Mirror symmetry of Abelian variety and Multi Theta functions

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§ 0 Introduction

In this paper, we study mirror symmetry of complex and symplectic tori as an example of homological mirror symmetry conjecture of Kontsevich [24], [25] between symplectic and complex manifolds. We discussed mirror symmetry of tori in [12] emphasizing its "noncommutative" generalization. In this paper, we concentrate on the case of a commutative (usual) torus. Our result is a generalization of one by Polishchuk and Zaslow [42], [41], who studied the case of elliptic curve.

The main results of this paper establish a dictionary of mirror symmetry between symplectic geometry and complex geometry in the case of tori of arbitrary dimension. We wrote this dictionary in the introduction of [12]. We present the argument in a way so that it suggests a possibility of its generalization. However there are various serious difficulties for the generalization, some of which we mention in this paper.

In this paper, we will define a new family of theta functions on complex tori, which we call multi theta function. It is a generating function of the numbers obtained by counting holomorphic polygons in tori and describe various product structures (Yoneda, and Massey Yoneda products) of the sheaf cohomology group on its mirror.

We recall that one famous application [7] of mirror symmetry is that a generating function of the number counting rational curves in a Calabi-Yau manifold is equal to the Yukawa coupling, a product structure of sheaf cohomology, of its mirror. In the case of complex tori, there is no rational curve. Hence the statement above is void. However, if we include Lagrangian submanifolds on symplectic side and coherent sheaves in complex side, we can derive many nontrivial consequences of mirror symmetry. Exploring them is the purpose of this paper. Namely we find relations between counting problem of holomorphic polygons (0 loop correlation function of topological open string) and product structures of sheaf cohomology in its mirror. We remark that including Lagrangian submanifolds and coherent sheaves correspond to including branes. So it is naturally related to the recent progress of string theory. (See for example [40].)

Let us describe the results of this paper.

In § 1, we show a way to construct a complex manifold which is a moduli space of Lagrangian submanifolds (plus line bundles on it) of a symplectic manifolds (M,ω), together with B field B, (that is a closed 2 form). (We put $\Omega = \omega + \sqrt{-1}B$.) This complex manifold is expected to be components of the moduli space of coherent sheaves (more precisely objects of the derived category of coherent sheaves) of the mirror $(M,\Omega)^\vee$. A component of this moduli space which is to correspond to the moduli space of the skyscraper sheaves is the mirror manifold $(M,\Omega)^\vee$ itself. This is an idea by Strominger-Yau-Zaslow [47]. There are various troubles to make this construction rigorous in the general situation. In the case of a torus, we can make it rigorous and define a mirror torus in this way.

In § 2, we show a way to associate an object of the derived category of coherent sheaves of the mirror $(M,\Omega)^\vee$ to a Lagrangian submanifold of $(M,\Omega)^\vee$. There are again troubles to make this construction rigorous in the general situation. We make it rigorous in the case of affine Lagrangian submanifolds of tori. Namely we construct a holomorphic vector bundle $\mathcal{E}(L,\mathcal{L})$ of $(T^{2n},\Omega)^\vee$ for each pair $(L,\mathcal{L})$ of an affine Lagrangian submanifold L of $(T^{2n},\Omega)^\vee$ and a flat line bundle $\mathcal{L}$ on L.

Sections 3 and 5 are devoted to the proof of :

Theorem 3.1 $\quad H^k((T^{2n},\Omega)^\vee,\mathcal{E}(L,\mathcal{L})) \cong HF^k((L_{st},0),(L,\mathcal{L}))$.

Here L_{st} is the Lagrangian submanifold of (T^{2n},Ω) which corresponds to the structure sheaf of $(T^{2n},\Omega)^\vee$ and HF is the Floer cohomology of Lagrangian intersection. ([11], [38], [17].)

We prove also in § 6 an isomorphism

Theorem 6.1 $\quad Ext^k(\mathcal{E}(L_1,\mathcal{L}_1),\mathcal{E}(L_2,\mathcal{L}_2)) \cong HF^k((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2))$.

In § 3, § 6, we also give explicit isomorphisms in Theorems 3.1 and 6.1 in case $k=0$, by using the relation between theta function and product structure of Floer homology. (We give explicit isomorphisms for higher cohomology in § 11.) We also prove the commutativity of the following diagram (Theorem 6.5.)

$$\begin{array}{ccc}
HF^0((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)) \otimes HF^0((L_2,\mathcal{L}_2),(L_3,\mathcal{L}_3)) & \rightarrow & HF^0((L_1,\mathcal{L}_1),(L_3,\mathcal{L}_3)) \\
\downarrow & & \downarrow \\
Hom(\mathcal{E}(L_1,\mathcal{L}_1),\mathcal{E}(L_2,\mathcal{L}_2)) \otimes Hom(\mathcal{E}(L_2,\mathcal{L}_2);\mathcal{E}(L_3,\mathcal{L}_3)) & \rightarrow & Hom(\mathcal{E}(L_1,\mathcal{L}_1),\mathcal{E}(L_3,\mathcal{L}_3))
\end{array}$$

Diagram A

Here the vertical arrows are the isomorphism is in Theorem 6.1. The horizontal arrow in the first line is the product structure m_2 in Floer homology. (See § 3.) The horizontal arrow in the second line is the composition of homomorphisms.

In § 7, we study a moduli space $\mathcal{M}(\tilde{L})$ of the pair $(L,\mathcal{L})$ and show that it is a component of the moduli space of vector bundles on the mirror $(T^{2n},\Omega)^\vee$. We also construct a universal bundle $\mathcal{P} \rightarrow \mathcal{M}(\tilde{L}) \tilde{\times} (T^{2n},\Omega)^\vee$. Namely the restriction of $\mathcal{P}$ to $\{(L,\mathcal{L})\} \times (T^{2n},\Omega)^\vee$ is $\mathcal{E}(L,\mathcal{L})$. We also discuss holomorphicity of the maps m_2 with respect to $(L,\mathcal{L})$.

In § 8, we study the case when pairs of affine Lagrangian submanifolds L_1, L_2 are not transversal to each other and generalize Theorem 3.1 to this case. We also discuss the case of disconnected Lagrangian submanifold. When several components coincide, we find an example of the phenomenon called enhanced gauge symmetry.

In §§ 9,10,11,12 , we study higher products m_k, $k \geq 3$. This operators m_k are defined by using multi theta functions. Multi theta function is a generating function of the counting problem of holomorphic polygons in $\boldsymbol{C}^n$ with affine boundary conditions. (The author would like to thank M. Gromov who introduced the problem counting holomorphic polygons in $\boldsymbol{C}^n$ to the author.) The number counting holomorphic polygons in $\boldsymbol{C}^n$ is the simplest nontrivial case of "open string analogue" of Gromov-Witten invariant. By the same reason as Gromov-Witten invariant, there is a transversality problem to define this number rigorously. In the case of open string version, the problem is more serious. Namely the methods developed to define Gromov-Witten invariants rigorously are not enough to establish its "open string analogue" rigorously. In fact, in the most naive sense, this number is ill-defined. Oh [38] discovered this trouble in a related context of Floer homology theory of Lagrangian intersections. The basic reason is similar to the wall crossing problem discovered by Donaldson [8] to define Donaldson invariant of 4 - manifolds with $b_2^+ = 1$. In our case, this problem is related to the fact that Massey product is well-defined only as an element of some coset space. Donaldson introduced a chamber structure to study this ill-definedness of Donaldson invariant. For our problem of counting holomorphic polygons, we need also to study a chamber structure. In our case, the wall (that is the boundary of the chamber) may also be ill-defined. Namely the point where the number of holomorphic polygons jumps may also *depend* on the perturbation. (This problem is pointed out in [12] § 5.) We will find that the "homology class of the wall" is well-defined, and will determines it. Figures 10 - 15 in § 10 are examples of the combinatorial structure of the chamber we found. The homology class of wall in turn contains enough information to determines m_k modulo homotopy equivalence of A^∞ category. (See [15] for its definition.) Our way to determine the coefficients of m_k is constructive. Namely there is an algorithm to calculate it. More precisely, we formulate Axioms (Axiom I,II,III,IV) which the number counting holomorphic polygons is expected to satisfy. We then prove the following :

Theorem 10.17 *There exists a coefficient function* c_k *satisfying Axioms* I,II,III,IV.

(Here the coefficient function is one which is supposed to be the number counting holomorphic $k+1$ gons and which will be a coefficient of the multi theta series we introduce.)

Theorem 10.18 *Let* c_k^1, c_k^2 *be two coefficient functions satisfying Axioms* I,II,III,IV. *Then* c_k^1 *is homologues to* c_k^2.

Using the coefficient functions c_k, obtained in Theorems 10.17, we define multi theta series by

$$\sum c_k[v_1+\gamma_1,\cdots,v_{k+1}+\gamma_{k+1}]\exp\Big(-2\pi Q(v_1+\gamma_1,\cdots,v_{k+1}+\gamma_{k+1}) + 2\pi\sqrt{-1}\sum \alpha_i(p_{i\,i+1}-p_{i-1\,i})\Big) \tag{0.1}$$

Here v_i parametrize the affine Lagrangian submanifold in $\boldsymbol{C}^n$ parallel to a given one. The sum is taken over all $(\gamma_1,\cdots,\gamma_{k+1})$ which run in certain lattice in $\boldsymbol{R}^{n(k-2)}$. $Q(v_1,\cdots,v_{k+1})$ is the symplectic area of the $k+1$-gon bounding the union of affine Lagrangian submanifolds parametrized by v_i and is a quadratic form of index $k-2$. α_i is a flat connection on the affine Lagrangian submanifold and $p_{i\,i+1}$ is the point where two affine Lagrangian submanifolds

(i-th and $i+1$-th) intersect. (See § 9 for precise definition.)

(0.1) gives a usual theta function in the case $k=2$. (In case $n=1$ this fact was observed by Kontsevich in [25].) In case $k=3$, (0.1) is an indefinite theta series which looks similar to those used by Götche-Zagier [21] to study Donaldson's polynomial invariant in the case when $b_2^+=1$. In case $k\ge 4$, it seems that (0.1) is a new family of theta series.

Using these multi theta functions as matrix elements, we obtain maps

(0.2)
$$m_k : HF((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2))\otimes\cdots\otimes HF((L_k,\mathcal{L}_k),(L_{k+1},\mathcal{L}_{k+1})) \to HF((L_1,\mathcal{L}_1),(L_{k+1},\mathcal{L}_{k+1})).$$

If we move L_i, $\mathcal{L}_i$ then m_k moves. Thus we may regard $v_1,\cdots,v_{k+1},\alpha_1,\cdots,\alpha_{k+1}$ as variables also.

Let us explain Axioms I,II,III,IV we put to coefficient function c_k briefly. The essential part of Axiom I asserts that $Q(v_1,\cdots,v_{k+1})$ is positive if $c_k[v_1,\cdots,v_{k+1}]$ is nonzero. It implies that (0.1) converges. On the other hand, it is a consequence of the positivity of the volume of holomorphic disk.

Axioms II,III are equivalent to Maurer-Cartan or Batalin-Vilkovisky master equation :

(0.3)
$$dc_k^{(\ell)} + \sum_{\substack{\ell_1+\ell_2=\ell+1\\ k_1+k_2=k+1}} \pm\, c_{k_1}^{(\ell_1)}\circ c_{k_2}^{(\ell_2)} = 0.$$

Here $c_k^{(\ell)}$ is a generalization of c_k and is a degree ℓ current valued version of it. ($c_k^{(0)}=c_k$ is a locally constant function.) d is the De-Rham operator with respect to v_i variable.

In the case $\ell=-1$, (0.3) reduces to the A^∞ formulae

(0.4)
$$\sum_{k_1+k_2=k} \pm\, m_{k_1}\circ m_{k_2} = 0,$$

introduced in [45], [13]. The origin of (0.3) and (0.4) in symplectic geometry is a degeneration of holomorphic polygons.

We remark that a differential equation (0.3) appears in many literatures recently. (See [1], [46] , [3], [28], [44] etc.) The L^∞ version appears mainly in those literatures. (0.3) is an A^∞ version. (Here L stands for Lie and A for associative.) We use (0.3) to prove

Theorem 10.49
$$\bar\partial m_k^{(0\,\ell)} + \sum_{\substack{\ell_1+\ell_2=\ell+1\\ k_1+k_2=k+1}} \pm\, m_{k_1}^{(0\,\ell_1)}\circ m_{k_2}^{(0\,\ell_2)} = 0.$$

Here we use $c_k^{(\ell)}$ in the same way as c_k to define $m_k^{(\ell)}$, that is a degree ℓ current whose value is in the homomorphism bundle (0.2). $m_k^{(0\,\ell)}$ in Theorem 10.49 is the 0ℓ part of it.

In the case when $k=2$, (0.3) will be $dc_2^{(\ell)}=0$. Hence $c_2^{(\ell)}$ define a De-Rham cohomology class of certain space. Axiom IV asserts that this cohomology class is a generator.

Using Morse homotopy [16] of quadratic function and [18], we prove that the counting function of holomorphic polygons satisfies Axiom IV. (Theorem 10.15.)

In § 11, using m_k we define isomorphisms in Theorems 3.1 and 6.1 explicitly in the case of higher cohomology (Theorem 11.28). (The proof of Theorem 3.1 and 6.1 in §§ 3,5,6, are based on Riemann-Roch's theorem and is not constructive in case $k > 0$.) We remark that in our situation, Floer cohomology $HF^k((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2))$ is easy to calculate and has a natural basis. Hence the isomorphism in Theorems 3.1 and 6.1 gives a canonical basis of sheaf cohomology. Using this isomorphism, we generalize Diagram A to

$$\begin{array}{ccc} HF^k((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)) \otimes HF^\ell((L_2,\mathcal{L}_2),(L_3,\mathcal{L}_3)) & \rightarrow & HF^{k+\ell}((L_1,\mathcal{L}_1),(L_3,\mathcal{L}_3)) \\ \downarrow & & \downarrow \\ Ext^k(\mathcal{E}(L_1,\mathcal{L}_1),\mathcal{E}(L_2,\mathcal{L}_2)) \otimes Ext^\ell(\mathcal{E}(L_2,\mathcal{L}_2);\mathcal{E}(L_3,\mathcal{L}_3)) & \rightarrow & Ext^{k+\ell}(\mathcal{E}(L_1,\mathcal{L}_1),\mathcal{E}(L_3,\mathcal{L}_3)) \end{array}$$

Diagram B

(Theorem 11.18.) We also prove that m_3 will become the triple Massey-Yoneda product in sheaf theory. (Theorem 11.23.) We can prove a similar results for higher Massey-Yoneda product. But it is rather hard to state it, because higher Massey product is defined only on certain subset and is well-defined only as an element of certain coset space. Theorem 12.5 is a better way to state it.

In §§ 1 - 11, we considered semi-homogeneous sheaves in the sense of [32]. In § 12, we consider more general sheaves, using its resolution by semi-homogeneous sheaves. Let $x_{ij} \in HF((L_i,\mathcal{L}_i),(L_j,\mathcal{L}_j))$. We consider equations of the form :

$$\sum_{k,i_0=i,\,i_k=j} \pm m_k\left(x_{i_0 i_1},\cdots,x_{i_{k-1} i_k}\right) = 0 \tag{0.5}$$

for each i, j. (See § 12 for precise notation and sign.) We prove :

Theorem 12.5 *There exists a family of the objects of derived category of coherent sheaves on* $(T^{2n},\Omega)^\vee$ *parametrized by the solution of* (0.5).

Note that (0.5) is a polynomial of x_{ij} and its coefficients are special values of multi theta functions. Roughly speaking, the object in Theorem 12.5 is the cohomology sheaf of the modified Dolbeault operator $\hat{\bar\partial} = \bar\partial + \sum \pm m_k^{(\ell)}(\bullet, x\cdots x)$. Theorem 10.49 and (0.5) imply $\hat{\bar\partial} \circ \hat{\bar\partial} = 0$. Theorem 12.5 seems to be related to the monad or quiver description of the moduli space of stable sheaves. (See Example 12.30.)

We next calculate the cohomology of the sheaf obtained in Theorem 12.5. Namely we show

Corollary 12.25 *The cohomology group of the objects in Theorem* 12.5, *is isomorphic to the vector space of the solutions of the following linear equations (of* s_i.*)*

$$\sum_{k,\, i_1 = i, i_k = j} \pm m_k\left(s_i, x_{i_1 i_2}, \cdots, x_{i_{k-1} i_k}\right) = 0 .$$
$$\sum_{k,\, i_1 = i, i_k = j} \pm m_k\left(x_{i_1 i_2}, \cdots, x_{i_{k-1} i_k}, s_j^*\right) = 0 .$$

We prove a similar results for an extension of two objects in Theorem 12.5. (Theorem 12.23.)

We explain in § 12 that a mirror of the system satisfying (0.5), is a smooth Lagrangian submanifold obtained from Lagrangian tori L_i by Lagrangian surgery. Thus, in various cases, a mirror of a sheaves which is not semi-homogeneous, is a Lagrangian submanifold which is not affine. Then Corollary 12.25 and Theorem 12.23 provide a way to calculate Floer homology between Lagrangian submanifolds in tori.

A conjecture of Mukai implies that every sheaf on $(T^{2n}, \Omega)^{\vee}$ is obtained as in Theorem 12.5. (See Conjecture 12.27.) Thus, if we assume Mukai's conjecture, Theorem 12.5 and Lagrangian surgery will give a correspondence between Lagrangian submanifold and objects of derived categories, that is the homological mirror conjecture of tori.

§ 1 Moduli space of Lagrangian submanifolds and construction of a mirror torus

In this section, we construct a mirror torus of a given symplectic torus (T^{2n},Ω) such as $T^{2n} = \mathbf{C}^n \big/ \left(\mathbf{Z} + \sqrt{-1}\mathbf{Z}\right)^n$, $\Omega \in \Lambda^{1,1}(T^{2n})$. (Note that the complex structure of the torus T^{2n} will *not* be used below. We use it only to set the condition $\Omega \in \Lambda^{1,1}(T^{2n})$.) As in some of the other sections, we first give an idea which the author expects to work in more general situations. We then will make it rigorous in the case of a torus.

Let (M,Ω) be a symplectic manifold (M,ω) together with a closed 2 form B on M. Here we put $\Omega = \omega + \sqrt{-1}B$. (Note $-B + \sqrt{-1}\omega$ is used in many of the literatures.)

Definition 1.1 $\mathcal{L}ag^{\sim +}(M,\Omega)$ is the set of all pairs $(L,\mathcal{L})$ with the following properties :

(1.2.1) L is a Lagrangian submanifold of (M,ω),

(1.2.2) $\mathcal{L} \to L$ is a line bundle together with a $U(1)$ connection $\nabla^{\mathcal{L}}$ such that $F_{\nabla^{\mathcal{L}}} = 2\pi\sqrt{-1}B|_L$.

We put the C^∞ topology on $\mathcal{L}ag^{\sim +}(M,\Omega)$. This space is of infinite dimensional. We will divide it by the group of Hamiltonian diffeomorphisms. The quotient space is a finite dimensional manifold. Let $f : M \times [0,1] \to \mathbf{R}$ be a smooth function and we put $f_t(x) = f(x,t)$. Let $X_{f,t}$ denote the Hamiltonian vector field associated to $f_t(x)$. It induces a one parameter family of symplectic diffeomorphisms $\varphi : M \times [0,1] \to M$ by :

$$\varphi(x,0) = x, \quad \frac{\partial}{\partial t}\varphi(x,t) = X_{f,t}(x). \tag{1.3}$$

We put $\varphi_1(x) = \varphi(x,1)$. The diffeomorphism φ_1 is called a *Hamiltonian diffeomorphism.*

Definition 1.4 Let $(L,\mathcal{L}),(L',\mathcal{L}') \in \mathcal{L}ag^{\sim +}(M,\Omega)$. We say that $(L,\mathcal{L})$ is *Hamiltonian equivalent* to $(L',\mathcal{L}')$ if the following holds. There exists $f : M \times [0,1] \to \mathbf{R}$ such that the map $\varphi : M \times [0,1] \to M$ solving (1.3) and satisfying $\varphi_1(L) = L'$. Also there exists a connection ∇ on $\pi_1^*\mathcal{L} \to L \times [0,1]$ with the following properties.

(1.5.1) $F_\nabla = 2\pi\sqrt{-1}\varphi^* B$,

(1.5.2) $\nabla|_{L\times\{0\}} = \nabla^{\mathcal{L}}$.

(1.5.3) There exists an isomorphism $(L,\nabla|_{L\times\{1\}}) \cong (L',\nabla^{\mathcal{L}'})$ covering φ_1.

It is easy to see that Hamiltonian equivalence defines an equivalence relation on $\mathcal{L}ag^{\sim +}(M,\Omega)$. Let $\mathcal{L}ag^{+}(M,\Omega)$ denote the quotient space with quotient topology.

Remark 1.6 In [47], Strominger-Yau-Zaslow proposed closely related but a bit different moduli space. Namely they proposed the moduli space of the pairs of special Lagrangian submanifolds and flat line bundles on it. It seems that, by taking a special Lagrangian submanifold, we take a representative of Hamiltonian equivalence. However one needs to study some open questions to clarify the relation between two moduli spaces. Let us mention some of them.

Problem 1.7 Let L, L' be special Lagrangian submanifolds of a Kähler manifold M. Suppose that there exists a Hamiltonian diffeomorphism φ_1 such that $\varphi_1(L) = L'$. When does it imply $L = L'$?

Problem 1.8 Let L be a Lagrangian submanifold in a Kähler manifold M. When does there exist a Hamiltonian diffeomorphism φ_1 such that $\varphi_1(L)$ is a special Lagrangian submanifold ?

There are examples where the answer is negative for Proposition 1.8. The moduli space of the pairs $(L, \mathcal{L})$ of special Lagrangian submanifold L in a Calabi-Yau manifold and a flat $U(1)$ bundle $\mathcal{L}$ has a complex structure. In a similar way, our moduli space has a complex structure as we will soon define. (In our case, we do not need to assume that M is a Calabi-Yau manifold and can start with a general symplectic manifold.) However, in fact, we do not know whether it is a manifold, since we do not know whether $\mathcal{L}ag^{+}(M,\Omega)$ is Hausdorff or not.

Problem 1.9 When is $\mathcal{L}ag^{+}(M,\Omega)$ Hausdorff ?

In fact, it is more natural to consider a local version of Problem 1.9. Let $(L, \mathcal{L}) \in \widetilde{\mathcal{L}ag}^{+}(M,\Omega)$. By Darbout-Weinstein theorem, a neighborhood U of L in M is symplectically diffeomorphic to a neighborhood of the zero section of $T^{*}L$. We denote it by $\psi : U \to T^{*}L$. Let ω' is the standard symplectic form on $T^{*}L$ and B' be a closed 2 form on $T^{*}M$ which coincides with $\psi_{*}B$ in a neighborhood of zero section.

Condition 1.10 ψ induces a homeomorphism from an open neighborhood $\mathcal{U}$ of $[L, \mathcal{L}] \in \mathcal{L}ag^{+}(M,\Omega)$ to an open neighborhoods of $[L, \mathcal{L}] \in \mathcal{L}ag^{+}(T^{*}L, \omega' + \sqrt{-1}B')$.

Furthermore, the following holds. For each ε there exists $\mathcal{U}_{\varepsilon}$ a neighborhood of $(L, \mathcal{L})$ in $\widetilde{\mathcal{L}ag}^{+}(M,\Omega)$, such that if $(L_1, \mathcal{L}_1), (L_2, \mathcal{L}_2) \in \mathcal{U}_{\varepsilon}$ and if they are Hamiltonian equivalent to each other, then the function f in Definition 1.4 can be chosen so that its C^1 norm is smaller than ε.

The reader who is familiar with symplectic geometry may find that Condition 1.10 is closely related to the flux conjecture. (See [26].)

Proposition 1.11 *Let* $K \subseteq \mathcal{L}ag^{+}(M,\Omega)$. *We assume that Condition* 1.10 *holds for each* $[L, \mathcal{L}] \in K$. *Then a neighborhood of* K *in* $\mathcal{L}ag^{+}(M,\Omega)$ *has a structure of complex manifold.*

Proof: Let $[L,\mathcal{L}]\in K$. We are going to construct a chart on its neighborhood. Let $\ell_1,\cdots,\ell_b$ be loops representing a basis of $H_1(L,\mathbf{Z})$ and $[L',\mathcal{L}']$ be in a neighborhood of $[L,\mathcal{L}]$. By Condition 1.10, we may assume that L' is C^1 close to L. Hence we may assume that it is a graph of a closed one form u on L. We put $L' = L_u$ = graph of u. We define $\phi_i : S^1\times[0,1]\to T^*L$ by $\phi_i(s,t) = tu(\ell_i(s))$. We put

$$h_i(L',\mathcal{L}') = h_{\phi_i(\cdot,1)}(\mathcal{L}')\exp\left(-2\pi\int\phi_i^*\Omega\right). \tag{1.12}$$

Lemma 1.13 *h_i defines a map from a neighborhood of $[L,\mathcal{L}]$ in $\mathcal{L}ag^+(M,\Omega)$ to $\mathbf{C}$.*

Proof: Suppose $[L',\mathcal{L}']=[L'',\mathcal{L}'']$. We need to prove $h_i(L',\mathcal{L}') = h_i(L'',\mathcal{L}'')$. To save notation, we assume $(L'',\mathcal{L}'') = (L,\mathcal{L})$. We may assume also that $L' = L_u$ = graph of u. Using Condition 1.10 we find that u is exact and that the function f in Definition 1.4 can be chosen to be independent of t. Moreover $u = df$. Therefore $\exp\left(-\int\phi_i^*\omega\right) = 1$. We put $\nabla^{\mathcal{L}} = d/ds + \alpha ds$, $\nabla^{\mathcal{L}'} = d/ds + \beta ds$ where $s\in[0,2\pi]$ is the coordinate of S^1 and α,β are $u(1) = \sqrt{-1}\mathbf{R}$ valued functions on S^1. Then (1.5.2) and (1.5.3) imply

$$\int_0^{2\pi}\alpha ds - \int_0^{2\pi}\beta ds = \int_0^{2\pi} ds\int_0^1 dt\, F_\nabla .$$

Therefore we have

$$h_{\phi_i(\cdot,1)}(\mathcal{L}')\exp\left(-2\pi\sqrt{-1}\int\phi_i^* B\right) = h_{\phi_i(\cdot,0)}(\mathcal{L}).$$

Lemma 1.13 follows.

By Lemma 1.13, $h = (h_1,\cdots,h_b)$ is a map from a neighborhood of $[L,\mathcal{L}]$ in $\mathcal{L}ag^+(M,\Omega)$ to $\mathbf{C}^b$. Then again Condition 1.10 implies that h is injective there. We take h as a coordinate around $[L,\mathcal{L}]$. It is straightforward to verify that the coordinate change is biholomorphic. We thus proved Proposition 1.11.

We now consider the case of a simplex torus, (T^{2n},Ω), where $\Omega = \omega + \sqrt{-1}B$ is a complexified symplectic form. We assume that Ω is homogeneous. We put $V = \tilde{T}^{2n}$ the universal cover. V is a $2n$ - dimensional real vector space. In this paper, we are studying the commutative case. It means that we *assume* the following :

Assumption 1.14 There exists an n- dimensional linear subspace $\tilde{L}_{pt}$ of V such that $\Omega|_{\tilde{L}_{pt}} = 0$ and that $\Gamma\cap\tilde{L}_{pt}\cong\mathbf{Z}^n$.

We write $\tilde{L}_{pt}$ since this will correspond to the points (skyscraper sheaves) in the mirror. As we remarked in [12], this assumption is satisfied if $T^{2n} = \mathbf{C}^n\Big/\left(\mathbf{Z}+\sqrt{-1}\mathbf{Z}\right)^n$

and Ω is of 1-1 type. Let $\mathcal{M}(\tilde{L}_{pt})$ be the set of all $[L,\mathcal{L}]\in \mathcal{L}ag^{+}(T^{2n},\Omega)$ such that the universal cover of L is parallel to $\tilde{L}_{pt}$. It is easy to see that Condition 1.10 is satisfied for this $K=\mathcal{M}(\tilde{L}_{pt})$. We also observe that $\mathcal{M}(\tilde{L}_{pt})$ is a connected component of $\mathcal{L}ag^{+}(T^{2n},\Omega)$. We are going to describe the complex structure we obtained on $\mathcal{M}(\tilde{L}_{pt})$. We put $V^{*}=Hom(V,\boldsymbol{R})$. Let $x\in V$. We define $I_x : V\oplus V^{*}\to \boldsymbol{C}$ by

$$I_x(v,\sigma)=\Omega(x,v)+\sqrt{-1}\sigma(x). \tag{1.15}$$

It is easy to see that there exists a unique complex structure on $V\oplus V^{*}$ such that $I_x : V\oplus V^{*}\to \boldsymbol{C}$ is complex linear for each $x\in V$.

Let $\tilde{L}$ be a Lagrangian linear subspace of (V,Ω). Then there exists a natural $\boldsymbol{R}$-linear surjection : $V\oplus V^{*}\to V/\tilde{L}\oplus\tilde{L}^{*}$, where $\tilde{L}^{*}=Hom_{\boldsymbol{R}}(\tilde{L},\boldsymbol{R})$. It is also easy to see that there exist a unique complex structure on $V/\tilde{L}\oplus\tilde{L}^{*}$ such that the map : $V\oplus V^{*}\to V/\tilde{L}\oplus\tilde{L}^{*}$ is complex linear.

Let $(v,\sigma)\in V/\tilde{L}_{pt}\oplus\tilde{L}^{*}_{pt}$. We obtain an affine subspace $\hat{L}_{pt}(v)=\tilde{L}_{pt}+v$ and its quotient $L_{pt}(v)\subseteq T^{2n}$. On the other hand, σ is regarded as a flat connection ∇_σ on the trivial bundle on $L_{pt}(v)$, by the isomorphism $\boldsymbol{R}\cong u(1)$, $\sigma\mapsto 2\pi\sqrt{-1}\sigma$. Let $\mathcal{L}(\sigma)$ denote the pair of trivial line bundle and the connection ∇_σ. Hence $(L_{pt}(v),\sigma)=(L_{pt}(v),\mathcal{L}(\sigma))$ is an element of $\mathcal{L}ag^{\sim+}(T^{2n},\Omega)$.

We put $\Gamma=\pi_1(T^{2n})$ and

$$\left(\Gamma\cap\tilde{L}_{pt}\right)^{\vee}=\left\{\mu\in\tilde{L}^{*}_{pt}\;\middle|\;\forall\gamma\in\Gamma\cap\tilde{L}_{pt}\quad \mu(\gamma)\in\boldsymbol{Z}\right\}. \tag{1.16}$$

It is easy to see that $(L_{pt}(v),\sigma)$ is Hamiltonian equivalent to $(L_{pt}(v'),\sigma')$ if and only if $v-v'\in\Gamma/\Gamma\cap\tilde{L}_{pt}$, $\sigma-\sigma'\in\left(\Gamma\cap\tilde{L}_{pt}\right)^{\vee}$. We define

Definition 1.17 $$\mathcal{M}(\tilde{L}_{pt})=\frac{V/\tilde{L}_{pt}\oplus\tilde{L}^{*}_{pt}}{\left(\Gamma/\Gamma\cap\tilde{L}_{pt}\right)\oplus\left(\Gamma\cap\tilde{L}_{pt}\right)^{\vee}}.$$

It is easy to see that the complex structure we defined by using (1.15) coincides with one by Proposition 1.11 in this case. Now we use Strominger-Yau-Zaslow's idea to define :

Definition 1.18 *A mirror* $(T^{2n},\Omega)^{\vee}$ of (T^{2n},Ω) is $\mathcal{M}(\tilde{L}_{pt})$.

We remark that $\mathcal{M}(\tilde{L}_{pt})$ may depend on the choice of $\tilde{L}_{pt}$. Hence there are many different mirrors of (T^{2n},Ω).

Remark 1.19 In Definition 1.1, we assumed $F_\nabla=2\pi\sqrt{-1}B|_L$. On the other hand, in the case of a torus we assumed $F_\nabla=0$, $\Omega|_L=0$. Note that there exists a line bundle on L

satisfying the condition $F_\nabla = 2\pi\sqrt{-1}B|_L$ if and only if $\left[B|_L\right] \in H^2(L,\boldsymbol{Z})$. Therefore, if we change B by adding a harmonic form B_0 representing an element of $H^2(T^{2n},\boldsymbol{Z})$, and if we replace $\mathcal{L}$ by $\mathcal{L} \otimes \mathcal{L}_0|_L$, where $\mathcal{L}_0$ is a complex line bundle on T^{2n} with connection such that $F_{\mathcal{L}_0} = -2\pi\sqrt{-1}B_0$, then we have $F_\nabla = 0$, $\Omega|_L = 0$. Thus, in our situation, we may assume $F_\nabla = 0$, $\Omega|_L = 0$ instead of $F_\nabla = 2\pi\sqrt{-1}B|_L$ without loosing generality.

