)\right)_0(b_0)$ we put :

$$\left(\mathcal{F}_\ell(y_1\cdots y_\ell)\right)_h(x_1\cdots x_h)\in Hom(C(a_0,b_0),C(a_\ell,b_k))$$

(10.26)
$$\left(\left(\mathcal{F}_\ell(y_1\cdots y_\ell)\right)_h(x_1\cdots x_h)\right)(z)=\eta_{h+\ell+1}\left(y_1\cdots y_\ell z x_1\cdots x_h\right).$$

In Definition 10.6 we wrote $\mathcal{F}_1(y)=T^y$. The following is a generalization of Lemma 10.7.

Proposition 10.27

$\mathcal{F}: C^o\to \mathcal{F}unc(C,\mathcal{C}h)$ defined above is an A^∞ functor,

Proof:

We use Φ for the composition in $\mathcal{F}unc(C,\mathcal{C}h)$. In fact, since $\eta_k=0$ for $k\geq 3$ in $\mathcal{C}h$) it follows that $\Phi_k=0$ for $k\geq 3$. Thus again using symbolic notations, we are only to verify

(10.28)
$$\begin{aligned}0=&\left(\left(\partial\left(\mathcal{F}(y\cdots y)\right)\right)(\cdots)\right)(z)+\sum\pm\left(\left(\mathcal{F}(y\cdots\partial y\cdots y)\right)(\cdots)\right)(z)\\&+\sum\pm\left(\left(\mathcal{F}(y\cdots y\eta(y\cdots y)y\cdots y)\right)(\cdots)\right)(z)\\&+\sum\pm\left(\left(\Phi_2\left(\mathcal{F}(y\cdots y)\mathcal{F}(y\cdots y)\right)\right)(\cdots)\right)(z)\end{aligned}$$

The first term of (10.28) is

(10.29)
$$\begin{aligned}&\left(\partial\left(\left(\mathcal{F}(y\cdots y)\right)(\cdots)\right)\right)(z)+\sum\pm\left(\left(\mathcal{F}(y\cdots y)(\cdots\partial x\cdots)\right)\right)(z)\\&\quad+\sum\pm\left(\left(\mathcal{F}(y\cdots y)(\cdots\eta(\cdots)\cdots)\right)\right)(z)\\&\quad+\sum\pm\left(\Phi\left(\mathcal{F}_0\otimes\mathcal{F}(y\cdots y)\right)(\cdots)\right)(z)+\left(\Phi\left(\mathcal{F}(y\cdots y)\otimes\mathcal{F}_0\right)(\cdots)\right)(z)\\=&\quad\partial\left(\left(\left(\mathcal{F}(y\cdots y)\right)(\cdots)\right)(z)\right)\pm\left(\left(\mathcal{F}(y\cdots y)\right)(\cdots)\right)(\partial z)\\&\quad+\sum\pm\eta(y\cdots yz\cdots\partial x\cdots)+\sum\pm\eta(y\cdots yz\cdots\eta(\cdots)\cdots)\\&\quad+\sum\pm\eta(y\cdots y\eta(z\cdots)\cdots)+\sum\pm\eta(\eta(y\cdots yz\cdots)\cdots)\\=&\quad\partial\left(\eta(y\cdots yz\cdots)\right)+\eta(y\cdots y(\partial z)\cdots)+\\&\quad+\sum\pm\eta(y\cdots yz\cdots\partial x\cdots)+\sum\pm\eta(y\cdots yz\cdots\eta(\cdots)\cdots)\\&\quad+\sum\pm\eta(y\cdots y\eta(z\cdots)\cdots)+\sum\pm\eta(\eta(y\cdots yz\cdots)\cdots)\end{aligned}$$

On the other hand, the second term of (10.28) is

(10.30)
$$\sum\pm\eta(y\cdots(\partial y)\cdots yz\cdots).$$

The 3rd term of (10.28) is

(10.31) $$\sum \pm\eta(y\cdots y\eta(y\cdots y)\cdots yz\cdots).$$

The 4th term of (10.28) is

(10.32) $$\sum \pm\eta(y\cdots y\eta(y\cdots yz\cdots)\cdots).$$

It is immediate from (2.1) that (10.29)+(10.30)+(10.30)+(10.31)=0. The proof of Proposition 10.27 is now complete.

§ 11 Homotopy equivalence

In this section, we define and study homotopy equivalences between two A^∞categories, two A^∞functors, and two objects of A^∞categories. This was essential in §§ 5,6,7, where we discussed well definedness of the relative Floer homology.

Definition 11.1 An A^∞category $\mathcal{C}$ is said to *have an identity* if there exists an element $1_a \in \mathcal{C}_0(a,a)$ such that

(11.1.1) $\partial 1_a = 0$.
(11.1.2) $\eta_2(1_b \otimes x) = x$, $\eta_2(y \otimes 1_b) = y$, for every $x \in \mathcal{C}(a,b)$, $y \in \mathcal{C}(b,c)$.
(11.1.3) $\eta_{k+\ell+1}(x_1 \otimes \cdots \otimes x_k \otimes 1_a \otimes x_1 \otimes \cdots \otimes x_\ell) = 0$ for $k+\ell \geq 2$.

The discussion in §10 can be generalized to the topological A^∞category with minor change. (We remark also that $\mathcal{F}unc(\mathcal{C},\mathcal{C}h)$ is an A^∞category if $\mathcal{C}$ is a topological A^∞category. Namely composition of two topological pre natural transformations is always well defined. This is because intersection of finitely many Bair sets is a Bair set.)
However we need to modify Definition 11.1 in a nontrivial way to generalize it to topological A^∞category. In fact, in our basic example $\mathcal{L}ag(X,\omega)$, the chain complex $\mathcal{C}_*(a,a)$ is never well defined. This is because the Lagrangian submanifold is never transversal to itself. We will discuss this point in §13.

Definition 11.2 Let $F : \mathcal{C}^1 \to \mathcal{C}^2$ be an A^∞functor. We assume that $\mathcal{C}^2$ has an identity. We then define the *identity transformation* 1_F from F to itself by

(11.2.1) $(1_F)_0(a) = 1_a$.
(11.2.2) $(1_F)_1(x) = x$
(11.2.3) $(1_F)_k(x_1 \otimes \cdots \otimes x_k) = 0$, $k > 1$.

Lemma 11.3

$$\partial 1_F = 0.$$

Proof:

$(\partial 1_F)_0 = (\partial 1_F)_1 = 0$ is immediate from (11.1.1). For $k > 1$, we have :

$$
\begin{aligned}
&(\partial 1_F)_k(x_1\otimes\cdots\otimes x_k)\\
&=\sum_m \sum_{\substack{k_1,k_2\\ k_1+k_2+m=k}} \pm\eta_{k_1+k_2+1}(F(x_1)\otimes\cdots\otimes F(x_{k_1})\\
&\qquad\qquad \otimes(1_F)_m(x_{k_1+1}\otimes\cdots\otimes x_{k_1+m})\otimes F(x_{k_1+m+1})\otimes\cdots\otimes F(x_{k_1+k_2+m})) \quad .\\
&=\sum_m \sum_{\substack{k_1,k_2\\ k_1+k_2+m=k}} \pm\,\eta_{k_1+k_2+1}(F(x_1)\otimes\cdots\otimes F(x_{k_1})\otimes 1_a\otimes F(x_{k_1+m+1})\otimes\cdots\otimes F(x_{k_1+k_2+m+1}))\\
&=0
\end{aligned}
$$

Definition 11.4 Let $F^i : C^1 \to C^2$, $i = 1,2$ be A^∞functors. We assume that C^2 has an identity. A natural transformation $T : F^1 \to F^2$ of degree 0, is said to be a *homotopy equivalence* if there exits another natural transformation $T' : F^2 \to F^1$ such that $\eta_2(T\otimes T') - 1_{F^1}$ and $\eta_2(T'\otimes T) - 1_{F^2}$ is exact. We say that F^1 is *homotopy equivalent* to F^2 if there exists a homotopy equivalence $T : F^1 \to F^2$.

We remark that the composition $\eta_2(T'\otimes T)$ of two natural transformations is again a natural transformation by Theorem 10.17.

It is easy to see from Theorem 10.17 that the composition of homotopy equivalences is also a homotopy equivalence.

We recall that a chain map $\varphi : C \to C'$ is said to be a chain homotopy equivalence if there exists a chain map $\varphi : C' \to C$ and a homomorphisms $H : C \to C$, $H' : C' \to C'$ such that $\varphi' \circ \varphi = 1 + \partial H$, $\varphi \circ \varphi' = 1 + \partial H'$.

Lemma 11.5

Let $F^i : C \to \mathcal{C}h$, $i = 1,2$ be A^∞functors and $T : F^1 \to F^2$ be a homotopy equivalence. Then for any object a of C, the map $T_0(a)$ induces a chain homotopy equivalence $T_0(a) : F^1(a) \to F^2(a)$.

The proof is immediate from the following :

Lemma 11.6

Let $F^i : C \to \mathcal{C}h$, $i = 1,2$ be A^∞functors and $T, T^1, T^2 : F^1 \to F^2$ are pre natural transformations.

(11.6.1) *If $\partial T = 0$ then $T_0(a) : F^1(a) \to F^2(a)$ is a chain map for any object a of C.*

(11.6.2) *If $T^1 - T^2 = \partial T$, $\partial T^1 = \partial T^2 = 0$, then $T_0(a) : F^1(a) \to F^2(a)$ is a chain homotopy from $T_0^1(a) : F^1(a) \to F^2(a)$ to $T_0^2(a) : F^1(a) \to F^2(a)$.*

Proof : Immediate from definition.

We furthermore find the following :

Lemma 11.7

Let $F^i : \mathcal{C} \to \mathcal{Ch}$, $i = 1,2$ be A^∞functors and $T : F^1 \to F^2$ be a homotopy equivalence. Then, for any A^∞functor $F : \mathcal{C} \to \mathcal{Ch}$, we have a chain homotopy equivalence

$$\mathcal{Func}(F, F^1) \to \mathcal{Func}(F, F^2),$$
$$\mathcal{Func}(F^2, F) \to \mathcal{Func}(F^1, F).$$

Proof: We define $\mathcal{T} : \mathcal{Func}(F, F^1) \to \mathcal{Func}(F, F^2)$ *by*

$$\mathcal{T}(S) = \eta_2(S \otimes T).$$

Then it is a chain map since $\partial T = 0$. We can construct $\mathcal{T}' : \mathcal{Func}(F, F^2) \to \mathcal{Func}(F, F^1)$ from $\mathcal{T}' : F^2 \to F^1$ in Definition 11.1. Using the fact that $\eta_2(T \otimes T') - 1_{F^1}$ and $\eta_2(T' \otimes T) - 1_{F^2}$ is exact, we can prove easily that $\mathcal{T}' \circ \mathcal{T}$ and $\mathcal{T} \circ \mathcal{T}'$ are homotopic to the identity. We can prove that $\mathcal{Func}(F^2, F) \to \mathcal{Func}(F^1, F)$ is a chain homotopy equivalence in a similar way.

Definition 11.8 Let $a, b \in \mathcal{Ob}(\mathcal{C})$, $x \in \mathcal{C}(a, b)$. We say that x is a *homotopy equivalence*, if $\mathcal{F}_1(x) : F^b \to F^a$ is a homotopy equivalence. We say that $a, b \in \mathcal{Ob}(\mathcal{C})$ are homotopy equivalent to each other if there exists a homotopy equivalence $x \in \mathcal{C}(a, b)$.

Remark 11.9

Theorem 12.2 implies that if F^a is homotopic to F^b, then a is homotopy equivalent to b.

We next define homotopy equivalence between A^∞categories. For this purpose we define composition of A^∞functors.

Definition 11.10

Let $F^i : \mathcal{C}^i \to \mathcal{C}^{i+1}$, $i = 1,2$ be A^∞functors. Its *composition* $F^2 \circ F^1$ is defined by

(11.10.1) $\left(F^2 \circ F^1\right)_0(a) = F_0^2(F_0^1(a)) \in \mathcal{Ob}(\mathcal{C}^3)$, for $a \in \mathcal{Ob}(\mathcal{C}^1)$.

(11.10.2)

$$\left(F^2 \circ F^1\right)_k(x_1 \otimes\cdots\otimes x_k) = \sum_{\substack{m_1+\cdots+m_\ell = k\\ m_j>0}} \pm F^2_\ell\left(F^1_{m_1}(x_1 \otimes\cdots\otimes x_{m_1}) \otimes\cdots\otimes F^1_{m_\ell}(x_{k-m_\ell+1} \otimes\cdots\otimes x_k)\right)$$

for $x_i \in C^1(a_{i-1},a_i)$.

Lemma 11.11 $F^2 \circ F^1$ *is an* A^∞*functor.*

Proof:

$$\begin{aligned}
&\partial\left(\left(F^2 \circ F^1\right)(\cdots)\right)\\
&\quad = \sum \pm\partial\left(F^2(F^1\cdots F^1)\right)\\
&\quad = \sum \pm F^2(F^1\cdots\left(\partial F^1\right)\cdots F^1) + \sum \pm F^2(F^1\cdots\eta(F^1\cdots F^1)\cdots F^1)\\
&\qquad\quad + \sum \pm\eta\left(\left(F^2(F^1\cdots F^1)\right)\cdots\left(F^2(F^1\cdots F^1)\right)\right)\\
&\quad = \sum \pm F^2(F^1\cdots\left(F^1(\cdots\partial x\cdots)\right)\cdots F^1) + \sum \pm F^2(F^1\cdots\left(F^1(\cdots\eta(\cdots)\cdots)\right)\cdots F^1)\\
&\qquad\qquad\quad + \sum \pm F^2(F^1\cdots\eta(F^1\cdots F^1)\cdots F^1)\\
&\qquad\quad + \sum \pm F^2(F^1\cdots\eta(F^1\cdots F^1)\cdots F^1)\\
&\qquad\quad + \sum \pm\eta\left(\left(F^2(F^1\cdots F^1)\right)\cdots\left(F^2(F^1\cdots F^1)\right)\right)\\
&\quad = \sum \pm\left(F^2 \circ F^1\right)(\cdots\partial x\cdots) + \sum \pm\left(F^2 \circ F^1\right)(\cdots\eta(\cdots)\cdots)\sum \pm\eta\left(\left(F^2 \circ F^1\right)\cdots\left(F^2 \circ F^1\right)\right)
\end{aligned}$$

.

Lemma 11.12 $F^3 \circ\left(F^2 \circ F^1\right) = \left(F^3 \circ F^2\right)\circ F^1$

This lemma is immediate from definition

Lemma 11.13 *Let* $G : C^1 \to C^2$ *be an* A^∞*functor. It induces an* A^∞*functors* $G^* : \mathcal{F}unc(C^2,C) \to \mathcal{F}unc(C^1,C)$, $G_* : \mathcal{F}unc(C,C^1) \to \mathcal{F}unc(C,C^2)$ *for any* A^∞*category* C.

Proof:

An object of $\mathcal{F}unc(C,C^1)$ is an A^∞*functor* $F : C \to C^1$. We put $G_{*0}(F) = G \circ F$. Let $F^i : C \to C^1$ be A^∞*functor*s and $T^i \in \mathcal{F}unc(F^{i-1},F^i)$. (Namely T^i is a pre natural transformation.) We put

$$\left(G_{*h}(T^1 \otimes\cdots\otimes T^h)\right)_0(a) = G_h\left(T^1_0(a) \otimes\cdots\otimes T^h_0(a)\right),$$
$$(G(T\cdots T))(\cdots) = \sum \pm G(F\cdots FTF\cdots FTF\cdots FTF\cdots F)$$

more precisely the later formula is :

$$\left(G_h(T^1\otimes\cdots\otimes T^h)\right)_k(x_1,\cdots,x_k)$$

$$\begin{aligned}
= \sum_{k_0,\cdots,k_h}\sum \pm G_{k_0+\cdots+k_h+h}\Bigg[\, & F^0_{\ell^0_1}\left(x_1\otimes\cdots\otimes x_{\ell^0_1}\right)\otimes\cdots\otimes F^0_{\ell^0_{k_1}}\left(x_{\ell^0_1+\cdots+\ell^0_{k_0-1}+1}\otimes\cdots\otimes x_{\ell^0_1+\cdots+\ell^0_{k_0}}\right)\\
& \otimes T^1_{m_1}\left(x_{\ell^0_1+\cdots+\ell^0_{k_0}+1}\otimes\cdots\otimes x_{\ell^0_1+\cdots+\ell^0_{k_0}+m_1}\right)\otimes\\
& F^1_{\ell^1_1}\left(x_{\ell^0_1+\cdots+\ell^0_{k_0}+m_1+1}\otimes\cdots\otimes x_{\ell^0_1+\cdots+\ell^0_{k_0}+m_1+\ell^1_1}\right)\otimes\\
& \qquad\cdots\otimes F^1_{\ell^1_{k_2}}\left(x_{\ell^0_1+\cdots+\ell^0_{k_0}+m_1+\ell^1_1+\cdots+\ell^1_{k_1-1}+1}\otimes\cdots\otimes x_{\ell^0_1+\cdots+\ell^0_{k_0}+m_1+\ell^1_1+\cdots+\ell^1_{k_1}}\right)\\
& \otimes T^2_{m_2}\left(x_{\ell^0_1+\cdots+\ell^0_{k_0}+m_1+\ell^1_1+\cdots+\ell^1_{k_1}+1}\otimes\cdots\otimes x_{\ell^0_1+\cdots+\ell^0_{k_0}+m_1+\ell^1_1+\cdots+\ell^1_{k_1}+m_2}\right)\otimes\\
& \qquad\qquad\bullet\\
& \qquad\qquad\bullet\\
& \qquad\qquad\bullet\\
& \otimes T^h_{m_h}\left(x_{\sum_{i=0}^{h-1}\sum_{j=1}^{k_i}\ell^i_j+\sum_{i=1}^{h-1}m_i+1}\otimes\cdots\otimes x_{\sum_{i=0}^{h-1}\sum_{j=1}^{k_i}\ell^i_j+\sum_{i=1}^{h}m_i}\right)\otimes\\
& F^h_{\ell^h_1}\left(x_{\sum_{i=0}^{h-1}\sum_{j=1}^{k_i}\ell^i_j+\sum_{i=1}^{h}m_i+1}\otimes\cdots\otimes x_{\sum_{i=0}^{h-1}\sum_{j=1}^{k_i}\ell^i_j+\sum_{i=1}^{h}m_i+\ell^h_1}\right)\otimes\\
& \qquad\cdots\otimes F^h_{\ell^h_{k_h}}\left(x_{\sum_{i=0}^{h-1}\sum_{j=1}^{k_i}\ell^i_j+\sum_{i=1}^{h}m_i+\sum_{j=1}^{k_h-1}\ell^h_j+1}\otimes\cdots\otimes x_{\sum_{i=0}^{h}\sum_{j=1}^{k_i}\ell^i_j+\sum_{i=1}^{h}m_i}\right)\Bigg]
\end{aligned}$$

(We remark that ℓ^j_i can be 0 in the above formula. But $m_i > 0$.)

Then

$$\begin{aligned}
&\left(\partial\left(G_*(T\otimes\cdots\otimes T)\right)\right)_0(a)\\
&\quad=\partial\left(G\left(T_0\cdots T_0\right)\right)\\
&\quad=\sum\pm G\left(T_0^1\cdots\partial T_0\cdots T_0^k\right)\\
&\qquad+\sum\pm G\left(T_0\cdots\eta\left(T_0\cdots T_0\right)\cdots T_0\right)\\
&\qquad+\sum\pm\eta\left(\left(G\left(T_0\cdots T_0\right)\right)\cdots\left(G\left(T_0\cdots T_0\right)\right)\right)
\end{aligned}$$

$$\begin{aligned}
&\sum\pm\left(G_*(T\otimes\cdots\otimes\partial T\cdots\otimes T)\right)_0(a)\\
&\quad=\sum\pm G\left(T_0^1\cdots\partial T_0\cdots T_0^k\right)
\end{aligned}$$

$$\begin{aligned}
&\sum\pm\left(G_*(T\cdots\eta(T\cdots T)\cdots T)\right)_0(a)\\
&\quad=\sum\pm G(T_0\cdots\left(\eta(T\cdots T)\right)_0\cdots T_0)\\
&\quad=\sum\pm G(T_0\cdots\eta\left(T_0\cdots T_0\right)\cdots T_0)
\end{aligned}$$

$$\begin{aligned}
&\sum\pm\left(\Phi\left(G_*(T\cdots\cdots T)\cdots G_*(T\cdots\cdots T)\right)\right)_0(a)\\
&\quad=\sum\pm\eta\left(\left(G_*(T\cdots\cdots T)\right)_0(a)\cdots\left(G_*(T\cdots\cdots T)\right)_0(a)\right).\\
&\quad=\sum\pm\eta\left(G\left(T_0\cdots T_0\right)\cdots G\left(T_0\cdots T_0\right)\right)
\end{aligned}$$

Hence

$$\begin{aligned}
&\left(\partial\left(G_*(T\cdots T)\right)\right)_0+\sum\pm\left(G_*(T\cdots\partial T\cdots T)\right)_0\\
&\quad+\sum\pm\left(G_*(T\cdots\eta(T\cdots T)\cdots T)\right)_0\\
&\quad+\sum\pm\eta\left(\left(G_*(T\cdots T)\right)\cdots\left(G_*(T\cdots T)\right)\right)_0=0
\end{aligned}.$$

For higher k, we calculate (we use Φ for (higher) compositions of pre natural transformations.)

$$
\begin{aligned}
&\left(\partial\left(G_*(T\cdots T)\right)\right)(\cdots)\\
&\quad =\partial\left(\left(G_*(T\cdots T)\right)(\cdots)\right)\\
&\qquad +\sum \pm\left(G_*(T\cdots T)\right)(\cdots\partial x\cdots)\\
&\qquad +\sum \pm\left(G_*(T\cdots T)\right)(\cdots\eta(\cdots)\cdots)\\
&\qquad +\sum \pm\eta\Big((G\circ F)\cdots(G\circ F)\left(\left(G_*(T\cdots T)\right)(\cdots)\right)(G\circ F)\cdots(G\circ F)\Big)\\
&\quad =\partial\left(\sum\pm(G(F\cdots FTF\cdots FTF\cdots F))\right)\\
&\qquad +\sum \pm\left(G(F\cdots FTF\cdots F\otimes F(\cdots\partial x\cdots)\otimes F\cdots F)\right)\\
&\qquad +\sum \pm\left(G(F\cdots FTF\cdots F\otimes T(\cdots\partial x\cdots)\otimes F\cdots F)\right)\\
&\qquad +\sum \pm\left(G(F\cdots FTF\cdots F\otimes F(\cdots\eta(\cdots)\cdots)\otimes F\cdots F)\right)\\
&\qquad +\sum \pm\left(G(F\cdots FTF\cdots F\otimes T(\cdots\eta(\cdots)\cdots)\otimes F\cdots F)\right)\\
&\qquad +\sum \pm\eta\Big(G(F\cdots F)\cdots G(F\cdots F)G(F\cdots FTF\cdots FTF\cdots F)G(F\cdots F)\cdots G(F\cdots F)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\sum\pm\left(G_*(T\cdots\partial T\cdots T)\right)(\cdots)\\
&\quad = \sum\pm G(F\cdots FTF\cdots F\otimes(\partial T)(\cdots)\otimes F\cdots FTF\cdots F)\\
&\quad = \sum\pm G(F\cdots FTF\cdots F\otimes\partial\left(T(\cdots)\right)\otimes F\cdots FTF\cdots F)\\
&\qquad +\sum\pm G(F\cdots FTF\cdots F\otimes T(\cdots\partial x\cdots))\otimes F\cdots FTF\cdots F)\\
&\qquad +\sum\pm G(F\cdots FTF\cdots F\otimes T(\cdots\eta(\cdots)\cdots))\otimes F\cdots FTF\cdots F)\\
&\qquad +\sum\pm G(F\cdots FTF\cdots F\otimes\eta(F\cdots FTF\cdots F))\otimes F\cdots FTF\cdots F)
\end{aligned}
$$

$$
\begin{aligned}
&\sum\pm\left(G_*(T\cdots\Phi(T\cdots T)\cdots\otimes T)\right)(\cdots)\\
&\quad = \sum\pm G(F\cdots FTF\cdots F\otimes(\Phi(T\cdots T))(\cdots)\otimes F\cdots FTF\cdots F)\\
&\quad = \sum\pm G(F\cdots FTF\cdots F\otimes\eta(F\cdots FTF\cdots FTF\cdots F)\otimes F\cdots FTF\cdots F)
\end{aligned}
$$

$$
\begin{aligned}
&\sum\pm\left(\Phi\left(G_*(T\cdots T)\cdots G_*(T\cdots T)\right)\right)(\cdots)\\
&\quad = \sum\pm\eta\left((G\circ F)\cdots(G\circ F)G_*(T\cdots T)(G\circ F)\cdots(G\circ F)G_*(T\cdots T)(G\circ F)\cdots(G\circ F)\right)\\
&\quad = \sum\pm\eta\Big(G(F\cdots F)\cdots G(F\cdots F)G(F\cdots FTF\cdots FTF\cdots F)G(F\cdots F)\\
&\qquad\qquad \cdots G(F\cdots F)G(F\cdots FTF\cdots FTF\cdots F)G(F\cdots F)\cdots G(F\cdots F)\Big)
\end{aligned}
$$

.

Therefore, using the fact that G, F^i are A^∞ functors , we obtain :

$$
\begin{aligned}
&\partial\big(G_*(T\cdots T)\big)+\sum \pm G_*(T\cdots\partial T\cdots T)\\
&\qquad+\sum \pm G_*(T\cdots T\eta(T\cdots T)T\cdots T)\\
&\qquad+\sum \pm\Phi\big(\big(G_*(T\cdots T)\big)\cdots\big(G_*(T\cdots T)\big)\big)\\
&=\partial\Big(\sum \pm G(F\cdots FTF\cdots FTF\cdots F)\Big)\\
&\qquad+\sum \pm G(F\cdots FTF\cdots F\otimes F(\cdots\partial x\cdots)\otimes F\cdots F)\\
&\qquad+\sum \pm G(F\cdots FTF\cdots F\otimes F(\cdots\eta(\cdots)\cdots)\otimes F\cdots F)\\
&\qquad+\sum \pm\eta\Big(G(F\cdots F)\cdots G(F\cdots F)G(F\cdots FTF\cdots FTF\cdots F)G(F\cdots F)\cdots G(F\cdots F)\Big)\\
&\qquad+\sum \pm G(F\cdots FTF\cdots F\otimes\partial\big(T(\cdots)\big)\otimes F\cdots FTF\cdots F))\\
&\qquad+\sum \pm G(F\cdots FTF\cdots F\eta(F\cdots FTF\cdots F))F\cdots FTF\cdots F)\\
&\qquad+\sum \pm G(F\cdots FTF\cdots F\eta(F\cdots FTF\cdots FTF\cdots F)F\cdots FTF\cdots F)\\
&\qquad+\sum \pm\eta\Big(G(F\cdots F)\cdots G(F\cdots F)G(F\cdots FTF\cdots FTF\cdots F)G(F\cdots F)\\
&\qquad\qquad\cdots G(F\cdots F)G(F\cdots FTF\cdots FTF\cdots F)G(F\cdots F)\cdots G(F\cdots F)\Big)\\
&=\partial\Big(\sum \pm G(F\cdots FTF\cdots FTF\cdots F)\Big)\\
&\qquad+\sum \pm G(F\cdots FTF\cdots F\otimes\partial\big(T(\cdots)\big)\otimes F\cdots FTF\cdots F))\\
&\qquad+\sum \pm G(F\cdots FTF\cdots F\otimes\partial\big(F(\cdots)\big)\otimes F\cdots FTF\cdots F))\\
&\qquad+\sum \pm G(F\cdots FTF\cdots F\eta(F\cdots F)F\cdots FTF\cdots F))\\
&\qquad+\sum \pm\eta\Big(G(F\cdots F)\cdots G(F\cdots F)G(F\cdots FTF\cdots FTF\cdots F)G(F\cdots F)\cdots G(F\cdots F)\Big)\\
&\qquad+\sum \pm G(F\cdots FTF\cdots F\eta(F\cdots FTF\cdots F))F\cdots FTF\cdots F)\\
&\qquad+\sum \pm G(F\cdots FTF\cdots F\eta(F\cdots FTF\cdots FTF\cdots F)F\cdots FTF\cdots F)\\
&\qquad+\sum \pm\eta\Big(G(F\cdots F)\cdots G(F\cdots F)G(F\cdots FTF\cdots FTF\cdots F)G(F\cdots F)\\
&\qquad\qquad\cdots G(F\cdots F)G(F\cdots FTF\cdots FTF\cdots F)G(F\cdots F)\cdots G(F\cdots F)\Big)\\
&=0
\end{aligned}
$$

The proof for $G^*:\mathcal{F}unc(C^2,C)\to\mathcal{F}unc(C^1,C)$ is similar. The proof of Lemma 11.13 is now complete.