Before studying the case of a torus more, we return to the general case and add a few remarks. In fact, the moduli space $\mathcal{L}ag^+(M,\Omega)$ we defined above, is too big for our purpose in the general case. For example if L is any compact Lagrangian submanifold of $\boldsymbol{C}^n$ then it is automatically contained in any M as a Lagrangian submanifold. We want to avoid such a “local” Lagrangian submanifold. In the case of $M = T^2$, [42] avoid a Lagrangian circle which is homologous to zero. One way to do so is to restrict ourselves to special Lagrangian submanifolds. Certainly Lagrangian submanifolds of $\boldsymbol{C}^n$ are not minimal and hence not special. However there are cases we do not want to restrict ourselves to special Lagrangian submanifolds. For example, in the case when $M = \Sigma_g$, a surface of higher genus, there is only one spacial Lagrangian submanifold (closed geodesic in this case) in each homology class. As a consequence, the moduli space of pairs of special Lagrangian submanifolds and flat line bundles on it, is odd (one) dimensional. In the case of Calabi-Yau manifold however such a phenomenon never happens by [30].

The way we are proposing here is to restrict ourselves to the Lagrangian submanifold for which Floer homology is well-defined. We discuss well-definedness of Lagrangian intersection Floer homology in [17]. We define there a series of obstructions in $H^{even}(L,\boldsymbol{Q})$. Using it and constructing generating functions in a similar way as the definition of the boundary operator ∂, we “obtain” elements of $H^{even}(L,\boldsymbol{C})$. (However there are troubles to establish the obstruction theory in this way. What we prove in [17] is somewhat weaker than that.) Formally (namely modulo convergence problem) this class gives a holomorphic map from $\mathcal{L}ag^+(M,\Omega)$ (if we include appropriate quantum correction of the complex structure on $\mathcal{L}ag^+(M,\Omega)$). It seems reasonable to expect that Condition 1.10 is satisfied for the Lagrangian submanifold for which Floer homology is well-defined.

Conjecture 1.20 Condition 1.10 is satisfied if the obstruction classes defined in [17] vanish for $(L,\mathcal{L})$.

It might be possible to use Floer homology to solve Conjecture 1.24 in a similar way as [39], [26].

We remark that in the case when $H_{n-even}(L,\boldsymbol{Q}) \to H_{n-even}(M,\boldsymbol{Q})$ is injective, the obstruction classes in [17] are automatically 0. Here, in our case of affine Lagrangian submanifolds in a torus, the obstruction class vanishes automatically.

Remark 1.21 The relation between Problem 1.8 (the existence of a special Lagrangian submanifold in a Hamiltonian diffeomorphism class) and vanishing of the obstruction classes of [17] is mysterious also.

We recall that Becker-Becker-Strominger [4] found a relation between D-brain and calibrated geometry by studying the condition for D-brain to preserve supper symmetry. Our condition in [17] is one so that $\partial\partial = 0$ holds after modification. These two conditions may be related to each other. We remark that speciality is a local condition while the vanishing of obstruction class is a global one. This might mean that, after adding appropriate correction terms, the BRST symmetry ($\partial\partial = 0$) is not broken in perturbation theory and soliton effect only can break it.

We denote by $\mathcal{Lag}(M,\Omega)$ the subspace of $\mathcal{Lag}^{+}(M,\Omega)$ consisting of the elements represented by the pairs $(L,\mathcal{L})$ such that the obstruction classes vanish. We used this notation $\mathcal{Lag}(M,\Omega)$ in the introduction of [12].

We remark that there is one very important point which is not mentioned above. Namely the author does not know how to compactify the moduli space $\mathcal{Lag}(M,\Omega)$. In the case of a torus, we do not need compactification, since our component of $\mathcal{Lag}(T^{2n},\Omega)$ is already compact. (See however § 8.) In general, we need to include singular Lagrangian submanifolds for the compactification. A related serious trouble is how to define Floer homology between such singular Lagrangian submanifolds.

We remark that the complex structure discussed in this section seems to be the same one as [47], [31], [20]. One needs some “quantum correction” to obtain a complex structure of the mirror in the case when one needs a compactification, we do not discuss it here since in the case of tori we do not need it.

§ 2 Construction of a sheaf from an affine Lagrangian submanifold

We next construct a sheaf from an affine Lagrangian submanifold. Again we first present an argument which might work in more general situations than the case of tori.

Let (M,Ω) be a symplectic manifold with complexified symplectic form as in § 1. We *assume* that there exists a component of $\mathcal{Lag}(M,\Omega)$ which is isomorphic to the mirror $(M,\Omega)^\vee$. We remark that this assumption is rather restrictive. A more realistic assumption is that an appropriate compactification of $\mathcal{Lag}(M,\Omega)$ is a mirror $(M,\Omega)^\vee$. Since the author does not know the way to work in this generality, he discuss only this restrictive case in this paper.

Let $(L,\mathcal{L})$ be another element of $\mathcal{Lag}(M,\Omega)$. We are going to find an object of a variant of the derived category of coherent sheaves on $(M,\Omega)^\vee$. Let us first explain what we mean by it. Let X be a complex manifold. We consider a system, $\left(U_i,\mathcal{F}_i^\bullet,\varphi_{ij}\right)$ such that :

(2.1.1) $X=\bigcup U_i$ is an open covering.

(2.1.2) For each i, $\mathcal{F}_i^\bullet$ is a cochain complex of coherent sheaves on U_i.

(2.1.3) For each i,j with $U_i\cap U_j\neq\emptyset$, $\varphi_{i,j}^\bullet$ is a morphisms of sheaves $\varphi_{i,j}^k:\mathcal{F}_i^k\to\mathcal{F}_j^k$. Such that $\delta^k\varphi_{i,j}^k=\varphi_{i,j}^{k+1}\delta^k$ and that $\varphi_{i,j}^k$ induces an isomorphism $\varphi_{i,j,*}^k:\mathcal{H}^k(\mathcal{F}_i^\bullet)\to\mathcal{H}^k(\mathcal{F}_j^\bullet)$ of cohomology sheaves. (Here we put $\mathcal{H}^k(\mathcal{F}_i^\bullet)=\mathrm{Ker}\delta^k\big/\mathrm{Im}\delta^{k-1}$.)

(2.1.4) $\varphi_{j,i,*}^k\circ\varphi_{i,j,*}^k$ is identity.

(2.1.5) $\varphi_{j,\ell,*}^k\circ\varphi_{i,j,*}^k=\varphi_{i,\ell,*}^k$ on $U_i\cap U_j\cap U_\ell$.

Two such systems are said to be equivalent to each other, if there exist chain maps of sheaves which are compatible with φ_{ij}'s and induce isomorphisms on cohomologies. We say an equivalence class of such a system $\left(U_i,\mathcal{F}_i^\bullet,\varphi_{ij}\right)$ an element of $Ob(\mathbf{D}(X))$, the derived category of the sheaves on X. This definition may be a bit different from the usual one, since usually one considers global chain complexes of sheaves. The problem to determine when our definition coincides with the usual one is delicate and is not discussed in this paper. Morphism between two objects is defined in the same way as the usual derived category. (See [22].)

Remark 2.2 In (2.1.4),(2.1.5), we assumed that the maps $\varphi_{i,j}^\bullet$ are compatible in cohomology level. In order to introduce A^∞ structure ([15]), it seems necessary to assume higher compatibility. Namely we need to assume : $\varphi_{i,j}^\bullet\circ\varphi_{j,\ell}^\bullet$ is chain homotopic to $\varphi_{i,\ell}^\bullet$ by a chain homotopy $H_{i,j,\ell}$: the composition $H_{i,j,\ell}\circ\varphi_{\ell,m}^\bullet$ is chain homotopic to $\varphi_{i,j}^\bullet\circ H_{j,\ell,m}$: and so on. Then the equivalence relation we need is also more strict. We will obtain A^∞ category rather than derived category in this way.

Now we sketch the way how an element $(L,\mathcal{L})$ of $\mathcal{Lag}(M,\Omega)$ defines an element of

$Ob(\mathbf{D}((M,\Omega)^{\vee}))$. The argument here is sketchy since the author does not know how to make it rigorous in the general situation. We will make it rigorous in the case of an affine Lagrangian submanifold in a torus later.

The basic idea is to use a family of Floer homologies. Let $x \in (M,\Omega)^{\vee}$. We identify it with a pair $(L_x, \mathcal{L}_x)$. (More precisely the equivalence class of $(L_x, \mathcal{L}_x)$ is x.) By changing the representative L_x if necessary, we may assume that L_x is transversal to L. We choose a small neighborhood U_x of x in $\mathcal{L}ag^{+}(M,\Omega)$ and also smooth family of representatives $(L_y, \mathcal{L}_y)$ for $y \in U_x$. We may assume that L_y is transversal to L. Now we define vector bundles $\bigcup_{y \in U_x} CF^k((L,\mathcal{L}),(L_y,\mathcal{L}_y)) \to U_x$ as follows.

(2.3) $$CF^k((L,\mathcal{L}),(L_y,\mathcal{L}_y)) = \bigoplus_{\substack{p \in L \cap L_y \\ \varepsilon(p)=k}} Hom(\mathcal{L}_p, \mathcal{L}_{yp}).$$

Here $k \in \boldsymbol{Z}/2\boldsymbol{Z}$ and $\varepsilon(p)$ is 1 if $T_pL \oplus T_pL_y \cong T_pM$ is orientation preserving and is -1 otherwise. $\mathcal{L}_p$ is the fiber of the bundle $\mathcal{L}$ at p and $\mathcal{L}_{yp}$ is the fiber of the bundle $\mathcal{L}_y$ at p.

Since L_y is transversal to L for each y, it is obvious that (2.3) defines a complex vector bundle on U_x. We need to define a holomorphic structure on this bundle to obtain an element of $Ob(\mathbf{D}((M,\Omega)^{\vee}))$. A problem to do so is "gauge fixing". Namely there is a trouble to choose a representative $\mathcal{L}_y$ in an equivalence classes, since $\mathcal{L}_y$ has a nontrivial automorphism $U(1)$. This problem, in fact, already appears to define (2.3) as a vector bundle. (See Remark 2.10.) This is a delicate point and will be discussed later in the case of a torus.

Next we use Floer's boundary operator with local coefficient (together with Kontsevich's modification) to define

(2.4) $$\delta^k : CF^k((L,\mathcal{L}),(L_y,\mathcal{L}_y)) \to CF^{k+1}((L,\mathcal{L}),(L_y,\mathcal{L}_y)).$$

Roughly speaking (2.4) is defined as follows. Let $p,q \in L \cap L_y$, such that $\varepsilon(p) = k+1$, $\varepsilon(q) = k$. We consider the moduli space of pseudoholomorphic disks $\varphi : D^2 \to M$ such that $\varphi(\partial_1 D^2) \subseteq L$, $\varphi(\partial_2 D^2) \subseteq L_y$, $\varphi(-1) = p$, $\varphi(1) = q$. Here $\partial_1 D^2$ (resp. $\partial_2 D^2$) is the part of ∂D^2 satisfying $\operatorname{Im} z \ge 0$ (resp. $\operatorname{Im} z \le 0$). Then the $Hom(Hom(\mathcal{L}_q, \mathcal{L}_{yq}), Hom(\mathcal{L}_p, \mathcal{L}_{yp}))$ component $\delta_{q,p}$ of (2.4) is

(2.5) $$\delta_{q,p} a = \sum_{\varphi} \pm \exp\left(-2\pi \int \varphi^* \Omega\right) P_{\mathcal{L}}(\partial_2 D^2) \circ a \circ P_{\mathcal{L}_y}(\partial_1 D^2).$$

Here $a \in Hom(\mathcal{L}_q, \mathcal{L}_{qy})$, $P_{\mathcal{L}}(\partial_1 D^2) : \mathcal{L}_p \to \mathcal{L}_q$ is the parallel transport along the path $\varphi(\partial_1 D^2) \subseteq L$ and $P_{\mathcal{L}_y}(\partial_2 D^2) : \mathcal{L}_{yq} \to \mathcal{L}_{yp}$ is the parallel transport along the path $\varphi(\partial_2 D^2) \subseteq L_q$. The sign $\pm$ is defined by the orientation of the moduli space of pseudoholomorphic disks. The same argument as Floer's [11], [38] "proves" $\delta\delta = 0$.

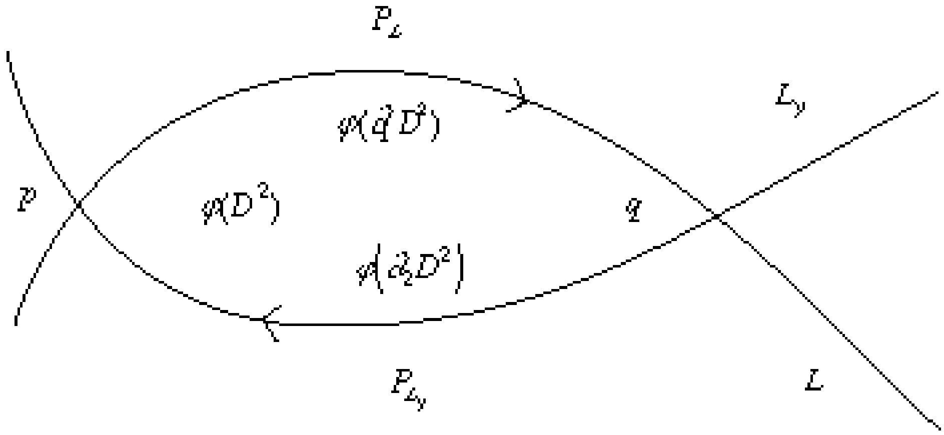

Figure 1

The argument here is not rigorous since we do not know the convergence of (2.5). Moreover, in fact, we need to modify the coboundary operator δ so that $\delta\delta = 0$ is satisfied and this modification can be done only in case when $(L_y, \mathcal{L}_y) \in \mathcal{L}ag(M,\Omega)$. (See [17].)

Our definition of complex structure on $\mathcal{L}ag^+(M,\Omega)$ is designed so that δ will become holomorphic if we put an appropriate holomorphic structure on $CF^k((L,\mathcal{L}),(L_y,\mathcal{L}_y))$. Thus we have a cochain complex of sheaves (holomorphic vector bundles) on U_x.

We remark here that the chain complex we obtained is $\boldsymbol{Z}_2$ graded rather than $\boldsymbol{Z}$ graded. We will define $\boldsymbol{Z}$ grading later in the case of a torus.

We thus constructed (2.1.1) and (2.1.2). The construction of the chain homomorphism (2.1.3) is roughly as follows. Let $U_x \cap U_{x'} \neq \emptyset$. We put $V = U_x \cap U_{x'}$. For each $y \in V$, we have two representatives $(L_{y,1}, \mathcal{L}_{y,1})$, $(L_{y,2}, \mathcal{L}_{y,2})$. Here $(L_{y,1}, \mathcal{L}_{y,1})$ is close to $(L_x, \mathcal{L}_x)$ and $(L_{y,2}, \mathcal{L}_{y,2})$ is close to $(L_{x'}, \mathcal{L}_{x'})$. We remark that $(L_{y,1}, \mathcal{L}_{y,1})$ is Hamiltonian equivalent to $(L_{y,2}, \mathcal{L}_{y,2})$. We recall that the Floer cohomology of Lagrangian intersection is invariant of Hamiltonian diffeomorphism. (See [17] for the proof.) Namely there exists a chain homomorphism

$$\varphi_{y,k} : CF^k((L,\mathcal{L}),(L_{y,1},\mathcal{L}_{y,1})) \to CF^k((L,\mathcal{L}),(L_{y,2},\mathcal{L}_{y,2})) \tag{2.6}$$

which induces an isomorphism to the cohomology. (The construction of (2.6) is not rigorous because of a convergence problem. In [17], we go around the convergence problem by introducing a formal power series ring, that is the Novikov ring [37].) The proof there "implies" that we can take $\varphi_{y,k}$ so that it depends smoothly on y. Moreover it is holomorphic if we define holomorphic structure in an appropriate way. (2.1.4) and (2.1.5) are consequences of the standard argument in Floer theory, which shows that (2.6) is canonical modulo chain homotopy. (See [11].) Thus we sketched an idea of a construction of an element of $Ob(\mathbf{D}((M,\Omega)^{\vee}))$. (In fact this object is the dual of one we associate to $(L,\mathcal{L})$. This will become clear from the remarks we will give later in this section.)

Now we make the above idea rigorous in the case of a torus. Let $\tilde{L}_{pt}$ be as in Assumption 1.14. In fact we need another Lagrangian linear subspace also. Namely we assume :

Assumption 2.7 $\tilde{L}_{st}$ is an n-dimensional linear subspace of V such that $\Omega|_{\tilde{L}_{st}} = 0$, $\Gamma \cap \tilde{L}_{st} \cong \mathbf{Z}^n$ and that $L_{st} \cap L_{pt}(0)$ is one point. Here $L_{st} = \tilde{L}_{st} / \tilde{L}_{st} \cap \Gamma$.

We write L_{st} since it will correspond to the structure sheaf of the mirror. The reason we need to fix $\tilde{L}_{st}$ will be explained later. It is easy to see that such $\tilde{L}_{st}$ exists (if $\tilde{L}_{pt}$ exists), but is not unique.

Now let $\tilde{L} \subseteq V$ be another n-dimensional linear subspace such that $\Omega|_{\tilde{L}} = 0$. We assume also that $\tilde{L} \cap \Gamma \cong \mathbf{Z}^n$. We take an affine space $\hat{L}$ parallel to $\tilde{L}$ and put $L = \hat{L} / \tilde{L} \cap \Gamma$. L is a closed Lagrangian submanifold of T^{2n}. Let $\alpha \in Hom(\tilde{L}, \mathbf{R})$ and we regard it as a connection of a trivial bundle on L. Hence (L, α) is regarded as an element of $\mathcal{Lag}(T^{2n}, \Omega)$. We assume, for simplicity, that $\tilde{L}$ is transversal to $\tilde{L}_{pt}$. (We remove this assumption in § 8.) We first construct a smooth complex vector bundle on $(T^{2n}, \Omega)^{\vee}$.

We will define a $\left(\Gamma / \Gamma \cap \tilde{L}_{pt}\right) \oplus \left(\Gamma \cap \tilde{L}_{pt}\right)^{\vee}$ action on the trivial bundle $\tilde{\mathfrak{E}}(L, \alpha)$ on $V / \tilde{L}_{pt} \oplus \tilde{L}_{pt}^{*}$. Let $(v, \sigma) \in V / \tilde{L}_{pt} \oplus \tilde{L}_{pt}^{*}$. We put $\hat{L}_{pt}(v) = \tilde{L}_{pt} + v$ and let $L_{pt}(v) \subseteq T^{2n}$ be its quotient . We put

$$\tilde{\mathfrak{E}}(L, \alpha)_{(v, \sigma)} = \bigoplus_{p \in L \cap L_{pt}(v)} \mathbf{C}[p].$$

Let $\gamma \in \left(\Gamma / \Gamma \cap \tilde{L}_{pt}\right)$. It is easy to see that $L_{pt}(v) = L_{pt}(v + \gamma)$. Therefore, by definition, $\tilde{\mathfrak{E}}(L, \alpha)_{(v, \sigma)}$ coincides with $\tilde{\mathfrak{E}}(L, \alpha)_{(\gamma + v, \sigma)}$. Thus we defined an action of $\Gamma / \Gamma \cap \tilde{L}_{pt}$ on $\tilde{\mathfrak{E}}(L, \sigma)$.

We next define an action of $\left(\Gamma \cap \tilde{L}_{pt}\right)^{\vee}$. Let $\mu \in \left(\Gamma \cap \tilde{L}_{pt}\right)^{\vee}$. μ is a homomorphism from $\tilde{L}_{pt}$ to $\mathbf{R}$. We regard it as a gauge transformation on $L_{pt}(v)$ as follows. We take the (unique) point $x_0(v) \in \hat{L}_{pt}(v) \cap \tilde{L}_{st}$. For $x \in \hat{L}_{pt}(v)$ we put :

$$g_{\mu, v}(x) = \exp\left(2\pi \sqrt{-1} \mu(x - x_0(v))\right).$$

$g_{\mu, v}$ is a $U(1)$ valued map and hence is a gauge transformation. Since $\mu(\gamma) \in \mathbf{Z}$ for $\gamma \in \Gamma \cap \tilde{L}_{st}$, it follows that $g_{\mu, v}$ induces a map $L_{pt}(v) \to U(1)$. We denote it by the same symbol. Then we define

$$\mu(c\,[p]) = g_{\mu, v}(p)\, c\,[p], \tag{2.8}$$

where $p \in L \cap L_0(v)$. Here we remark that we regard the right hand side as an element of $\tilde{\mathfrak{E}}(L, \beta)_{(v, \mu + \sigma)}$.

Lemma 2.9 *The actions of* $\gamma \in \left(\Gamma / \Gamma \cap \tilde{L}_{pt}\right)$ *and* $\mu \in \left(\Gamma \cap \tilde{L}_{pt}\right)^{\vee}$ *on* $\tilde{\mathfrak{E}}(L, \alpha)_{(v, \sigma)}$ *we defined above commute to each other.*

Proof: We remark that $\# L_{st} \cap L_{pt}(0) = 1$ implies that $\left(\Gamma \cap \tilde{L}_{pt}\right) \oplus \left(\Gamma \cap \tilde{L}_{st}\right) = \Gamma$. Hence we may regards $\gamma \in \Gamma \cap \tilde{L}_{st}$. Then, by definition, we have $g_{\mu, v+\gamma}(x+\gamma) = g_{\mu, v}(x)$. Lemma 2.9 follows from the definition.

Thus we defined an action of $\left(\Gamma / \Gamma \cap \tilde{L}_{pt}\right) \oplus \left(\Gamma \cap \tilde{L}_{pt}\right)^{\vee}$ on $\tilde{\mathcal{E}}(L, \beta)$. Let $\mathcal{E}(L, \beta) \to (T^{2n}, \Omega)^{\vee}$ be the quotient bundle.

Remark 2.10 In the above construction, we used L_{st} to regard elements of $\left(\Gamma \cap \tilde{L}_{pt}\right)^{\vee}$ as gauge transformations on $L_{pt}(v)$. Namely we require that the gauge transformation is identity at $L_{st} \cap L_{pt}(v)$. This is the way we kill the authomorphism group $U(1)$ of the flat bundle on $L_{pt}(v)$.

We next are going to construct a holomorphic structure on $\mathcal{E}(L, \beta)$. It suffices to construct its local (holomorphic) frame for this purpose. We use a term of a theta series for this purpose as follows. Let $(v, \sigma) \in V / \tilde{L}_{pt} \oplus \tilde{L}^{*}_{pt}$. We take $p \in L \cap L_0(v)$. We will define a frame $\mathbf{e}_{\tilde{p}}$ whose value at (v, σ) is $[p]$. Here $\tilde{p}$ is a lift of p to $\hat{L}_{pt}(v)$. Let $(v', \sigma') \in V / \tilde{L}_{pt} \oplus \tilde{L}^{*}_{pt}$ be in a small neighborhood of (v, σ). We find $p' \in L \cap L_0(v')$ and its lift $\tilde{p}'$ which lies in a small neighborhood of p and $\tilde{p}$ respectively. We define

(2.11)
$$e_{\tilde{p}, \sigma}(v', \sigma') = \exp\Big(2\pi \int_{D(\tilde{p}, x_0(v), x_0(v'), \tilde{p}')} \Omega - 2\pi\sqrt{-1}\big(\sigma(x_0(v) - \tilde{p}) + \sigma'(\tilde{p}' - x_0(v')) + \alpha(\tilde{p} - \tilde{p}')\big)\Big).$$

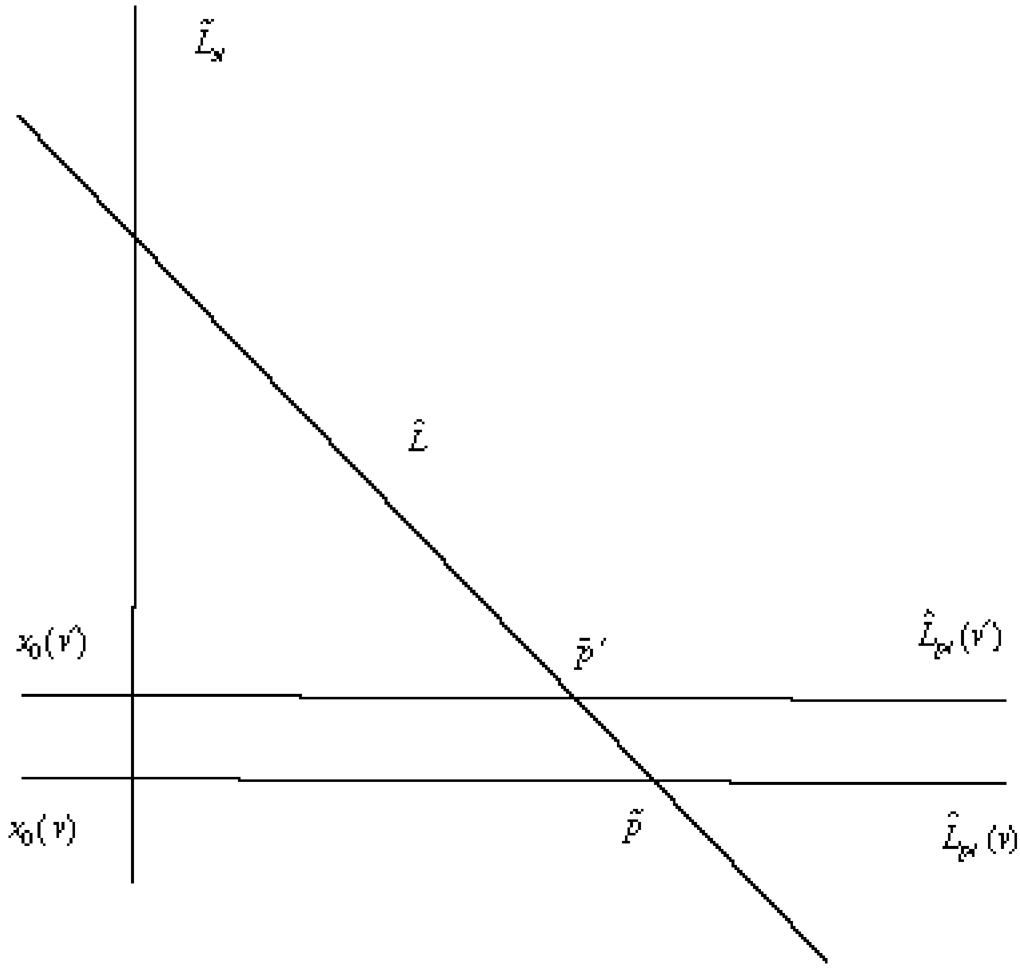

Figure 2

Here $D(\tilde{p}, x_0(v), x_0(v'), \tilde{p}')$ in (2.11) is the union of two triangles $\Delta_{\tilde{p}\, x_0(v)\, x_0(v')}$ and $\Delta_{x_0(v')\, \tilde{p}'\, \tilde{p}}$. Hereafter we write

(2.12) $$Q(a,b,c,d) = \int_{D(a,b,c,d)} \Omega.$$

Using Stokes' theorem we can prove $Q(a,b,c,d) = Q(b,c,d,a)$. We put

(2.13) $$\mathbf{e}_{\tilde{p},\sigma}(v',\sigma') = e_{\tilde{p},\sigma}(v',\sigma')[p'].$$

Lemma 2.14 below implies that $\mathbf{e}_{\tilde{p},\sigma}$ is a section of $\mathcal{E}(L,\alpha)$ in a neighborhood of $(v,\sigma) \in (T^{2n},\Omega)^{\vee}$. If we take $\tilde{p}$ for each $p \in L \cap L_0(v)$, then $\mathbf{e}_{\tilde{p},\sigma}$, $p \in L \cap L_0(v)$ is a local frame of the bundle $\mathcal{E}(L,\beta)$.

Lemma 2.14 *If* $\gamma \in \left(\Gamma \cap \tilde{L}_{pt}\right)$ *and* $\mu \in \left(\Gamma \cap \tilde{L}_{pt}\right)^{\vee}$, *then there exists a holomorphic function* $g(v',\sigma')$ *such that* $\mathbf{e}_{\tilde{p},\sigma}(v',\sigma') = g(v',\sigma')\mathbf{e}_{\tilde{p}+\gamma,\sigma+\mu}(v',\sigma'+\mu)$.

Proof: We put $g(v',\sigma') = e_{\tilde{p},\sigma}(v',\sigma')/e_{\tilde{p}+\gamma,\sigma}(v',\sigma')$. By (2.11), we have

(2.15) $$\log e_{\tilde{p}+\gamma}(v',\sigma') - \log e_{\tilde{p}}(v',\sigma') = -2\pi I_{\gamma}(v'-v,\sigma'-\sigma).$$

Here I_γ is as in (11.5).

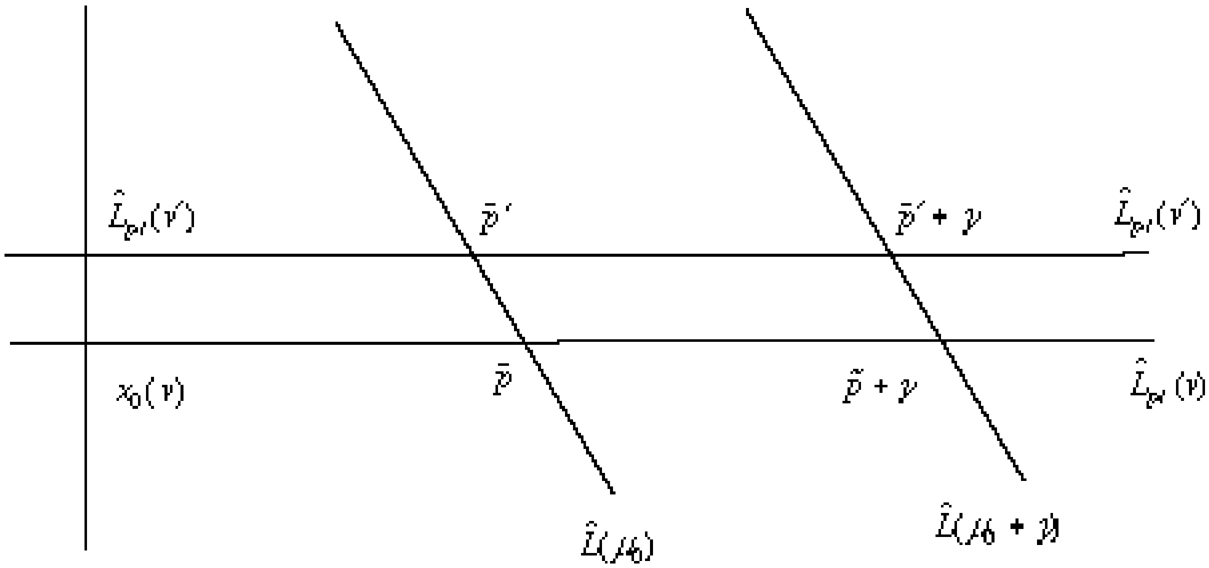

Figure 3

By Lemma 1.17 and the construction of complex structure, (2.15) implies that $g(v',\sigma')$ is a holomorphic function of (v',σ'). On the other hand, we have

$$e_{\tilde p,\sigma+\mu}(v',\sigma'+\mu)\big/ e_{\tilde p,\sigma}(v',\sigma') = \exp\Big(2\pi\sqrt{-1}\mu(\tilde p' - x_0(v') - \tilde p + x_0(v))\Big) = g_{\mu,v}(\tilde p')\, g_{\mu,v}(\tilde p)^{-1}.$$

Hence $\mu\Big(\mathbf{e}_{q,\sigma}(v',\sigma')\Big) = g_{\mu,v}(p)\mathbf{e}_{q,\sigma+\mu}(v',\sigma'+\mu)$. The proof of Lemma 2.14 is now complete.

Lemma 2.14 implies that there exists a unique holomorphic structure on $\mathcal{E}(L,\beta)\to(T^{2n},\Omega)^\vee$ such that $\mathbf{e}_{\tilde p}$ is a local holomorphic section. We thus constructed a holomorphic vector bundle $\mathcal{E}(L,\beta)\to(T^{2n},\Omega)^\vee$.