It follows immediately from Lemma 11.13 that :

Corollary 11.14

Let $F^i:C^1\to C^2$, $i=1,2$ be A^∞ functors. We assume that C^2 has an identity. Let a natural transformation $T:F^1\to F^2$ be a homotopy equivalence.

(11.14.1) *Let* $G : \mathcal{C}^2 \to \mathcal{C}^3$ *be an* A^∞*functor. We assume that* $\mathcal{C}^3$ *has an identity and* G *sends identity to identity. Then* $G_{*1}(T)$ *is a homotopy equivalence from* $G \circ F^1$ *to* $G \circ F^2$.
(11.14.2) $G : \mathcal{C}^0 \to \mathcal{C}^1$ *be an* A^∞*functor. Then* $G_1^*(T)$ *is a homotopy equivalence from* $F^1 \circ G$ *to* $F^2 \circ G$.

We now define homotopy equivalence between A^∞categories with identity. We first introduce some trivial notations.

Definition 11.15
Let $\mathcal{C}$ be an A^∞category. We define an A^∞functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ by $(1_{\mathcal{C}})_0(a) = a$, $(1_{\mathcal{C}})_1(x) = x$, $(1_{\mathcal{C}})_k(x_1 \otimes \cdots \otimes x_k) = 0$, $k > 1$. We call it the *identity functor.*

Definition 11.16
Let $F : \mathcal{C}^1 \to \mathcal{C}^2$, $i = 1,2$ be an A^∞functor. We assume that $\mathcal{C}^1, \mathcal{C}^2$ have identities and F preserves identity. We say that F is a *homotopy equivalence* if there exists an A^∞functor $F' : \mathcal{C}^2 \to \mathcal{C}^1$ such that $F' \circ F$, $F \circ F'$ are both homotopic to identity functors.

We say that two A^∞category with identities are *homotopy equivalent* if there exists a homotopy equivalence between them.

The following lemmata can be easily proved from what we already proved.

Lemma 11.17 *The composition of two homotopy equivalences is a homotopy equivalence.*

Lemma 11.18 *If* $F : \mathcal{C}^1 \to \mathcal{C}^2$ *is a homotopy equivalence. then* $F^* : \mathcal{F}unc(\mathcal{C}^2, \mathcal{C}) \to \mathcal{F}unc(\mathcal{C}^1, \mathcal{C})$, $F_* : \mathcal{F}unc(\mathcal{C}, \mathcal{C}^1) \to \mathcal{F}unc(\mathcal{C}, \mathcal{C}^2)$ *are homotopy equivalences.*

Lemma 11.19 *If* $F : \mathcal{C}^1 \to \mathcal{C}^2$ *is a homotopy equivalence then for each* $a, b \in \mathcal{O}b(\mathcal{C}^1)$ $F_1 : \mathcal{C}^1(a,b) \to \mathcal{C}^2(F_0(a), F_0(b))$ *is a chain homotopy equivalence.*

§ 12 Yoneda's lemma

One important result in category theory is Yoneda's lemma, which enables us to embed a category $\mathcal{C}$ to a category of functors from $\mathcal{C}$ (to the category of abelian group for example in the case of additive category). The purpose of this section is prove an A^∞version of it.

Definition 12.1 Let $\mathcal{C}$ be an A^∞ category and $\Xi \subseteq \mathcal{Ob}(\mathcal{C})$. The full subcategory $\mathcal{C}(\Xi)$ is the A^∞ category such that $\mathcal{Ob}(\mathcal{C}(\Xi)) = \Xi$ and morphisms, boundary operator, (higher) compositions are the same as $\mathcal{C}$.

We define a full subcategory $\mathcal{Rep}(\mathcal{C},\mathcal{Ch})$ of $\mathcal{Func}(\mathcal{C},\mathcal{Ch})$ such that $\mathcal{Ob}(\mathcal{Rep}(\mathcal{C},\mathcal{Ch}))$ is the set of all representable functors.

Theorem 12.2 *Let $\mathcal{C}$ be an A^∞category with identity. Then the A^∞functor $\mathcal{F} : \mathcal{C}^o \to \mathcal{Rep}(\mathcal{C},\mathcal{Ch})$ constructed in Theorem* 10.17 *is a homotopy equivalence.*

We remark that 3rd and higher compositions are zero in $\mathcal{Rep}(\mathcal{C},\mathcal{Ch})$. Hence Theorem 12.2 implies the following corollary, which simplifies the study of A^∞category.

Corollary 12.3 *Any A^∞ category with identity is homotopy equivalent to an A^∞ category whose 3rd and higher compositions vanish.*

We remark that there is an analogue of Corollary 12.3 for A^∞ spaces based on Bar construction. See [1].

Proof of Theorem 12.2*:* We are going to construct an A^∞functor $\mathcal{G} : \mathcal{Rep}(\mathcal{C},\mathcal{Ch}) \to \mathcal{C}^o$ such that $\mathcal{G} \circ \mathcal{F}$ and $\mathcal{F} \circ \mathcal{G}$ are homotopic to identity.

The map $\mathcal{G}_0 : \mathcal{Ob}(\mathcal{Rep}(\mathcal{C},\mathcal{Ch})) \to \mathcal{Ob}(\mathcal{C})$ is obvious.

We construct $\mathcal{G}_1 : \mathcal{Rep}(\mathcal{C},\mathcal{Ch})(F^{b_0}, F^{b_1}) \to \mathcal{C}(b_1, b_0)$. Let $T \in \mathcal{Rep}(\mathcal{C},\mathcal{Ch})(F^{b_0}, F^{b_1})$. (Namely $T : F^{b_0} \to F^{b_1}$ is a pre natural transformation.) We put

(12.4.1) $$\mathcal{G}_1(T) = T_0(1_{b_0}) \in \mathcal{C}(b_1, b_0).$$

(Here we recall $T_0(b_0) : F^{b_0}(b_0) \to F^{b_1}(b_0)$ and $F^{b_0}(b_0) = \mathcal{C}(b_0, b_0)$, $F^{b_1}(b_0) = \mathcal{C}(b_1, b_0)$.)

$\mathcal{G}_1 : \mathcal{Func}(F^{b_1}, F^{b_0}) \to \mathcal{C}(b_1, b_0)$ is a chain map, since

$$\mathcal{G}_1(\partial T) = (\partial T)_0(1_{b_0}) = \partial\left(T_0(1_{b_0})\right) = \partial\left(\mathcal{G}_1(T)\right)$$

Next, let $T^i \in \mathcal{Rep}(C,\mathcal{Ch})(F^{b_{i-1}},F^{b_i})$. We define $\mathcal{G}_k(T^1\otimes\cdots\otimes T^k)\in C(b_k,b_0)$, by induction as follows:

(12.4.2) $$\mathcal{G}_2(T^1,T^2) = \left(T_1^2\left(\mathcal{G}_1(T^1)\right)\right)(1_{b_1}) \in C(b_2,b_0).$$

$$\begin{aligned}&\mathcal{G}_{k+1}(T^1\otimes\cdots\otimes T^{k+1})\\ (12.4.k+1)\quad &= \sum_{k_1+\cdots k_\ell=k}\pm\left(T_\ell^{k+1}\left(\mathcal{G}_{k_1}(T^{k-k_1+1}\otimes\cdots\otimes T^k)\otimes\cdots\otimes\mathcal{G}_{k_\ell}(T^1\otimes\cdots\otimes T^{k_\ell})\right)\right)(1_{b_k}).\\ &\in C(b_{k+1},b_1)\end{aligned}$$

Lemma 12.5

$\mathcal{G}:\mathcal{Rep}(C,\mathcal{Ch})\to C^o$ is an A^∞ functor.

Proof:

We are going to verify that $\mathcal{G}_k$ satisfy (2.2.k) by induction on k. We already verified (2.2.1). To simplify the notation we write (12.4.k+1) as

$$\mathcal{G}(T^1\cdots T^{k+1}) = \sum\pm\left(T^{k+1}\left(\mathcal{G}(T\cdots T)\cdots\mathcal{G}(T\cdots T)\right)\right)(1_{b_k})$$

Now we calculate :

$$\begin{aligned}&\partial\left(\mathcal{G}(T^1\cdots T^{k+1})\right)+\sum\pm\mathcal{G}(\cdots\partial T\cdots T^{k+1})\pm\mathcal{G}(\cdots\partial T^{k+1})\\ &\qquad+\sum\pm\mathcal{G}(\cdots\Phi(T,T)\cdots T^{k+1})+\sum\pm\mathcal{G}(\cdots\Phi(T^k,T^{k+1}))\\ &\qquad+\sum\pm\eta\left(\mathcal{G}(T\cdots T)\cdots\mathcal{G}(T\cdots T)\right)\\ =\quad&\partial\left(\sum\pm\left(T^{k+1}\left(\mathcal{G}(T\cdots T)\cdots\mathcal{G}(T\cdots T)\right)\right)(1_{b_k})\right)\\ &\qquad+\sum\pm\left(T^{k+1}\left(\mathcal{G}(T\cdots T)\cdots\mathcal{G}(T\cdots\partial T\cdots T)\cdots\mathcal{G}(T\cdots T)\right)\right)(1_{b_k})\\ &\qquad+\sum\pm\left(\left(\partial T^{k+1}\right)\left(\mathcal{G}(T\cdots T)\cdots\mathcal{G}(T\cdots T)\right)\right)(1_{b_k})\\ &\qquad+\sum\pm\left(T^{k+1}\left(\mathcal{G}(T\cdots T)\cdots\mathcal{G}(T\cdots\Phi(T,T)\cdots T)\cdots\mathcal{G}(T\cdots T)\right)\right)(1_{b_k})\\ &\qquad+\sum\pm\left(\left(\Phi(T^k,T^{k+1})\right)\left(\mathcal{G}(T\cdots T)\cdots\mathcal{G}(T\cdots T)\right)\right)(1_{b_{k-1}})\\ &\qquad+\sum\pm\eta\left(\mathcal{G}(T\cdots T)\cdots\mathcal{G}(T\cdots T)\right)\end{aligned}$$

$$
\begin{aligned}
= \quad & \partial\Big(\sum \pm\Big(T^{k+1}(\mathcal{G}(T\cdots T)\cdots \mathcal{G}(T\cdots T))\Big)(1_{b_k})\Big) \\
& +\sum \pm\Big(T^{k+1}\big(\mathcal{G}(T\cdots T)\cdots \mathcal{G}(T\cdots \partial T\cdots T)\cdots \mathcal{G}(T\cdots T)\big)\Big)(1_{b_k}) \\
& +\partial\Big(\sum \pm\Big(T^{k+1}(\mathcal{G}(T\cdots T)\cdots \mathcal{G}(T\cdots T))\Big)(1_{b_k})\Big) \\
& +\sum \pm\Big(T^{k+1}\big(\mathcal{G}(T\cdots T)\cdots \otimes \partial\big(\mathcal{G}(T\cdots T)\big)\otimes\cdots \mathcal{G}(T\cdots T)\big)\Big)(1_{b_k}) \\
& +\sum \pm\Big(T^{k+1}\big(\mathcal{G}(T\cdots T)\cdots \otimes \eta\big(\mathcal{G}(T\cdots T)\cdots \mathcal{G}(T\cdots T)\big)\otimes\cdots \mathcal{G}(T\cdots T)\big)\Big)(1_{b_k}) \\
& +\sum \pm\Big(T^{k+1}\big(\mathcal{G}(T\cdots T)\cdots \mathcal{G}(T\cdots T)\big)\Big)\Big(F^{b_k}\big(\mathcal{G}(T\cdots T)\cdots \mathcal{G}(T\cdots T)\big)(1_{b_{k-1}})\Big) \\
& +\sum \pm\Big(F^{b_{k+1}}\big(\mathcal{G}(T\cdots T)\cdots \mathcal{G}(T\cdots T)\big)\Big)\Big(T^{k+1}\big(\mathcal{G}(T\cdots T)\cdots \mathcal{G}(T\cdots T)\big)(1_{b_{k-1}})\Big) \\
& +\sum \pm\Big(T^{k+1}\big(\mathcal{G}(T\cdots T)\cdots \otimes \mathcal{G}(T\cdots \Phi(T,T)\cdots T)\otimes\cdots \mathcal{G}(T\cdots T)\big)\Big)(1_{b_k}) \\
& +\sum \pm\Big(T^{k+1}\big(\mathcal{G}(T\cdots T)\cdots \mathcal{G}(T\cdots T)\big)\Big)\big(\mathcal{G}(T\cdots T)\big) \\
& +\sum \pm\eta\big(\mathcal{G}(T\cdots T)\cdots \mathcal{G}(T\cdots T)\big) \qquad . \\
= \quad & \sum \pm\Big(T^{k+1}\big(\mathcal{G}(T\cdots T)\cdots \otimes \mathcal{G}(T\cdots \partial T\cdots T)\otimes\cdots \mathcal{G}(T\cdots T)\big)\Big)(1_{b_k}) \\
& +\sum \pm\Big(T^{k+1}\big(\mathcal{G}(T\cdots T)\cdots \otimes \partial\big(\mathcal{G}(T\cdots T)\big)\otimes\cdots \mathcal{G}(T\cdots T)\big)\Big)(1_{b_k}) \\
& +\sum \pm\Big(T^{k+1}\big(\mathcal{G}(T\cdots T)\cdots \otimes \eta\big(\mathcal{G}(T\cdots T)\cdots \mathcal{G}(T\cdots T)\big)\otimes\cdots \mathcal{G}(T\cdots T)\big)\Big)(1_{b_k}) \\
& +\sum \pm\Big(T^{k+1}\big(\mathcal{G}(T\cdots T)\cdots \mathcal{G}(T\cdots T)\big)\Big)\big(\mathcal{G}(T\cdots T)\big) \\
& +\sum \pm\eta\big(\mathcal{G}(T\cdots T)\cdots \mathcal{G}(T\cdots T)\big) \\
& +\sum \pm\Big(T^{k+1}\big(\mathcal{G}(T\cdots T)\cdots \mathcal{G}(T\cdots \Phi(T,T)\cdots T)\cdots \mathcal{G}(T\cdots T)\big)\Big)(1_{b_k}) \\
& +\sum \pm\Big(T^{k+1}\big(\mathcal{G}(T\cdots T)\cdots \mathcal{G}(T\cdots T)\big)\Big)\big(\mathcal{G}(T\cdots T)\big) \\
& +\sum \pm\eta\big(\mathcal{G}(T\cdots T)\cdots \mathcal{G}(T\cdots T)\big) \\
= \quad & 0
\end{aligned}
$$

We thus proved that $\mathcal{G}$ is an A^∞ functor.

We next prove that $\mathcal{G}\circ\mathcal{F}$ and $\mathcal{F}\circ\mathcal{G}$ are homotopic to identity. It is easy to see that $\mathcal{G}\circ\mathcal{F}$ is the identity functor. (We remark that $\mathcal{G}_k\big(\mathcal{F}(\cdots)\cdots\mathcal{F}(\cdots)\big)=0$ for $k\ge 2$.)

We are going to find a natural transformation $\mathcal{T}$ from the identity functor to $\mathcal{F}\circ\mathcal{G}$ and prove that $\mathcal{T}$ is a homotopy equivalence.

Observing that $(\mathcal{F}\circ\mathcal{G})_0 : \mathcal{O}b\to\mathcal{O}b$ is the identity, we put $\mathcal{T}_0 = \text{identity map}$.

Let $T^i \in \mathcal{F}unc(F^{b_{i-1}},F^{b_i})$. $x_i\in C(a_{i-1},a_i)$, $z\in F^{b_0}(a_0)=C(b_0,a_0)$. We define :

$$
\Big(\big(\mathcal{T}_1(T^1)\big)_0(a_0)\Big)(z)=\big(T_1^1(z)\big)\big(1_{b_0}\big)\in F^{b_1}(a_0),
$$

$$\left(\left(\mathcal{T}_1(T^1)\right)_h(x_1\cdots x_h)\right)(z)=\left(T^1_{h+1}(zx_1\cdots x_h)\right)\left(1_{b_0}\right)\in F^{b_1}(a_h)\,.$$

Let us verify $\left(\partial\mathcal{T}\right)_1=0$. We first calculate

$$\begin{aligned}
&\left(\left(\partial\mathcal{T}\right)_1(T^1)\right)_0(z)\\
&\quad=\left(\Phi\left(\left(1_{\mathcal{F}unc(C,C\hbar)}\right)_1\left(T^1\right)\otimes\mathcal{T}_0(F^{b_1})\right)\right)_0(z)\pm\left(\Phi\left(\mathcal{T}_0(F^{b_0})\otimes\left(\mathcal{F}\circ\mathcal{G}\right)_1(T^1)\right)\right)_0(z)\\
&\qquad\pm\left(\partial\left(\mathcal{T}_1(T^1)\right)\right)_0(z)\pm\left(\mathcal{T}_1(\partial T^1)\right)_0(z)\qquad\qquad(12.6)\\
&\quad=T^1_0(z)\pm\left(\left(\mathcal{F}\circ\mathcal{G}\right)_1(T^1)\right)_0(z)\pm\partial\left(\left(\mathcal{T}_1(T^1)\right)_0(z)\right)\\
&\qquad\pm\left(T^1_1(\partial z)\right)(1_{b_0})\pm\left(\left(\partial T^1\right)_1(z)\right)(1_{b_0})
\end{aligned}$$

On the other hand, we have

$$\left(\left(\mathcal{F}\circ\mathcal{G}\right)_1(T^1)\right)_0(z)=\eta\left(T^1_0(1_{b_0})\otimes z\right).$$

Hence, (12.6) is equal to :

$$\begin{aligned}
&T^1_0(z)\pm\eta\left(T^1_0(1_{b_0})\otimes z\right)\pm\partial\left(\left(T^1_1(z)\right)(1_{b_0})\right)\\
&\qquad\pm\left(T^1_1(\partial z)\right)(1_{b_0})\pm\partial\left(\left(T^1_1(z)\right)(1_{b_0})\right)\pm\left(T^1_1(\partial z)\right)(1_{b_0})\,.\\
&\qquad\pm T^1_0\left(\left(F^{b_0}_1(z)\right)(1_{b_0})\right)\pm F^{b_1}_1(z)\left(T^1_0(1_{b_0})\right)\\
&=0
\end{aligned}$$

We next calculate

$$
\begin{aligned}
&\Big(\big((\partial\mathcal{T})_1(T^1)\big)_h(x\cdots x)\Big)(z)\\
&= \pm\Big(\big(\partial\big(\mathcal{T}_1(T^1)\big)\big)_h(x\cdots x)\Big)(z) \pm \Big(\big(\mathcal{T}_1(\partial T^1)\big)_h(x\cdots x)\Big)(z)\\
&\quad \pm\Big(\big(\Phi\big(T^1\otimes\mathcal{T}_0(F^{b_1})\big)\big)_h(x\cdots x)\Big)(z)\\
&\quad \pm\Big(\big(\Phi\big(\mathcal{T}_0(F^{b_0})\otimes(\mathcal{F}\circ\mathcal{G})_1(T^1)\big)\big)_h(x\cdots x)\Big)(z)\\
&= \partial\Big(\big((\mathcal{T}_1(T^1))_h(x\cdots x)\big)(z)\Big) \pm \Big(\big(\mathcal{T}_1(T^1)\big)_h(x\cdots x)\Big)(\partial z) \qquad (12.7)\\
&\quad +\sum\pm\Big(\big(\mathcal{T}_1(T^1)\big)_h(x\cdots\partial x\cdots x)\Big)(z)\\
&\quad +\sum\pm\Big(\big(\mathcal{T}_1(T^1)\big)_h(x\cdots\eta(x\cdots x)\cdots x)\Big)(z)\\
&\quad +\sum\pm\Big(\big(\mathcal{T}_1(T^1)\big)(x\cdots x)\Big)\Big(\big(F^{b_0}(x\cdots x)\big)(z)\Big)\\
&\quad +\sum\pm\Big(F^{b_1}(x\cdots x)\Big)\Big(\big(\big(\mathcal{T}_1(T^1)\big)(x\cdots x)\big)(z)\Big)\\
&\quad \pm\Big(\big(\partial T^1\big)_{h+1}(zx\cdots x)\Big)(1_{b_0}) \pm \Big(T^1_h(x\cdots x)\Big)(z) + \Big(\big((\mathcal{F}\circ\mathcal{G})_1(T^1)\big)_h(x\cdots x)\Big)(z)
\end{aligned}
$$

We have

$$
\Big(\big((\mathcal{F}\circ\mathcal{G})_1(T^1)\big)_h(x\cdots x)\Big)(z) = \eta\Big(T^1_0(1_{b_0})\otimes z\otimes x_1\otimes\cdots\otimes x_h\Big).
$$

Hence (12.7) is equal to

$$
\begin{aligned}
&= \partial\Big(\big(T^1_{h+1}(zx\cdots x)\big)(1_{b_0})\Big) \pm \Big(T^1_{h+1}(\partial z\otimes x\cdots x)\Big)(1_{b_0})\\
&\quad +\sum\pm\Big(T^1_{h+1}(zx\cdots\partial x\cdots x)\Big)(1_{b_0}) + \sum\pm\Big(T^1_{h+1}(zx\cdots\eta(x\cdots x)\cdots x)\Big)(1_{b_0})\\
&\quad +\sum\pm\Big(T^1\big(\eta(zx\cdots x)x\cdots x\big)\Big)(1_{b_0}) + \eta\Big(\big(T^1(zx\cdots x)\big)(1_{b_0})\otimes x\cdots x\Big)\\
&\quad \pm\partial\Big(\big(T^1_{h+1}(zx\cdots x)\big)(1_{b_0})\Big) \pm \Big(T^1_{h+1}(\partial z\otimes x\cdots x)\Big)(1_{b_0})\\
&\quad +\sum\pm\Big(T^1_{h+1}(zx\cdots\partial x\cdots x)\Big)(1_{b_0}) + \sum\pm\Big(T^1_{h+1}(zx\cdots\eta(x\cdots x)\cdots x)\Big)(1_{b_0})\\
&\quad +\sum\pm T^1\big(\eta(z\cdots)\cdots\big)\\
&\quad \pm\Big(T^1_h(x\cdots x)\Big)(z) \pm \sum\pm\eta\Big(\big(T^1(zx\cdots x)\big)(1_{b_0})\otimes x\cdots x\Big)\\
&\quad \pm\eta\Big(T^1_0(1_{b_0})\otimes zx\cdots x\Big)\\
&\quad \pm\Big(T^1_h(x\cdots x)\Big)(z) \pm \eta\Big(T^1_0(1_{b_0})\otimes zx\cdots x\Big)\\
&= 0
\end{aligned}
$$

We thus verified $(\partial\mathcal{T})_1 = 0$.

We next put

$$\begin{aligned}&\left(\mathcal{T}_2(T^1\otimes T^2)\right)_0(z)\\&\quad=\left(T_2^2\left(\mathcal{G}_1(T^1)\otimes z\right)\right)(1_{b_1})+\left(\mathcal{T}_1(T^2)\right)_0\left(\left(\mathcal{T}_1(T^1)\right)_0(z)\right)\\&\quad=\left(T_2^2\left(T_0^1(1_{b_0})\otimes z\right)\right)(1_{b_1})+\left(T_1^2\left(\left(T_1^1(z)\right)(1_{b_0})\right)\right)(1_{b_1})\end{aligned}$$

$$\begin{aligned}&\left(\left(\mathcal{T}_2(T^1\otimes T^2)\right)_h(x\cdots x)\right)(z)\\&\quad=\left(T_{h+2}^2\left(\mathcal{G}_1(T^1)\otimes z\otimes x_1\otimes\cdots\otimes x_h\right)\right)(1_{b_1})\\&\qquad+\sum_{i=0}^{h}\pm\left(T_{h-i+1}^2\left(\left(\mathcal{T}_1(T^1)_i(x_1\cdots x_i)\right)(z)\otimes x_{i+1}\cdots x_h\right)\right)(1_{b_1})\\&\quad=\left(T_{h+2}^2\left(T_0^1(1_{b_0})\otimes z\otimes x_1\otimes\cdots\otimes x_h\right)\right)(1_{b_1})\\&\qquad+\sum_{i=0}^{h}\pm\left(T_{h-i+1}^2\left(\left(T_{i+1}^1(zx_1\cdots x_i)\right)(1_{b_0})\otimes x_{i+1}\cdots x_h\right)\right)(1_{b_1})\end{aligned}$$

For general k,h, we define $\mathcal{T}_{k+1}$ by induction :

$$\begin{aligned}&\left(\mathcal{T}_{k+1}(T^1\otimes\cdots\otimes T^{k+1})\right)_0(z)\\&\quad=\sum_{\substack{m\ge 0\\ \ell\ge 0}}\pm T_{\ell+1}^{k+1}\Big(\mathcal{G}_{k_1}\left(T^{k-k_1+1}\otimes\cdots\otimes T^k\right)\otimes\cdots\\&\qquad\otimes\mathcal{G}_{k_\ell}\left(T^{m+1}\otimes\cdots\otimes T^{m+k_\ell}\right)\otimes\mathcal{T}_m\left(T^1\otimes\cdots\otimes T^m\right)_0(z)\Big)(1)\end{aligned}.$$

$$\begin{aligned}&\left(\mathcal{T}_{k+1}(T^1\otimes\cdots\otimes T^{k+1})\right)_h\left(x_1\otimes\cdots\otimes x_h\right)(z)\\&\quad=\sum_{\substack{m\ge 0\\ \ell\ge 0\\ i\ge 0}}\pm T_{\ell+n+1}^{k+1}\Big(\mathcal{G}_{k_1}\left(T^{k-k_1+1}\otimes\cdots\otimes T^k\right)\otimes\cdots\otimes\mathcal{G}_{k_\ell}\left(T^{m+1}\otimes\cdots\otimes T^{m+k_\ell}\right)\\&\qquad\otimes\mathcal{T}_m\left(T^1\otimes\cdots\otimes T^m\right)_i\left(x_1\otimes\cdots\otimes x_i\right)(z)\otimes x_{i+1}\otimes\cdots\otimes x_h\Big)(1)\end{aligned}$$

Lemma 12.8

$\mathcal{T}:1\to\mathcal{F}\circ\mathcal{G}$ *is a natural transformation.* (*Namely* $\partial\mathcal{T}=0$.)