Proposition 2.16 *If* (L,α) *is Hamiltonian equivalent to* (L',α') *then* $\mathcal{E}(L,\alpha)$ *is isomorphic to* $\mathcal{E}(L',\alpha')$.

Proof: We suppose $(L,\alpha)=(L(w),\alpha)$, $(L',\alpha')=(L(w+\xi),\alpha+\zeta)$. Here $\xi\in\left(\Gamma/\Gamma\cap\tilde L\right)$ and $\zeta\in\left(\Gamma\cap\tilde L\right)^\vee$. Since $L(w+\xi)=L(w)$ it follows that $\mathcal{E}(L(w),\alpha)\cong\mathcal{E}(L(w+\xi),\alpha)$. Choose and fix $y\in L(w)$. Let $p\in L(w)\cap L_{pt}(v)$, $\tilde p\in\hat L(w)\cap\hat L_{pt}(v)$. We let $\tilde y$ be a lift of y in $\hat L(w)$. We define

(2.17) $$\tilde\Psi([p],(L(v),\sigma)) = \Big(\exp\big(2\pi\sqrt{-1}\zeta(\tilde p-\tilde y)\big)[p],(L(v),\sigma)\Big).$$

Here $([p],(L(v),\sigma)) \in \tilde{\mathcal{E}}(L(w),\alpha)_{(v,\sigma)}$. It is straightforward to see that (2.17) is compatible with the actions of $\left(\Gamma/\Gamma \cap \tilde{L}_{pt}\right) \oplus \left(\Gamma \cap \tilde{L}_{pt}\right)^{\vee}$ and is independent of the lift $\tilde{y}$. We can also verify easily that $\tilde{\Psi}$ define an isomorphism $\Psi : \mathcal{E}(L(w),\alpha) \to \mathcal{E}(L(w),\alpha + \zeta)$. Hence Proposition 2.16.

We will prove a converse of Proposition 2.16 in § 7.

Before going further we add several remarks on our construction. First the way we constructed the bundle $\mathcal{E}(L,\alpha)$ is a consequence of the dictionary between symplectic and complex geometry itself. To see this, we first recall that, by the construction of Strominger-Yau-Zaslow, the pair $(L_{pt}(v),\sigma)$ is to correspond to the skyscraper sheaf at the point $(L_{pt}(v),\sigma) \in (T^{2n},\Omega)^{\vee}$. (We write it $\mathcal{E}(L_{pt}(v),\sigma)$.) Namely an n-brane $(L_{pt}(v),\sigma)$ in (T^{2n},Ω) corresponds to a 0- brane in $(T^{2n},\Omega)^{\vee}$. Let $(L,\alpha) \in \mathcal{L}ag(T^{2n},\Omega)$ be another element. Suppose that it corresponds to a sheaf $\mathcal{E}(L,\alpha)$ on $(T^{2n},\Omega)^{\vee}$. Then our dictionary implies

(2.18) $$HF((L,\alpha),(L_{pt}(v),\sigma)) \cong Ext(\mathcal{E}(L,\alpha),\mathcal{E}(L_{pt}(v),\sigma)).$$

In our case, $\mathcal{E}(L,\alpha)$ is a vector bundle. Hence we can identify $Ext(\mathcal{E}(L,\alpha),\mathcal{E}(L_{pt}(v),\sigma))$ to the dual vector space of the fiber of $\mathcal{E}(L,\alpha)$ at $(L_{pt}(v),\sigma) \in (T^{2n},\Omega)^{\vee}$. This was the way we *defined* $\mathcal{E}(L,\alpha)$.

We next explain the reason why we need to fix $\tilde{L}_{st}$ to define $\mathcal{E}(L,\alpha) \to (T^{2n},\Omega)^{\vee}$. Our purpose is to construct a functor

(2.19) $$\mathcal{L}ag(T^{2n},\Omega) \to \mathbf{D}\left((T^{2n},\Omega)^{\vee}\right)$$

so that the Lagrangian submanifolds parallel to $\tilde{L}_{pt}$ are mapped to the skyscraper sheaves. Note that the automorphism group of the category $\mathbf{D}\left((T^{2n},\Omega)^{\vee}\right)$ is rather big. Mukai constructed (see [33], [34]) a symmetry, called Fourier-Mukai transformation. In fact, we can see such a symmetry from mirror symmetry itself. Namely the "mirror" of a Fourier-Mukai transformation (or S-duality) is realized by a symplectic diffeomorphism of (T^{2n},Ω). This phenomenon, that is S-duality will become easier duality in the mirror, is observed by physicists in more general situations and is called the duality of duality.

So there can be many possible ways to construct the functor (2.19). The ambiguity is described by Fourier-Mukai transformation which sends skyscraper sheaves to skyscraper sheaves. If we see such transformation in the mirror (T^{2n},Ω), they are (linear) symplectic diffeomorphisms which preserve $\tilde{L}_{pt}$. For example, if we consider the case when $n=1$, then the group of linear symplectic diffeomorphisms of T^2 is an extension of T^2 by $SL(2,\boldsymbol{Z})$. This group $SL(2,\boldsymbol{Z})$ will become the S-duality group of the mirror $T^{2\vee}$. The element of $SL(2,\boldsymbol{Z})$ which preserves $\boldsymbol{R} \subseteq \boldsymbol{C}$ is a matrix of the form $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. To kill this symmetry we need to fix a direction transversal to $\boldsymbol{R}$. This is equivalent to fix the

Lagrangian submanifold which becomes the structure sheaf of the mirror tori.

We remark that here we put the trivial bundle on L_{st}. But we can in fact put any flat line bundle on L_{st} instead. The change of the choice of the flat bundle on L_{st} (and changing L_{st} to another affine Lagrangian submanifold $L_{st}(v)$ parallel to L_{st}) corresponds to the connected component of the automorphism group of $\mathbf{D}\left((T^{2n},\Omega)^{\vee}\right)$ (that it the group $(T^{2n},\Omega)^{\vee}$ itself) . We defined an action of $\left(\Gamma\cap\tilde{L}_{pt}\right)^{\vee}$ to $\mathcal{E}(L,\alpha)$ in such a way that the bundle $\mathcal{E}(L_{st},\alpha)$ is trivial as a complex vector bundle. Also the construction of the holomorphic structure on $\mathcal{E}(L,\alpha)$ is designed so that $\mathcal{E}(L_{st},0)$ is a trivial as a holomorphic bundle. (Namely $s_{\tilde{p}}(v',\alpha')\equiv 1$ in that case.)

We next define a lift of $\boldsymbol{Z}_2$- degree of the Floer cohomology to $\boldsymbol{Z}$. We first recall the following fact which we mentioned in [12] § 4. The Floer degree $\eta(p)\in\boldsymbol{Z}$ of $p\in L_1\cap L_2$ is not well-defined in the general situation. In general, only the difference $\eta(p)-\eta(q)$ is well defined, (modulo twice of the minimal Chern number. See [38].) In the case of a pair of mutually transversal affine Lagrangian submanifolds $\hat{L}_1$, $\hat{L}_2$ in a torus, $\eta(p)-\eta(q)$ is always zero. But we do not have a canonical way to define $\eta(\hat{L}_1,\hat{L}_2)=\eta(p)$ as an integer. ($\eta(p)\in\boldsymbol{Z}_2$ is well defined.) However, for three affine Lagrangian submanifolds $\hat{L}_i$, the Maslov index (Kashiwara class, see [23]) is well-defined as follows. We choose a complex structure J on V and a Lagrangian linear subspace $\tilde{L}_0$ such that $\tilde{L}_0, J\tilde{L}_0$ are transversal to $\tilde{L}_1$, $\tilde{L}_2$, $\tilde{L}_3$. Using J, we regards $V=T^*\tilde{L}_0$. (Here we regards $J\tilde{L}_0$ as the fiber.) Then $\tilde{L}_i$, $i=1,2,3$ are graphs of exact 1 forms $dV(\tilde{L}_0,\tilde{L}_i)$. Here $V(\tilde{L}_0,\tilde{L}_i)$ is a quadratic functions on $\tilde{L}_i$. Let $\overline{\eta}^*(\tilde{L}_i,\tilde{L}_j)$ be the index of the quadratic form $V(\tilde{L}_0,\tilde{L}_j)-V(\tilde{L}_0,\tilde{L}_i)$. Then we define

$$\eta\left(\hat{L}_1,\hat{L}_2,\hat{L}_3\right)=2n-\left(\overline{\eta}^*\left(\hat{L}_1,\hat{L}_2\right)+\overline{\eta}^*\left(\hat{L}_2,\hat{L}_3\right)+\overline{\eta}^*\left(\hat{L}_3,\hat{L}_1\right)\right) \tag{2.20}$$

and can verify that (2.20) is independent of $\tilde{L}_0$. J.

Remark 2.21 In the general case, $\eta\left(\hat{L}_1,\hat{L}_2,\hat{L}_3\right)$ (more precisely $\eta\left(p_{12},p_{23},p_{31}\right)$ where $p_{ij}\in L_i\cap L_j$) is well-defined modulo twice of the minimal Maslov number. In fact, it is the minus of the virtual dimension of the moduli space of pseudoholomorphic triangles. In our case, minimal Maslov number is 0 since $\pi_2(T^{2n},L)=0$. Hence $\eta\left(\hat{L}_1,\hat{L}_2,\hat{L}_3\right)$ is well-defined as an integer.

In our situation, we already fixed two Lagrangian linear subspaces $\tilde{L}_{pt}$ and $\tilde{L}_{st}$. Using them, we can define $\boldsymbol{Z}$ grading of the Floer homology $HF((L_{st},0),(L,\beta))$. Namely we *define* so that $HF^k((L_{st},0),(L,\alpha))$ is nonzero only if :

$$k=\eta^*(\tilde{L}_{st},\tilde{L}) \tag{2.22}$$

(when $\tilde{L}$ is transversal to $\tilde{L}_{st}$ and $\tilde{L}_{pt}$) and

Definition 2.23 $\eta^*(\tilde{L}_{st},\tilde{L}) = \eta(\tilde{L}_{st},\tilde{L},\tilde{L}_{pt})$.

We are here using the cohomology degree η^* which is related to the homology degree η by $\eta = n - \eta^*$. Let $\tilde{L}_1$, $\tilde{L}_2$ be a pair of mutually transversal affine Lagrangian submanifolds. We assume also that they are transversal to $\tilde{L}_{pt}$ and $\tilde{L}_{st}$.

Definition 2.24 $\eta^*(\tilde{L}_1,\tilde{L}_2) = \eta^*(\tilde{L}_{st},\tilde{L}_2) - \eta^*(\tilde{L}_{st},\tilde{L}_1) + \eta(\tilde{L}_{st},\tilde{L}_1,\tilde{L}_2)$.

Lemma 2.25

(2.26.1) $\eta^*(\tilde{L}_1,\tilde{L}_2) + \eta^*(\tilde{L}_2,\tilde{L}_3) = \eta^*(\tilde{L}_1,\tilde{L}_3) + \eta(\tilde{L}_1,\tilde{L}_2,\tilde{L}_3)$.

(2.26.2) $\eta^*(\tilde{L}_1,\tilde{L}_2) + \eta^*(\tilde{L}_2,\tilde{L}_1) = n$.

Proof: By definition we have

(2.27)
$$\begin{aligned}&\eta^*(\tilde{L}_1,\tilde{L}_2) + \eta^*(\tilde{L}_2,\tilde{L}_3) - \eta^*(\tilde{L}_1,\tilde{L}_3)\\ &= \eta(\tilde{L}_{st},\tilde{L}_1,\tilde{L}_2) + \eta(\tilde{L}_{st},\tilde{L}_2,\tilde{L}_3) - \eta(\tilde{L}_{st},\tilde{L}_1,\tilde{L}_3).\end{aligned}$$

(2.20) and $\bar{\eta}^*(\tilde{L}_i,\tilde{L}_j) = n - \bar{\eta}^*(\tilde{L}_j,\tilde{L}_i)$ implies :

(2.28)
$$\eta(\tilde{L}_{st},\tilde{L}_1,\tilde{L}_2) + \eta(\tilde{L}_{st},\tilde{L}_2,\tilde{L}_3) = \eta(\tilde{L}_1,\tilde{L}_2,\tilde{L}_3) + \eta(\tilde{L}_{st},\tilde{L}_1,\tilde{L}_3).$$

(2.27) and (2.28) imply (2.26.1).

To prove (2.26.2) we recall

(2.29.1)
$$\eta(\tilde{L}_{st},\tilde{L}_1,\tilde{L}_2) = 2n - \left(\bar{\eta}^*(\tilde{L}_{st},\tilde{L}_1) + \bar{\eta}^*(\tilde{L}_1,\tilde{L}_2) + \bar{\eta}^*(\tilde{L}_2,\tilde{L}_{st})\right),$$

(2.29.2)
$$\eta(\tilde{L}_{st},\tilde{L}_2,\tilde{L}_1) = 2n - \left(\bar{\eta}^*(\tilde{L}_{st},\tilde{L}_2) + \bar{\eta}^*(\tilde{L}_2,\tilde{L}_1) + \bar{\eta}^*(\tilde{L}_1,\tilde{L}_{st})\right).$$

(2.29), $\bar{\eta}^*(\tilde{L}_i,\tilde{L}_j) = n - \bar{\eta}^*(\tilde{L}_j,\tilde{L}_i)$, and definition imply

$$\eta^*(\tilde{L}_1,\tilde{L}_2) + \eta^*(\tilde{L}_2,\tilde{L}_1) = \eta(\tilde{L}_{st},\tilde{L}_1,\tilde{L}_2) + \eta(\tilde{L}_{st},\tilde{L}_2,\tilde{L}_1) = n.$$

The proof of Lemma 2.25 is complete.

Let us explain the reason why we defined $\eta^*(\tilde{L}_1,\tilde{L}_2)$ as in Definitions 2.23 and 2.24. It again comes from our dictionary. The dictionary requires :

(2.30)
$$HF^k((L_{st},0),(L_{pt}(v),\sigma)) \cong Ext^k(\mathcal{O},\mathcal{E}(L_{pt}(v),\sigma)).$$

Here $\mathcal{O}$ is the structure sheaf and $\mathcal{E}(L_{pt}(v),\sigma)$ is a skyscraper sheaf. It is easy to see that the right hand side of (2.30) is nonzero only for $k=0$. Therefore, we need to choose

(2.31)
$$\eta^*(\tilde{L}_{st},\tilde{L}_{pt}) = 0.$$

On the other hand, since $\mathcal{E}(L,\alpha)$ is locally free (in the case when $\tilde{L}$ is transversal to $\tilde{L}_{pt}$), it follows that $HF^k((L,\alpha),(L_{pt}(v),\sigma)) \cong Ext^k(\mathcal{E}(L,\alpha),\mathcal{E}(L_{pt}(v),\sigma))$ is nonzero only for $k=0$. Therefore we choose

(2.32) $$\eta^*(\tilde{L},\tilde{L}_{pt}) = 0\,.$$

We also require (2.26.2). ((2.26.2) is the mirror of Serre duality. (See Remark 3.3.))

Definitions 2.23 and 2.24 follow from (2.26),(2.31) and (2.32).

We remark that the mod 2 degree of Floer homology is canonically defined *provided the Lagrangian submanifolds are oriented.* We have chosen orientations of $\tilde{L}_{st}$ and $\tilde{L}_{pt}$ so that the intersection number $L_{st} \bullet L_{pt}(v)$ is *plus* 1. For third Lagrangian subspace $\tilde{L}$ we *choose* its orientation so that the intersection number $L \bullet L_{pt}(v)$ is positive. Hence $\eta_*(\tilde{L},\tilde{L}_{pt}) = 0$ is consistent with mod 2 degree of Floer homology.

The vector bundle we constructed in this section is a semi-homogeneous vector bundle in the sense of [32]. In fact, it is obtained from a line bundle by push forward. (See § 4.) More general bundles and sheaves will be discussed in §§ 8,12.

§ 3 Sheaf cohomology and Floer cohomology 1 (Construction of a homomorphism)

Theorem 3.1 $HF^k((L_{st},0),(L,\alpha)) \cong H^k((T^{2n},\Omega)^\vee, \mathcal{E}(L,\alpha))$ *if* $\tilde L$ *is transversal to* $\tilde L_{st}, \tilde L_{pt}$.

In §§ 3,5, we are mainly concern with the case when $k = \eta(\tilde L_{st}, \tilde L, \tilde L_{pt}) = 0$. In that case, we will construct an explicit map $HF^0((L_{st},0),(L,\alpha)) \to H^0((T^{2n},\Omega)^\vee, \mathcal{E}(L,\alpha))$ in this section and will prove that it is an isomorphism in §5. The main idea of the proof is again to use our dictionary itself to associate a section of the bundle $\mathcal{E}(L,\alpha)$ to each element of $HF^0((L_{st},0),(L,\alpha))$. We first remark that (2.26.2) implies

$$CF^k((L,\alpha),(L',\alpha')) \cong CF^{n-k}((L',\alpha'),(L,\alpha)).$$

The boundary operators are zero in our case. But in fact it will be dual in the general situation. Hence we have a perfect bilinear pairing (3.2) which will be denoted by $\langle , \rangle$.

(3.2) $$HF^k((L,\alpha),(L',\alpha')) \otimes HF^{n-k}((L',\alpha'),(L,\alpha)) \to \mathbf{C}.$$

Remark 3.3 In B-model, the pairing (3.2) corresponds to Serre duality as follows. Let $\mathcal{F}, \mathcal{G}$ be two locally free sheaves on a Kähler manifold M. Then we have a perfect pairing :

$$Ext^k(\mathcal{F},\mathcal{G}) \otimes Ext^{n-k}(\mathcal{G}, \mathcal{F} \otimes \mathcal{O}(\Lambda^n TM)) \to \mathbf{C}$$

Here $\mathcal{O}(\Lambda^n TM)$ is the sheaf of holomorphic n- forms. If we assume that M is a Calabi-Yau manifold then $\mathcal{O}(\Lambda^n TM)$ is trivial. Hence we have a pairing similar to (3.2). They correspond to each other by mirror symmetry. We will prove it in the case of tori in § 11. (Theorem 11.40).

We recall that $[p]$, $p \in L_{st} \cap L$ is a basis of $HF^0((L_{st},0),(L,\alpha))$. So we are to going to define a section $s_p \in \Gamma((T^{2n},\Omega)^\vee, \mathcal{E}(L,\alpha))$ for each $p \in L_{st} \cap L$. Let $[v,\sigma] \in (T^{2n},\Omega)^\vee$. We will define the value $s_p(v,\sigma) \in \mathcal{E}(L,\alpha)_{[v,\sigma]}$. Here $\mathcal{E}(L,\alpha)_{[v,\sigma]}$ is the fiber of $\mathcal{E}(L,\alpha)$ at (v,σ). By definition we have :

(3.4) $$\mathcal{E}(L,\alpha)^*_{[v,\sigma]} = HF^0((L,\alpha),(L_{pt}(v),\sigma)).$$

Here $*$ denotes the dual vector space. Hence $s_p(v,\sigma) \in \left(HF^0((L,\alpha),(L_{pt}(v),\sigma))\right)^*$.

We recall that two Lagrangian submanifolds L_{st} and $L_{pt}(v)$ intersect to each other at a unique point $x(v) \in T^{2n}$. Hence we have a canonical element

(3.5) $$[x(v)]\in HF^n((L_{pt}(v),\sigma),(L_{st},0))\cong HF^0((L_{st},0),(L_{pt}(v),\sigma))^*.$$

Definition 3.6 $$\begin{aligned} s_p(v,\sigma) &= m_2([x(v)],[p])\in HF^n((L_{pt}(v),\sigma),(L,\alpha)) \\ &\cong HF^0((L,\alpha),(L_{pt}(v),\sigma))^*\cong \mathcal{E}(L,\alpha)_{[v,\sigma]}. \end{aligned}$$

The map $m_2: HF^n((L_{pt}(v),\sigma);(L_{st},0))\otimes HF^0((L_{st},0),(L,\alpha))\to HF^n((L_{pt}(v),\sigma),(L,\alpha))$ in Definition 3.6 is the product structure of Floer homology which will be defined later in this section. The more precise statement of Theorem 3.1 in the case $k=0$ is the following Theorems 3.7 and 3.8.

Theorem 3.7 s_p *defined in Definition* 3.6 *is holomorphic.*

Hence we obtain a map $HF^0((L_{st},0),(L,\alpha))\to H^0((T^{2n},\Omega)^\vee;\mathcal{E}(L,\alpha))$ $[p]\mapsto s_p$, which we denote by $\Phi_{(L,\alpha)}$.

Theorem 3.8 *The map* $\Phi_{(L,\alpha)}$ *is an isomorphism* $HF^0((L_{st},0),(L,\alpha))\cong H^0((T^{2n},\Omega)^\vee,\mathcal{E}(L,\alpha))$ *in case* $\eta(\tilde L_{st},\tilde L,\tilde L_{pt})=0$.

To prove Theorem 3.7, we begin with the definition of m_2. The definition is similar to the proof of [12] Theorem 4.37. We need a modification since we include a line bundle. Let $L_1,\ L_2,\ L_3$ be three mutually transversal affine Lagrangian submanifolds and $\hat L_i\subseteq V$ be connected components of there inverse images. Let $\{\bar a_i\}=L_1\cap L_2$, $\{\bar b_j\}=L_2\cap L_3$, $\{\bar c_k\}=L_3\cap L_1$. Let $\mathcal{L}_i$ be a flat line bundle on L_i. We find

(3.9) $$\begin{aligned} HF((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)) &= \bigoplus_i Hom(\mathcal{L}_{1,\bar a_i},\mathcal{L}_{2,\bar a_i}) \\ HF((L_2,\mathcal{L}_2),(L_3,\mathcal{L}_3)) &= \bigoplus_j Hom(\mathcal{L}_{2,\bar b_j},\mathcal{L}_{3,\bar b_j}) \\ HF((L_1,\mathcal{L}_1),(L_3,\mathcal{L}_3)) &= \bigoplus_k Hom(\mathcal{L}_{1,\bar c_k},\mathcal{L}_{3,\bar c_k}) \end{aligned}$$

where $\mathcal{L}_{1,\bar a_i}$ is the fiber of $\mathcal{L}_1$ at $\bar a_i$. For $\gamma\in \mathbf{Z}^n=\pi_1(L_3)$, let $\{a(\gamma)\}=\hat L_1\cap\hat L_2(\gamma)$, $\{b(\gamma)\}=\hat L_2(\gamma)\cap\hat L_3$. We choose $\hat L_1,\ \hat L_3$ such that $\{c\}=\hat L_1\cap\hat L_3$ $\pi(c)=\bar c_k$. (Figure 4.) We put

(3.10) $$Q(a,b,c)=\int_{\Delta_{abc}}\left(\omega+\sqrt{-1}B\right)$$

where Δ_{abc} is the geodesic triangle in $\mathbf{C}^n$ whose vertices are a,b,c.

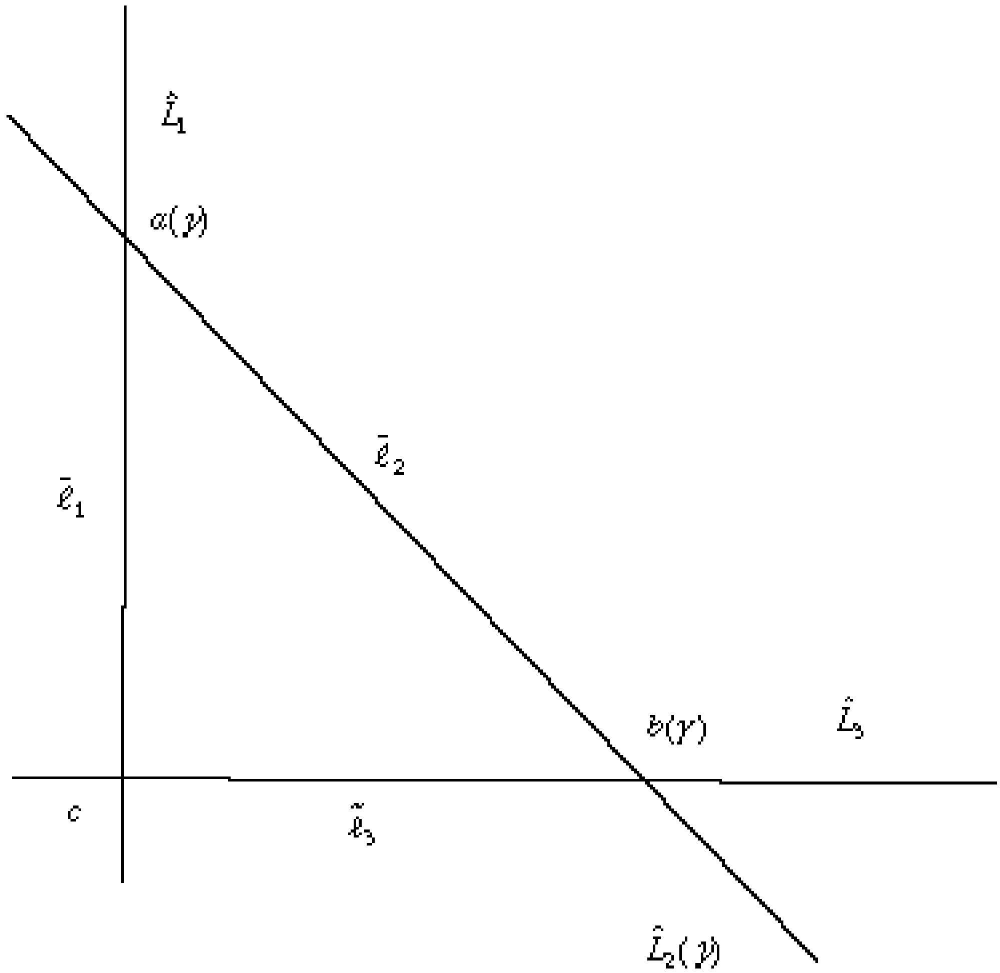

Figure 4.

Definition 3.11

$$Z_{ijk}(L_1, L_2, L_3)(v_{12} \otimes v_{23}) = \sum_{\substack{\pi(a(\gamma)) = \bar{a}_i \\ \pi(b(\gamma)) = \bar{b}_j}} \pm \exp\left(-2\pi Q(a(\gamma), b(\gamma), c)\right) P_{(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3), ijk, \gamma}(v_{12} \otimes v_{23}).$$

Here $v_{12} \in Hom(\mathcal{L}_{1,\bar{a}_i}, \mathcal{L}_{2,\bar{a}_i})$, $v_{23} \in Hom(\mathcal{L}_{2,\bar{b}_j}, \mathcal{L}_{3,\bar{b}_j})$. We explain the sign later during the proof of Theorem 3.16. We define

$$P_{(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3), ijk, \gamma} : Hom(\mathcal{L}_{1,\bar{a}_i}, \mathcal{L}_{2,\bar{a}_i}) \otimes Hom(\mathcal{L}_{2,\bar{b}_j}, \mathcal{L}_{3,\bar{b}_j}) \to Hom(\mathcal{L}_{1,\bar{c}_k}, \mathcal{L}_{3,\bar{c}_k})$$

below. Let $v_{12} \in Hom(\mathcal{L}_{1,\bar{a}_i}, \mathcal{L}_{2,\bar{a}_i})$, $v_{23} \in Hom(\mathcal{L}_{2,\bar{b}_j}, \mathcal{L}_{3,\bar{b}_j})$. We choose path $\tilde{\ell}_i$ such that $\tilde{\ell}_2 \subseteq \hat{L}_2(\gamma)$, $\tilde{\ell}_i \subseteq \hat{L}_i$ $(i = 1,3)$ and that $\tilde{\ell}_1, \tilde{\ell}_2, \tilde{\ell}_3$ joins c to $a(\gamma)$, $a(\gamma)$ to $b(\gamma)$, and $b(\gamma)$ to c respectively. (See Figure 4.) We put $\ell_i = \pi \circ \tilde{\ell}_i$. Then ℓ_1, ℓ_2, ℓ_3 joins $\bar{c}_k$ to $\bar{a}_i$, $\bar{a}_i$ to $\bar{b}_j$, and $\bar{b}_j$ to $\bar{c}_k$ respectively. We now define :

$$\text{(3.12)} \qquad P_{(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3), ijk, \gamma}(v_{12} \otimes v_{23}) = P_{\ell_3, \mathcal{L}_3} \circ v_{23} \circ P_{\ell_2, \mathcal{L}_2} \circ v_{12} \circ P_{\ell_1, \mathcal{L}_1}.$$

Here $P_{\ell_1,\mathcal{L}_1} : \mathcal{L}_{1,\bar{c}_k} \to \mathcal{L}_{1,\bar{a}_i}$ is the parallel transportation of the flat bundle $\mathcal{L}_1$ along the path ℓ_1. The definitions of $P_{\ell_2,\mathcal{L}_2}$, $P_{\ell_3,\mathcal{L}_3}$ are similar. (A similar but a bit more complicated construction is used in [14] § 1. In fact, the one we are discussing here is the genus 0 version of the construction there.) We thus explained the notation of Definition 3.9. We also remark the following lemma.

Lemma 3.13 *If* $\eta(L_1, L_2, L_3) = 0$ *then the right hand side of Definition* 3.11 *converges.*

The proof is similar to the proof of [12] Theorem 4.12. (Note that the difference between the construction in this section and one in [12] § 4, is only the phase factor, that is multiplication of complex numbers of absolute value one.)

By Definition 3.9 and (3.7) we obtain

$$m_2 : HF((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)) \otimes HF((L_2,\mathcal{L}_2),(L_3,\mathcal{L}_3)) \to HF((L_1,\mathcal{L}_1),(L_3,\mathcal{L}_3)), \tag{3.14}$$

in the case L_i are transversal to each other. We state the following basic properties of m_2.

Theorem 3.15 *Let* $x_{ij} \in HF((L_i,\mathcal{L}_i),(L_j,\mathcal{L}_j))$ *then*

$$m_2(x_{12}, m_2(x_{23}, x_{34})) = m_2(m_2(x_{12}, x_{23}), x_{34}).$$

The proof is the same as the proof of [12] "Theorem 5.1", which is rigorous in the case of a torus.

Theorem 3.16 *Let* $x_{ij} \in HF((L_i,\mathcal{L}_i),(L_j,\mathcal{L}_j))$. *If* $\deg x_{12} + \deg x_{23} + \deg x_{31} = n$ *then we have*

$$\begin{aligned} \langle m_2(x_{12}, x_{23}), x_{31}\rangle &= (-1)^{(\deg x_{12} + \deg x_{23})\deg x_{31}} \langle m_2(x_{31}, x_{12}), x_{23}\rangle \\ &= (-1)^{(\deg x_{31} + \deg x_{12})\deg x_{23}} \langle m_2(x_{23}, x_{31}), x_{12}\rangle. \end{aligned}$$

The proof is the same as the proof of [12] Theorem 4.8. Let us explain the sign here. We choose $\tilde{L}_0$ and J, such that $\tilde{L}_0$ and $J\tilde{L}_0$ are transversal to $\tilde{L}_i$. We then regards $\tilde{L}_j$ as a graph of $dV(\tilde{L}_0, \tilde{L}_j)$ where $V(\tilde{L}_0, \tilde{L}_j)$ is a quadratic function and put $V(\tilde{L}_i, \tilde{L}_j) = V(\tilde{L}_0, \tilde{L}_j) - V(\tilde{L}_0, \tilde{L}_i)$. We fix an orientation of unstable manifold $U(\tilde{L}_i, \tilde{L}_j)$ (negative eigenspace) of $V(\tilde{L}_i, \tilde{L}_j)$. We find that $U(\tilde{L}_1, \tilde{L}_2) \cap U(\tilde{L}_2, \tilde{L}_3) \cap U(\tilde{L}_3, \tilde{L}_1)$ is $\{0\}$. Orientations of $U(\tilde{L}_i, \tilde{L}_j)$ and one of V determines a sign of the intersection $U(\tilde{L}_1, \tilde{L}_2) \cap U(\tilde{L}_2, \tilde{L}_3) \cap U(\tilde{L}_3, \tilde{L}_1)$. This sign by definition is the sign in Definition 3.11. (We discuss the sign more in § 10.) The sign in Theorem 3.16 is then immediate.

We now start the proof of Theorem 3.7. For this purpose, we write more explicitly the map in Definition 3.11 in the case when $(L_1,\mathcal{L}_1) = (L_{st}, 0)$, $(L_2,\mathcal{L}_2) = (L, \alpha)$, $(L_3,\mathcal{L}_3) = (L_{pt}(v), \sigma)$. We remark that $L_{st} \cap L_{pt}(v) = \{x(v)\}$ consists of one point. Hence there is only one choice of k that is $k=1$. (Namely we put $c=x(v)$.) We are considering $p\in L_{st}\cap L$. We put $p=\bar{a}_j$. To prove Theorem 3.7 we only need to study s_p locally. We fix $q\in L_{pt}(v)\cap L$. Then for each w close to v, we have $q'\in L_{pt}(w)\cap L$ close to q. We recall $\mathcal{E}(L,\alpha)_{[v,\tau]}=\oplus_{x\in L_{pt}(v)\cap L}\boldsymbol{C}[x]$. Using this basis we put :

$$s_p([v',\sigma'])=s_{p,q}([v',\sigma'])[q']^*+\cdots$$

and study $s_{p,q}([v',\sigma'])$. Here we regards $s_p\in HF^n((L(v),\tau),(L,\alpha))$ and regard $[q']^*$ as the dual basis of the basis $[q']$ of $HF^0((L,\alpha),(L(v),\tau))$.)

We fix a component $\hat{L}$ of the inverse image of L in V. Let $\Gamma(\hat{L})=\Gamma\big/\Gamma\cap\tilde{L}$. ($\tilde{L}$ is the linear subspace of V parallel to $\hat{L}$.) For $\gamma\in\Gamma(\hat{L})$, we put $\hat{L}(\gamma)=\hat{L}+\gamma$ and $\{\tilde{p}(\gamma)\}=\tilde{L}_{st}\cap\hat{L}(\gamma)$. Set

$$\Gamma_0(\hat{L})=\left\{\gamma\in\Gamma(\hat{L})\,\middle|\,\pi(\tilde{p}(\gamma))=p,\pi(\tilde{q}(\gamma))=q\right\}.$$

Here $\pi:V\to T^{2n}$ is the projection. We remark that $\pi(\tilde{q}'(\gamma))=q'$ for $\gamma\in\Gamma_0(\hat{L})$. Definitions 3.6 and 3.11 imply (See Figure 5)

$$\begin{aligned}s_{pq}(v',\sigma')=\sum_{\gamma\in\Gamma_0(\hat{L})}\exp\Big(&-2\pi Q(\tilde{p}(\gamma)\,\tilde{q}'(\gamma)\,x_0(v'))+\\&2\pi\sqrt{-1}\big(\sigma'(x_0(v')-\tilde{q}'(\gamma))+\alpha(\tilde{q}'(\gamma)-\tilde{p}(\gamma))\big)\Big)\end{aligned}\tag{3.17}$$

Let $\sum_\gamma s_{p,q}(v',\sigma',\gamma)$ be the right hand sides of (3.17). We put

$$g_\gamma(v',\sigma')=\log s_{pq}(v',\sigma',\gamma)-\log s_{pq}(v,\sigma,\gamma)-2\pi\log e_{\tilde{p}}(v',\sigma').\tag{3.18}$$

Here $e_{\tilde{p}}(v',\sigma')$ is defined by (2.11).

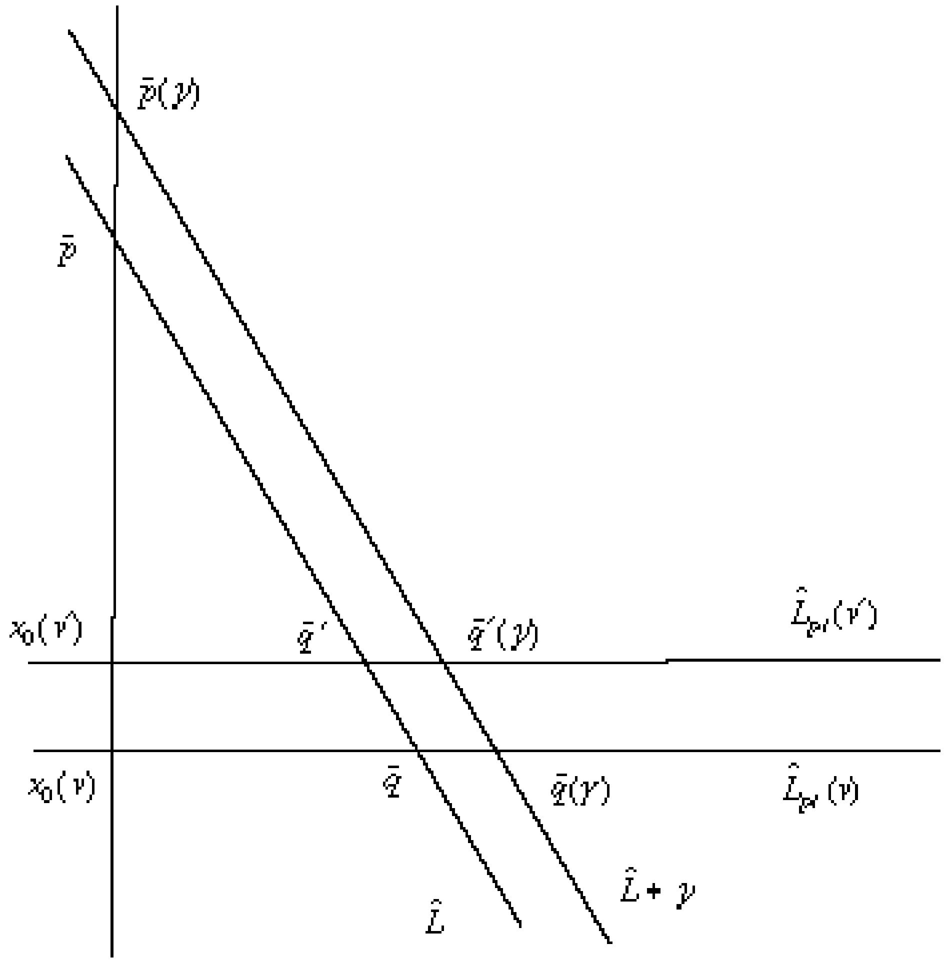

Figure 5

Lemma 3.19 *$g_\gamma(v',\sigma')$ is a holomorphic function of (v',σ').*

Proof: By definition and (2.11), we have :

$$
\begin{aligned}
g_\gamma(v',\sigma') &= 2\pi Q(x_0(v), x_0(v'), \tilde{q}'(\gamma), \tilde{q}(\gamma)) \\
&\quad -2\pi\sqrt{-1}\Big(\sigma(x_0(v)-\tilde{q}(\gamma)) + \sigma'(\tilde{q}'(\gamma)-x_0(v')) \\
&\qquad + \alpha(\tilde{q}(\gamma)-\tilde{q}'(\gamma))\Big) \\
&\quad -\log e_{\tilde{q},\sigma}(v',\sigma') \\
&= -2\pi I_\gamma(v'-v, \sigma'-\sigma)
\end{aligned}
$$

Here $Q(x_0(v), x_0(v'), \tilde{q}'(\gamma), \tilde{q}(\gamma)))$ is as in (2.12) (see Figure 5), and $\gamma = \tilde{q}(\gamma) - \tilde{q} = \tilde{q}'(\gamma) - \tilde{q}'$. Lemma 3.19 follows.

Lemma 3.19, (3.17),(3.18) and the definitions imply that s_p is holomorphic. The proof of Theorem 3.7 is complete.

§ 4 Isogeny

In this section we use the idea of Polishchuk and Zaslow in [42] §5.3 to reduce the proof of Theorems 3.1 to the case of line bundles. Let (L,α) be as in § 3. We remark that the rank of the vector bundle $\mathcal{E}(L,\alpha)$ is $L \bullet L_{pt}(v)$. Hence there exists a finite group $G(L) \subseteq L_{pt}(0) \subseteq T^{2n}$ with the following property :

(4.1.1) The order of $G(L)$ is $L \bullet L_{pt}(v)$.
(4.1.2) L is $G(L)$ invariant.
(4.1.3) $G(L)$ acts transitively on $L \cap L_{pt}(v)$.

Let G be a subgroup of $G(L)$. We put $(T^{2n},\Omega)/G = (\overline{T}^{2n},\overline{\Omega})$. We use $\tilde{L}_{st} \subseteq V =$ the universal cover of $\overline{T}^{2n}$ to define a mirror $(\overline{T}^{2n},\overline{\Omega})^{\vee}$. Let $G^{\vee} = Hom(G,U(1))$ be the dual group.

Lemma 4.2 *$G^{\vee}$ acts on $(\overline{T}^{2n},\overline{\Omega})^{\vee}$ such that $(\overline{T}^{2n},\overline{\Omega})^{\vee}/G^{\vee} = (T^{2n},\Omega)^{\vee}$.*

Proof: The universal cover of $(\overline{T}^{2n},\overline{\Omega})^{\vee}$ is identified to the universal cover $V/\tilde{L}_{pt} \oplus \tilde{L}^{*}_{pt}$ of $(\overline{T}^{2n},\overline{\Omega})^{\vee}$. We remark that $\Gamma' = \pi_1(\overline{T}^{2n},\overline{\Omega})$ contains $\Gamma = \pi_1(T^{2n},\Omega)$ as an index $\# G$ subgroup. It is easy to see $\Gamma'/\Gamma' \cap \tilde{L}_{pt} \cong \Gamma/\Gamma \cap \tilde{L}_{pt}$, $\Gamma' \cap \tilde{L}_{pt}/\Gamma \cap \tilde{L}_{pt} \cong G$. Hence $\left(\Gamma \cap \tilde{L}_{pt}\right)^{\vee} / \left(\Gamma' \cap \tilde{L}_{pt}\right)^{\vee} = G^{\vee}$. Lemma 4.2 follows.

By (4.1.2), there exists a Lagrangian submanifold $\overline{L} = L/G$ of $(\overline{T}^{2n},\overline{\Omega})$. There is a flat connection $\overline{\alpha}$ on $\overline{L}$ such that $\pi^{*}\overline{\alpha} = \alpha$. Here $\pi : L \to \overline{L}$ is the covering map. (4.1.3) implies that $\left|\overline{L} \bullet \overline{L}_{pt}\right| = \# G(L)/G$. Hence rank $\mathcal{E}(\overline{L},\overline{\alpha}) = \# G(L)/G$. Let $\pi : (\overline{T}^{2n},\overline{\Omega})^{\vee} \to (T^{2n},\Omega)^{\vee}$ be the $G^{\vee}$ covering constructed by Lemma 4.2.

Proposition 4.3 *There exists an isomorphism $\pi_*\left(\mathcal{E}(\overline{L},\overline{\alpha})\right) \cong \mathcal{E}(L,\alpha)$. Here π_* is the push forward of the bundle.*

Proof: We put

(4.4) $$A(\alpha) = \left\{\alpha + \gamma \;\middle|\; \gamma \in \left(\Gamma \cap \tilde{L}_{pt}\right)^{\vee}\right\}$$

(4.5) $$B(v) = \left\{ p \in \hat{L}_{pt}(v) \;\middle|\; \pi(p) = L_{pt}(v) \cap L \right\}.$$

Let $\hat{E}(\alpha, v)$ be the vector space consisting of all maps $u(\lambda, p)$: $A(\alpha) \times B(v) \to \mathbf{C}$. For $(\gamma,\mu) \in \left(\Gamma'/\Gamma' \cap \tilde{L}\right) \times \left(\Gamma \cap \tilde{L}_{pt}\right)^{\vee}$, we put

(4.6) $$\big((\gamma,\mu)u\big)(\lambda+\mu,\gamma+p)=\exp\Big(2\pi\sqrt{-1}\big(\mu(p-x_0(v))+\lambda(\gamma)\big)\Big)u(\lambda,p).$$

(4.6) defines actions of $\left(\Gamma'/\Gamma'\cap\tilde{L}\right)\times\left(\Gamma'\cap\tilde{L}_{pt}\right)^{\vee}$ and of $\left(\Gamma/\Gamma\cap\tilde{L}\right)\times\left(\Gamma\cap\tilde{L}_{pt}\right)^{\vee}$. The definition of $\mathcal{E}(L,\alpha)$ (see (2.8)) implies the following :

(4.7) $$\pi_*\left(\mathcal{E}(\bar{L},\bar{\alpha})\right)_{(L_{pt}(v),\mathcal{L})}\cong\left\{u\in\hat{E}(\alpha,v)\;\middle|\;\forall(\gamma,\mu)\in\left(\Gamma'/\Gamma'\cap\tilde{L}\right)\times\left(\Gamma'\cap\tilde{L}_{pt}\right)^{\vee}(\gamma,\mu)u=u\right\},$$

(4.8) $$\mathcal{E}(L,\alpha)_{(L_{pt}(v),\mathcal{L})}\cong\left\{u\in\hat{E}(\alpha,v)\;\middle|\;\forall(\gamma,\mu)\in\left(\Gamma/\Gamma\cap\tilde{L}\right)\times\left(\Gamma\cap\tilde{L}_{pt}\right)^{\vee}(\gamma,\mu)u=u\right\}.$$

We are going to construct an isomorphism between (4.7) and (4.8) by a Fourier transformation. Let γ_i be the representative of $\left(\Gamma'/\Gamma'\cap\tilde{L}\right)/\left(\Gamma/\Gamma\cap\tilde{L}\right)$ and μ_j be the representatives of $\left(\Gamma\cap\tilde{L}_{pt}\right)^{\vee}/\left(\Gamma'\cap\tilde{L}_{pt}\right)^{\vee}$. For $u\in\mathcal{E}(L,\alpha)_{(L_{pt}(v),\mathcal{L})}$, $u'\in\pi_*\left(\mathcal{E}(\bar{L},\bar{\alpha})\right)_{(L_{pt}(v),\mathcal{L})}$ we put :

(4.9) $$\mathcal{F}(u)(\lambda,p)=\sum_i u(\lambda,\gamma_i+p).$$

(4.10) $$\mathcal{F}'(u')(\lambda,p)=\sum_j\exp\Big(-2\pi\sqrt{-1}\mu_j(p-x_0(v))\Big)u'(\lambda+\mu_j,p).$$

Lemma 4.11 $\mathcal{F}(u)\in\pi_*\left(\mathcal{E}(\bar{L},\bar{\alpha})\right)_{(L_{pt}(v),\mathcal{L})}$, $\mathcal{F}'(u')\in\mathcal{E}(L,\alpha)_{(L_{pt}(v),\mathcal{L})}$, $\mathcal{F}'\mathcal{F}(u)=u$, $\mathcal{F}\mathcal{F}'(u')=u'$.

Proof: It is easy to see $\mathcal{F}(u)(\lambda,\gamma'+p)=\mathcal{F}(u)(\lambda,p)$ for $\gamma'\in\Gamma'$. Let $\mu'\in\left(\Gamma'\cap\tilde{L}_{pt}\right)^{\vee}$. We have

$$\begin{aligned}\mathcal{F}(u)(\mu'+\lambda,p)&=\sum_i u(\mu'+\lambda,\gamma_i+p)\\&=\sum_i\exp\Big(2\pi\sqrt{-1}\mu'(\gamma_i+p-x_0(v))\Big)u(\lambda,\gamma_i+p)\\&=\exp\Big(2\pi\sqrt{-1}\mu'(p-x_0(v))\Big)\mathcal{F}(u)(\lambda,p).\end{aligned}$$

Therefore $\mathcal{F}(u)\in\pi_*\left(\mathcal{E}(\bar{L},\bar{\alpha})\right)_{(L_{pt}(v),\mathcal{L})}$. The proof of $\mathcal{F}'(u')\in\mathcal{E}(L,\alpha)_{(L_{pt}(v),\mathcal{L})}$ is similar.

On the other hand, we have

(4.12) $$\begin{aligned}\mathcal{F}'\mathcal{F}(u)(\lambda,p)&=\sum_{i,j}\exp\Big(-2\pi\sqrt{-1}\mu_j(p-x_0(v))\Big)u(\lambda+\mu_j,p+\gamma_i)\\&=\sum_{i,j}\exp\Big(2\pi\sqrt{-1}\mu_j(\gamma_i)\Big)u(\lambda,p+\gamma_i)\end{aligned}.$$

We remark that we may choose γ_i and μ_j so that $\gamma_1=0$, $\mu_1=0$ and

(4.13) $$\sum_j\exp\Big(2\pi\sqrt{-1}\mu_j(\gamma_i)\Big)=0\ \text{ unless }\ i=1.\quad\sum_i\exp\Big(2\pi\sqrt{-1}\mu_j(\gamma_i)\Big)=0\ \text{ unless }\ j=1.$$

(4.12) and (4.13) imply $\mathcal{J}'\mathcal{J}(u) = u$. The proof of $\mathcal{J}\mathcal{J}'(u') = u'$ is similar. The proof of Lemma 4.11 is complete.

It is easy to see that $\mathcal{J}$, $\mathcal{J}'$ give isomorphisms asserted by Proposition 4.3.

Proposition 4.14 $H^*((\overline{T}^{2n}, \overline{\Omega})^\vee, \mathcal{E}(\overline{\mathcal{L}}, \overline{\alpha})) \cong H^*((T^{2n}, \Omega)^\vee, \mathcal{E}(L, \alpha))$.

Proof: It suffices to prove the proposition in the case when $G = G(L)$. Then $\mathcal{E}(\overline{L}, \overline{\mathcal{L}})$ is a line bundle. Therefore $H^k((\overline{T}^{2n}, \overline{\Omega})^\vee, \mathcal{E}(\overline{\mathcal{L}}, \overline{\alpha}))$ is nontrivial for only one k. (See [35] § 16.) Proposition 4.14 follows from this fact, Proposition 4.3 and Leray spectral sequence.

We next discuss the relation between the isomorphism in Proposition 4.14 and the homomorphism we constructed in § 3. Since $G \subseteq L_{pt}$ it follows that $L_{st} \bullet L = \overline{L}_{st} \bullet \overline{L}$. Here $\overline{L}_{st} = \tilde{L}_{st} / \Gamma' \subseteq \overline{T}^{2n}$. Moreover we can identify $L_{st} \cap L$ and $\overline{L}_{st} \cap \overline{L}$. Hence there exists a canonical isomorphism

(4.15) $$HF(L_{st}, L) \cong HF(\overline{L}_{st}, \overline{L}).$$

Lemma 4.16 *The following diagram commutes :*

$$\begin{array}{ccc} HF^0(L_{st}, L) & \rightarrow & H^0((\overline{T}^{2n}, \overline{\Omega})^\vee, \mathcal{E}(\overline{L}, \overline{\alpha})) \\ \downarrow & & \downarrow \\ HF^0(\overline{L}_{st}, \overline{L}) & \rightarrow & H^0((T^{2n}, \Omega), \mathcal{E}(L, \alpha)) \end{array}$$

Here the vertical arrows are the isomorphisms (4.15) *and Proposition* 4.14. *The horizontal arrow is the map* $\Phi_{(L,\alpha)}$, $\Phi_{(\overline{L},\overline{\alpha})}$ *in Definition* 3.4.

Proof: We remark that the isomorphism in Proposition 4.14 is given by $\mathcal{J}$ in (4.9). The lemma then follows immediately from the definitions.

We discuss isogeny more in § 6.

§ 5 Sheaf cohomology and Floer cohomology 2 (Proof of isomorphism)

In this section, we prove Theorems 3.1 and 3.8. We first prove that the map $\Phi_{L,\alpha} : HF^0((L_{st},0),(L,\alpha)) \to H^0((T^{2n},\Omega)^\vee,\mathcal{E}(L,\alpha))$ in Definition 3.6 is injective. We use an inner product on $H^0((T^{2n},\Omega)^\vee,\mathcal{E}(L,\alpha)) = \Gamma(\mathcal{E}(L,\alpha))$ for this purpose. To define a hermitian inner product on our vector bundle $\mathcal{E}(L,\alpha)$, we first remark that the bundle $\tilde{\mathcal{E}}(L,\alpha)$ on the universal cover $V/\tilde{L}_{pt} \oplus \tilde{L}^*_{pt}$ (which we defined by (2.7)) has a hermitian inner product in an obvious way. Since the action $g_{\mu,v}$ defined by (2.8) is unitary on each fiber, it induces a hermitian inner product on $\mathcal{E}(L,\alpha)$. We denote it by $(,)$. We next take a flat Riemannian metric on $(T^{2n},\Omega)^\vee$ and fix it. The hermitian inner product $(,)$ on $\Gamma(\mathcal{E}(L,\alpha))$ is induced by the metric and the hermitian inner product on $\mathcal{E}(L,\alpha)$.

Proposition 5.1 $\left(\Phi_{L,\alpha}([p]),\Phi_{L,\alpha}([p])\right) > 0$. $\left(\Phi_{L,\alpha}([p]),\Phi_{L,\alpha}([p'])\right) = 0$ *for* $p,p' \in L_{st} \cap L$, $p \ne p'$.

Proof: Let $v \in V/\tilde{L}^*_{pt}$ and $\tilde{p},\tilde{p}' \in \tilde{L}_{st}$ be an inverse image of p,p' in V. Let $\hat{L}(p)$, $\hat{L}(p')$ are the connected components of inverse images of L in V such that $\{\tilde{p}\} = \tilde{L}_{st} \cap \hat{L}(p)$, $\{\tilde{p}'\} = \tilde{L}_{st} \cap \hat{L}(p')$. We put

(5.2) $$g_{p,q,v,\gamma}(\sigma) = \exp\left(-2\pi Q(\tilde{p}(\gamma),\tilde{q}(\gamma),x_0(v)) + 2\pi\sqrt{-1}\left(\sigma(x_0(v) - \tilde{q}(\gamma)) + \alpha(\tilde{q} - \tilde{p}(\gamma))\right)\right).$$

Here the notation is similar to (3.17) and is as follows. $\sigma \in \tilde{L}^*_{pt}$, $\{x_0(v)\} = \tilde{L}_{st} \cap \hat{L}_{pt}(v)$, $\{\tilde{q}\} \in \hat{L}_{pt}(v) \cap \hat{L}(p)$, $\gamma \in \Gamma$, $\{\tilde{q}(\gamma)\} \in \hat{L}_{pt}(v) \cap \hat{L}(p+\gamma)$, $\{\tilde{p}(\gamma)\} \in \tilde{L}_{st} \cap \hat{L}(p+\gamma)$. (See Figure 5.) We define $g_{p',q,v,\gamma}(\sigma)$ in a similar way.

By (3.17) we have

(5.3) $$\Phi_{L,\alpha}([p])(v,\sigma) = \sum_{\gamma \in \Gamma_0(\hat{L})} g_{p,q,v,\gamma}(\sigma), \quad \Phi_{L,\alpha}([p'])(v,\sigma) = \sum_{\gamma \in \Gamma_0(\hat{L})} g_{p',q,v,\gamma}(\sigma).$$

Lemma 5.4

(5.4.1) $$\int_{\sigma \in \tilde{L}^*_{pt}/\left(\Gamma \cap \tilde{L}_{pt}\right)^\vee} g_{p,q,v,\gamma}(\sigma)\overline{g_{p,q,v,\gamma'}(\sigma)}\, d\sigma = 0, \text{ if } \gamma,\gamma' \in \Gamma_0(\hat{L}),\ \gamma \ne \gamma'.$$

(5.4.2) $$\int_{\sigma \in \tilde{L}^*_{pt}/\left(\Gamma \cap \tilde{L}_{pt}\right)^\vee} g_{p,q,v,\gamma}(\sigma)\overline{g_{p',q,v,\gamma'}(\sigma)}\, d\sigma = 0, \text{ if } \gamma,\gamma' \in \Gamma_0(\hat{L}),\ p \ne p'.$$

Proof: We remark that

$$g_{p,q,v,\gamma}(\sigma)\overline{g_{p,q,v,\gamma'}(\sigma)} = C\exp\left(2\pi\sqrt{-1}\sigma(\tilde{q}(\gamma') - \tilde{q}(\gamma))\right)$$

where C is independent of σ. (5.4.1) follows. The proof of (5.4.2) is similar.

By (5.4.1), we have

$$\int_{\sigma\in\tilde L^*_{pt}/(\Gamma\cap\tilde L_{pt})^\vee}\Phi_{L,\alpha}([p])(v,\sigma)\overline{\Phi_{L,\alpha}([p])(v,\sigma)}\,d\sigma$$
$$=\sum_{\gamma\in\Gamma_0(\hat L)}\int_{\sigma\in\tilde L^*_{pt}/(\Gamma\cap\tilde L_{pt})^\vee}g_{p,q,v,\gamma}(\sigma)\overline{g_{p,q,v,\gamma}(\sigma)}\,d\sigma>0.$$

$\left(\Phi_{L,\alpha}([p]),\Phi_{L,\alpha}([p])\right)>0$ follows. (5.4.2) implies $\left(\Phi_{L,\alpha}([p]),\Phi_{L,\alpha}([p'])\right)=0$ in a similar way. The proof of Proposition 5.1 is complete.

We remark that Proposition 5.1 implies the injectivity of $\Phi_{L,\alpha}$.

To prove Theorems 3.1 and 3.8, we use Proposition 4.14 and Lemma 4.16, and we may assume that $L\bullet L_{pt}(0)=1$ namely $\mathcal{E}(L,\alpha)$ is a line bundle. By Proposition 5.1, the map $\Phi_{L,\alpha}$ is injective. Hence Theorem 3.1 implies Theorem 3.8. We now are going to prove Theorem 3.1 in the case when $\mathcal{E}(L,\alpha)$ is a line bundle. We use Riemann-Roch's theorem for this purpose. Namely we are going to calculate the first Chern class of $\mathcal{E}(L,\alpha)$. To state it, we need some notations. Let $\tilde L,\tilde L_{st},\tilde L_{pt}$ be as before. We assume that they are transversal to each other. Hence $\tilde L$ may be regarded as a graph of a linear isomorphism : $\tilde L_{st}\to\tilde L_{pt}$. We write it as $\phi_L:\tilde L_{st}\to\tilde L_{pt}$. We next remark that there exists an isomorphism $V/\tilde L_{pt}\cong\tilde L_{st}$. We have $\tilde L_{st}\oplus\tilde L^*_{pt}\cong V/\tilde L_{pt}\oplus\tilde L^*_{pt}$, $\Gamma/\Gamma\cap\tilde L_{pt}\cong\Gamma\cap\tilde L_{st}\subseteq\tilde L_{st}$.

Definition 5.5 Let $\gamma,\gamma'\in\Gamma/\Gamma\cap\tilde L_{pt}\cong\Gamma\cap\tilde L_{st}$, $\mu,\mu'\in\left(\Gamma\cap\tilde L_{pt}\right)^\vee$. We define :

$$E_L((\gamma,\mu),(\gamma',\mu'))=\mu(\phi_L(\gamma'))-\mu'(\phi_L(\gamma)).$$

Since $L\bullet L_{pt}(0)=1$, it follows that $\phi_L\left(\Gamma/\Gamma\cap\tilde L_{pt}\right)\subseteq\Gamma\cap\tilde L_{pt}$. Therefore E_L is integer valued. By definition, E_L is anti-symmetric. We can extend E_L to an $\boldsymbol{R}$- bilinear anti-symmetric form on $\tilde L_{st}\oplus\tilde L^*_{pt}\cong V/\tilde L_{pt}\oplus\tilde L^*_{pt}$, we denote it by the same symbol.

Lemma 5.6 $E_L(J_\Omega x,J_\Omega y)=E_L(x,y)$, *where* J_Ω *is the complex structure on* $V/\tilde L_{pt}\oplus\tilde L^*_{pt}$ *introduced in §1.*

We prove Lemma 5.6 later.

Theorem 5.7 $c_1(\mathcal{E}(L,\alpha))=E_L$. *Here we regard an anti-symmetric map* E_L *on* $\left(\Gamma/\Gamma\cap\tilde L_{pt}\right)\oplus\left(\Gamma\cap\tilde L_{pt}\right)=\pi_1((T^{2n},\Omega)^\vee)$ *as an element of* $H^2((T^{2n},\Omega)^\vee,\boldsymbol{Z})$.

Note that Lemma 5.6 implies that $E_L\in H^{1,1}((T^{2n},\Omega)^\vee)$. We prove Theorem 5.7 later in this section.

We next show that Theorem 5.7 implies Theorem 3.1. For this purpose, we need to recall some standard results on the cohomology of line bundles on complex tori. We define a hermitian form $H_{L,\Omega}$ on $V/\tilde L_{pt}\oplus\tilde L^*_{pt}$ by

(5.8) $$H_{L,\Omega}(x,y) = E_L(J_\Omega x, y) + \sqrt{-1}E(x,y).$$

Lemma 5.6 implies that $H_{L,\Omega}$ is hermitian. We recall the following classical result. (See [35] § 16, [27] Theorem 5.5.)

Theorem 5.9 $$H^k((T^{2n},\Omega)^\vee, \mathcal{E}(L,\alpha)) = \begin{cases} 0 & k \neq \text{index } H_{L,\Omega} \\ \mathbf{Z}^{|Pf\,E_L|} & k = \text{index } H_{L,\Omega} \end{cases}.$$

Here $Pf\ E_L$ is the Pfaffian of the anti-symmetric form E_L and index $H_{L,\Omega}$ is the number of negative eigenvalues of the hermitian form $H_{L,\Omega}$. It is easy to see that, in our case

(5.10) $$Pf\ E_L = \#\frac{\Gamma \cap \tilde{L}_{pt}}{\phi_L\left(\Gamma/\Gamma \cap \tilde{L}_{pt}\right)} = |L \bullet L_{st}|.$$

We next need the following :

Lemma 5.11 index $H_{L,\Omega}(x,y) = \eta^*(\tilde{L}_{st},\tilde{L})$, *where* $\eta^*(\tilde{L}_{st},\tilde{L})$ *is defined by Definition* 2.23.

Theorem 3.1 follows from Theorems 5.7, 5.9, Lemma 5.11 and (5.10).

We now prove Lemmata 5.6, 5.11. We put

(5.12) $$U = \left(\Gamma \cap \tilde{L}_{pt}\right) \oplus \left(\Gamma/\Gamma \cap \tilde{L}_{pt}\right)^\vee,\ U_1 = \Gamma \cap \tilde{L}_{pt},\ U_2 = \left(\Gamma/\Gamma \cap \tilde{L}_{pt}\right)^\vee.$$

We first calculate the complex structure J_Ω. We first note that the symplectic form ω defines an isomorphism $I_\omega : \tilde{L}_{st} \to \tilde{L}_{pt}^*$ by $I_\omega(v)(x) = \omega(x,v)$. Similarly the closed 2 form B defines $I_B : \tilde{L}_{st} \to \tilde{L}_{pt}^*$ by $I_B(v)(x) = B(x,v)$. We then find, from the definition, that $v + \sigma \mapsto I_\omega v + \sqrt{-1}\left(I_B v + \sigma\right)$ is a complex linear isomorphism : $\tilde{L}_{st} \oplus \tilde{L}_{pt}^* \to \tilde{L}_{pt}^* \otimes_{\mathbf{R}} \mathbf{C}$, where $v,v' \in \tilde{L}_{st}$, $\sigma,\sigma' \in \tilde{L}_{pt}^*$. Hence

(5.13) $$\begin{aligned} J_\Omega\begin{pmatrix} v \\ \sigma \end{pmatrix} &= \begin{pmatrix} I_\omega & 0 \\ I_B & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} I_\omega & 0 \\ I_B & 1 \end{pmatrix}\begin{pmatrix} v \\ \sigma \end{pmatrix} \\ &= \begin{pmatrix} -I_\omega^{-1}I_B & -I_\omega^{-1} \\ I_\omega + I_B I_\omega^{-1} I_B & I_B I_\omega^{-1} \end{pmatrix}\begin{pmatrix} v \\ \sigma \end{pmatrix}. \end{aligned}$$

Therefore, we have :

$$
(5.14)\quad \begin{aligned}
E_L(J_\Omega v, J_\Omega v') &= E_L(I_\omega(v) + I_B I_\omega^{-1} I_B(v), -I_\omega^{-1} I_B(v')) \\
&\quad + E_L(-I_\omega^{-1} I_B(v), I_\omega(v') + I_B I_\omega^{-1} I_B(v')) \\
&= -\omega\left(\phi_L\left(I_\omega^{-1} I_B(v')\right), v\right) - B\left(\phi_L\left(I_\omega^{-1} I_B(v')\right), I_\omega^{-1} I_B(v)\right) \\
&\quad + \omega\left(\phi_L\left(I_\omega^{-1} I_B(v)\right), v'\right) + B\left(\phi_L\left(I_\omega^{-1} I_B(v)\right), I_\omega^{-1} I_B(v')\right).
\end{aligned}
$$

On the other hand, since $\omega\big|_{\tilde{L}} = B\big|_{\tilde{L}} = 0$ if follows that

$$
(5.15)\qquad \omega\left(v, \phi_L\left(v'\right)\right) = -\omega\left(\phi_L\left(v\right), v'\right), \quad B\left(v, \phi_L\left(v'\right)\right) = -B\left(\phi_L\left(v\right), v'\right).
$$

Therefore

$$
B\left(\phi_L\left(I_\omega^{-1} I_B(v')\right), I_\omega^{-1} I_B(v)\right) - B\left(\phi_L\left(I_\omega^{-1} I_B(v)\right), I_\omega^{-1} I_B(v')\right) = 0.
$$

Moreover we have

$$
\begin{aligned}
&\omega\left(\phi_L\left(I_\omega^{-1} I_B(v)\right), v'\right) - \omega\left(\phi_L\left(I_\omega^{-1} I_B(v')\right), v\right) \\
&\quad = \omega\left(\phi_L\left(v'\right), I_\omega^{-1} I_B(v)\right) - \omega\left(\phi_L\left(v\right), I_\omega^{-1} I_B(v')\right) \\
&\quad = B\left(\phi_L\left(v'\right), v\right) - B\left(\phi_L\left(v\right), v'\right) = 0.
\end{aligned}
$$

Hence

$$
(5.16)\qquad E_L(J_\Omega v, J_\Omega v') = 0 = E_L(v, v').
$$

We can prove

$$
(5.17)\qquad E_L(J_\Omega \sigma, J_\Omega \sigma') = 0 = E_L(\sigma, \sigma')
$$

in a similar way. We next calculate using (5.13), (5.15) :

$$
(5.18)\quad \begin{aligned}
E_L(J_\Omega v, J_\Omega \sigma) &= E_L(I_\omega(v) + I_B I_\omega^{-1} I_B(v), -I_\omega^{-1}(\sigma)) + E_L(-I_\omega^{-1} I_B(v), I_B I_\omega^{-1}(\sigma)) \\
&= -\omega\left(\phi_L I_\omega^{-1}(\sigma), v\right) - B\left(\phi_L I_\omega^{-1}(\sigma), I_\omega^{-1} I_B(v)\right) + B\left(\phi_L I_\omega^{-1} I_B(v), I_\omega^{-1}(\sigma)\right) \\
&= -\omega\left(\phi_L(v), I_\omega^{-1}(\sigma)\right) = -\sigma\left(\phi_L(v)\right) = E_L(v, \sigma).
\end{aligned}
$$

Lemma 5.6 follows from (5.16) , (5.17) and (5.18).