Proof:

We prove by induction that $(\partial\mathcal{T})_{k+1}=0$. We already proved the case when $k=-1,0$. We use Φ to denote the composition in $\mathcal{F}unc(\mathcal{C},\mathcal{C}h)$ and Ψ to denote the composition in $\mathcal{Func}(\mathcal{Func}(C,\mathcal{Ch}),\mathcal{Func}(C,\mathcal{Ch}))$. We calculate

$$
\begin{aligned}
&\left((\partial\mathcal{T})_{k+1}\left(T^1\cdots T^{k+1}\right)\right)_0(z)\\
&\qquad=\pm\partial\left(\left(\mathcal{T}_{k+1}\left(T^1\cdots T^{k+1}\right)\right)_0(z)\right)\pm\left(\mathcal{T}_{k+1}\left(T^1\cdots T^{k+1}\right)\right)_0(\partial z)\\
&\qquad\qquad+\sum_{i\neq k+1}\pm\left(\mathcal{T}_{k+1}\left(T^1\cdots\partial T^i\cdots T^{k+1}\right)\right)_0(z)+\left(\mathcal{T}_{k+1}\left(T^1\cdots\partial T^{k+1}\right)\right)_0(z)\\
&\qquad\qquad+\sum_{i\neq k}\pm\left(\mathcal{T}_{k+1}\left(T^1\cdots\Phi(T^i\otimes T^{i+1})\cdots T^{k+1}\right)\right)_0(z)\\
&\qquad\qquad\pm\left(\mathcal{T}_{k+1}\left(T^1\cdots\Phi(T^k\otimes T^{k+1})\right)\right)_0(z)\\
&\qquad\qquad+\sum_{i\neq k}\pm\Psi\left(\left(1_{\mathcal{Func}(\mathcal{Func}(C,\mathcal{Ch}),\,\mathcal{Func}(C,\mathcal{Ch}))}\right)_1(T^1)\otimes\left(\mathcal{T}_k\left(T^2\cdots T^{k+1}\right)\right)\right)_0(z)\\
&\qquad\qquad+\sum_{i\neq k}\pm\Psi\left(\left(\mathcal{T}(T\cdots T)\right)\otimes\left(\mathcal{F}\circ\mathcal{G}\right)(T\cdots T)\right)_0(z)
\end{aligned}
\tag{12.9}
$$

The sum of 1st, 2nd, 3rd and 5th terms of (12.9) is

$$
\begin{aligned}
&\sum\pm\partial\left(T^{k+1}\left(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\right)(1)\right)\\
&\qquad+\sum\pm T^{k+1}\left(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(\partial z)\right)(1)\\
&\qquad+\sum\pm T^{k+1}\left(\mathcal{G}\cdots\mathcal{G}(\cdots\partial T\cdots)\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\right)(1)\\
&\qquad+\sum\pm T^{k+1}\left(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(\cdots\partial T\cdots)_0(z)\right)(1)\\
&\qquad+\sum\pm T^{k+1}\left(\mathcal{G}\cdots\mathcal{G}(\cdots\Phi(TT)\cdots)\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\right)(1)\\
&\qquad+\sum\pm T^{k+1}\left(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(\cdots\Phi(TT)\cdots)_0(z)\right)(1)
\end{aligned}
\tag{12.10}
$$

The 4th term of (12.9) is

$$
\begin{aligned}
&\sum \pm\left(\partial T^{k+1}\right)\left(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\right)(1)\\
&\quad = \sum \pm\partial\left(T^{k+1}\left(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\right)(1)\right)\\
&\qquad +\sum \pm T^{k+1}\left(\mathcal{G}\cdots\partial\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\right)(1)\\
&\qquad +\sum \pm T^{k+1}\left(\mathcal{G}\cdots\mathcal{G}\otimes\partial\left(\mathcal{T}(T\cdots T)_0(z)\right)\right)(1)\\
&\qquad +\sum \pm T^{k+1}\left(\mathcal{G}\cdots\otimes\eta(\mathcal{G}\cdots\mathcal{G})\otimes\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\right)(1).\\
&\qquad +\sum \pm T^{k+1}\left(\mathcal{G}\cdots\otimes\eta\left(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\right)\right)(1)\\
&\qquad +\sum \pm\Phi\left(F(\mathcal{G})\otimes T^{k+1}\left(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\right)\right)(1)\\
&\qquad \pm\Phi\left(F\left(\mathcal{T}_k\left(T^1\cdots T^k\right)_0\right)\otimes T_0^{k+1}\right)(z)\\
&\qquad +\sum \pm\Phi\left(T^{k+1}(\mathcal{G}\cdots\mathcal{G})\otimes F\left(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\right)\right)(1)
\end{aligned}
\tag{12.11}
$$

(We remark that $F(y_1\cdots y_k)(1)=0$ for $k\ge 2$.) The 6th term of (12.9) is

$$
\begin{aligned}
&\sum \pm\Phi(T^k\otimes T^{k+1})\left(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\right)(1)\\
&\quad = \sum \pm T^{k+1}\left(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\right)\left(T^k(\mathcal{G}\cdots\mathcal{G})(1)\right).\\
&\qquad +\sum \pm T_0^{k+1}\left(T^k\left(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\right)(1)\right)
\end{aligned}
\tag{12.12}
$$

The 7th and 8th term of (12.9) is

$$
\begin{aligned}
&\sum \pm T^{k+1}\left(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0\left(T_0^1(z)\right)\right)(1)\\
&\qquad +\sum \pm\eta\left(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\right).
\end{aligned}
\tag{12.13}
$$

We next use the induction hypothesis to calculate the term $\sum \pm T^{k+1}\left(\mathcal{G}\cdots\mathcal{G}\otimes\partial\left(\mathcal{T}(T\cdots T)_0(z)\right)\right)(1)$ in (12.11). Then (12.11) is equal to

(12.14)
$$
\begin{aligned}
&\sum \pm\partial\Big(T^{k+1}\big(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\big)(1)\Big)\\
&\quad+\sum\pm T^{k+1}\big(\mathcal{G}\cdots\partial\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\big)(1)\\
&\quad+\sum\pm T^{k+1}\big(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(\partial z)\big)(1)\\
&\quad+\sum\pm T^{k+1}\big(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots\partial T\cdots T)_0(z)\big)(1)\\
&\quad+\sum\pm T^{k+1}\big(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots\Phi(TT)\cdots T)_0(z)\big)(1)\\
&\quad+\sum\pm T^{k+1}\Big(\mathcal{G}\cdots\mathcal{G}\otimes\Psi\big(T^1\otimes(\mathcal{T}(T\cdots T))\big)_0(z)\Big)(1)\\
&\quad+\sum\pm T^{k+1}\Big(\mathcal{G}\cdots\mathcal{G}\otimes\Psi\big((\mathcal{T}(T\cdots T))\otimes(\mathcal{F}\circ\mathcal{G})(T\cdots T)\big)_0(z)\Big)(1)\\
&\quad+\sum\pm T^{k+1}\big(\mathcal{G}\cdots\otimes\eta(\mathcal{G}\cdots\mathcal{G})\otimes\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\big)(1)\\
&\quad+\sum\pm T^{k+1}\Big(\mathcal{G}\cdots\otimes\eta\big(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\big)\Big)(1)\\
&\quad+\sum\pm T^{k+1}\big(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\big)(\mathcal{G})\\
&\quad\pm T_0^{k+1}\Big(\mathcal{T}_k\big(T^1\cdots T^k\big)_0(z)\Big)\\
&\quad+\sum\pm\eta\Big(T^{k+1}(\mathcal{G}\cdots\mathcal{G})(1)\otimes\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\Big)
\end{aligned}
.
$$

We remark that, by (12.4), we have

(12.15)
$$
\begin{aligned}
&\sum\pm T^{k+1}(\mathcal{G}(T^a\cdots T)\cdots\mathcal{G}(T\cdots T^k))(1)=\mathcal{G}(T^a\cdots T^{k+1})\\
&\sum\pm T^{k}(\mathcal{G}(T^a\cdots T)\cdots\mathcal{G}(T\cdots T^{k-1}))(1)=\mathcal{G}(T^a\cdots T^{k}).
\end{aligned}
$$

Therefore most of the terms of (1210)+(12.12)+(12.13)+(12.14) cancels and this sum is equal to :

(12.16)
$$
\begin{aligned}
&T_0^{k+1}\Big(T^k\big(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\big)\Big)\\
&\quad+\sum\pm T^{k+1}\Big(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0\big(T_0^1(z)\big)\Big)(1)\\
&\quad+\sum\pm T^{k+1}\Big(\mathcal{G}\cdots\mathcal{G}\otimes\big(T^1\otimes\mathcal{T}(T\cdots T)_0\big)(z)\Big)(1)\\
&\quad+\sum\pm T^{k+1}\Big(\mathcal{G}\cdots\mathcal{G}\otimes\Psi\big((\mathcal{T}(T\cdots T))\otimes(\mathcal{F}\circ\mathcal{G})(T\cdots T)\big)_0(z)\Big)(1)\\
&\quad+\sum\pm T^{k+1}\Big(\mathcal{G}\cdots\otimes\eta\big(\mathcal{G}\cdots\mathcal{T}(T\cdots T)_0(z)\big)\Big)(1)\\
&\quad\pm T_0^{k+1}\Big(T^k\big(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\big)\Big)
\end{aligned}
.
$$

It is easy to see that (12.16) is zero. We thus proved $\Big((\partial\mathcal{T})_{k+1}\big(T^1\cdots T^{k+1}\big)\Big)_0=0$. The proof that $\Big((\partial\mathcal{T})_{k+1}\big(T^1\cdots T^{k+1}\big)\Big)_h=0$ for $h>0$ is similar and hence is left to the reader.

We next define a natural transformation $\mathcal{T}':\mathcal{F}\circ\mathcal{G}\to 1$ as follows :

(12.17.0) $\mathcal{T}_0 = \mathcal{T}'_0$,
(12.17.1) $\mathcal{T}_1 = \pm\mathcal{T}'_1$
(12.17.2) $\mathcal{T}'_2\left(T^1 \otimes T^2\right)_0(z) = \pm T_2^2\left(T_0^1(1_{b_0}) \otimes z\right)(1_{b_1})$

$$\mathcal{T}'_2\left(T^1 \otimes T^2\right)_h(x_1 \otimes \cdots \otimes x_h)(z) = \pm T_{h+2}^2\left(T_0^1(1_{b_0}) \otimes z \otimes x_1 \otimes \cdots \otimes x_h\right)(1_{b_1})$$

(12.17.h) $\mathcal{T}'_{k+1}\left(T^1 \otimes \cdots \otimes T^{k+1}\right)_0(z) = \sum \pm T^{k+1}\left(\mathcal{G}(T\cdots T) \otimes \cdots \otimes \mathcal{G}(T\cdots T) \otimes z\right)(1_{b_k})$

$$\mathcal{T}'_{k+1}\left(T^1 \otimes \cdots \otimes T^{k+1}\right)_h\left(x_1 \otimes \cdots \otimes x_h\right)(z) = \sum \pm T^{k+1}\left(\mathcal{G}(T\cdots T) \otimes \cdots \otimes \mathcal{G}(T\cdots T) \otimes z \otimes x_1 \otimes \cdots \otimes x_h\right)(1_{b_k}).$$

Lemma 12.18 *$\mathcal{T}' : \mathcal{F} \circ \mathcal{G} \to 1$ is a natural transformation. Namely $\partial\mathcal{T}' = 0$.*

Proof:

The proof of $(\partial\mathcal{T}')_0 = 0$ and $(\partial\mathcal{T}')_1 = 0$ is the same as the proof of $(\partial\mathcal{T})_0 = 0$ and $(\partial\mathcal{T})_1 = 0$. We calculate

$$\begin{aligned}
&\left((\partial\mathcal{T}')_{k+1}\left(T^1\cdots T^{k+1}\right)\right)_0(z) \\
&= \pm\partial\left(\left(\mathcal{T}'_{k+1}\left(T^1\cdots T^{k+1}\right)\right)_0(z)\right) \pm \left(\mathcal{T}'_{k+1}\left(T^1\cdots T^{k+1}\right)\right)_0(\partial z) \\
&\quad + \sum_{i\ne k+1} \pm\left(\mathcal{T}'_{k+1}\left(T^1\cdots\partial T^i\cdots T^{k+1}\right)\right)_0(z) + \left(\mathcal{T}'_{k+1}\left(T^1\cdots\partial T^{k+1}\right)\right)_0(z) \\
&\quad + \sum_{i\ne k} \pm\left(\mathcal{T}'_{k+1}\left(T^1\cdots\Phi(T^i \otimes T^{i+1})\cdots T^{k+1}\right)\right)_0(z) \qquad (12.19)\\
&\quad \pm\left(\mathcal{T}'_{k+1}\left(T^1\cdots\Phi(T^k \otimes T^{k+1})\right)\right)_0(z) \\
&\quad + \sum_{i\ne k} \pm\Psi\left(\left(\mathcal{F}\circ\mathcal{G}\right)(T\cdots T) \otimes \left(\mathcal{T}(T\cdots T)\right)\right)_0(z) \\
&\quad + \sum_{i\ne k} \pm\Psi\left(\left(\mathcal{T}\left(T^1\cdots T^k\right)\right) \otimes \left(1_{\mathcal{F}unc(\mathcal{F}unc(C,\mathcal{C}h),\mathcal{F}unc(C,\mathcal{C}h))}\right)_1(T^{k+1})\right)_0(z)
\end{aligned}.$$