We turn to the proof of Lemma 5.11. We put $\Omega_s = \omega + s\sqrt{-1}B$. We remark that the J_{Ω_s} hermitian form H_{L,Ω_s} is well-defined and nondegenerate for each s. Hence its index is independent of s. So to prove Lemma 5.11, we may assume $B = 0$. Then we have

$$
(5.19)\qquad H_{L,\Omega_0}(v, v) = E_L\left(J_{\Omega_0}\left(v\right), v\right) = E_L\left(I_\omega(v), v\right) = \omega(\phi_L(v), v).
$$

for $v \in \tilde{L}_{st}$. Lemma 5.11 follows from (5.19) and the definition of $\eta^*(\tilde{L}_{st},\tilde{L})$.

The rest of this section is devoted to the proof of Theorem 5.7. We remark that the first Chern class of $\mathcal{E}(L,\alpha)$ is independent of α. Hence we put $\alpha = 0$ for simplicity. We are going find a 1-cocycle $e_u(z): U \times \left(\tilde{L}_{st} \oplus \tilde{L}^*_{pt}\right) \to \boldsymbol{C} - \{0\}$ representing our line bundle $\mathcal{E}(L,0)$. For this purpose we will find a holomorphic trivialization of the pull back bundle $\tilde{\mathcal{E}}(L,0)$ on $V \big/ \tilde{L}_{pt} \oplus \tilde{L}^*_{pt}$. Note that there is an obvious trivialization $\tilde{\mathcal{E}}(L,0) \cong \left(V \big/ \tilde{L}_{pt} \oplus \tilde{L}^*_{pt}\right) \times \boldsymbol{C}$. Namely, by choosing a lift $\hat{L} \cong \boldsymbol{R}^n$ of L, we define a global frame s' of $\tilde{\mathcal{E}}(L,0)$ by $s'(v,\sigma) = [p(v)]$ for $(v,\sigma) \in V \big/ \tilde{L}_{pt} \oplus \tilde{L}^*_{pt}$. Here $\{p(v)\} = \hat{L} \cap \hat{L}_{pt}(v)$. We recall $\tilde{\mathcal{E}}(L,0)_{(v,\sigma)} = \boldsymbol{C}[x_0(v)]$, since we assumed that $\tilde{\mathcal{E}}(L,0)$ is a line bundle. However this frame does not respect the holomorphic structure introduced in § 2. A holomorphic global frame of $\tilde{\mathcal{E}}(L,0)$ is obtained by

$$s(v,\sigma) = \exp\Big(2\pi Q\big(p(0), 0, v, p(v)\big) - 2\pi\sqrt{-1}\sigma\big(p(v) - v\big)\Big)\, s'(v,\sigma). \tag{5.20}$$

Note $\{v\} = \hat{L}_{st} \cap \hat{L}(v)$. Other notations are as in (2.11). We use the action of U on $\tilde{\mathcal{E}}(v,0)$ defined in § 2 and obtain the following formulae for $u_1 \in U_1$, $u_2 \in U_2$.

$$\begin{aligned} u_1 \bullet s(v,\sigma) = \exp\Big(&2\pi Q\big(p(0), 0, u_1 + v, p(u_1 + v)\big) \\ &-2\pi\sqrt{-1}\sigma\big((u_1 + v) - p(u_1 + v)\big) \\ &-2\pi Q\big(p(0), 0, z_1, p(v)\big) + 2\pi\sqrt{-1}\sigma\big(v - p(v)\big)\Big)\, s(u_1 + v, \sigma). \end{aligned} \tag{5.21.1}$$

$$u_2 \bullet s(v,\sigma) = \exp\Big(-2\pi\sqrt{-1}u_2\big(v - p(v)\big)\Big)\, s(v, u_2 + \sigma). \tag{5.21.2}$$

(5.21) follows from (5.20) and

$$\big(u_1 \bullet s'\big)(v + u_1, \sigma) = s'(v,\sigma), \tag{5.22.1}$$

$$\big(u_2 \bullet s'\big)(v, \sigma + u_2) = \exp\Big(2\pi\sqrt{-1}u_2(x_0(v))\Big)\, s'(v,\sigma). \tag{5.22.2}$$

(5.22) is a consequence of the definition in § 2. By the definition of ϕ_L we have

$$x_0(v) - v = x_0(0) + \phi_L(v). \tag{5.23}$$

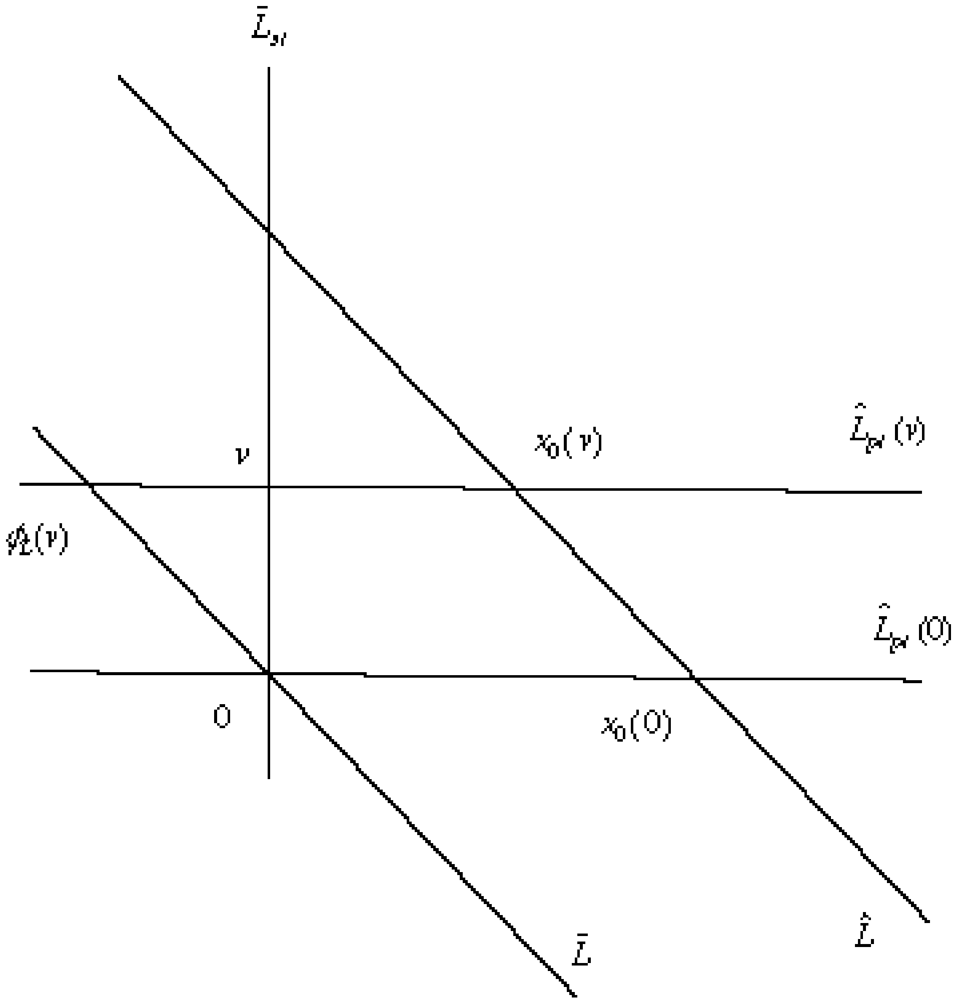

Figure 6

Here we regards $v \in V/\tilde{L}_{pt} \cong \tilde{L}_{st}$. Therefore a 1 cocycle $e_u(z)$ defining $\mathcal{L}(L,0)$ is :

(5.24.1) $$e_{u_1}(v,\sigma) = \exp\Big(2\pi Q\big(p(v), v, v+u_1, p(v+u_1)\big) + 2\pi\sqrt{-1}\sigma(\phi_L(u_1))\Big).$$

(5.24.2) $$e_{u_2}(v,\sigma) = \exp(2\pi\sqrt{-1}u_2(x_0(0)+\phi_L(v))) .$$

We put

(5.25.1) $$f_{u_1}(v,\sigma) = -\sqrt{-1}Q\big(x_0(v), v, v+u_1, x_0(v+u_1)\big) + \sigma(\phi_L(u_1)) .$$

(5.25.2) $$f_{u_2}(v,\sigma) = u_2(x_0(0)+\phi_L(v)) .$$

Then, by a standard result (see Proposition in page 18 of [35]), we find that the first Chern class of $\mathcal{L}(L,0)$ is represented by :

(5.26) $$E(u,u') = f_u(z+u') + f_u(z) - f_{u'}(z+u) - f_{u'}(z).$$

We remark that $Q\big(p(z_1), z_1, z_1+u_1, p(z_1+u_1)\big)$ is affine with respect to z_1. Using this fact and (5.25), (5.26), we find $E(u,u') = E_L(u,u')$. The proof of Theorem 5.7 is now complete.

We remark that Theorem 5.7 and Appel-Humbert theorem (see [35]) imply the following.

Corollary 5.27 *Any line bundle on* $(T^{2n},\Omega)^{\vee}$ *is isomorphic to* $\mathcal{E}(L,\alpha)$ *for some affine Lagrangian subspace* L *and* α.

§ 6 Extension and Floer cohomology 1 (0 th cohomology)

Let $\tilde{L}_1$ and $\tilde{L}_2$ be Lagrangian linear subspaces of (V,Ω). We assume that they are transversal to each other and to $\tilde{L}_{pt}$, $\tilde{L}_{st}$. Let $v_i \in V/\tilde{L}_i$ and $\alpha_i \in \tilde{L}_i^*$. We obtain a holomorphic vector bundle $\mathcal{E}(L_i(v_i),\alpha_i)$ on $(T^{2n}\Omega)^\vee$. We put $(L_i(v_i),\alpha_i) = (L_i,\mathcal{L}_i)$.

Theorem 6.1 $HF^k((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)) \cong Ext^k(\mathcal{E}(L_1,\mathcal{L}_1),\mathcal{E}(L_2,\mathcal{L}_2))$.

In the case when $k = \eta(\tilde{L}_1,\tilde{L}_2) = 0$ our result is more explicit. Namely we construct an explicit isomorphism also in this section. (In the case $k > 0$, we will construct an explicit isomorphism in § 11.) We assume $\eta(\tilde{L}_1,\tilde{L}_2) = 0$. Let $[p] \in HF^0((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2))$, where $p \in L_1 \cap L_2$. Let $(L_{pt}(v),\sigma) \in (T^{2n},\Omega)^\vee$. We define a homomorphism $S_p(L_{pt}(v),\sigma) : \mathcal{E}(L_1,\mathcal{L}_1)_{(L_{pt}(v),\sigma)} \to \mathcal{E}(L_2,\mathcal{L}_2)_{(L_{pt}(v),\sigma)}$ by

(6.2) $$S_p(L_{pt}(v),\sigma)(x) = m_2(x,[p]) \in \mathcal{E}(L_2,\mathcal{L}_2)_{(L_{pt}(v),\sigma)} \cong HF^n\big((L_{pt}(v),\sigma),(L_2,\mathcal{L}_2)\big).$$

Here $x \in \mathcal{E}(L_1,\mathcal{L}_1)_{(L_{pt}(v),\sigma)} \cong HF^n\big((L_{pt}(v),\sigma),(L_1,\mathcal{L}_1)\big)$.

Lemma 6.3 *$S_p(L_{pt}(v),\sigma)$ is holomorphic with respect to $(L_{pt}(v),\sigma)$. Hence $S_p \in Hom((\mathcal{E}(L_1,\mathcal{L}_1),\mathcal{E}(L_2,\mathcal{L}_2))$.*

Lemma 6.3 follows from Theorem 7.22 which is proved in the next section. By Lemma 6.3, we obtain a homomorphism

$$\Phi_{(L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)} : HF^0((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)) \to Hom((\mathcal{E}(L_1,\mathcal{L}_1),\mathcal{E}(L_2,\mathcal{L}_2)).$$

Theorem 6.4 *$\Phi_{(L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)}$ is an isomorphism.*

We prove Theorems 6.4 and 6.1 later in this section. We next prove the following :

Theorem 6.5 *If $\eta(\tilde{L}_1,\tilde{L}_2) = \eta(\tilde{L}_2,\tilde{L}_3) = 0$, then the following diagram commutes.*

$$\begin{array}{ccc} HF^0((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)) \otimes HF^0((L_2,\mathcal{L}_2),(L_3,\mathcal{L}_3)) & \to & HF^0((L_1,\mathcal{L}_1),(L_3,\mathcal{L}_3)) \\ \downarrow & & \downarrow \\ Hom(\mathcal{E}(L_1,\mathcal{L}_1),\mathcal{E}(L_2,\mathcal{L}_2)) \otimes Hom(\mathcal{E}(L_2,\mathcal{L}_2);\mathcal{E}(L_3,\mathcal{L}_3)) & \to & Hom(\mathcal{E}(L_1,\mathcal{L}_1),\mathcal{E}(L_3,\mathcal{L}_3)) \end{array}$$

Diagram 1

where the vertical arrows are $\Phi_{(L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)} \otimes \Phi_{(L_2,\mathcal{L}_2),(L_3,\mathcal{L}_3)}$ and $\Phi_{(L_1,\mathcal{L}_1),(L_3,\mathcal{L}_3)}$, and the horizontal arrows are m_2 and the composition product.

Proof: Let $[p_{i\,i+1}] \in HF^0((L_i(v_i),\sigma_i),(L_{i+1}(v_{i+1}),\sigma_{i+1}))$, $(L_{pt}(v),\sigma) \in (T^{2n},\Omega)^\vee$ and $x \in \mathcal{E}(L_1(v_1),\sigma_1)_{(L_{pt}(v),\sigma)}$. We have

(6.6) $$\left(\Phi_{(L_1(v_1),\sigma_1),(L_3(v_3),\sigma_3)}\left(m_2([p_{12}],[p_{23}])\right)\right)(x) = m_2\left(x, m_2([p_{12}],[p_{23}])\right),$$

(6.7) $$\Phi_{(L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)}([p_{12}])\left(\Phi_{(L_2,\mathcal{L}_2),(L_3,\mathcal{L}_3)}([p_{23}])(x)\right) = m_2\left(m_2(x,[p_{12}]),[p_{23}]\right).$$

The associativity relation (Theorem 3.15) then implies Theorem 6.5.

By putting $L_1 = L_{st}$ in Theorem 6.5, we obtain the following :

Corollary 6.8 *If* $\eta(\tilde{L}_1,\tilde{L}_2) = 0$ *the following diagram commutes for* $k = \ell = 0$.

$$\begin{array}{ccc}
HF^0((L_{st},0),(L_1,\mathcal{L}_1)) \otimes HF^0((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)) & \rightarrow & HF^0((L_{st},0),(L_2,\mathcal{L}_2)) \\
\downarrow & & \downarrow \\
H^0((T^{2n},\Omega)^\vee, \mathcal{E}(L_1,\mathcal{L}_1)) \otimes Hom(\mathcal{E}(L_1,\mathcal{L}_1),\mathcal{E}(L_2,\mathcal{L}_2)) & \rightarrow & H^0((T^{2n},\Omega)^\vee, \mathcal{E}(L_2,\mathcal{L}_2))
\end{array}$$

Diagram 2

Here the vertical arrows are $\Phi_{(L_1,\mathcal{L}_1)} \otimes \Phi_{(L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)}$ *and* $\Phi_{(L_2,\mathcal{L}_2)}$ *and the horizontal arrows are* m_2 *and evaluation map.*

In § 11, we generalize Theorem 6.5 and Corollary 6.8 to higher cohomology.

We now start the proof of Theorems 6.1 and 6.4. We first show the following :

Lemma 6.9 *The map* $\Phi_{(L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)}$ *is injective.*

Proof: The proof is similar to one of injectivity of $\Phi_{(L,\mathcal{L})}$ we gave in § 5. We first define an inner product of the bundle $\mathcal{H}om\left(\mathcal{E}(L_1,\mathcal{L}_1),\mathcal{E}(L_2,\mathcal{L}_2)\right)$ using ones on $\mathcal{E}(L_1,\mathcal{L}_1)$ and $\mathcal{E}(L_2,\mathcal{L}_2)$. Then we prove

(6.10) $$\left(\Phi_{(L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)}[p], \Phi_{(L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)}[p]\right) > 0,$$

(6.11) $$\left(\Phi_{(L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)}[p], \Phi_{(L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)}[q]\right) = 0,$$

for $p,q \in L_1 \cap L_2$, $p \neq q$. The proof of (6.10) and (6.11) is the same as the proof of Proposition 5.1 and is omitted. (6.10) and (6.11) imply Lemma 6.9.

By Lemma 6.9, Theorem 6.1 implies Theorem 6.4.

Lemma 6.12 *Theorem* 6.1 *holds if* $\mathcal{E}(L_1,\mathcal{L}_1)$ *and* $\mathcal{E}(L_2,\mathcal{L}_2)$ *are line bundles.*

Proof: We have

(6.13) $$Ext^k(\mathcal{E}(L_1,\mathcal{L}_1),\mathcal{E}(L_2,\mathcal{L}_2)) \cong H^k((T^{2n},\Omega)^\vee,\mathcal{E}(L_1,\mathcal{L}_1)^* \otimes \mathcal{E}(L_2,\mathcal{L}_2))$$

in this case. By Theorem 5.7, we have

(6.14) $$c^1\left(\mathcal{E}(L_1,\mathcal{L}_1)^* \otimes \mathcal{E}(L_2,\mathcal{L}_2)\right) = -E_{L_1} + E_{L_2}.$$

Let ϕ_{L_i} be as in § 5. Then we find a Lagrangian linear subspace L_3 such that $-\phi_{L_1} + \phi_{L_2} = \phi_{L_3}$. Therefore

(6.15) $$Pf\ E_{L_3} = \#\left(L_{st} \cap L_3(0)\right) = \#\left(L_1 \cap L_2\right).$$

Hence Theorem 5.9 , (6.14) and (6.15) imply Theorem 6.1 in this case. (In fact, we can prove $\mathcal{E}(L_1,\mathcal{L}_1)^* \otimes \mathcal{E}(L_2,\mathcal{L}_2) \cong \mathcal{E}(L_3,\mathcal{L}_3)$. Here to define $\mathcal{L}_3$ we identify $\tilde{L}_1 \cong \tilde{L}_2 \cong V/\tilde{L}_{pt}$ and put $\tilde{\mathcal{L}}_3 \cong \tilde{\mathcal{L}}_1^* \otimes \tilde{\mathcal{L}}_2$. We omit the proof since we do not use it.)

In the rest of this section, we reduce Theorem 6.1 to the case when $\mathcal{E}(L_i,\mathcal{L}_i)$ are line bundles. We further study isogeny for this purpose. Let $G(L_1)$, $G(L_2)$ be as in § 4. Let $G \subseteq L_{pt}$ be a finite subgroup. We define $(\overline{T}^{2n},\overline{\Omega})$ as in § 4. We remark that Lemma 4.2 holds for our G also. We use the notations in § 4. We study the following three cases :

Case 1 : $G \subseteq G(L_1) \cap G(L_2))$. Let $\overline{\mathcal{L}}_i$ be a flat connection on $\overline{L}_i$ such that $\pi^*\overline{\mathcal{L}}_i \cong \mathcal{L}_i$. We remark that $\pi^*\left(\overline{\mathcal{L}}_i + \mu\right) \cong \mathcal{L}_i$ for $\mu \in G^\vee$.

Lemma 6.16 $\pi^*\mathcal{E}(L_i,\mathcal{L}_i) \cong \bigoplus_{\mu \in G^\vee} \mathcal{E}(\overline{L}_i,\overline{\mathcal{L}}_i + \mu)$.

Proof: By Proposition 4.4, $\mathcal{E}(L_i,\mathcal{L}_i) = \pi_*\left(\mathcal{E}(\overline{L}_i,\overline{\mathcal{L}}_i)\right)$. Note that $G^\vee$ is the deck transformation group of the covering $\pi : (\overline{T}^{2n},\overline{\Omega})^\vee \to (T^{2n},\Omega)^\vee$. Hence

$$\pi^*\mathcal{E}(L_i,\mathcal{L}_i) \cong \bigoplus_{\mu \in G^\vee} \mu^*\mathcal{E}(\overline{L}_i,\overline{\mathcal{L}}_i).$$

Here we regard $\mu : (\overline{T}^{2n},\overline{\Omega})^\vee \to (\overline{T}^{2n},\overline{\Omega})^\vee$. It is easy to see $\mu^*\mathcal{E}(\overline{L}_i,\overline{\mathcal{L}}_i) \cong \mathcal{E}(\overline{L}_i,\overline{\mathcal{L}}_i + \mu)$. Lemma 6.16 follows.

Lemma 6.17 *Suppose that there exists* k_0 *such that* $Ext^k(\mathcal{E}(\overline{L}_1,\overline{\mathcal{L}}_1),\mathcal{E}(\overline{L}_2,\overline{\mathcal{L}}_2 + \mu))$ *vanishes for* $k \neq k_0$ *then*

$$Ext^k(\mathcal{E}(L_1,\mathcal{L}_1),\mathcal{E}(L_2,\mathcal{L}_2)) \cong \bigoplus_{\mu \in G^\vee} Ext^k\left(\mathcal{E}(\overline{L}_1,\overline{\mathcal{L}}_1),\mathcal{E}(\overline{L}_2,\overline{\mathcal{L}}_2 + \mu)\right)$$

Proof: Proposition 4.3 and Lemma 6.16 implies

$$
\begin{aligned}
Ext^k(\mathcal{E}(L_1,\mathcal{L}_1),\mathcal{E}(L_2,\mathcal{L}_2)) &\cong Ext^k\Big(\pi_*\big(\mathcal{E}(\overline{L}_1,\overline{\mathcal{L}}_1)\big),\mathcal{E}(L_2,\mathcal{L}_2)\Big)\\
&\cong \bigoplus_{\mu\in G^\vee} Ext^k\big(\mathcal{E}(\overline{L}_1,\overline{\mathcal{L}}_1),\mathcal{E}(\overline{L}_2,\overline{\mathcal{L}}_2+\mu)\big)
\end{aligned}
$$

We next consider the Floer cohomology. The assumption $G\subseteq G(L_1)\cap G(L_2)$ implies that L_1 and L_2 are both G invariant. Hence G acts on $L_1\cap L_2$ freely. Therefore we have

$$
HF^k\big((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)\big)\cong HF^k\big((\overline{L}_1,\overline{\mathcal{L}}_1),(\overline{L}_2,\overline{\mathcal{L}}_2)\big)\otimes R(G). \tag{6.18}
$$

Lemma 6.17 and (6.18) implies that Theorem 6.1 holds for $(L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)$ if it holds for $(\overline{L}_1,\overline{\mathcal{L}}_1),(\overline{L}_2,\overline{\mathcal{L}}_2)$.

Case 2 : $G\subseteq G(L_1)$, $G\cap G(L_2)=\{1\}$. We put $G=\{\gamma_1,\cdots,\gamma_g\}$ and $L_2(\gamma_i)=\gamma_i L_2$. By assumption $G\cap G(L_2)=\{1\}$, we have $\gamma_i L\cap\gamma_j L_2=\varnothing$ for $i\neq j$. Let $\overline{L}_2=\pi(L_2)$. π induces an isomorphism $L_2\cong\overline{L}_2$. Using this isomorphism we define $\overline{\mathcal{L}}_2$ on $\overline{L}_2$.

Lemma 6.19 $\mathcal{E}(\overline{L}_2,\overline{\mathcal{L}}_2)\cong\pi^*\mathcal{E}(L_2,\mathcal{L}_2)$.

Proof: Let $(L_{pt}(v),\sigma)\in(T^{2n},\Omega)^\vee$. Lemma 6.19 follows from :

$$
\mathcal{E}(L_2,\mathcal{L}_2)_{(L_{pt}(v),\sigma)}=\bigoplus_{p\in L_2\cap L_{pt}(v)}\mathbf{C}[p]\cong\bigoplus_{p\in\overline{L}_2\cap\overline{L}_{pt}(v)}\mathbf{C}[\overline{p}]=\mathcal{E}(\overline{L}_2,\mathcal{L}_2)_{(\overline{L}_{pt}(v),\sigma)}. \tag{6.20}
$$

Lemma 6.21 *Suppose that there exists* k_0 *such that* $Ext^k(\mathcal{E}(\overline{L}_1,\overline{\mathcal{L}}_1),\mathcal{E}(\overline{L}_2,\overline{\mathcal{L}}_2))$ *vanishes for* $k\neq k_0$ *then* $Ext^k(\mathcal{E}(L_1,\mathcal{L}_1),\mathcal{E}(L_2,\mathcal{L}_2))\cong Ext^k\big(\mathcal{E}(\overline{L}_1,\overline{\mathcal{L}}_1),\mathcal{E}(\overline{L}_2,\overline{\mathcal{L}}_2)\big)$.

Proof: Lemma 6.19 and Proposition 4.3 imply :

$$
\begin{aligned}
Ext^k(\mathcal{E}(L_1,\mathcal{L}_1),\mathcal{E}(L_2,\mathcal{L}_2)) &\cong Ext^k\Big(\pi_*\big(\mathcal{E}(\overline{L}_1,\overline{\mathcal{L}}_1)\big),\mathcal{E}(L_2,\mathcal{L}_2)\Big)\\
&\cong Ext^k\big(\mathcal{E}(\overline{L}_1,\overline{\mathcal{L}}_1),\pi^*\mathcal{E}(L_2,\mathcal{L}_2)\big)\\
&\cong Ext^k\big(\mathcal{E}(\overline{L}_1,\overline{\mathcal{L}}_1),\mathcal{E}(\overline{L}_2,\overline{\mathcal{L}}_2)\big)
\end{aligned}
$$

π induces an isomorphism $L_1\cap L_2\cong\overline{L}_1\cap\overline{L}_2$. Hence

$$
HF^k\big((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)\big)\cong HF^k\big((\overline{L}_1,\overline{\mathcal{L}}_1),(\overline{L}_2,\overline{\mathcal{L}}_2)\big). \tag{6.22}
$$

Lemma 6.21 and (6.22) implies that Theorem 6.1 holds for $(L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)$ if it holds for $(\overline{L}_1,\overline{\mathcal{L}}_1),(\overline{L}_2,\overline{\mathcal{L}}_2)$.

Case 3: $G\subseteq G(L_2)$, $G\cap G(L_1)=\{1\}$. We have $\mathcal{E}(\overline{L}_1,\overline{\mathcal{L}}_1)\cong\pi^*\mathcal{E}(L_1,\mathcal{L}_1)$ and

$\mathcal{E}(L_2, \mathcal{L}_2) \cong \pi_* \mathcal{E}(\bar{L}_2, \bar{\mathcal{L}}_2)$. Hence

$$\begin{aligned} Ext^k(\mathcal{E}(L_1, \mathcal{L}_1), \mathcal{E}(L_2, \mathcal{L}_2)) &\cong Ext^k\left(\mathcal{E}(L_1, \mathcal{L}_1), \pi_*\left(\mathcal{E}(\bar{L}_2, \bar{\mathcal{L}}_2)\right)\right) \\ &\cong Ext^k\left(\pi^*\mathcal{E}(L_1, \mathcal{L}_1), \mathcal{E}(\bar{L}_2, \bar{\mathcal{L}}_2)\right) \\ &\cong Ext^k\left(\mathcal{E}(\bar{L}_1, \bar{\mathcal{L}}_1), \mathcal{E}(\bar{L}_2, \bar{\mathcal{L}}_2)\right) \end{aligned}$$

if $Ext^k(\mathcal{E}(\bar{L}_1, \bar{\mathcal{L}}_1), \mathcal{E}(\bar{L}_2, \bar{\mathcal{L}}_2))$ vanishes for $k \neq k_0$. On the other hand, (6.22) holds also in this case. Therefore Theorem 6.1 holds for $(L_1, \mathcal{L}_1), (L_2, \mathcal{L}_2)$ if it holds for $(\bar{L}_1, \bar{\mathcal{L}}_1), (\bar{L}_2, \bar{\mathcal{L}}_2)$.

Combining these three cases and Lemma 6.12 the proof of Theorem 6.1 is complete.

§ 7 Moduli space of holomorphic vector bundles on a mirror torus

Let $\tilde L$ be a Lagrangian linear subspace of (V,Ω) transversal to $\mathcal{L}_{pt}$ and satisfying $\tilde L \cap \Gamma \cong \mathbf{Z}^n$. We constructed a complex manifold (torus) $\mathcal{M}(\tilde L)$ in § 2. On the other hand, for each element $[L(w),\alpha] \in \mathcal{M}(\tilde L)$, we constructed a holomorphic vector bundle $\mathcal{E}(L(w),\alpha)$ on $(T^{2n},\Omega)^\vee$. In this section, we construct a universal family of vector bundles on $\mathcal{M}(\tilde L)$. One delicate point in doing so is gauge fixing which we mentioned in § 2. In fact, during the proof of Proposition 2.16, we need to choose a base point on $L(w)$. In other words, the isomorphism $\mathcal{E}(L(w),\alpha) \cong \mathcal{E}(L(w),\alpha+\mu)$ for $\mu \in \left(G \cap \tilde L\right)^\vee$ depends on the choice of the base point on $L(w)$ and is not canonical. To choose a base point on $L(w)$ systematically, we need additional data. Namely we fix other affine Lagrangian submanifold. More precisely we start with the following situation.

Assumption 7.1 Let $\tilde L_1, \tilde L_2$ and $\tilde M_1, \tilde M_2$ be Lagrangian linear subspaces of (V,Ω) such that

(7.1.1) $\tilde L_i \cap \Gamma \cong \mathbf{Z}^n$, $\tilde M_i \cap \Gamma \cong \mathbf{Z}^n$.

(7.1.2) $\tilde L_i$ is transversal to $\tilde M_i$.

(7.1.3) $\tilde L_1$ is transversal to $\tilde L_2$.

We put $M_i = \tilde M_i \big/ \left(\tilde M_i \cap \Gamma\right) \cong T^n \subseteq T^{2n}$.