The sum of 1st, 2nd, 3rd, and 5th terms of (12.19) is equal to

$$\begin{aligned}
&\sum \pm\partial\left(T^{k+1}(\mathcal{G}\cdots\mathcal{G}z)(1)\right) \\
&\quad + \sum \pm T^{k+1}(\mathcal{G}\cdots\mathcal{G} \otimes \partial z)(1) \\
&\quad + \sum \pm T^{k+1}(\mathcal{G}\cdots\mathcal{G}(\cdots\partial T\cdots)\cdots\mathcal{G}z)(1) \qquad (12.20)\\
&\quad + \sum \pm T^{k+1}(\mathcal{G}\cdots\mathcal{G}(\cdots\Phi(TT)\cdots)\cdots\mathcal{G}z)(1)
\end{aligned}.$$

The 4th term of (12.19) is

(12.21)
$$
\begin{aligned}
&\sum \pm\left(\partial T^{k+1}\right)(\mathcal{G}\cdots\mathcal{G}z)(1)\\
&\quad=\sum \pm\partial\left(T^{k+1}(\mathcal{G}\cdots\mathcal{G}z)(1)\right)\\
&\qquad+\sum \pm T^{k+1}(\mathcal{G}\cdots\partial\mathcal{G}\cdots\mathcal{G}z)(1)\\
&\qquad+\sum \pm T^{k+1}(\mathcal{G}\cdots\mathcal{G}\otimes\partial z)(1)\\
&\qquad+\sum \pm T^{k+1}(\mathcal{G}\cdots\eta(\mathcal{G}\cdots\mathcal{G})\cdots\mathcal{G}z)(1)\\
&\qquad+\sum \pm T^{k+1}(\mathcal{G}\cdots\eta(\mathcal{G}\cdots\mathcal{G}z))(1)\\
&\qquad+\sum \pm\Phi\left(F(\mathcal{G}),T^{k+1}(\mathcal{G}\cdots\mathcal{G}z)\right)(1)\\
&\qquad+\sum \pm\Phi\left(T^{k+1}(\mathcal{G}\cdots\mathcal{G}),F(\mathcal{G}\cdots\mathcal{G}z)\right)(1)
\end{aligned}
.
$$

The 6th terms of (12.19) is

(12.22)
$$
\begin{aligned}
&\sum \pm T^{k+1}(\mathcal{G}\cdots\mathcal{G}z)\left(T^{k}(\mathcal{G}\cdots\mathcal{G})(1)\right)\\
&\quad+\sum \pm T_0^{k+1}\left(T^{k}(\mathcal{G}\cdots\mathcal{G}z)(1)\right)
\end{aligned}
$$

The sum of 7th and 8th terms of (12.19) is

(12.23)
$$
\begin{aligned}
&\sum \pm T^{k+1}(\mathcal{G}\cdots\mathcal{G}\eta(\mathcal{G}\cdots\mathcal{G}z))(1)\\
&\quad+\sum \pm T_0^{k+1}\left(T^{k}(\mathcal{G}\cdots\mathcal{G}z)(1)\right)
\end{aligned}
.
$$

Using (12.15) , we find (12.20)+(12,21)+(12.22)+(12.23)=0. We proved $\left((\partial\mathcal{T}')_{k+1}\left(T^1\cdots T^{k+1}\right)\right)_0=0$. The proof that $\left((\partial\mathcal{T}')_{k+1}\left(T^1\cdots T^{k+1}\right)\right)_h=0$ for $h>0$ is similar and hence is left to the reader.

To complete the proof of Theorem 12.2 we are only to show the following :

Lemma 12.24 $\Psi(\mathcal{T}\otimes\mathcal{T}')=1_{1_{\mathcal{F}unc(\mathcal{F}unc(\mathcal{C},\mathcal{C}h),\mathcal{F}unc(\mathcal{C},\mathcal{C}h))}}$. $\Psi(\mathcal{T}'\otimes\mathcal{T})=1_{\mathcal{F}\circ\mathcal{G}}$.

We remark that to show that $\mathcal{T}$ is a homotopy equivalence we need only to show that $\Psi(\mathcal{T}\otimes\mathcal{T}')$ and $\Psi(\mathcal{T}'\otimes\mathcal{T})$ are homotopic to identity. But in fact we can prove that it is equal to identity.

Proof:

$\Psi(\mathcal{T}\otimes\mathcal{T}')_0=\text{identity}$ is obvious from definition. We have

$$\Psi(\mathcal{T}\otimes\mathcal{T}')_1(T)=\pm\Phi(\mathcal{T}(T)\otimes\mathcal{T}_0')\pm\Phi(\mathcal{T}_0\otimes\mathcal{T}'(T))=0.$$

For higher k, we calculate

$$\begin{aligned}&\Psi(\mathcal{T}\otimes\mathcal{T}')_{k+1}(T^1\cdots T^{k+1})\\ &\quad=\pm\Phi(\mathcal{T}_{k+1}(T^1\cdots T^{k+1})\otimes\mathcal{T}'_0)\pm\Phi(\mathcal{T}_0\otimes\mathcal{T}'_{k+1}(T^1\cdots T^{k+1}))\\ &\qquad+\sum_{i=1}^{k}\Phi(\mathcal{T}_i(T^1\cdots T^i)\otimes\mathcal{T}'_{k+1-i}(T^{i+1}\cdots T^{k+1}))\\ &\quad=\pm\mathcal{T}_{k+1}(T^1\cdots T^k)\pm\mathcal{T}'_{k+1}(T^1\cdots T^{k+1})+\sum_{i=1}^{k}\Phi(\mathcal{T}_i(T^1\cdots T^i)\otimes\mathcal{T}'_{k+1-i}(T^{i+1}\cdots T^{k+1}))\end{aligned}.$$

We then have

$$(12.25)\qquad\begin{aligned}&\left(\Psi(\mathcal{T}\otimes\mathcal{T}')_{k+1}(T^1\cdots T^{k+1})\right)_0(z)\\ &=\sum\pm T^{k+1}\big(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\big)(1)\\ &\qquad+\sum\pm T^{k+1}\big(\mathcal{G}\cdots\mathcal{G}\otimes z\big)(1)\\ &\qquad+\sum_{i=1}^{k}\pm T^{k+1}\Big(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}_i\big(T^1\cdots T^i\big)_0(z)\Big)(1)\end{aligned}.$$

We remark that

$$\sum\pm T^{k+1}\big(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}_0(z)\big)(1))=\sum\pm T^{k+1}\big(\mathcal{G}\cdots\mathcal{G}\otimes z\big)(1).$$

Hence (12.25) implies $\left(\Psi(\mathcal{T}\otimes\mathcal{T}')_{k+1}(T^1\cdots T^{k+1})\right)_0(z)=0$. The proof of $\left(\Psi(\mathcal{T}\otimes\mathcal{T}')_{k+1}(T^1\cdots T^{k+1})\right)_h=0$ is similar. We thus proved $\Psi(\mathcal{T}\otimes\mathcal{T}')=1_{1_{\mathcal{F}unc(\mathcal{F}unc(\mathcal{C},\mathcal{C}h),\mathcal{F}unc(\mathcal{C},\mathcal{C}h))}}$.

We turn to the proof of $\Psi(\mathcal{T}'\otimes\mathcal{T})=1_{\mathcal{F}\circ\mathcal{G}}$. $\Psi(\mathcal{T}'\otimes\mathcal{T})_0=\text{identity}$ and $\Psi(\mathcal{T}'\otimes\mathcal{T})_1=0$ is easy to show. We have

$$\begin{aligned}&\Psi(\mathcal{T}'\otimes\mathcal{T})_{k+1}(T^1\cdots T^{k+1})\\ &\quad=\pm\Phi(\mathcal{T}'_{k+1}(T^1\cdots T^{k+1})\otimes\mathcal{T}_0)\pm\Phi(\mathcal{T}'_0\otimes\mathcal{T}_{k+1}(T^1\cdots T^{k+1}))\\ &\qquad+\sum_{i=1}^{k}\Phi(\mathcal{T}'_i(T^1\cdots T^i)\otimes\mathcal{T}_{k+1-i}(T^{i+1}\cdots T^{k+1}))\\ &\quad=\pm\mathcal{T}'_{k+1}(T^1\cdots T^{k+1})\pm\mathcal{T}'_{k+1}(T^1\cdots T^{k+1})+\sum_{i=1}^{k}\Phi(\mathcal{T}_i(T^1\cdots T^i)\otimes\mathcal{T}_{k+1-i}(T^{i+1}\cdots T^{k+1}))\end{aligned}.$$

We are going to prove

$$\begin{aligned}&\pm\mathcal{T}'_{k+1}(T^1\cdots T^{k+1})\pm\mathcal{T}'_{k+1}(T^1\cdots T^{k+1})\\ &\quad+\sum_{i=1}^{k}\Phi(\mathcal{T}'_i(T^1\cdots T^i)\otimes\mathcal{T}_{k+1-i}(T^{i+1}\cdots T^{k+1}))=0\end{aligned}\tag{12.26.k}$$

by an induction on k. The case when $k=0$ is already proved. Suppose that (12.26. $k-1$) is correct then we have

$$\begin{aligned}&\left(\Psi(\mathcal{T}'\otimes\mathcal{T})_{k+1}(T^1\cdots T^{k+1})\right)_0(z)\\ &=\sum\pm T^{k+1}\left(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0(z)\right)(1)\\ &\qquad+\sum\pm T^{k+1}\left(\mathcal{G}\cdots\mathcal{G}\otimes z\right)(1)\\ &\qquad+\sum_{i>0}\pm T^{k+1}\left(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}(T\cdots T)_0\left(\mathcal{T}'_i\left(T^1\cdots T^i\right)\right)\right)(1))\\ &=\sum_{i>0}\pm T^{k+1}\left(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}_i\left(T^1\cdots T^i\right)_0(z)\right)(1)\\ &\qquad+\sum_{i>0}\pm T^{k+1}\left(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}'_i\left(T^1\cdots T^i\right)_0(z)\right)(1)\\ &\qquad+\sum_{\substack{i>0\\ j>0}}\pm T^{k+1}\left(\mathcal{G}\cdots\mathcal{G}\otimes\mathcal{T}_j\left(T^{i+1}\cdots T^{i+j}\right)_0\left(\mathcal{T}'_i\left(T^1\cdots T^i\right)\right)\right)(1))\end{aligned}\tag{12.27}$$

We use induction hypothesis and find that (12.27) vanishes. The proof of $\left(\Psi(\mathcal{T}'\otimes\mathcal{T})_{k+1}(T^1\cdots T^{k+1})\right)_h=0$ is similar.

The proof of Lemma 12.24 is complete. The proof of Theorem 12.2 is now complete.

We finally prove the following lemma used in § 9.

Lemma 12.28 *Let C be an A^∞ category with identity, $a\in\mathcal{O}b(C)$, $F\in\mathcal{F}unc(C,\mathcal{C}h)$. We assume that F preserves identity. Then, $\mathcal{F}unc(F^a,F)$ is chain homotopy equivalent to $F(a)$.*

We define

$$\mathcal{T}:F(a)\to\mathcal{F}unc(F^a,F)\tag{12.29}$$

as follows. Let $y\in F(a)$. We define pre natural transformation $\mathcal{T}(y):F^a\to F$. For $z\in F^a(b_0)\in C(a,b_0)$ we put

$$\mathcal{T}(y)(z)=\left(F_1(z)\right)(y)\in F(b_0).\tag{12.30}$$

For $x_i \in C(b_{i-1}, b_i)$, we put

$$(12.31) \qquad \big(\mathcal{T}(y)\big)_k (x_1 \cdots x_k)(z) = \big(F_{k+1}(zx_1 \cdots x_k)\big)(y) \in F(b_k).$$

We find

$$\begin{aligned}
&\big(\mathcal{T}(y)\big)(\cdots)(z) \\
&= \sum \pm\partial\big((F_{k+1}(z\cdots))(y)\big) + \sum \pm\big(F_{k+1}(\partial z \otimes x_1 \cdots x_k)\big)(y) \\
&\qquad + \sum \pm\big(F_{k+1}(z \otimes \cdots \partial x \cdots)\big)(y) + \sum \pm\big(F(z \otimes \cdots \eta(\cdots)\cdots)\big)(y) \\
&\qquad + \sum \pm F(\eta(z\cdots)\cdots)(y) + \sum \pm F(\cdots)\big(F(z\cdots)(y)\big)
\end{aligned}.$$

By (2.2.k) this is equal to $T^{\partial y}(\cdots)(z)$. Thus we constructed a chain map (12.29). We define

$$\Theta : \mathcal{F}unc(F^a, F) \to F(a),$$

by

$$(12.32) \qquad \Theta(T) = \big(T_0(a)\big)(1_a) \in F(a).$$

We have $\Theta \circ \mathcal{T} = identity$. To see that $\mathcal{T} \circ \Theta$ is homotopic to identity, we take a short cut by using Corollary 12.3. Using it, we may reduce to the case when higher compositions in $\mathcal{C}$ are trivial. In that case Θ is obviously an isomorphism, which complete the proof.

We remark that, in fact, we do not need to introduce $\mathcal{T}$ for the proof of Lemma 12.28 itself. But we need explicit formula (12.31) in the discussion of § 9. Namely we use the fact that Formula (9.25) coincides with Formula (12.31). See Remark 13.17 also.