Definition 7.2 $\mathcal{M}(\tilde L_i, \tilde M_i)$ is the set of pairs $([L_i(w_i),\alpha_i], p_i) \in \mathcal{M}(\tilde L_i) \times T^{2n}$ such that $p_i \in M_i \cap L_i(w_i)$.

$\mathcal{M}(\tilde L_i, \tilde M_i) \to \mathcal{M}(\tilde L_i)$, $([L_i(w_i),\alpha_i], p_i) \mapsto [L_i(w_i),\alpha_i]$ is a $\left|M_i \bullet L(w)\right|$ hold covering. Hence the complex structure on $\mathcal{M}(\tilde L_i)$ induces one on $\mathcal{M}(\tilde L_i, \tilde M_i)$. We are going to define a holomorphic vector bundle $\mathcal{E}(\tilde L_1, \tilde L_2; \tilde M_1, \tilde M_2) \to \mathcal{M}(\tilde L_1, \tilde M_1) \times \mathcal{M}(\tilde L_2, \tilde M_2)$ such that the fiber at $(([L_1(w_1),\alpha_1], p_1), ([L_2(w_2),\alpha_2], p_2))$ is identified with $HF((L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2))$. Let

(7.3) $$\left((w_1,\alpha_1),(w_2,\alpha_2)\right) \in \left(V\big/\tilde L_1 \times \tilde L_1^*\right) \times \left(V\big/\tilde L_2 \times \tilde L_2^*\right), \quad p_i \in M_i \cap L_i(w_i).$$

Let $V(\tilde L_1, \tilde L_2; \tilde M_1, \tilde M_2)$ be the totality of all $\left((w_1,\alpha_1),(w_2,\alpha_2), p_1, p_2\right)$ satisfying (7.3). We put

(7.4) $$\tilde{\mathcal{E}}(\tilde L_1, \tilde L_2; \tilde M_1, \tilde M_2)_{((w_1,\alpha_1),(w_2,\alpha_2),p_1,p_2)} = \bigoplus_{x \in L_1(w_1) \cap L_2(w_2)} \mathbf{C}[x].$$

$\tilde{\mathcal{E}}(\tilde L_1, \tilde L_2; \tilde M_1, \tilde M_2)$ is a complex vector bundle on $V(\tilde L_1, \tilde L_2; \tilde M_1, \tilde M_2)$. We put

(7.5) $$\Gamma(\tilde L_1,\tilde L_2;\tilde M_1,\tilde M_2)=\left(\Gamma/\Gamma\cap\tilde L_1\right)\times\left(\Gamma\cap\tilde L_1\right)^\vee\times\left(\Gamma/\Gamma\cap\tilde L_2\right)\times\left(\Gamma\cap\tilde L_2\right)^\vee.$$

$\Gamma(\tilde L_1,\tilde L_2;\tilde M_1,\tilde M_2)$ acts on $V(\tilde L_1,\tilde L_2;\tilde M_1,\tilde M_2)$ in an obvious way and the quotient space is $\mathcal{M}(\tilde L_1,\tilde M_1)\times\mathcal{M}(\tilde L_2,\tilde M_2)$. We define an action of $\Gamma(\tilde L_1,\tilde L_2;\tilde M_1,\tilde M_2)$ on $\tilde{\mathcal{P}}(\tilde L_1,\tilde L_2;\tilde M_1,\tilde M_2)$ as follows. Let $\left((w_1,\alpha_1),(w_2,\alpha_2),p_1,p_2\right)\in V(\tilde L_1,\tilde L_2;\tilde M_1,\tilde M_2)$, $x\in L_1(w_1)\cap L_2(w_2)$. $[x]\in\tilde{\mathcal{P}}(\tilde L_1,\tilde L_2;\tilde M_1,\tilde M_2)_{((w_1,\alpha_1),(w_2,\alpha_2),p_1,p_2)}$. Let $\left(\gamma_1,\mu_1,\gamma_2,\mu_2\right)\in\Gamma(\tilde L_1,\tilde L_2;\tilde M_1,\tilde M_2)$. We choose a lift $\tilde p_i\in\hat L_i(w_i)\cap\hat M_i(u_i)$ of p_i. (Here $u_i\in\Gamma$.) Let $\tilde x\in\hat L_1(w_1)\cap\hat L_2(w_2)$ be the lift of x. We then put

(7.6) $$\begin{aligned}\left(\gamma_1,\mu_1,\gamma_2,\mu_2\right)[x] &= \exp\left(-2\pi\sqrt{-1}\mu_1\left(\tilde x-\tilde p_1\right)+2\pi\sqrt{-1}\mu_2\left(\tilde x-\tilde p_2\right)\right)\\ &\in \tilde{\mathcal{P}}(\tilde L_1,\tilde L_2;\tilde M_1,\tilde M_2)_{((w_1+\gamma_1,\alpha_1+\mu_1),(w_2+\gamma_2,\alpha_2+\mu_2),p_1,p_2)}.\end{aligned}$$

Lemma 7.7 (7.6) *defines an action of* $\Gamma(\tilde L_1,\tilde L_2;\tilde M_1,\tilde M_2)$ *on* $V(\tilde L_1,\tilde L_2;\tilde M_1,\tilde M_2)$.

The proof is straightforward and is omitted.

Remark 7.8 The reader may wonder why we can not simply take $\tilde p_i\in\hat L_i(w_i)\cap\tilde M_i$. (If we could do it we would be unnecessary to take a covering $\mathcal{M}(\tilde L_i,\tilde M_i)\to\mathcal{M}(\tilde L_i)$.) However if we do it, the analogue of Lemma 2.9 does not hold. Using our choice $\tilde p_i\in\hat L_i(w_i)\cap\hat M_i(u_i)$, we can prove an analogue of Lemma 2.9 in our situation.

Let $\mathcal{P}(\tilde L_1,\tilde L_2;\tilde M_1,\tilde M_2)\to\mathcal{M}(\tilde L_1,\tilde M_1)\times\mathcal{M}(\tilde L_2,\tilde M_2)$ be the bundle obtained by taking the quotient $\tilde{\mathcal{P}}(\tilde L_1,\tilde L_2;\tilde M_1,\tilde M_2)/\Gamma(\tilde L_1,\tilde L_2;\tilde M_1,\tilde M_2)$.

We next construct a holomorphic structure on $\mathcal{P}(\tilde L_1,\tilde L_2;\tilde M_1,\tilde M_2)$. We again construct a holomorphic local frame. We use the same notation as above. Let $\left((w_1',\alpha_1'),(w_2',\alpha_2'),p_1',p_2'\right)\in V(\tilde L_1,\tilde L_2;\tilde M_1,\tilde M_2)$ be in a neighborhood of $\left((w_1,\alpha_1),(w_2,\alpha_2),p_1,p_2\right)$. There exists a point $\tilde x'\in\hat L_1(w_1')\cap\hat L_2(w_2')$ in a neighborhood of $\tilde x$. (See Figure 7.)

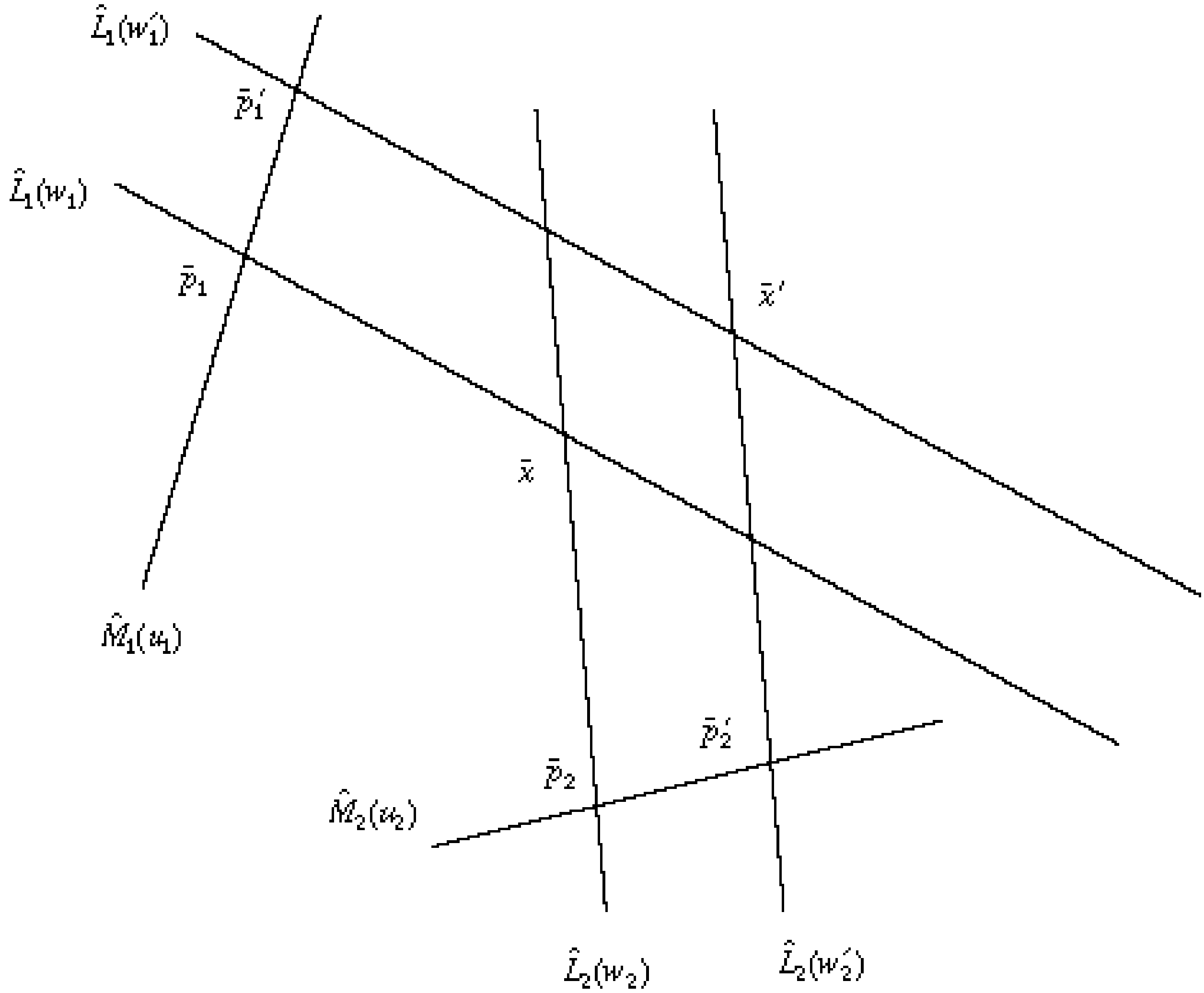

Figure 7

We define :

(7.9)
$$
\begin{aligned}
&e_{((w_1,\alpha_1),(w_2,\alpha_2),\tilde{p}_1,\tilde{p}_2,\tilde{x})}\big((w_1',\alpha_1'),(w_2',\alpha_2')\big)\\
&\quad= \exp\Big(-2\pi Q\big(\tilde{x},\tilde{p}_1,\tilde{p}_1',\tilde{x}',\tilde{p}_2',\tilde{p}_2\big)\\
&\qquad+2\pi\sqrt{-1}\big(\alpha_1(\tilde{p}_1-\tilde{x})+\alpha_1'(\tilde{x}'-\tilde{p}')+\alpha_2'(\tilde{p}_2'-\tilde{x}')+\alpha_2(\tilde{x}-\tilde{p}_2)\big)\Big)
\end{aligned}
$$

where $Q\big(\tilde{x},\tilde{p}_1,\tilde{p}_1',\tilde{x}',\tilde{p}_2',\tilde{p}_2\big)$ is a integration of Ω over union of 4 triangles $\Delta_{\tilde{x}\tilde{p}_1\tilde{p}_1'}$, $\Delta_{\tilde{x}\tilde{p}_1\tilde{x}'}$, $\Delta_{\tilde{x}\tilde{x}\tilde{p}_2'}$, $\Delta_{\tilde{x}\tilde{p}_2'\tilde{p}_2}$. Then the frame is

(7.10)
$$
\begin{aligned}
&\mathbf{e}_{((w_1,\alpha_1),(w_2,\alpha_2),\tilde{p}_1,\tilde{p}_2,\tilde{x})}\big((w_1',\alpha_1'),(w_2',\alpha_2')\big)\\
&\qquad= e_{((w_1,\alpha_1),(w_2,\alpha_2),\tilde{p}_1,\tilde{p}_2,\tilde{x})}\big((w_1',\alpha_1'),(w_2',\alpha_2')\big)[x'].
\end{aligned}
$$

Lemma 7.11 *There exists a unique holomorphic structure on* $\mathcal{P}(\tilde{L}_1,\tilde{L}_2;\tilde{M}_1,\tilde{M}_2)$ *such that* $\mathbf{e}_{((w_1,\alpha_1),(w_2,\alpha_2),\tilde{p}_1,\tilde{p}_2,\tilde{x})}$ *is a local holomorphic section.*

The proof is a straightforward analogue of the proof of Lemma 2.14 and is omitted. We thus obtained a holomorphic vector bundle

$$\mathcal{E}(\tilde L_1,\tilde L_2;\tilde M_1,\tilde M_2)\to \mathcal{M}(\tilde L_1,\tilde M_1)\times\mathcal{M}(\tilde L_2,\tilde M_2) \tag{7.12}$$

under Assumption 7.1.

We put $\tilde L_1=\tilde L_{pt}$, $\tilde L_2=\tilde L$, $\tilde M_1=\tilde L_{st}$, $\tilde M_2=\tilde L_{st}$. We obtain

$$\mathcal{P}(\tilde L_{pt},\tilde L;\tilde L_{st},\tilde L_{st})\to (T^{2n},\Omega)^\vee\times\mathcal{M}(\tilde L,\tilde L_{st})\,. \tag{7.13}$$

(Note $L_{pt}\bullet L_{st}=1$ implies $\mathcal{M}(\tilde L_{pt},\tilde L_{st})=\mathcal{M}(\tilde L_{pt})=(T^{2n},\Omega)^\vee$.) Let us consider the group $G=L_{st}\cap L(0)$. It is easy to see $\mathcal{M}(\tilde L,\tilde L_{st})\big/G=\mathcal{M}(\tilde L)$. Since $(T^{2n},\Omega)^\vee\cong L_{st}\big/(\Gamma\cap L_{st})\times L^*_{pt}\big/(\Gamma\cap L_{pt})^\vee$ as a group, G acts on $(T^{2n},\Omega)^\vee$. (We let G act on the first factor $L_{st}\big/(\Gamma\cap L_{st})$.) It is easy to find a G action on $\mathcal{E}(\tilde L,\tilde L_{pt};\tilde L_{st},\tilde L_{st})$ such that (7.13) is equivariant. Thus we divide (7.13) by G and obtain

$$\mathcal{P}(\tilde L_{pt},\tilde L)\to\frac{(T^{2n},\Omega)^\vee\times\mathcal{M}(\tilde L,\tilde L_{st})}{G}\,. \tag{7.14}$$

Note that we have a fiber bundle

$$(T^{2n},\Omega)^\vee\to\frac{(T^{2n},\Omega)^\vee\times\mathcal{M}(\tilde L,\tilde L_{st})}{G}\xrightarrow{\ pr\ }\mathcal{M}(\tilde L)\,. \tag{7.15}$$

It is easy to verify the following :

Proposition 7.16 *Let $(L,\alpha)\in\mathcal{M}(\tilde L)$. The restriction of $\mathcal{P}(\tilde L_{pt},\tilde L)$ to $pr^{-1}(L,\alpha)$ is isomorphic to $\mathcal{E}(L,\alpha)$. (Here pr is as in* (7.15).*)*

Propositions 7.16 and the following Proposition 7.17 imply that we may regard (7.14) as the universal bundle.

Proposition 7.17 *Let $(L,\alpha),(L',\alpha')\in\mathcal{M}(\tilde L)$. Assume that $\mathcal{E}(L,\alpha)$ is isomorphic to $\mathcal{E}(L',\alpha')$. Then $(L,\alpha)=(L',\alpha')$.*

We prove Proposition 7.17 later in this section.

We next put $\tilde L_1=\tilde L_{pt}$, $\tilde L_2=\tilde L_{st}$, $\tilde L_2=\tilde L_{st}$, $\tilde M_2=\tilde L_{pt}$ and obtain

$$\mathcal{P}\to(T^{2n},\Omega)^\vee\times\mathcal{M}(\tilde L_{st}). \tag{7.18}$$

We remark that since $L_{pt}\bullet L_{st}=1$ it follows that $\mathcal{M}(\tilde L_{st},\tilde L_{pt})=\mathcal{M}(\tilde L_{st})$.

Proposition 7.19 *$\mathcal{M}(\tilde L_{st})$ is the dual torus $(T^{2n},\Omega)^{\vee\wedge}$ of $(T^{2n},\Omega)^\vee$. The bundle* (7.18) *is the Poincaré bundle.*

The proof is easy and is omitted.

We next study the holomorphicity of m_2. We suppose that $\tilde{L}_1, \tilde{L}_2, \tilde{M}_1, \tilde{M}_2$ and $\tilde{L}_2$, $\tilde{L}_3, \tilde{M}_2, \tilde{M}_3$ and $\tilde{L}_1, \tilde{L}_3, \tilde{M}_1, \tilde{M}_3$ all satisfy Assumption 7.1. We obtain bundles (7.12) and $\mathcal{P}(\tilde{L}_2, \tilde{L}_3; \tilde{M}_2, \tilde{M}_3) \to \mathcal{M}(\tilde{L}_2, \tilde{M}_2) \times \mathcal{M}(\tilde{L}_3, \tilde{M}_3)$, $\mathcal{P}(\tilde{L}_1, \tilde{L}_3; \tilde{M}_1, \tilde{M}_3) \to \mathcal{M}(\tilde{L}_1, \tilde{M}_1) \times \mathcal{M}(\tilde{L}_3, \tilde{M}_3)$. We consider product $\mathcal{M}(\tilde{L}_1, \tilde{M}_1) \times \mathcal{M}(\tilde{L}_2, \tilde{M}_2) \times \mathcal{M}(\tilde{L}_3, \tilde{M}_3)$ and pull back bundles $\pi_{12}^*\left(\mathcal{P}(\tilde{L}_1, \tilde{L}_2; \tilde{M}_1, \tilde{M}_2)\right)$, $\pi_{23}^*\left(\mathcal{P}(\tilde{L}_2, \tilde{L}_3; \tilde{M}_2, \tilde{M}_3)\right)$, $\pi_{13}^*\left(\mathcal{P}(\tilde{L}_1, \tilde{L}_3; \tilde{M}_1, \tilde{M}_3)\right)$ on it.

Theorem 7.22 *If* $\eta(\tilde{L}_1, \tilde{L}_2, \tilde{L}_3) = 0$, *then* m_2 *defines a holomorphic map*

$$\pi_{12}^*\left(\mathcal{P}(\tilde{L}_1, \tilde{L}_2; \tilde{M}_1, \tilde{M}_2)\right) \otimes \pi_{23}^*\left(\mathcal{P}(\tilde{L}_2, \tilde{L}_3; \tilde{M}_2, \tilde{M}_3)\right) \to \pi_{13}^*\left(\mathcal{P}(\tilde{L}_1, \tilde{L}_1; \tilde{M}_2, \tilde{M}_3)\right).$$

Proof: We define notations by the following Figure 8.

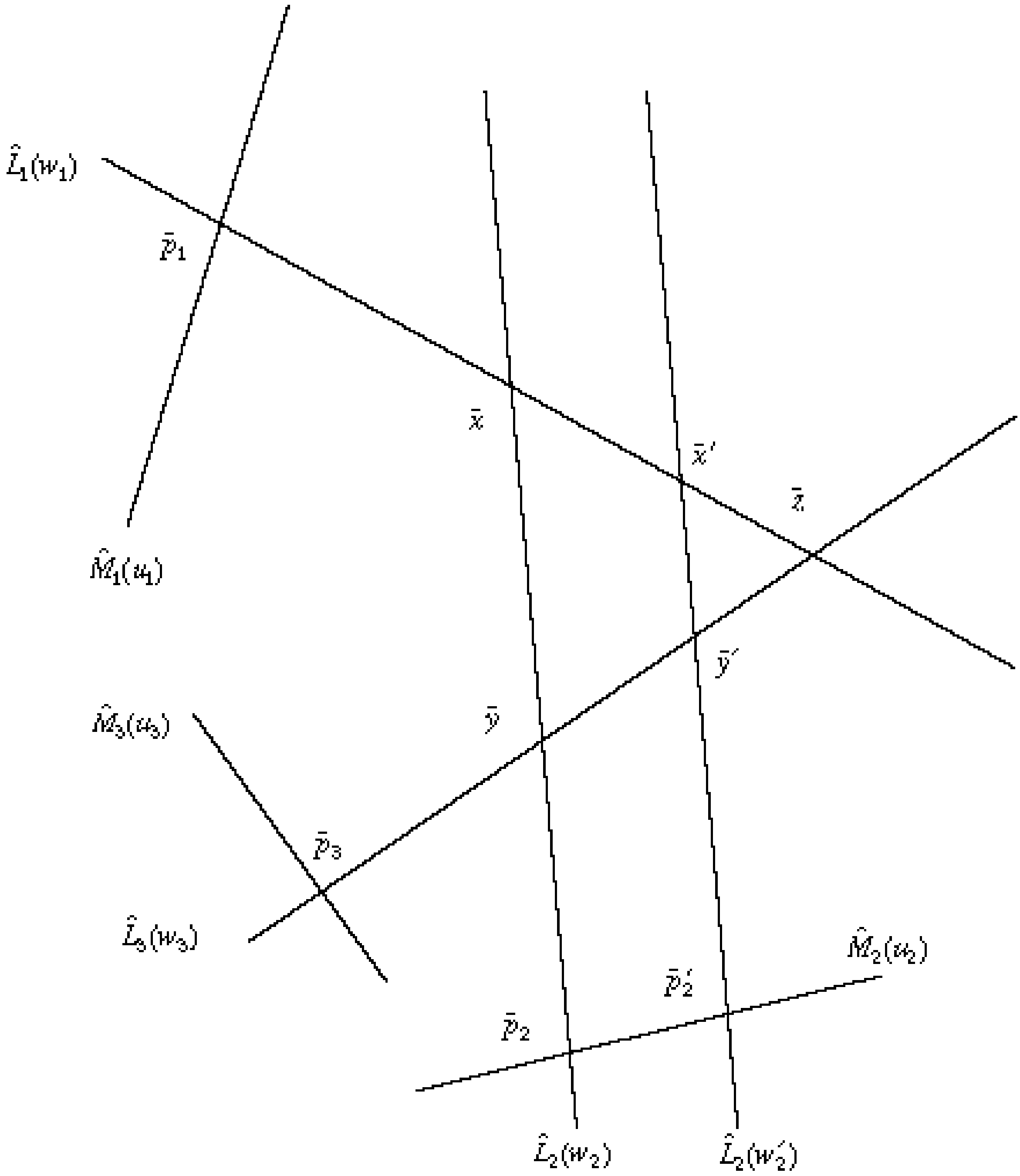

Figure 8

We prove holomorphicity on the second factor $\mathcal{M}(\tilde{L}_2, \tilde{M}_2)$. We put

$$
\begin{aligned}
& m_2\big([x],[y]\big) = \sum f_{[x],[y],[z]}(w_1,\alpha_1,w_2,\alpha_2,w_3,\alpha_3)[z] \\
& m_2\big([x'],[y']\big) = \sum f_{[x],[y],[z]}(w_1,\alpha_1,w_2',\alpha_2',w_3,\alpha_3)[z], \\
& h_{[x],[y],[z]}(w_1,\alpha_1,w_2',\alpha_2',w_3,\alpha_3) \\
& \qquad\qquad = f_{[x],[y],[z]}(w_1,\alpha_1,w_2',\alpha_2',w_3,\alpha_3) \\
& \qquad\qquad\quad \times e_{((w_1,\alpha_1),\,(w_2,\alpha_2),\,\tilde{p}_1,\,\tilde{p}_2,\tilde{x})}\Big((w_1,\alpha_1),(w_2',\alpha_2')\Big)^{-1} \\
& \qquad\qquad\quad \times e_{((w_2,\alpha_2),(w_3,\alpha_3),\,\tilde{p}_2,\,\tilde{p}_3,\tilde{z})}\Big((w_2',\alpha_2'),(w_3,\alpha_3)\Big)
\end{aligned}
$$

It suffices to show that $h_{[x][y],[z]}(w_1,\alpha_1,w_2',\alpha_2',w_3,\alpha_3)$ is holomorphic with respect to $\left(w_2',\alpha_2'\right)$. By definition $f_{[x],[y],[z]}(w_1,\alpha_1,w_2',\alpha_2',w_3,\alpha_3)$ is a sum of the terms such as

$$\exp\left(-2\pi Q(\tilde{x}',\tilde{y}',z)+2\pi\sqrt{-1}\left(\alpha_2'(\tilde{y}'-\tilde{x}')+\alpha_3(\tilde{z}-\tilde{y}')+\tilde{\alpha}_1(\tilde{x}'-\tilde{z})\right)\right). \tag{7.23}$$

on the other hand

$$\begin{aligned}\log e_{((w_1,\alpha_1),(w_2,\alpha_2),\tilde{p}_1,\tilde{p}_2,\tilde{x})}&\left((w_1,\alpha_1),(w_2',\alpha_2')\right)\\ &=-2\pi Q(\tilde{p}_2,\tilde{x},\tilde{x}',\tilde{p}_2')+2\pi\sqrt{-1}\left(\alpha_2(\tilde{x}-\tilde{p}_2)+\alpha_1(\tilde{x}'-\tilde{x})+\alpha_2'(\tilde{p}_2'-\tilde{x}')\right)\end{aligned} \tag{7.24}$$

$$\begin{aligned}\log e_{((w_2,\alpha_2),(w_3,\alpha_3),\tilde{p}_2,\tilde{p}_3,\tilde{z})}&\left((w_2',\alpha_2'),(w_3,\alpha_3)\right)\\ &=-2\pi Q(\tilde{y},\tilde{p}_2,\tilde{p}_2',\tilde{y}')+2\pi\sqrt{-1}\left(\alpha_2(\tilde{p}_2-\tilde{y})+\alpha_2'(\tilde{y}'-\tilde{p}_2')+\alpha_3(\tilde{y}-\tilde{y}')\right)\end{aligned} \tag{7.25}$$

We find that the logarithm of (7.23) minus (7.24) plus (7.25) is a complex linear map plus constant. The proof of Theorem 7.22 is complete.

We remark that Lemma 6.3 is a special case of Theorem 7.22.

We finally prove Proposition 7.17. It suffices to show the following :

Proposition 7.26 *If* $(L,\mathcal{L}(\alpha)),(L',\mathcal{L}(\alpha'))\in\mathcal{M}(\tilde{L})$ *and if* $(L,\mathcal{L}(\alpha))\neq(L',\mathcal{L}(\alpha'))$ *then* $Ext(\mathcal{E}(L,\mathcal{L}(\alpha))),\mathcal{E}(L',\mathcal{L}(\alpha')))=0$.

Proof: We use isogeny in a way similar to § 6 and find that it suffices to show the proposition in the case when $\mathcal{E}(L,\mathcal{L}(\alpha))$ is a line bundle. Let $(L_{st}(u),\mathcal{L}(\beta))\in\mathcal{M}(\tilde{L}_{st})$. We study $\mathcal{E}(L,\mathcal{L}(\alpha))\otimes\mathcal{E}(L_{st}(u),\mathcal{L}(\beta))$. Using a splitting $V=\tilde{L}_{st}\oplus\tilde{L}_{pt}$ we obtain a map $\psi_L:\hat{L}\to\tilde{L}_{st}$. Since $\left|L\bullet L_{pt}\right|=1$ it follows that ϕ_L induces $\psi_L:L\to L_{st}(0)$. We remark that $\psi_L:L\to L_{st}(0)$ is $\left|L\bullet L_{st}\right|$:1 map. We regards $u\in\tilde{L}_{pt}$ since $V=\tilde{L}_{st}\oplus\tilde{L}_{pt}$. We then obtain $\bar{u}\in V\big/\tilde{L}$.

Lemma 7.27 $\mathcal{E}(L,\mathcal{L}(\alpha))\otimes\mathcal{E}(L_{st}(u),\mathcal{L}(\beta))\cong\mathcal{E}(L+\bar{u},\mathcal{L}(\alpha+\psi_L^*(\beta)))$.

Proof: Let $(L_{pt}(v),\sigma)\in(T^{2n},\Omega)^{\vee}$, $x\in L_{pt}(v)\cap L$, $y\in L_{st}(u)\cap L$. We remark $x+u\in L_{pt}(v)\cap(L+\bar{u})$. We put $\Psi\left([x]\otimes[y]\right)=[x+u]$. It is easy to check that Ψ defines the required isomorphism.

Now we consider the map $\mathcal{M}(\tilde{L}_{st})\to\mathcal{M}oduli(\tilde{L})$, $\mathcal{E}(L,\mathcal{L}(\alpha))\mapsto\mathcal{E}(L,\mathcal{L}(\alpha))\otimes\mathcal{E}(L_{st}(u),\mathcal{L}(\beta))$, where $\mathcal{M}oduli(\tilde{L})$ is the component of the moduli space of line bundles containing $\mathcal{E}(L,\mathcal{L}(\alpha))$. It is classical that this map is $\mathrm{rank}H((T^{2n},\Omega)^{\vee},\mathcal{E}(L,\mathcal{L}(\alpha)))$ hold covering. (See Proposition (iii) in [35] 84 page.) On the other hand, we know that $(L,\mathcal{L}(\alpha))\mapsto(L+\bar{u},\mathcal{L}(\alpha+\psi_L^*(\beta)))$: $\mathcal{M}(\tilde{L})\to\mathcal{M}(\tilde{L})$ is $\left|L\bullet L_{st}\right|$ hold covering also. Hence by Proposition 2.16 $\mathcal{M}oduli(\tilde{L})\cong\mathcal{M}(\tilde{L})$. Namely

$\mathcal{E}(L, \mathcal{L}(\alpha)) \neq \mathcal{E}(L', \mathcal{L}(\alpha'))$. Since $c^1\left(\mathcal{E}(L, \mathcal{L}(\alpha))\right) = c^1\left(\mathcal{E}(L', \mathcal{L}(\alpha'))\right)$ by Theorem 5.6, we have $Ext(\mathcal{E}(L, \mathcal{L}(\alpha))), \mathcal{E}(L', \mathcal{L}(\alpha'))) = 0$. The proof of Proposition 7.26 is complete.

§ 8 Nontransversal or disconnected Lagrangian submanifolds.

In this section, we discuss two generalizations of the construction in §§ 3 - 7. One is to the case when Lagrangian submanifolds are not transversal to each other. The other is the case when Lagrangian submanifolds are disconnected or the case of a limit of a sequence of disconnected Lagrangian submanifolds. Our argument here is a bit sketchy. They are not used in §§ 9,10,11,12.

First we generalize the construction to include the case when the linear Lagrangian subspace $\tilde{L}$ may not be transversal to $\tilde{L}_{pt}$. In this case, we obtain a coherent sheaf which is not a vector bundle. We consider $\left(\tilde{L}+\tilde{L}_{pt}\right)/\tilde{L}_{pt}$ and $\left(\tilde{L}\cap\tilde{L}_{pt}\right)^{\perp}=\left\{\sigma\in\tilde{L}_{pt}^{*}\ \middle|\ \sigma|_{\tilde{L}\cap\tilde{L}_{pt}}=0\right\}$. The sum $\left(\tilde{L}+\tilde{L}_{pt}\right)/\tilde{L}_{pt}\oplus\left(\tilde{L}\cap\tilde{L}_{pt}\right)^{\perp}$ is a subspace of the universal cover $V/\tilde{L}_{pt}\oplus\tilde{L}_{pt}^{*}$ of $(T^{2n},\Omega)^{\vee}$. It is easy to see that this subspace is complex linear. The sheaf $\mathcal{E}(L(w),\alpha)$ we will obtain has a support on a subtorus parallel to $\left(\tilde{L}+\tilde{L}_{pt}\right)/\tilde{L}_{pt}\oplus\left(\tilde{L}\cap\tilde{L}_{pt}\right)^{\perp}$. Here $w\in V/\tilde{L}$, $\alpha\in\tilde{L}^{*}$. To explain the reason, we first recall the following calculation of Floer homology. Let $L_i(w)$ be affine Lagrangian submanifolds.