§ 13 Approximate identity

Our purpose in this section is to describe a modification we need to handle the case of topological A^{∞} category which does not have an identity. We use the notion, approximate identity for this purpose. We also discuss the existence of approximate identity for our example $\mathcal{Lag}(X,\omega)$.

Definition 13.1 Let $\mathcal{C}$ be a topological A^{∞} category. We say that $\mathcal{C}$ has an *approximate identity* if for each $a \in \mathcal{Ob}(\mathcal{C})$ there exists an open neighborhood U_a of a in $\mathcal{Ob}(\mathcal{C})$ and an element $1_{a;b} \in \mathcal{C}_0(a,b)$, $1'_{a;b} \in \mathcal{C}_0(b,a)$ for each element b in a Bair subset V_a of U_a, which have the following properties.

Let n be an positive integer. Let $n^{\mathcal{Ob}(\mathcal{C})}$ be the set of all subset Ξ of order n of $\mathcal{Ob}(\mathcal{C})$. Then for any Ξ in a Bair subset of $n^{\mathcal{Ob}(\mathcal{C})}$, there exists $V_a(\Xi)$ which is a Bair subset of an open subset $U_a(\Xi)$ of U_a, such that the following holds for each $b \in V_a(\Xi)$.

(13.2.1) $\partial 1_{a;b} = 0$, $\partial 1'_{a;b} = 0$.

(13.2.2) If $c \in \Xi$, then $\eta_2 : \mathcal{C}(c,b) \otimes \mathcal{C}(b,a) \to \mathcal{C}(c,a)$ and $\eta_2 : \mathcal{C}(c,a) \otimes \mathcal{C}(a,b) \to \mathcal{C}(c,b)$ are well defined and $\eta_2\big(\eta_2\big(x \otimes 1'_{a;b}\big) \otimes 1_{a;b}\big) = x$, $\eta_2\big(\eta_2\big(x' \otimes 1_{a;b}\big) \otimes 1'_{a;b}\big) = x'$, for $x \in \mathcal{C}(c,b)$, $x' \in \mathcal{C}(c,a)$.

(13.2.3) If $c \in \Xi$, then $\eta_2 : \mathcal{C}(b,a) \otimes \mathcal{C}(a,c) \to \mathcal{C}(b,c)$ and $\eta_2 : \mathcal{C}(a,b) \otimes \mathcal{C}(b,c) \to \mathcal{C}(a,c)$ are well defined and $\eta_2\big(1'_{a;b} \otimes \eta_2\big(1_{a;b} \otimes y\big)\big) = y$, $\eta_2\big(1_{a;b} \otimes \eta_2\big(1'_{a;b} \otimes y'\big)\big) = y'$ for $y \in \mathcal{C}(b,c)$, $y' \in \mathcal{C}(a,c)$.

(13.2.4) If $c_i \in \Xi$, $k \ge 0$, $\ell \ge 0$, $k+\ell \ge 3$ then

$$\begin{aligned}\eta_{k+\ell+1} : \mathcal{C}(a_1,a_2) \otimes \cdots \otimes \mathcal{C}(a_{k-1},a_k) \otimes \mathcal{C}(a_k,b) \otimes \mathcal{C}(b,a)\\ \otimes \mathcal{C}(a,a_{k+1}) \otimes \mathcal{C}(a_{k+1},a_{k+2}) \cdots \otimes \mathcal{C}(a_{k+\ell-1},a_{k+\ell}) \to \mathcal{C}(a_1,a_{k+\ell}),\end{aligned}$$

$$\begin{aligned}\eta_{k+\ell+1} : \mathcal{C}(a_1,a_2) \otimes \cdots \otimes \mathcal{C}(a_{k-1},a_k) \otimes \mathcal{C}(a_k,a) \otimes \mathcal{C}(a,b)\\ \otimes \mathcal{C}(b,a_{k+1}) \otimes \mathcal{C}(a_{k+1},a_{k+2}) \cdots \otimes \mathcal{C}(a_{k+\ell-1},a_{k+\ell}) \to \mathcal{C}(a_0,a_{k+\ell}).\end{aligned}$$

are well defined. (In case $k=0$ the target is $\mathcal{C}(b,a_{k+\ell})$ and $\mathcal{C}(a,a_{k+\ell})$. If $\ell = 0$ the target is $\mathcal{C}(a_1,a)$ and $\mathcal{C}(a_1,b)$.)

(13.2.5) In the situation of (13.2.4),

$$\eta_{k+\ell}(x_1 \otimes \cdots \otimes x_{k-1} \otimes 1'_{a;b} \otimes y_1 \otimes \cdots \otimes y_{\ell-1}) = 0,$$
$$\eta_{k+\ell}(x_1 \otimes \cdots \otimes x_{k-1} \otimes 1_{a;b} \otimes y_1 \otimes \cdots \otimes y_{\ell-1}) = 0.$$

Remark 13.3

In Definition 13.1, we assumed that U_a is a neighborhood of a. In case we will try to include non simply connected Lagrangian submanifolds (in future), this assumption seems not appropriate. This is because if Λ' is C^1 close to Λ but is not a Hamiltonian perturbation of it, then $HF(\Lambda',\Lambda)$ has no natural identity element in general. Since we do not consider nonsimply connected Lagrangian in this paper, we put that assumption here.

Let $\mathcal{C}$ be a topological A^∞ category with approximate identity and $\Xi \in n^{\mathcal{O}b(\mathcal{C})}$ be a generic subset of $\mathcal{O}b(\mathcal{C})$ order n. We define an "A^∞ category $\mathcal{C}(\Xi)$" as follows. (In fact $\mathcal{C}(\Xi)$ is not an A^∞ category as we will soon explain.) The set of objects of $\mathcal{C}(\Xi)$ is Ξ. By choosing Ξ generic, we may assume that for each $a_i \in \Xi$, $a_i \ne a_j$ $(i \ne j)$, the k-th composition

$$\eta_k : \mathcal{C}(a_0,a_1) \otimes \cdots \otimes \mathcal{C}(a_{k-1},a_k) \to \mathcal{C}(a_0,a_k),$$

is well defined. We choose $i(a) \in V_a(\Xi - \{a\})$, for each $a \in \Xi$. (We take Ξ generic such that $V_a(\Xi - \{a\})$ exists.) We define

$$\mathcal{C}(\Xi)(a,a') = \mathcal{C}(a,a') \text{ if } a \ne a'.$$
$$\mathcal{C}(\Xi)(a,a) = \mathcal{C}(a,i(a)).$$

We now define k-th composition η' in $\mathcal{C}(\Xi)$. If $a_i \ne a_j$ $(i \ne j)$, then $\eta'_k : \mathcal{C}(\Xi)(a_0,a_1) \otimes \cdots \otimes \mathcal{C}(\Xi)(a_{k-1},a_k) \to \mathcal{C}(\Xi)(a_0,a_k)$ is equal to η_k.

The k-th composition $\eta'_k(x_1 \otimes \cdots \otimes x_k)$ is defined only for $x_i \in C(a_{i-1},a_i)$ such that there is no i_1,i_2,i_3 with $a_{i_1} = a_{i_2} = a_{i_3}$. (Thus $\mathcal{C}(\Xi)$ is not an A^∞ category.) In that case the definition is as follows.

We put $\bar{x}_i = \eta_2(1'_{a_i,i(a_i)}, x_i)$ if $a_{i-1} = a_{i-2}$, and $\bar{x}_i = x_i$ otherwise. We put

$$\eta'_k(x_1 \otimes \cdots \otimes x_k) = \eta_k(\bar{x}_1 \otimes \cdots \otimes \bar{x}_k). \tag{13.4}$$

We also define $1_a \in \mathcal{C}(\Xi)(a,a) = \mathcal{C}(a,a')$ by $1_a = 1_{a;a'}$.

Lemma 13.5

If there is no i_1, i_2, i_3 *with* $a_{i_1} = a_{i_2} = a_{i_3}$ *then* η' *satisfy Formula* (2.1) *for* $x_i \in C(a_{i-1}, a_i)$.

Proof:

We calculate

$$\begin{aligned}
&\sum \pm\eta'(x \otimes \cdots \otimes \eta'(x \cdots x) \otimes \cdots \otimes x) \\
= &\sum_{c_{i-1}=c_{i-2}} \pm\eta(\bar{x} \otimes \cdots \otimes \eta_2(1 \otimes \eta'(x_i \cdots x)) \otimes \cdots \otimes \bar{x}) \\
&\qquad + \sum_{c_{i-1}\neq c_{i-2}} \pm\eta(\bar{x} \otimes \cdots \otimes \eta'(x_i \cdots x) \otimes \cdots \otimes \bar{x}) \\
= &\sum_{c_{i-1}=c_{i-2}} \pm\eta(\bar{x} \otimes \cdots \otimes \eta'(\eta_2(1 \otimes x_i) \otimes \cdots x) \otimes \cdots \otimes \bar{x}) \\
&\qquad + \sum_{c_{i-1}\neq c_{i-2}} \pm\eta(\bar{x} \otimes \cdots \otimes \eta'(x_i \cdots x) \otimes \cdots \otimes \bar{x}) \\
= &\sum \pm\eta(\bar{x} \otimes \cdots \otimes \eta(\bar{x} \cdots \bar{x}) \otimes \cdots \otimes \bar{x})
\end{aligned}$$

The lemma then follows easily.

Lemma 13.6

1_a *satisfies conditions in Definition* 12.1. *More precisely :*

(13.6.2) *If* , $a \neq b$ $x \in C(\Xi)(a,b)$ *we have*

$$\eta'_2(1_a \otimes x) = x\ ,\ \ \eta'_2(x \otimes 1_b) = x.$$

(13.6.3) *If* $x_i \in C(\Xi)(a_{i-1}, a_i)$ *and if non of the three elements among* $a_0, \cdots, a_k, a$ *coincides then we have*

$$\eta'_{k+1}(x_1 \otimes \cdots \otimes x_i \otimes 1_{a_i} \otimes x_{i+1} \otimes \cdots \otimes x_k) = 0$$

for $k \geq 2$.

The lemma follows easily from definition.

We can then repeat the proof of Theorem 12.2 and obtain the following. Let $\mathcal{Rep}(\Xi)(\mathcal{C}, \mathcal{Ch})$ be the full subcategory of $\mathcal{Func}(\mathcal{C}, \mathcal{Ch})$ such that the set of its objects is the A^∞ functors represented by elements of Ξ.

Proposition 13.7

For every generic finite set $\Xi \in n^{\mathcal{Ob}(C)}$, *there exists an "$A^\infty$ functors"* $\mathcal{F} : C(\Xi)^o \to \mathcal{Rep}(\Xi)(C, \mathcal{Ch})$, $\mathcal{G} : \mathcal{Rep}(\Xi)(C, \mathcal{Ch}) \to C^o(\Xi)$ *such that* $\mathcal{G} \circ \mathcal{F}$ *is an identity functor and* $\mathcal{F} \circ \mathcal{G}$ *is homotopic to identity.*

Remark 13.8

Since higher composition of $C(\Xi)$ is not defined somewhere, $\mathcal{G}$ is not defined somewhere. This is the reason we write "A^∞ functors" in Proposition 14.6. The precise statement will become clear during the proof.

Proof:

The proof is similar to the proof of Theorem 12.2. So we only gives necessary change.

We define

$$\mathcal{F} : C(\Xi) \to \mathcal{Rep}(\Xi)(C, \mathcal{Ch})$$

in the same way as § 11, using composition η' namely :

$$\left(\mathcal{F}_\ell(y_1 \otimes \cdots \otimes y_k)_h(x_1 \otimes \cdots \otimes x_h)\right)(z) = \eta'\left(y_1 \otimes \cdots \otimes y_k \otimes z \otimes x_1 \otimes \cdots \otimes x_h\right)$$

This is well defined if $y_i \in C(b_i, b_{i-1})$, $x_i \in C(a_{i-1}, a_i)$, $z \in C(b_0, a_0)$ and non of the three elements among b_i, a_j, coincides.

We define

$$\mathcal{G} : \mathcal{Rep}(\Xi)(C, \mathcal{Ch}) \to C(\Xi),$$

again in the same way. Namely $T^i \in \mathcal{Fun}c(T^{b_{i-1}}, T^{b_i})$. We put

$$\mathcal{G}_1(T^1) = \eta\left(1_{b_0; i(b_0)} \otimes T_0^1(1'_{b_0; i(b_0)})\right) \in C(b_1, b_0) = C(\Xi)(b_1, b_0),$$

if $b_1 \neq b_0$. Otherwise it is not defined. $\mathcal{G}_k$ defined by the same induction formula as in §12. $\mathcal{G}_k(T^1 \cdots T^k)$, $T^i \in \mathcal{Fun}c(T^{b_{i-1}}, T^{b_i})$ is well defined if b_i are all distinct.

Then $\mathcal{G} \circ \mathcal{F}$ is identity functor. However it is not everywhere defined. Namely $(\mathcal{G} \circ \mathcal{F})_k(y_1 \otimes \cdots \otimes y_k)$ is defined only if $y_i \in C(b_i, b_{i-1})$ and b_i are all distinct.

We next consider $\mathcal{F} \circ \mathcal{G}$. We remark that $\left(\left((\mathcal{F} \circ \mathcal{G})_k(T^1 \cdots T^k)\right)_h(x_1 \otimes \cdots \otimes x_h)\right)(z)$ is defined if $T^i \in \mathcal{F}unc(T^{b_{i-1}}, T^{b_i})$, $x_i \in \mathcal{C}(a_{i-1}, a_i)$, $z \in \mathcal{C}(b_0, a_0)$, b_i are all distinct and if non of the three elements among b_i, a_j, coincides. Therefore, if b_i are all distinct, then $(\mathcal{F} \circ \mathcal{G})_k(T^1 \cdots T^k)$ is well defined as a topological A^∞ functor. Hence it is a morphism in $\mathcal{R}ep(\Xi)(\mathcal{C}, \mathcal{C}h)$.

We define $\mathcal{T}: 1 \to \mathcal{F} \circ \mathcal{G}$ and $\mathcal{T}': \mathcal{F} \circ \mathcal{G} \to 1$ by the same formula as the proof of Theorem 12.2. Then $\mathcal{T}_0 = \mathcal{T}_0' =$ identity map is defined and $\mathcal{T}_k(T^1 \cdots T^k)$, $\mathcal{T}_k'(T^1 \cdots T^k)$ is defined if $T^i \in \mathcal{F}unc(T^{b_{i-1}}, T^{b_i})$ and b_i are all distinct. $\Psi(\mathcal{T} \otimes \mathcal{T}') = 1$, $\Psi(\mathcal{T}' \otimes \mathcal{T}) = 1$ is proved in exactly the same way. Note that $(\Psi(\mathcal{T} \otimes \mathcal{T}')_k)(T^1 \cdots T^k) = 0$ and $(\Psi(\mathcal{T} \otimes \mathcal{T}')_k)(T^1 \cdots T^k) = 0$ holds only for $T^i \in \mathcal{F}unc(T^{b_{i-1}}, T^{b_i})$ with b_i are all distinct.

Definition 13.9 Let $\mathcal{C}$, $\mathcal{C}'$ be topological A^∞categories with approximate identity. Let $F: \mathcal{C} \to \mathcal{C}'$ be a topological A^∞functor. We say that F *preserves an approximate identity*, if for each $a \in \mathcal{O}b(c)$ in a Bair subset, there exists an open neighborhood $U_a' \subseteq U_a$ of a, where U_a is as in Definition 13.1, such that $F(1_{a:b}) = 1_{F_0(a):F_0(b)}$ and $F(1_{a:b}') = 1_{F_0(a):F_0(b)}'$ for each b in a Bair subset of U_a'.

Let $\mathcal{C}$, $\mathcal{C}'$ be topological A^∞categories with approximate identity. Let $F: \mathcal{C} \to \mathcal{C}'$ be a topological A^∞functor. We say that F is a *weak homotopy equivalence,* if it preserves an approximate identity and if there exists $F': \mathcal{C}' \to \mathcal{C}$ such that if $F(\Xi) = \mathcal{R}ep(\Xi)(\mathcal{C}, \mathcal{C}h)^o \to \mathcal{C} \xrightarrow{F} \mathcal{C}' \to \mathcal{R}ep(F(\Xi))(\mathcal{C}', \mathcal{C}h)^o$ and $F'(\Xi): \mathcal{R}ep(\Xi)(\mathcal{C}, \mathcal{C}h)^o \to \mathcal{C} \xrightarrow{F'} \mathcal{C}' \to \mathcal{R}ep(F(\Xi))(\mathcal{C}', \mathcal{C}h)^o$ are homotopy equivalences for every finite set Ξ of objects.

The following is an immediate corollary to Proposition 13.7.

Lemma 13.10

Let $F: \mathcal{C} \to \mathcal{C}'$ be a weak homotopy equivalence. Then we have

(13.10.1) $F_{1*}: \mathcal{C}(a_0, a_1) \to \mathcal{C}'(F(a_0), F(a_1))$ *is a chain homotopy equivalence for* (a_0, a_1) *in a Bair subset of* $Ob(\mathcal{C})^2$.

(13.10.2) *The following diagram commutes up to chain homotopy for* (a_0, a_1, a_2) *in a Bair subset of* $Ob(\mathcal{C})^3$.

$$\begin{array}{ccc} C(a_0,a_1)\otimes C(a_1,a_2) & \xrightarrow{\eta_2} & C(a_0,a_2) \\ \downarrow F\otimes F & & \downarrow F \\ C(F(a_0),F(a_1)) & & \\ \otimes C(F(a_1),F(a_2)) & \xrightarrow{\eta_2} & C(F(a_0),F(a_2)) \end{array}.$$

Diagram 13.11

Remark 13.12 There is one point in which our discussion so far on approximate identity is unsatisfactory. Let a and V_a as in Definition 13.1. It seems that it is not automatic from our definition of approximate identity that chain homotopy type of $C(a,b)$ is independent of the choice of b in V_a. However this fact holds (and seems important) in our examples. In fact, in the case when $C = \mathcal{MS}(M)$, this facts means that chain homotopy type of Morse-Witten complex is independent of the choice of Morse function. In the case when $C = \mathcal{Lag}(M,\omega)$, it implies a similar well definiedness of Floer homology.

It should also be possible to show that $C(a,b)$ is chain homotopy equivalent to $\mathcal{Func}(F^a,F^a)$ if $b \in V_a$. (We remark that the chain complex $\mathcal{Func}(F^a,F^a)$ is well defined while $C(a,a)$ may not be well defined.)

These points are related to the problem of transversality along diagonal and may be essential. There might be a way to find a good axiom from which they follow automatically. Since the author could not find it, we do not discuss this point in this paper.

Now we consider the case of our basic example $\mathcal{Lag}(X,\omega)$ and construct the approximate identity of it (modulo analytic detail).

Our basic tool is a method of proof of [22].

Let X be a symplectic manifold with $c^1(X) = N[\omega]$, $N \ge 2$, and Λ be a simply connected Lagrangian submanifold in X. A neighborhood of Λ can be identified to a neighborhood of zero of the cotangent bundle $T^*\Lambda$ of Λ. Hence a Lagrangian C^1-close to Λ is identified with a graph of a exact one form in Λ. Therefore a neighborhood of Λ in $\mathcal{Ob}(\mathcal{Lag}(X,\omega))$ is identified to a neighborhood of zero of $C^\infty(\Lambda)$.

For $f \in C^\infty(\Lambda)$ let Λ_f denote the corresponding element in $\mathcal{Ob}(\mathcal{Lag}(X,\omega))$. We remark that $\mathcal{Lag}(\Lambda,\Lambda_f)$ is isomorphic to $\mathcal{MS}(f)$ as an abelian group.

We remark that the 0-th homology of Morse-Witten complex $\mathcal{MS}(f)$ has a canonical generator, that is $\sum_{\mu(p)=0}[p]$. We let this element be $1_{\Lambda,\Lambda_f}$. Also $\mathcal{Lag}(\Lambda_f,\Lambda)\cong\mathcal{MS}(-f)$ has a canonical generator, which we put $1'_{\Lambda,\Lambda_f}$.

Our main result is :

Theorem 13.13 *If* $N\ge 2$ *for* X *then* $1_{\Lambda,\Lambda_f}$, $1'_{\Lambda,\Lambda_f}$ *are approximate identity.*

Outline of the proof:

Let us verify (13.2.2), (13.2.3) and (13.2.5). Let Λ' be another simply connected Lagrangian submanifold. We may assume that they are transversal to Λ. Let $q_1,\cdots,q_n=\Lambda\cap\Lambda'$. By choosing f enough small, we may assume that Λ' is also transversal to Λ_f and that there is a canonical one to one correspondence $\Lambda\cap\Lambda'\cong\Lambda_f\cap\Lambda'$. Let $q'_1,\cdots,q'_n=\Lambda_f\cap\Lambda'$.

Next we remark that the union of stable manifold St_p of p for $\mu(p)=0$ is dense in Λ. Hence we may assume that $q_i\in\bigcup_{\mu(p)=0}St_p$. We then have :

Lemma 13.14 $\eta(1_{\Lambda;\Lambda_f}\otimes[q_i])=[q'_i]$, $\eta([q'_i]\otimes 1'_{\Lambda;\Lambda_f})=[q_i]$.

This follows from Theorem 13.15 below and the same statement with Λ and Λ_f exchanged.

Theorem 13.15*

Let p *be a critical point of* f *with* $\mu(p)=0$. *Then, for sufficiently small* f *we have :*

$$\begin{aligned}&\mathcal{M}(\Lambda_f,\Lambda,\Lambda';p,q_i,q'_j)=\emptyset \quad \text{if } i\ne j\\ &\mathcal{M}(\Lambda_f,\Lambda,\Lambda';p,q_i,q'_i)=\emptyset \quad \text{if } q_i\notin St(p)\\ &\mathcal{M}(\Lambda_f,\Lambda,\Lambda';p,q_i,q'_i)=\{\text{one point}\} \quad \text{if } q_i\in St(p).\end{aligned}$$

Sketch of the Proof:

The first equality is easier. In fact suppose that $\mathcal{M}(\Lambda_{f_n},\Lambda,\Lambda';p,q_i,q_j)\ne\emptyset$ for $f_n\to 0$. Then we have a pseudo holomorphic disk which bounds $\Lambda\cup\Lambda'$ and contains q_i and q_j on the boundary. We remark that $\mu(q_i)=\mu(q_j)$. Hence using the assumption of simply connectivity of Λ, Λ' and $N\ge 2$, we find that this pseudo holomorphic disk belongs to the moduli space of dimension ≥ 4. This is impossible since we are considering the limit of the element $\mathcal{M}(\Lambda_{f_n},\Lambda,\Lambda';p,q_i,q_j)$ which is of 0 dimensional.

The construction part of the third equality follows the method of [22]. But we need to modify a bit. First we need to construct a gluing data at q_i. In a neighborhood of q_i, we scale everything then 3 Lagrangian submanifolds Λ, Λ_f, Λ' look like a triple of linear Lagrangian submanifolds in $\boldsymbol{C}^n$. Namely $\Lambda \approx \boldsymbol{R}^n$, $\Lambda_f \cong \boldsymbol{R}^n + \sqrt{-1}\vec{v}$. Here $\vec{v}$ is the gradient vector field of f at q_i. Using the fact that Λ' is transversal to Λ we can find a linear holomorphic disk in $\boldsymbol{C}^n$ whose boundaries are in these three linear Lagrangian submanifolds and which is $\left(t\vec{v},\sqrt{-1}s\vec{v}\right)$ $(s \in [0,1],\ \ t \in (-\infty,-R])$ outside a compact set. Uniqueness of such a Lagrangian submanifold is immediate since our equation together with its boundary value is linear.

So we can use this pseudo holomorphic disk as gluing data around q_i.

The other gluing date is constructed by using gradient line which goes from q_i to p in exactly the same way as [22].

The method to glue them is again the same as [22].

The proof that there is no other solution and the proof of the second equality is the same as the uniqueness part of the proof of [22] Part I. This complete the outline of the proof of Theorem 13.14. (The detail will be in the subsequent paper.)

We continue the proof of Theorem 13.13 Lemma 13.14 implies (13.2.2), (13.2.3). (13.2.5) is a consequence of the following Lemma 13.16. Let Λ_i be a simply connected Lagrangian submanifold such that Λ_i are transversal to each other and to Λ. Let $x_i \in \Lambda_{i-1} \cap \Lambda_i$. Here we regard $\Lambda_0 = \Lambda$, $\Lambda_{k+1} = \Lambda_f$.

Lemma 13.16* *Let* $\sum_{i=1}^{k+1}\mu(x_i) + (k+1) - 3 = 0$ *and* $\mu(p) = 0$. *Then for generic* Λ_i *and small* f, $\mathcal{M}(\Lambda_f,\Lambda,\Lambda_1,\cdots,\Lambda_k;p,x_1,\cdots,x_{k+1}) = \emptyset$.

Sketch of the proof:

Let $x'_{k+1} \in \Lambda \cap \Lambda_k$ be an element close to $x_{k+1} \in \Lambda_k \cap \Lambda_f$. Then by dimension counting, we have $\mathcal{M}(\Lambda,\Lambda_1,\cdots,\Lambda_k;x_1,\cdots x_k,x'_{k+1}) = \emptyset$. Lemma 13.16 then follows from the limit argument similar to the proof of the first equality of Theorem 13.15. The detail will be in a subsequence paper.

Lemma 13.16 implies

$$\langle \eta_k([x_{m+1}]\otimes\cdots\otimes[x_{k+1}]\otimes[p]\otimes[x_i]\otimes\cdots\otimes[x_{m-1}]),[x_m]\rangle = 0\ .$$

(13.2.5) follows immediately. We thus gave a sketch of the proof of Theorem 13.13. The detail will appear together with other analytic detail of the symplecitic part of the story of this paper.

Remark 13.17 We remark that we used Lemma 12.28 in the proof of Theorem 9.3. To generalize it to the situation we need we have to show the following :

Theorem 13.18* *Let N, E, Σ be as in § 3. Then the topological A^∞ functor $HF(N,E) : \mathcal{Lag}(R(\Sigma),\mathcal{Ch})$ preserves approximate identity.*

The proof of Theorem 13.18 is similar to the proof of Theorem 13.13 and will be given in a subsequent paper.

We finally remark one consequence of Theorem 13.18. Let $\Lambda_1,\Lambda_2 \in \mathcal{Ob}(\mathcal{Lag}(X,\omega))$. We first remark that

Proposition 13.19* *If Λ_2 is a Hamiltonian perturbation of Λ_1, then the topological A^∞ functor represented by Λ_1 is homotopy equivalent to the topological A^∞ functor represented by Λ_2.*

The proof is similar to the Floer's result on the independence of the Lagrangian intersection Floer homology by Hamiltonian perturbation. We remark that Theorem 13.18, Proposition 13.19 together with Lemma 12.18, imply the following :

Theorem 13.20 *Let $\Lambda_1,\Lambda_2 \in \mathcal{Ob}(\mathcal{Lag}(R(\Sigma,E))$, and let Λ_2 is a Hamiltonian perturbation of Λ_1. Then $HF(N,E)(\Lambda_1)$ is chain homotopy equivalent to $HF(N,E)(\Lambda_2)$.*

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