Proposition 8.1

$$HF^{k}\left(\mathcal{E}(L_1(w),\alpha),\mathcal{E}(L_2(v),\beta)\right)=H^{k-\mu}\left(L_1(w)\cap L_2(v),\beta|_{L_1(w)\cap L_2(v)}-\alpha|_{L_1(w)\cap L_2(v)}\right)$$

Here μ is a constant depending only on $\tilde{L}_1$, $\tilde{L}_2$ and the right hand sides is the cohomology with local coefficient.

Proof: One can prove Proposition 8.1 in the same way as [43], or by using the perturbation mentioned in [12]. We omit the detail.

We remark that the cohomology in the right hand side is trivial unless the flat connection $\beta|_{L_1(w)\cap L_2(v)}-\alpha|_{L_1(w)\cap L_2(v)}$ is trivial. Hence we have the following :

Lemma 8.2 $HF\left((L(w),\alpha),(L_{pt}(v),\sigma)\right)=0$ *unless* $w-v\in\left(\tilde{L}+\tilde{L}_{pt}\right)\mod\Gamma$ *and* $\alpha|_{\tilde{L}\cap\tilde{L}_{pt}}-\sigma|_{\tilde{L}\cap\tilde{L}_{pt}}=\mu|_{\tilde{L}\cap\tilde{L}_{pt}}$ *for some* $\mu\in\left(\tilde{L}_{pt}\cap\Gamma\right)^{\vee}$.

We put

(8.3) $$T(L(w),\alpha)=\left\{[v,\sigma]\ \middle|\ w-v\in\tilde{L}+\tilde{L}_{pt},\alpha|_{\tilde{L}\cap\tilde{L}_{pt}}-\sigma|_{\tilde{L}\cap\tilde{L}_{pt}}=0\right\}\subseteq(T^{2n},\Omega)^{\vee}.$$

We remark that $T(L(w),\alpha)$ depends only on $[w]\in\left(V/\tilde{L}\right)/\left(\Gamma/\tilde{L}\cap\Gamma\right)$,

$[\alpha] \in \tilde{L}^*\big/\left(\tilde{L} \cap \Gamma\right)^\vee$. We can prove that $T(L(w),\alpha)$ is a complex subtorus of $\left(T^{2n},\Omega\right)^\vee$. We put

(8.4.1) $$V' = \left(\tilde{L} + \tilde{L}_{pt}\right)\Big/\left(\tilde{L} \cap \tilde{L}_{pt}\right),\ \Gamma' = \left(\Gamma + \tilde{L} \cap \tilde{L}_{pt}\right)\Big/\left(\tilde{L} \cap \tilde{L}_{pt}\right).$$

(8.4.2) $$\tilde{L}' = \tilde{L}\Big/\left(\tilde{L} \cap \tilde{L}_{pt}\right),\ \tilde{L}'_{pt} = \tilde{L}_{pt}\Big/\left(\tilde{L} \cap \tilde{L}_{pt}\right).$$

(8.4.3) $$\tilde{L}'_{st} = \left(\tilde{L}_{st} \cap \left(\tilde{L} + \tilde{L}_{pt}\right)\right)\Big/\left(\tilde{L}_{st} \cap \tilde{L} \cap \tilde{L}_{pt}\right) = \tilde{L}_{st} \cap \left(\tilde{L} + \tilde{L}_{pt}\right).$$

Since $\tilde{L},\tilde{L}_{pt}$ are both Lagrangian linear subspaces it follows that V' has a symplectic structure and L', $\tilde{L}'_{pt}$ are Lagrangian subspaces of V'. In other words, V' is a symplectic reduction of V with respect to $\tilde{L} \cap \tilde{L}_{pt}$. (See [29] Chapter 2.) Γ' is a lattice of V'. Hence we obtain a mirror torus $(V'/\Gamma',\Omega')^\vee$ using $\tilde{L}'_{pt}$. We can easily find that $\tilde{L}'_{st}$ is also a Lagrangian linear subspace of V'. It is easy to see

(8.5.1) $$V'\big/\tilde{L}'_{pt} \cong \tilde{L}\big/\left(\tilde{L} \cap \tilde{L}_{pt}\right) \subseteq V\big/\tilde{L}_{pt},$$

(8.5.2) $$\left(\tilde{L}'_{pt}\right)^* \cong \left(\tilde{L} \cap \tilde{L}_{pt}\right)^\perp \subseteq \tilde{L}^*_{pt}.$$

Hence we may regard $(V'/\Gamma',\Omega')^\vee$ as a subgroup of $\left(T^{2n},\Omega\right)^\vee$. It is easy to see that $T(L(w),\alpha)$ is an orbit of $(V'/\Gamma',\Omega')^\vee$. We fix $(v_0,\sigma_0) \in T(L(w),\alpha)$ and define an isomorphism $I_{(v_0,\sigma_0)} : (V'/\Gamma',\Omega')^\vee \to T(L(w),\alpha)$ by $I_{(v_0,\sigma_0)}(g) = g(v_0,\sigma_0)$.

We next construct affine Lagrangian subspaces $L'(\overline{w};v_0,j)$ on $(V'\big/\Gamma',\Omega')$ for $(v_0,\sigma_0) \in T(L(w),\sigma)$. Let us consider $L(w) \cap L_{pt}(v_0) \subseteq T^{2n}$. It is a disjoint union of affine subtori. Let $\left(L(w) \cap L_{pt}(v_0)\right)_j$, $j = 1,\cdots,J$ be its connected components. Let $v_j \in \left(L(w) \cap L_{pt}(v_0)\right)_j$ and $\tilde{v}_j \in V$ be its lift. We may assume $\tilde{v}_j - v_0 \in \tilde{L} + \tilde{L}_{pt}$. Let $\overline{v}_j \in V'$ be the $\tilde{L} \cap \tilde{L}_{pt}$ equivalence class of $\tilde{v}_j$. $\overline{v}_j$ depends only on the component $\left(L(w) \cap L_{pt}(v_0)\right)_j$ (and v_0) and is independent of the choice of the point $v_j \in \left(L(w) \cap L_{pt}(v_0)\right)_j$. We put

(8.6) $$L'(\overline{w};v_0,j) = \hat{L}'(\overline{v}_j)\Big/\left(\Gamma' \cap \tilde{L}'\right).$$

Using the splitting $V = \tilde{L}_{st} \oplus \tilde{L}_{pt}$ we have a projection $\pi_{\tilde{L}_{pt}} : V \to \tilde{L}_{pt}$. We put

(8.7) $$\overline{\alpha}_{\sigma_0} = \alpha - \pi^*_{\tilde{L}_{pt}}\left(\sigma_0\right) \in \tilde{L}'^* \subseteq \tilde{L}^*.$$

We remark that $\tilde{L}'$ is transversal to $\tilde{L}_{pt}$. Hence by the construction of § 2, we obtain a holomorphic vector bundle $\mathcal{E}(L'(\overline{w};v_0,j),\overline{\alpha}_{\sigma_0})$ on $(V'\big/\Gamma',\Omega')^\vee$.

Lemma 8.8 *The holomorphic vector bundle* $I_{[v_j,\sigma_j]*}\mathcal{E}(L'(\overline{w};v_0,j),\overline{\alpha}_{\sigma_0})$ *on*

$T(L(w),\alpha)$ *is independent of the choice of* $(v_0,\sigma_0) \in T(L(w),\alpha)$.

The proof is straightforward and is omitted.

Definition 8.9 $\mathcal{E}(L(w),\alpha) = \bigoplus_{j=1}^{J} i_* I_{[v_j,\sigma_j]*} \mathcal{E}(L'(\overline{w};v_0,j),\overline{\alpha}_{\sigma_0})$. Where i is inclusion $T(L(w),\alpha) \subseteq \left(T^{2n},\Omega\right)^\vee$.

We can verify easily the following :

Lemma 8.10

$$Ext^k\left(\mathcal{E}(L(w),\alpha), \mathcal{E}(L_{pt}(v),\sigma)\right) = \begin{cases} 0 & \text{if } (v,\sigma) \notin \left(T^{2n},\Omega\right)^\vee \\ \bigoplus_j H^{k-\eta^*(\tilde{L}',\tilde{L}'_{pt})}(T(L(w),\alpha)_j) & \text{if } (v,\sigma) \in \left(T^{2n},\Omega\right)^\vee. \end{cases}$$

This is consistent with Proposition 8.1 and hence justify our definition.

We next consider the case when $\tilde{L}$ is not necessary transversal to $\tilde{L}_{st}$. We recall that, in the construction of $\mathcal{E}(L(w),\alpha)$ in § 2 and above, we did not assume that $\tilde{L}$ is transversal to $\tilde{L}_{st}$. But in the calculation of cohomology in sections 3 and 5, we assumed that $\tilde{L}$ is transversal to $\tilde{L}_{st}$. We remove this assumption and prove the following.

Theorem 8.11 $HF^k((L_{st},0),(L,\alpha)) \cong H^k((T^{2n},\Omega)^\vee, \mathcal{E}(L,\alpha))$.

Proof: We first show that it suffices to prove in the case when $\tilde{L}$ is transversal to $\tilde{L}_{pt}$. In fact, using the notation above, we find that

(8.12) $$H^*\left((T^{2n},\Omega)^\vee, \mathcal{E}(L(w),\alpha)\right) = \bigoplus_{j=1}^{J} H^*\left((V'/\Gamma',\Omega')^\vee, \mathcal{E}(L'(\overline{w};v_0,j),\overline{\alpha}_{\sigma_0})\right).$$

On the other hand we can easily find an isomorphism

(8.13) $$L_{st} \cap \mathcal{E}(L(w),\alpha) \cong \bigcup_j L'_{st} \cap L'(\overline{w};v_0,j).$$

Furthermore if we consider connection α on the left hand side and $\overline{\alpha}_{\sigma_0}$ on the right hand side of (8.13), then they are isomorphic to each other also. Hence by Proposition 8.1, we have

(8.14) $$HF\left((L_{st},0),(L(w),\alpha)\right) = \bigoplus_{j=1}^{J} HF\left((L'_{st},0),(L'(\overline{w};v_0,j),\overline{\alpha}_{\sigma_0})\right).$$

Thus Theorem 8.11 for $\mathcal{E}(L'(\overline{w};v_0,j),\overline{\alpha}_{\sigma_0})$ implies Theorem 8.11 for $\mathcal{E}(L(w),\alpha)$.

Hence we may and will assume that $\tilde{L}$ is transversal to $\tilde{L}_{pt}$. We put

(8.15.1) $$V'' = \left(\tilde{L} + \tilde{L}_{st}\right) \Big/ \left(\tilde{L} \cap \tilde{L}_{st}\right),\ \Gamma'' = \left(\Gamma + \tilde{L} \cap \tilde{L}_{st}\right) \Big/ \left(\tilde{L} \cap \tilde{L}_{st}\right).$$

(8.15.2) $$L'' = \tilde{L} \Big/ \left(\tilde{L} \cap \tilde{L}_{st}\right),\ L''_{st} = \tilde{L}_{st} \Big/ \left(\tilde{L}_{st} \cap \tilde{L}\right),$$

(8.15.3) $$L''_{pt} = \left(\tilde{L}_{pt} \cap \left(\tilde{L} + \tilde{L}_{st}\right)\right) \Big/ \left(\tilde{L}_{pt} \cap \tilde{L} \cap \tilde{L}_{st}\right) = \tilde{L}_{pt} \cap \left(\tilde{L} + \tilde{L}_{st}\right).$$

V'' is a symplectic reduction of V. We can prove also that L'', L''_{pt}, L''_{st} are Lagrangian linear subspaces of it. We obtain $(V''/\Gamma'', \Omega'')^{\vee}$. We remark that

(8.16.1) $$\frac{V''}{\tilde{L}''_{pt}} \cong \frac{\tilde{L}_{pt} + \left(\tilde{L} + \tilde{L}_{st}\right)}{\tilde{L}_{pt} + \left(\tilde{L} \cap \tilde{L}_{st}\right)} \cong \frac{V}{\tilde{L}_{pt} + \left(\tilde{L} \cap \tilde{L}_{st}\right)}.$$

(8.16.2) $$\tilde{L}''^{*}_{pt} = \left(\tilde{L}_{pt} \cap \left(\tilde{L} + \tilde{L}_{st}\right)\right)^{*}.$$

Therefore there exists a surjective linear map

(8.17) $$\pi : \frac{V}{\tilde{L}_{pt}} \oplus \tilde{L}^{*}_{pt} \to \frac{V''}{\tilde{L}''_{pt}} \oplus \tilde{L}''^{*}_{pt}.$$

It is easy to see that (8.17) is complex linear and induces a map $\pi : (T^{2n}, \Omega)^{\vee} \to (V''/\Gamma'', \Omega'')^{\vee}$. We then have :

Lemma 8.18 $\mathcal{L}(L(0),0) \cong \pi^{*} \mathcal{L}(L''(0),0)$.

The proof is straightforward and is omitted. We next compare $\mathcal{L}(L(0),0)$ with $\mathcal{L}(L(w),\alpha)$. Let $\bar{u} \in L(w) \cap L_{pt}(0)$ we lift it to $u \in \tilde{L}_{pt} \cong V/\tilde{L}_{st}$. Since $V = \tilde{L}_{st} \oplus \tilde{L}_{pt}$, we have an isomorphism $I : \tilde{L}_{st} \to \tilde{L}$. Let $\alpha' = \alpha \circ I$. We consider the line bundle $\mathcal{L}(L_{st}(u),\alpha')$ (we remark $c^1 \mathcal{L}(L_{st}(u),\alpha') = 0$).

Lemma 8.19 $\mathcal{L}(L(w),\alpha) \cong \mathcal{L}(L_{st}(u),\alpha') \otimes \mathcal{L}(L(0),0)$.

The proof is straightforward and is omitted. Using Lemmata 8.18 and 8.19, we can prove Theorem 8.11 by using Theorem 3.1. (The argument for it is standard one which is used to study degenerate line bundle in the theory of Abelian variety. See [35], [27].) The proof of Theorem 8.11 is now complete.

We can also generalize Theorem 6.1 in a similar way to the case when L is not transversal to L'. We omit it.

Next we study disjoint union of finitely many parallel affine Lagrangian submanifolds, $(L, \mathcal{L}) = \bigcup_{j=1}^{J} (L(w_j),\alpha_j)$. In case when $L(w_j) \neq L(w_{j'})$ we define

(8.20) $$\mathcal{E}(L,\mathcal{L}) = \oplus \mathcal{E}(L(w_j),\alpha_j).$$

(8.20) is a trivial generalization of the construction in § 2. Theorems 3.1 and 6.1 will be generalized also in a trivial way. However something interesting happens in case when $L(w_{j_1}) = L(w_{j_2})$, $\alpha_{j_1} = \alpha_{j_2}$. Our conclusion here is that those case correspond to the case when there is an indecomposable flat vector bundle on a Lagrangian submanifold.

We briefly recall the definition of Lagrangian intersection Floer homology in the case where there are flat *vector* bundles on it. Let L_1, L_2 be Lagrangian submanifolds and $\mathcal{L}_i \to L_i$ be flat vector bundles. We define Floer homology $HF((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2))$ as follows. We assume that L_1 is transversal to L_2 for simplicity. (The general case can be handled in the same way as the first half of this section.) Let $L_1 \cap L_2 = \{p_1,\cdots,p_N\}$. We put

$$CF((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)) = \bigoplus_{\{p_1,\cdots,p_N\}} Hom(\mathcal{L}_{1p_i},\mathcal{L}_{2p_i}).$$

In the general case, the boundary operator is defined in a way similar to § 2. However it is zero in our case. Hence we obtain the Floer homology $HF((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2))$. Let us use its family version. Let L be an affine Lagrangian submanifold in a torus and $\mathcal{L} \to L$ be a flat line bundle on it. We define

(8.21) $$\mathcal{E}(L,\mathcal{L})_{(v,\sigma)} = HF^n((L(v),\sigma),(L,\mathcal{L})).$$

It is a straightforward analogue of the argument of § 2, to construct a complex vector bundle $\mathcal{E}(L,\mathcal{L})$ such that (8.21) is the fiber. Let us define a holomorphic structure on it.

We may assume without loosing generality that the flat bundle $\mathcal{L} \to L$ is indecomposable. Since the fundamental group of L is abelian it follows that there exist subbundles $\mathcal{L}_j$ of $\mathcal{L}$ such that $\mathcal{L}$ has a filtration by flat bundles

(8.22) $$0 = \mathcal{L}_0 \subseteq \cdots \subseteq \mathcal{L}_J = \mathcal{L}$$

such that $\mathcal{L}_i / \mathcal{L}_{i-1} \cong \mathcal{L}(\alpha)$. Here $\mathcal{L}(\alpha)$ is the flat line bundle with holonomy α. (Note α is independent of i.) In order to define a holomorphic structure on $\mathcal{E}(L,\mathcal{L})$ we need to modify (2.11). Let us use the notation in Figure 2. We remark that $Hom(L(\sigma)_p,\mathcal{L}_p) = \mathbf{C}^J$. It has a filtration $Hom(L(\sigma)_p,\mathcal{L}_{jp}) = \mathbf{C}^j \subseteq \mathbf{C}^J$. Let us choose $\vec{e}_j[p]$ so that

(8.23.1) $$\vec{e}_j \in Hom(L(\sigma)_p,\mathcal{L}_{jp}) = \mathbf{C}^j \subseteq \mathbf{C}^J$$

(8.23.2) $$\vec{e}_j \notin Hom(L(\sigma)_p,\mathcal{L}_{j-1\,p}) = \mathbf{C}^{j-1} \subseteq \mathbf{C}^J.$$

We use it to define

(8.24)
$$\mathbf{e}_j(v',\sigma') = \exp\left(2\pi \int_{D(\tilde p, x_0(v), x_0(v'), \tilde q)} \Omega - 2\pi\sqrt{-1}\left(\sigma(x_0(v)-\tilde p) + \sigma'(\tilde q - x_0(v'))\right)\right) P_{\mathcal{L}}(\vec{e}_j)[\tilde q].$$

Here $P_{\mathcal{L}}$ is the parallel transport of $\mathcal{L}$ along the line $\tilde p \tilde q$. We use (8.24) instead of (2.11) to define a local holomorphic frame of $\mathcal{E}(L,\mathcal{L})$.

A holomorphic vector bundle $\mathcal{E}(L,\mathcal{L})$ constructed in this way is classical. Atiyah [2] found one on elliptic curves. Mukai observed that such bundles $\mathcal{E}(L,\mathcal{L})$ is transformed to an Artinian sheaf by appropriate Fourier-Mukai transformation. Artinian sheaf is an element of Hilbert scheme of points. It is related to our story in the following way.

We recall that we first considered the case when there are finitely many parallel affine Lagrangian submanifolds. The sheaf we obtained in this way is parametrized by the set $Sym_J\mathcal{M}(\tilde L)_{reg}$ of smooth points of J-th symmetric power of a mirror. There are various ways to compactify $Sym_J\mathcal{M}(\tilde L)_{reg}$. Off course $Sym_J\mathcal{M}(\tilde L)$ itself is a compactification. But more natural compactification in our context is the Hilbert scheme $Hilb_J\mathcal{M}(\tilde L)$, of points $\mathcal{M}(\tilde L)$. The relation between Hilbert scheme of points and a bundle like our $\mathcal{M}(\tilde L)$ is known. We can find it from the description of Hilber scheme near the singularity. (See [36] Theorem 1.14.) We remark that the relation between Hilbert scheme and nilpotent bundle is observed by [5].

What we are discussing here may also be regarded as a mathematically rigorous way to describe the relation between T-duality, D-brane and Chan-Paton Factor. In other words when several branes coincide to each other then enhancement of gauge symmetry occurs. (See [40] § 2.3.) In our case, we have a flat $U(J)$ bundle $\mathcal{L}$ on $L(w)$ when J Lagrangian submanifolds $L(w_j)$ coincides with $L(w)$. It seems interesting to try to generalize the story here to flat orbifold and relate it to more general gauge group than $U(J)$. It seems interesting to describe the complex structure of $Hilb_J\mathcal{M}(\tilde L)$ directly from symplectic geometry side, since this case (the phenomenon where a finitely many parallel Lagrangian submanifolds coincide in the limit) is the easiest example of the phenomenon that Lagrangian submanifolds becomes singular in the limit.

§ 9 Multi Theta series 1 (Definition and A^∞ Formulae.)

In this section we add "imaginary part" to the map m_k introduced in [12] § 5. As we mentioned there, the coefficients $c_k\left(L_1(v_1),\cdots,L_{k+1}(v_{k+1})\right)$ of multi theta series are obtained by counting holomorphic polygons whose boundaries are in $L_1(v_1)\cup\cdots\cup L_{k+1}(v_{k+1})$. Hence to define multi theta function rigorously in this way, we need to prove [12] Conjecture 5.33. In this and the next sections, we describe a way to go around this trouble and to define multi theta functions rigorously without assuming [12] Conjecture 5.33. In the next section, we also present a way to calculate $c_k\left(L_1(v_1),\cdots,L_{k+1}(v_{k+1})\right)$.

The way we proceed to do so is as follows. We will first find the properties we expect $c_k\left(L_1(v_1),\cdots,L_{k+1}(v_{k+1})\right)$ (obtained by counting holomorphic disks) satisfies. Some of them we prove rigorously (using Morse homotopy and [18]) but some others we can prove only in a heuristic way. We find an algorithm to find numbers $c_k\left(L_1(v_1),\cdots,L_{k+1}(v_{k+1})\right)$ satisfying these properties. We next prove that these properties are enough powerful to determine $c_k\left(L_1(v_1),\cdots,L_{k+1}(v_{k+1})\right)$ up to boundary. (We define what we mean by "up to boundary" later in this section.) We then use the algorithm to *define* $c_k\left(L_1(v_1),\cdots,L_{k+1}(v_{k+1})\right)$. We remark that the fact that the number $c_k\left(L_1(v_1),\cdots,L_{k+1}(v_{k+1})\right)$ is well-defined only up to "boundary" is related to the fact that (higher) Massey product is well-defined only as an element of some coset space.

Let us first define some notations. We take finitely many Lagrangian linear subspaces $\tilde L_j \subseteq V$, $j\in J$ such that $\tilde L_j \cap \Gamma \cong \boldsymbol{Z}^n$ and fix it. For simplicity we assume that they are pairwise transversal. We assume also that $st, pt \in J$. Namely $\tilde L_{st}$, $\tilde L_{pt}$ are one of the Lagrangian linear subspaces $\tilde L_j$ we consider. Let $\eta_k(L_{j_1},\cdots,L_{j_{k+1}})$ be Maslov index (Kashiwara class). It satisfies

(9.1.1) $$\eta(\tilde L_{j_1},\cdots,\tilde L_{j_{k+1}}) = \eta(\tilde L_{j_2},\cdots,\tilde L_{j_{k+1}},\tilde L_{j_1}),$$

(9.1.2) $$\eta(\tilde L_{j_1},\cdots,\tilde L_{j_{k+1}}) = \eta(\tilde L_{j_1},\cdots,\tilde L_{j_\ell},\tilde L_{j_m},\cdots,\tilde L_{j_{k+1}}) + \eta(\tilde L_{j_\ell},\cdots,\tilde L_{j_m}),$$

and $\eta(L_{j_1},L_{j_2},L_{j_3})$, $\eta^*(L_{j_1},L_{j_2}) = n-\eta(L_{j_1},L_{j_2})$ satisfy Lemma 2.25. More explicitly we define

(9.2) $$\eta(\tilde L_{j_1},\cdots,\tilde L_{j_{k+1}}) = \eta^*\left(\tilde L_{j_1},\tilde L_{j_2}\right)+\cdots+\eta^*\left(\tilde L_{j_k},\tilde L_{j_{k+1}}\right)-\eta^*\left(\tilde L_{j_1},\tilde L_{j_{k+1}}\right).$$

(Compare [23] appendix.) We put

(9.3.1) $$\mathcal{J}(n,k,d) = \left\{(j_1,\cdots,j_{k+1}) \,\middle|\, k-2-\eta(\tilde L_{j_1},\cdots,\tilde L_{j_{k+1}})+d=0\right\},$$

(9.3.2) $$(j_1,\cdots,j_{k+1})\in \mathcal{J}(n,k,\deg(j_1,\cdots,j_{k+1})),$$

(9.3.3) $$\deg(j_1,j_2) = \eta^*(L_{j_1},L_{j_2}).$$

Hereafter we write $\deg(1,\cdots,k+1)$ etc. in place of $\deg(j_1,\cdots,j_{k+1})$ etc. in case no confusion can occur. We remark that $k-2-\eta(j_1,\cdots,j_{k+1})$ is the virtual dimension of the moduli space of holomorphic polygons. More precisely we consider the following moduli space :

$$\tilde{\mathcal{M}}\left(\hat{L}_{j_1}(v_1),\cdots,\hat{L}_{j_{k+1}}(v_{k+1})\right)=\left\{(\varphi;z_1,\cdots,z_{k+1})\;\middle|\;\begin{array}{l}\varphi: D^2\to \mathbf{C}^n \text{ is holomorhpic}\\ z_i\in\partial D^2, \left(z_1,\cdots,z_{k+1}\right) \text{ respects the}\\ \text{cyclic order of } \partial D^2\\ \varphi(z_i)=p_{i,i+1}, \varphi(\partial_i D^2)\subseteq \hat{L}_{j_i}(v_i)\end{array}\right\}$$

Here $\partial_i D^2$ is a part of ∂D^2 between z_i and z_{i+1}. $PSL(2,\mathbf{R})$ acts on $\tilde{\mathcal{M}}\left(\hat{L}_{j_1}(v_1),\cdots,\hat{L}_{j_{k+1}}(v_{k+1})\right)$. Let $\mathcal{M}\left(\hat{L}_{j_1}(v_1),\cdots,\hat{L}_{j_{k+1}}(v_{k+1})\right)$ be the quotient space. Then $k-2-\eta\left(j_1,\cdots,j_{k+1}\right)$ is the virtual dimension of $\mathcal{M}\left(\hat{L}_{j_1}(v_1),\cdots,\hat{L}_{j_{k+1}}(v_{k+1})\right)$. We put

(9.4) $$c_k^{hol}[v_1,\cdots,v_{k+1}] = \#\,\mathcal{M}\left(\hat{L}_{j_1}(v_1),\cdots,\hat{L}_{j_{k+1}}(v_{k+1})\right).$$

The right hand side of (9.4) is the number counted with sign of $\mathcal{M}(\hat{L}_{j_1}(v_{j_1}),\cdots,\hat{L}_{j_{k+1}}(v_{j_{k+1}}))$. There is a trouble to make (9.4) to a rigorous definition. We explained this trouble in [12] § 5. We are going to study the property of $c_k^{hol}[v_1,\cdots,v_{k+1}]$ and use it as the axioms to define the number we actually use.

For $j_1,\cdots,j_{k+1}$, we put $\tilde{L}(j_1,\cdots,j_{k+1})(=\tilde{L}(1,\cdots,k+1))=\prod_{i=1}^{k+1}V/\tilde{L}_{j_i}$. For $(v_1,\cdots v_{k+1})\in\tilde{L}(1,\cdots,k+1)$, we obtain k affine Lagrangian subspaces $\hat{L}_{j_i}(v_i)$. We may regard $V\subseteq\tilde{L}(1,\cdots,k+1)$ by $v\mapsto(v,\cdots,v)$. We put $L(1,\cdots,k+1)=\tilde{L}(1,\cdots,k+1)/V$. Let $[v_1,\cdots v_{k+1}]\in L(1,\cdots,k+1)$ denote the equivalence class of $(v_1,\cdots v_{k+1})\in\tilde{L}(1,\cdots,k+1)$. For $(v_1,\cdots v_{k+1})\in\tilde{L}(1,\cdots,k+1)$ let $\tilde{p}_{i\,i+1}$ be the unique intersection point of $\hat{L}_{j_i}(v_i)$ and $\hat{L}_{j_{i+1}}(v_{i+1})$. We write $\tilde{p}_{i\,i+1}(v_1,\cdots,v_{k+1})=\tilde{p}_{i\,i+1}(\vec{v})$ in case we need to specify v_i. We define

(9.5) $$Q(v_1,\cdots,v_{k+1})=\sum_{i=1}^{k-1}Q(\tilde{p}_{k+1\,1}\,\tilde{p}_{i\,i+1}\,\tilde{p}_{i+1\,i+2})\in\mathbf{C}\,.$$

In other words, $Q(v_1,\cdots v_{k+1})$ is the integration of Ω over the $k+1$-gon $p_{12}\cdots p_{k\,k+1}\,p_{k+1\,1}$. Q is a quadratic function on $\tilde{L}(1,\cdots,k+1)$. Using Stokes theorem and the fact that $\tilde{L}_j$ are Lagrangian subspaces, we find

$$Q(v_1,\cdots,v_{k+1})=Q(v_{k+1},v_1,\cdots,v_k)\,.$$

We remark that Q is invariant of V and hence defines a map : $L(1,\cdots,k+1)\to\mathbf{C}$. We denote it by the same symbol Q.

We will consider a pair c_k and $\mathcal{W}$ for each $\deg(1,\cdots,k+1)\in 0$ such that

(9.6.1) $\mathcal{W}(1,\cdots,k+1)\subseteq L(1,\cdots,k+1)$,

(9.6.2) $c_k : L(1,\cdots,k+1)-\mathcal{W}(1,\cdots,k+1)\to\mathbf{Z}$.

We fix a norm $\| \ \|$ on $L(1,\cdots,k+1)$.

Axiom I

(9.7.1) There exists a positive number δ such that $c_k[v_1,\cdots,v_{k+1}]=0$, if $\mathrm{Re}\,Q(v_1,\cdots,v_{k+1})<\delta\|[v_1,\cdots,v_{k+1}]\|^2$.

(9.7.2) $c_k[cv_1,\cdots,cv_{k+1}]=c_k[v_1,\cdots,v_{k+1}]$ if $c\in\boldsymbol{R}-\{0\}$. In particular $\mathcal{W}(1,\cdots,k+1)$ is independent of $[v_1,\cdots,v_{k+1}]\mapsto[cv_1,\cdots,cv_{k+1}]$.

(9.7.3) $c_k[v_1,\cdots,v_{k+1}]$ is constant on each connected component of $L(1,\cdots,k+1)-\mathcal{W}(1,\cdots,k+1)$.

(9.7.4) $\mathcal{W}(1,\cdots,k+1)$ is a codimension one real analytic subset of $L(1,\cdots,k+1)$.

(9.7.5) $c_k[v_1,\cdots,v_{k+1}]=(-1)^{\mu}c_k[v_{k+1},v_1,\cdots,v_k]$, where $\mu=(\deg v_1+\cdots+\deg v_k)\deg v_{k+1}+k$.

We call $\mathcal{W}$ the *wall*, c_k the *coefficient function.* We remark that all of these properties are likely to be satisfied by the number c_k^{hol}. In fact, [12] Conjecture 5.33 is (9.7.4) and the most essential property (9.7.1) is a consequence of the fact that the symplectic area of holomorphic map is positive. Also the system obtained by Morse homotopy (see [16], [18]) satisfies these axioms.

Using $(\mathcal{W},c)$ satisfying Axiom I, we define a multi theta series. We put $\tilde{L}^*(1,\cdots,k+1)=\prod_{i=1}^{k+1}\tilde{L}^*_{j_i}$ and $L^*(1,\cdots,k+1)=\tilde{L}^*(1,\cdots,k+1)/V^*$. For $(\alpha_1,\cdots,\alpha_{k+1})\in\tilde{L}^*(1,\cdots,k+1)$, $(v_1,\cdots v_{k+1})\in\tilde{L}(1,\cdots,k+1)$, we put

$$H(\alpha_1,\cdots,\alpha_{k+1};v_1,\cdots v_{k+1})=\sum_{i=1}^{k+1}\alpha_i(\tilde{p}_{i\,i+1}-\tilde{p}_{i-1\,i}). \tag{9.8}$$

(See Figure 9.)

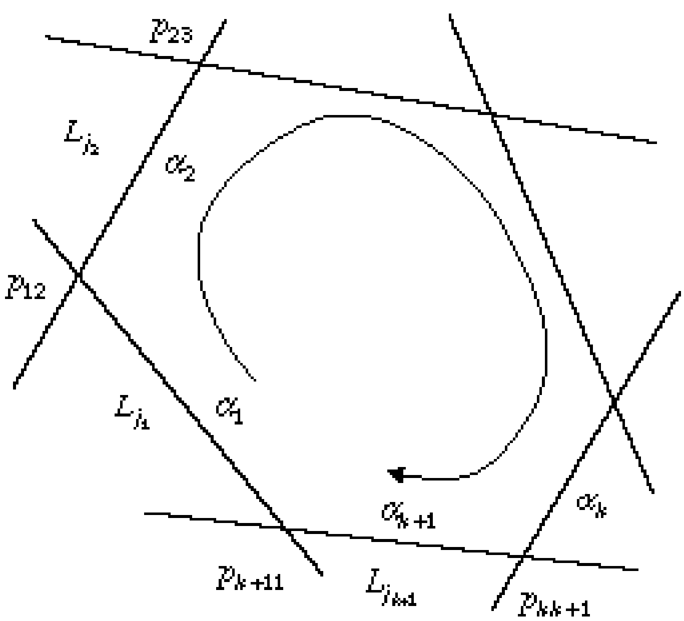

Figure 9

Here we recall that $\tilde{p}_{i\,i+1}$ is the unique intersection point of $\hat{L}_{j_i}(v_i)$ and $\hat{L}_{j_{i+1}}(v_{i+1})$ and $v_{k+2} = v_1$ etc. by convention. We remark that H is well defined as the map : $L^*(1,\cdots,k+1) \to \boldsymbol{R}$. We put

(9.9) $$\Gamma(1,\cdots,k+1) = \prod_{i=1}^{k+1} \Gamma \Big/ \left(\Gamma \cap \tilde{L}_{j_i}\right).$$

$\Gamma(1,\cdots,k+1)$ acts on $L(1,\cdots,k+1)$ by $(\gamma_1,\cdots,\gamma_{k+1})[v_1,\cdots,v_{k+1}] = [v_1+\gamma_1,\cdots,v_{k+1}+\gamma_{k+1}]$. Let $\bar{L}(1,\cdots,k+1)$ be the quotient space. We use the symbol $[[v_1,\cdots,v_{k+1}]]$ for elements of $\bar{L}(1,\cdots,k+1)$. For $[[v_1,\cdots,v_{k+1}]] \in \bar{L}(1,\cdots,k+1)$, we put

(9.10) $$\vartheta[[v_1,\cdots,v_{k+1}]] = \prod_{i=1}^{k+1}\left(L_{j_i}(v_i) \cap L_{j_{i+1}}(v_{i+1})\right) \subseteq \left(T^{2n}\right)^{k+1}.$$

$\vartheta[[v_1,\cdots,v_{k+1}]]$ is a finite set. We put $\pi : V \to T^{2n}$.

Definition 9.11 Let $(q_{12},\cdots,q_{k+11}) \in \vartheta[[v_1,\cdots,v_{k+1}]]$. We put

(9.12) $$V(q_{12},\cdots,q_{k+11}) = \left\{[v_1',\cdots,v_{k+1}'] \in L(1,\cdots,k+1) \,\middle|\, \pi(v_i) = \pi(v_i'), \pi\left(\tilde{p}_{i\,i+1}(\vec{v}')\right) = q_{i\,i+1}\right\}.$$

Note that there exists a subgroup $\Gamma_0 \subseteq \Gamma(1,\cdots,k+1)$ acting transitively on $V(q_{12},\cdots,q_{k+11})$ and

$$\left[\Gamma(1,\cdots,k+1):\Gamma_0\right] = \prod_{i=1}^{i}\left|L_{j_i}(v_i) \bullet L_{j_{i+1}}(v_{i+1})\right|.$$

Definition 9.13

(9.13) $$\begin{aligned}&\Theta_k\left([[v_1,\cdots,v_{k+1}]];[\alpha_1,\cdots,\alpha_{k+1}];\Omega\right)_{(q_{12},\cdots,q_{k+11})}\\ &= \sum_{[v_1',\cdots,v_{k+1}'] \in V(q_{12},\cdots,q_{k+11})} c_k[v_1',\cdots,v_{k+1}'] \exp\left(-2\pi Q(v_1',\cdots,v_{k+1}') + 2\pi\sqrt{-1}H(\alpha_1,\cdots,\alpha_{k+1};v_1',\cdots,v_{k+1}')\right).\end{aligned}$$

We remark that, in the case when $d=1$, $\alpha_i = 0$, $B=0$, (9.13) coincides with [12] (5.49), and in the case when $k=2$ (9.13) is m_2 in Theorem 7.22, that is a usual theta function. We also remark that the right hand side of (9.13) is *discontinuous* at the point $[[v_1,\cdots,v_{k+1}]]$ where $V(q_{12},\cdots,q_{k+11})$ intersects with $\mathcal{W}(1,\cdots,k+1)$. In the general situation, this can happen at a dense subset of $\bar{L}(1,\cdots,k+1)$ $(= T^{2n(k+1)})$. It seems likely that we can choose an appropriate perturbation so that the image $\overline{\mathcal{W}}(1,\cdots,k+1)$ of $\mathcal{W}(1,\cdots,k+1)$ in $\bar{L}(1,\cdots,k+1)$ is a union of finitely many compact submanifolds. (The author can prove it in case $k+1 \le 4$.) In that case, the set where Θ_k is disconnected is a union of finitely many codimension 1 compact submanifolds of $\bar{L}(1,\cdots,k+1)$. However in general the proof of it looks cumbersome and we do not need it in the application of later sections. So we do not try to proof it .

Definition 9.14 The set of all points $[[v_1,\cdots,v_{k+1}]]$ where $V(q_{12},\cdots,q_{k+11})$ intersects with $\mathcal{W}(1,\cdots,k+1)$ for some $(q_{12},\cdots,q_{k+11})\in\vartheta[[v_1,\cdots,v_{k+1}]]$ is called the *wall* also and is written as $\overline{\mathcal{W}}(1,\cdots,k+1)$.

Proposition 9.15 *If Axiom* I *is satisfied, then* (9.13) *converges on* $\overline{L}(1,\cdots,k+1)-\overline{\mathcal{W}}(1,\cdots,k+1)$ *pointwise and on* $\overline{L}(1,\cdots,k+1)$ *as a distribution.*

Using (9.7.1) the proof is easy and is omitted. We choose $\tilde{M}_j$ which is transversal to $\tilde{L}_j$ and satisfies $\tilde{M}_j\cap\Gamma\cong\boldsymbol{Z}^n$. We consider $\prod\mathcal{M}(\tilde{L}_{j_i},\tilde{M}_{j_i})$ introduced in §7, and the pull backs : $\pi_{12}^*\mathcal{P}(\tilde{L}_{j_1},\tilde{L}_{j_2};\tilde{M}_{j_1},\tilde{M}_{j_2})$ etc. of the bundles defined in § 7. Using Θ_k we define

$$
m_k:\pi_{12}^*\mathcal{P}(\tilde{L}_{j_1},\tilde{L}_{j_2};\tilde{M}_{j_1},\tilde{M}_{j_2})\otimes\cdots\otimes\pi_{k\,k+1}^*\mathcal{P}(\tilde{L}_{j_k},\tilde{L}_{j_{k+1}};\tilde{M}_{j_k},\tilde{M}_{j_{k+1}})
\to\pi_{1\,k+1}^*\mathcal{P}(\tilde{L}_{j_1},\tilde{L}_{j_{k+1}};\tilde{M}_{j_1},\tilde{M}_{j_{k+1}}). \tag{9.16}
$$

Definition 9.17

$$
\begin{aligned}
&m_k\big([q_{12}]\otimes\cdots\otimes[q_{k\,k+1}]\big)\\
&\quad=\sum_{q_{k+11}}\Theta_k\big([v_1,\cdots,v_{k+1}];[\alpha_1,\cdots,\alpha_{k+1}];\Omega\big)_{(q_{12},\cdots,q_{k+11})}[q_{1\,k+1}].
\end{aligned}
$$

We remark that $\deg m_k=2-k$.

In a similar way as the proof of Theorem 7.22, we find that "m_k is holomorphic outside $\overline{\mathcal{W}}(1,\cdots,k+1)$". However since $\overline{\mathcal{W}}(1,\cdots,k+1)$ may be dense, we need to be a bit careful to state it. We take a sequence of compact subsets $\mathcal{W}_{(e)}\subseteq\overline{\mathcal{W}}(1,\cdots,k+1)$ such that $\mathcal{W}_{(e)}\subseteq\operatorname{Int}\mathcal{W}_{(e+1)}$ and $\bigcup_e\mathcal{W}_{(e)}=\overline{\mathcal{W}}(1,\cdots,k+1)$ Let $\overline{\mathcal{W}}_{(e)}\subseteq\overline{L}(1,\cdots,k+1)$ be the image of $\mathcal{W}_{(e)}$. Then $\overline{\mathcal{W}}_{(e)}$ is compact and can be chosen to be a union of finitely many codimension 1 submanifolds with smooth boundary. For $[[v_1,\cdots,v_{k+1}]]\in\overline{\mathcal{W}}_{(e)}$, we choose a normal vector $\vec{n}=(n_1,\cdots,n_{k+1})$ to $\overline{\mathcal{W}}_{(e)}$ and put

$$
\big(\Delta_{(e)}c_k\big)[v_1',\cdots,v_{k+1}']=\lim_{\varepsilon\to0}\big(c_k[v_1'+\varepsilon n_1,\cdots,v_{k+1}'+\varepsilon n_{k+1}]-c_k[v_1'-\varepsilon n_1,\cdots,v_{k+1}'-\varepsilon n_{k+1}]\big). \tag{9.18}
$$

We use it to define :

$$
\begin{aligned}
&\big(\Delta_{(e)}\Theta_k\big)\big([[v_1,\cdots,v_{k+1}]];[\alpha_1,\cdots,\alpha_{k+1}];\Omega\big)_{(q_{12},\cdots,q_{k+11})}\\
&\quad=\sum_{[v_1',\cdots,v_{k+1}']\in\mathcal{W}_{(e)}\cap V(q_{12},\cdots,q_{k+11})}\big(\Delta_{(e)}c_k\big)[v_1',\cdots,v_{k+1}']\\
&\qquad\exp\big(-2\pi Q(v_1',\cdots,v_{k+1}')+2\pi\sqrt{-1}H(\alpha_1,\cdots,\alpha_{k+1};v_1',\cdots,v_{k+1}')\big).
\end{aligned} \tag{9.19}
$$

Using $\Delta_{(e)}\Theta_k$ we define $\Delta_{(e)}m_k$ in the same way as Definition 9.17.

For each $[[v_1,\cdots,v_{k+1}]]\in\overline{\mathcal{W}}_{(e)}$, let $Hev_{(e)}$ be a Heaviside function defined in a neighborhood of $[[v_1,\cdots,v_{k+1}]]$ such that $[[v_1,\cdots,v_{k+1}]]$ takes value in $\{0,1\}$ and jumps at $\overline{\mathcal{W}}_{(e)}$. Its

Dolbeault derivative $\bar\partial\, Hev_{(e)}$ is a (0 1) current on $\bar L(1,\cdots,k+1)$.

Proposition 9.20 $\bar\partial m_k = \lim_{e\to\infty} \Delta_{(e)} m_k \wedge \bar\partial\, Hev_{(e)}$. *Here the right hand side converges as a current.*

Proof: By the proof of Theorem 7.22, we find that $\bar\partial m_k$ is nonzero only because of the discontinuity of m_k. The proposition then follows from the definitions.

Remark 9.21 At first sight it may look strange to consider m_k which is discontinuous at a dense subset. I hope that after the arguments of § 11,12, where we regards it as current and use it in Dolbeault complex, it looks more natural to do so.

The operator m_k in Definition 9.17 is a family version of the higher multiplication of Floer cohomology which is introduced in [13] in the case when the coefficient function is c_k^{hol}, (by extending an idea of the definition of m_2 due to Donaldson [9].) As we remarked already, for the coefficient function c_k^{hol}, our m_k may be ill-defined on a (countably) infinitely many union of codimension one submanifolds. Namely higher multiplication of Floer cohomology is well-defined only at a Bair subset. This is what we asserted in [13]. For the "family version" we are discussing here, we need more and we regard m_k as a distribution.

Our next purpose is to define the notion that two coefficient functions are homologous to each other and show that m_k in Definition 9.17, up to appropriate chain homotopy, depends only on the homology class of coefficient functions. This is important for us since the author can calculate the coefficient function c^{hol} only up to boundary. We need to discuss A^∞ formulae for this purpose. One messy matter in introducing A^∞ structure is sign. The sign is related to suppersymmetry and is in fact an important matter. To simplify the sign we use a trick due to Getzler-Jones [19]. We order J and let $Ts(J)$ be the graded vector space spanned by the symbols $\left[e_{j_1j_2}\middle|\cdots\middle|e_{j_kj_{k+1}}\right]$, $j_1<\cdots<j_{k+1}$. (We write sometimes $\left[e_{12}\middle|\cdots\middle|e_{kk+1}\right]$ for simplicity.) We put

$$\deg e_{j_1,j_2} = \deg(\tilde L_{j_1},\tilde L_{j_2}),\ \ \deg\left[e_{j_1j_2}\middle|\cdots\middle|e_{j_kj_{k+1}}\right] = \sum \deg e_{j_ij_{i+1}} + k.$$

(Here we shift the degree of $\left[e_{j_1j_2}\middle|\cdots\middle|e_{j_kj_{k+1}}\right]$ by k. This construction, the suspension in the terminology of [19], is the main idea to simplify the sign.)

Suppose we have integers $b_{j_1,\cdots,j_{k+1}} (= b_{12\cdots k+1})$ for each $\deg(1,\cdots,k+1)=0$. We use it to obtain a map

$$\left[e_{12}\middle|\cdots\middle|e_{kk+1}\right] \mapsto b_{1,\cdots,k+1}[e_{1k+1}], \tag{9.22}$$

of degree -1. (Note $\deg(1,2)+\cdots+\deg(k,k+1)-\deg(1,k+1)=\eta(1,\cdots,k+1) = 2-k$.) We extend it to a map $b: Ts(J)\to Ts(J)$ of degree -1 by

$$(9.23)\qquad b\left[e_{12}\middle|\cdots\middle|e_{k\,k+1}\right]=\sum_{\ell<m}(-1)^{\sum_{i=1}^{\ell-1}\deg e_{ii}+1}b_{\ell,\cdots,m}\left[e_{12}\middle|\cdots\middle|e_{\ell-1\,\ell}\middle|e_{\ell m}\middle|e_{m\,m+1}\middle|\cdots\middle|e_{k\,k+1}\right].$$

Here the sum is taken for all ℓ, m such that $\deg(\ell,\cdots,m)=0$. (Note that the sign in (9.23) is usual one since degree of b is -1.) We say that b is a *derivative* if $b\circ b=0$. Next let $T(J)$ be the graded vector space spanned by $e_{12}\otimes\cdots\otimes e_{k\,k+1}$. Here $\deg\left(e_{12}\otimes\cdots\otimes e_{kk+1}\right)=\sum\deg e_{i\,i+1}$. (No degree shift this time.) We define s by :

$$se_{12}\otimes\cdots\otimes se_{kk+1}=\left[e_{12}\middle|\cdots\middle|e_{kk+1}\right].$$

We then find

$$(9.24)\qquad \left(s^{-1}\otimes\cdots\otimes s^{-1}\right)\left[e_{12}\middle|\cdots\middle|e_{k\,k+1}\right]=(-1)^{\mu}\left(e_{12}\otimes\cdots\otimes e_{k\,k+1}\right),$$

where $\mu(1,\cdots,k+1)=(k-1)\deg e_{12}+(k-2)\deg e_{23}+\cdots+\deg e_{k-1\,k}+k(k-1)/2$. (Note that sign is determined from the fact that $\deg s=1$.) We define

$$(9.25)\qquad b_k=s\circ c_k\circ\left(s^{-1}\otimes\cdots\otimes s^{-1}\right)$$

and

$$(9.26)\qquad c_k\left(e_{12}\otimes\cdots\otimes e_{k\,k+1}\right)=c_{1,\cdots,k+1}e_{1\,k+1}.$$

In [19], Getzler-Jones write $c_k=s^{-1}\circ b_k\circ\left(s\otimes\cdots\otimes s\right)$. However to have the sign $(-1)^{\mu(1,\cdots,k+1)}$ they obtained, it seems that (9.25) is a correct definition. Note $\left(s^{-1}\otimes\cdots\otimes s^{-1}\right)\circ\left(s\otimes\cdots\otimes s\right)=(-1)^{k(k-1)/2}$.

By definition $c_{1\cdots k+1}=(-1)^{\mu(1,\cdots,k+1)}b_{1\cdots k+1}$. $b\circ b=0$ is equivalent to an equation

$$\sum(-1)^{\mu(1,\cdots,k+1;\,\ell,m)}c_{1,\cdots\ell,m,\cdots,\,k+1}c_{\ell,\cdots,m}=0.$$

Here the sum is taken for all ℓ, m such that $\deg(\ell,\cdots,m)=\deg(1,\cdots,\ell,m,\cdots,k+1)=0$, and

$$(9.27)\qquad \begin{aligned}\mu(1,\cdots,k+1;\ell,m)&=\mu(1,\cdots,k+1)+\mu(\ell,\cdots,m)\\&\quad+\mu(1,\cdots,\ell,m,\cdots,k+1)+\sum_{i=1}^{\ell-1}\deg e_{ii+1}.\end{aligned}$$

The sign here is messy and complicated. But in fact we do not need to calculate it so much, since most of the calculation will be done by using b in place of c. (The reason we introduced c (and m) is that the degree coincides with natural one (in sheaf cohomology) for them.)

Now we go back to our situation. We consider coefficient functions $c_k[v_1,\cdots,v_{k+1}]$ for $\deg(1,\cdots,k+1)=0$, $[v_1,\cdots,v_{k+1}]\in L(1,\cdots,k+1)$.

Axiom II

$$\sum (-1)^{\mu(1,\cdots,k+1;\ell,m)} c_{k-m+\ell}\left[v_1,\cdots,v_\ell,v_m,\cdots,v_{k+1}\right] c_{m-\ell+1}\left[v_\ell,\cdots,v_m\right] = 0,$$

where $\mu(1,\cdots,k+1;\ell,m)$ *is as in* (9.27).

Lemma 9.28 *If the coefficient functions satisfies Axiom* II, *then we have*

$$\sum (-1)^{\mu(1,\cdots,k+1;\ell,m)} m_{k-m+\ell+1}\left(x_{12},\cdots,x_{\ell-1\,\ell},m_{m-\ell+1}\left(x_{\ell\,\ell+1},\cdots,x_{m-1m}\right),x_{m\,m+1},\cdots,x_{k\,k+1}\right) = 0.$$

Proof: Immediate from the definition and

$$\begin{aligned} &Q(v_1,\cdots,v_{k+1}) = Q(v_1,\cdots,v_\ell,v_m,\cdots v_k) + Q(v_\ell,\cdots,v_m),\\ &H\left(\alpha_1,\cdots,\alpha_{k+1};v_1,\cdots,v_{k+1}\right) = H(\alpha_1,\cdots,\alpha_\ell,\alpha_m,\cdots\alpha_{k+1};v_1,\cdots,v_\ell,v_m,\cdots v_k)\\ &\qquad\qquad +H\left(\alpha_\ell,\cdots,\alpha_m;v_\ell,\cdots,v_m\right). \end{aligned}$$

We next define the notion that two coefficient functions to be homologous. Let $f_{j_1,\cdots,j_{k+1}} (= f_{1,\cdots,k+1}) \in \boldsymbol{Z}$ for $\deg(1,\cdots,k+1) = 1$. We assume

(9.29) $$f_{j_1,j_2} = 1 \text{ for } (j_1,j_2) \in \mathcal{J}(n,2,1).$$

We have a map $\left[e_{12}|\cdots|e_{k\,k+1}\right] \mapsto f_{1,\cdots,k+1}[e_{1\,k+1}]$ of degree 0. We extend it to $f : Ts(J) \to Ts(J)$ by

(9.30) $$f\left[e_{12}|\cdots|e_{k\,k+1}\right] = \sum f_{a(1)\cdots a(2)} f_{a(2)\cdots a(3)} \cdots f_{a(e-1)\cdots a(e)} \left[e_{a(1)a(2)}|\cdots|e_{a(e-1)a(e)}\right].$$

Here the sum is taken over all $1 = a(1) < \cdots < a(e) = k+1$ such that $\deg(a(e),\cdots,a(e+1)) = 1$. We remark that there is no sign in (9.30) since f is of degree 0. We then put

(9.31) $$f_k = s \circ d_k \circ \left(s^{-1} \otimes \cdots \otimes s^{-1}\right).$$

Let $c_k^1[v_1,\cdots,v_{k+1}]$ and $c_k^2[v_1,\cdots,v_{k+1}]$ be two coefficient functions satisfying Axioms I and II. Let $b_k^1[v_1,\cdots,v_{k+1}]$, $b_k^1[v_1,\cdots,v_{k+1}]$ be functions corresponding to them by (9.25).

Definition 9.32 We say that c^1 is *homologous* to c^2 if there exists integer valued function $f_{1,\cdots,k+1}[v_1,\cdots,v_{k+1}]$ for $\deg(1,\cdots,k+1) = 1$ satisfying (9.29) such that

(9.33.1) $f \circ b^1 = b^2 \circ f$.

(9.33.2) If $b_k[v_1,\cdots,v_{k+1}] \neq 0$, $k \neq 2$, then $\operatorname{Re} Q(v_1,\cdots,v_{k+1}) < \delta \|[v_1,\cdots,v_{k+1}]\|^2$.

(9.33.3) $b_k[cv_1,\cdots,cv_{k+1}] = b_k[v_1,\cdots,v_{k+1}]$.

Let m_k^1 and m_k^2 be maps obtained from c_k^1 and c_k^2. We define d_k by (9.31). d_k

also defines a map n_k. (We use (9.33.2) to show the convergence.) We find that (9.33.1) implies

$$\sum \pm n_{k-m+\ell}\left(x_{12},\cdots,x_{\ell-1\,\ell},m^1_{m-\ell+1}\left(x_{\ell\ell+1},\cdots,x_{m-1\,m}\right),x_{m\,m+1},\cdots,x_{k\,k+1}\right)$$
$$(9.34) \qquad = \sum \pm m^2_i\left(n_{a(2)-a(1)+1}\left(x_{a(1)\,a(1)+1},\cdots,x_{a(2)-1\,a(2)}\right),\right.$$
$$\left.\cdots,n_{a(i)-a(i-1)+1}\left(x_{a(i-1)\,a(i-1)+1},\cdots,x_{a(i)-1\,a(i)}\right)\right).$$

(9.34) means that n defines an A^∞ functor from the A^∞ category determined by m^1_k to one determined by m^2_k. (See [15] for its definition.) (9.29) implies that n_1 is identity. Hence n is a homotopy equivalence in the sense of [15].

We explain its implication by an example. Consider the case $k=3$. We assume

$$(9.35) \qquad m_2(x_{12},x_{23}) = m_2(x_{23},x_{34}) = 0\,.$$

This is the situation where we can define Massey triple product. In our case, it is represented by $m^1_3(x_{12},x_{23},x_{34})$ or $m^2_3(x_{12},x_{23},x_{34})$. (We assume $m^1_2 = m^2_2$ for simplicity.) (9.34) implies

$$m^2_3(x_{12},x_{23},x_{34}) - m^1_3(x_{12},x_{23},x_{34})$$
$$(9.36) \qquad = \pm n_2(x_{12},m_2(x_{23},x_{34})) \pm n_2(m_2(x_{12},x_{23}),x_{34})$$
$$\pm m_2(x_{12},n_2(x_{23},x_{34})) \pm m_2(n_2(x_{12},x_{23}),x_{34})$$
$$= \pm m_2(x_{12},n_2(x_{23},x_{34})) \pm m_2(n_2(x_{12},x_{23}),x_{34})$$

It follows that $m^1_3(x_{12},x_{23},x_{34})$ coincides with $m^2_3(x_{12},x_{23},x_{34})$ modulo elements of the form $m_2(x_{12},\bullet)+m_2(\bullet,x_{34})$. This is consistent with the usual definition of Massey triple product. Thus the maps n determines the ambiguity of (higher) Massey products systematically. We will apply (9.34) more systematically in §§ 11 and 12.

§ 10 Multi Theta Series 2 (Calculation of the coefficients).

Now we determine the homology class of coefficient function. For this purpose, we add more axioms so that it is enough to characterize the homology class. To find appropriate axioms, we further study the counting problem of holomorphic polygons. Namely we study the structure of the wall of c_k^{hol}. For this purpose, we recall [12] Lemma 5.36. It implies Lemma 10.3 below, if we assume transversality. We need some notations and remarks to state the lemma.

Let $\deg(1,\cdots,k+1)=0$. In § 9 we considered the wall $\mathcal{W}(1,\cdots,k+1)$ as a subset of $L(1,\cdots,k+1)$. We can regard it also as a V invariant subset of $\prod_{i=1}^{k+1} V \big/ \tilde{L}_{j_i} = \tilde{L}(1,\cdots,k+1)$. We write $\tilde{\mathcal{W}}(1,\cdots,k+1)$ in the later case.

We next remark that, in case when $\deg(1,\cdots,k+1)=0$ and $k+1>3$, the quadratic function $\operatorname{Re} Q : L(1,\cdots,k+1) \to \boldsymbol{R}$ has negative eigenvalue. (See Lemma 10.20 below.) It follows that there exists a domain in $L(1,\cdots,k+1)$ where the coefficient function c_k must vanish by Axiom I. This implies that if we know the wall $\tilde{\mathcal{W}}(1,\cdots,k+1)$ as a cycle then we can determine c_k. More precisely we regard the wall $\tilde{\mathcal{W}}(1,\cdots,k+1)$ as a cycle as follows. We first triangulate $\tilde{\mathcal{W}}(1,\cdots,k+1)$. Let Δ be one of its top dimensional simplex. We assume that Δ is oriented. Since $\tilde{\mathcal{W}}(1,\cdots,k+1)$ is codimension one, we have an oriented normal vector $\vec{n}(p)$ for $p \in \Delta$. We consider an integer

$$c(\Delta) = \lim_{\varepsilon\to 0}\big(c_k(p+\varepsilon\vec{n}(p)) - c_k(p-\varepsilon\vec{n}(p))\big).$$

The sum $\sum c(\Delta)\Delta$ is a cycle. We denote this cycle by $\tilde{\mathcal{W}}(1,\cdots,k+1)$ by abuse of notation. In a similar way, we may regard c_k as a top dimensional chain in $\tilde{L}(1,\cdots,k+1)$ as follows. Let U_l be connected components of $\tilde{L}(1,\cdots,k+1) - \tilde{\mathcal{W}}(1,\cdots,k+1)$. We put

$$C(1,\cdots,k+1;c) = \sum c_k(U_l)[U_l].$$

Here $c_k(U_l)$ is the value of c_k at a point on U_l. ($c_k(U_l)$ is independent of the point on U_l by (9.7.3).) $C(1,\cdots,k+1;c)$ is a top dimensional chain and we have

$$\partial C(1,\cdots,k+1;c) = \tilde{\mathcal{W}}(1,\cdots,k+1), \tag{10.1}$$

as chains. Obviously $\tilde{\mathcal{W}}(1,\cdots,k+1)$ and (10.1) determine m_k if Q is negative somewhere on $\tilde{L}(1,\cdots,k+1)$. We remark that in the case when $\deg(1,2,3)=0$, Q is positive definite on $\tilde{L}(1,2,3)$. So $\tilde{\mathcal{W}}(1,2,3)$ determines c_2 only up to constant. However, in this case, we already know that c_2 is ± 1 everywhere and $\tilde{\mathcal{W}}(1,2,3)$ is empty, by [12] Theorem 4.18.

We generalize the definition of $C(1,\cdots,k+1;c^{hol})$ to the case when $\deg(1,\cdots,k+1)=d$ with $d>0$ by

$$C(1,\cdots,k+1;c^{hol}) = \left\{(v_1,\cdots,v_{k+1}) \in \tilde{L}(1,\cdots,k+1) \,\middle|\, \tilde{\mathcal{M}}(\hat{L}_{j_1}(v_1),\cdots,\hat{L}_{j_{k+1}}(v_{k+1})) \neq \emptyset\right\}. \tag{10.2}$$

"Lemma 10.3" *Let* $\deg(1,\cdots,k+1)=0$. *The wall* $\tilde{\mathcal{W}}^{hol}(1,\cdots,k+1)$ *of* c^{hol} *is a sum of*

$$C(j_\ell,\cdots,j_m;c^{hol})\times C(j_1,\cdots,j_\ell,j_m,\cdots,j_{k+1};c^{hol}) \tag{10.4}$$

here we take the sum over ℓ,m *such that* $\deg(\ell,\cdots,m)=1$ *or* $\deg(1,\cdots,\ell,m,\cdots,k+1)=1$.

We remark that $\deg(\ell,\cdots,m)=1$ implies that the virtual dimension of $\tilde{\mathcal{M}}(\hat{L}_{j_\ell}(v_\ell),\cdots,\hat{L}_{j_m}(v_m))$ is -1. Hence $C(\ell,\cdots,m;c^{hol})$ is a codimension one chain of $\tilde{L}(j_\ell,\cdots,j_m)$ in that case. (In case we regard $C(\ell,\cdots,m;c^{hol})$ as a chain, (10.4) is a bit imprecise, since we need to consider multiplicity and sign.)

As mentioned above, we do not prove Lemma 10.3 rigorously because of transversality problem. (One can certainly find a perturbation so that Lemma 10.3 holds after perturbation. But we do not need to work out this heavy job.) So instead we take it as an axiom. However to motivate the axiom, we explain the idea of the "proof" of "Lemma 10.3".

In fact, we already explained the most essential part of the "proof" in [12] § 5. Namely [12] Lemma 5.36 "implies" that if $(v_1,\cdots,v_{k+1})\in\tilde{\mathcal{W}}^{hol}(1,\cdots,k+1)$ then there exist ℓ,m such that

$$\tilde{\mathcal{M}}\Big(\hat{L}_{j_\ell}(v_\ell),\cdots,\hat{L}_{j_m}(v_m)\Big)\times\tilde{\mathcal{M}}\Big(\hat{L}_{j_1}(v_1),\cdots,\hat{L}_{j_\ell}(v_\ell),\hat{L}_{j_m}(v_m),\cdots,\hat{L}_{j_{k+1}}(v_{k+1})\Big) \tag{10.5}$$

is nonempty. Namely if $(v_1,\cdots,v_{k+1})=\lim_{e\to\infty}(v_1^{(e)},\cdots,v_{k+1}^{(e)})$ then the $k+1$ gons in $\tilde{\mathcal{M}}\Big(\hat{L}_{j_1}(v_1^{(e)}),\cdots,\hat{L}_{j_{k+1}}v_{k+1}^{(e)})\Big)$ splits into a union of $m-\ell+1$-gon and $k-m+\ell+2$ gon. By dimension counting, we find that the virtual dimension of (10.5) is -1. We consider the following three cases.

(10.6.1) $\deg(\ell,\cdots,m)=1$, $\deg(1,\cdots,\ell,m,\cdots,k+1)=0$.
(10.6.2) $\deg(\ell,\cdots,m)=0$, $\deg(1,\cdots,\ell,m,\cdots,k+1)=1$.
(10.6.3) $\deg(\ell,\cdots,m)>1$ or $\deg(1,\cdots,\ell,m,\cdots,k+1)>1$.

It is easy to see that if (10.6.1) or (10.6.2) holds then we have a term like (10.4). On the other hand, in the case when (10.6.3) is satisfied, the set of all $(v_1,\cdots,v_{k+1})$ such that (10.5) is nonempty is of codimension higher than one. Hence to find the wall as a codimension one cycle, we do not need to consider (10.6.3). This completes the "proof" of "Lemma 10.3".

To find a good axiom, we need to study the boundary of $C(1,\cdots,k+1;c^{hol})$ in the case when $\deg(1,\cdots,k+1)=d$, $d>0$ also. The "result" is the following "Lemma 10.8". We remark that if $\deg(1,\cdots,k+1)=d$ then the (virtual) codimension of $C(\ell,\cdots,m;c^{hol})$ is d. The following lemma follows from (9.1.2).

Lemma 10.7 *If* $\deg(1,\cdots,k+1)=d$, $\deg(\ell,\cdots,m)=d_1$, *and* $(1,\cdots,\ell,m,\cdots,k+1)\in d_2$ *then* $d+1=d_1+d_2$.

We next show :

"Lemma 10.8" *Let* $\deg(1,\cdots,k+1)=d$. *The boundary of the chain* $C(1,\cdots,k+1;c^{hol})$ *is a sum of*

$$\pm C(j_\ell,\cdots,j_m;c^{hol})\times C(j_1,\cdots,j_\ell,j_m,\cdots,j_{k+1};c^{hol}), \tag{10.9}$$

where the sum is taken over all ℓ,m *such that* $\deg(\ell,\cdots,m)=d_1\geq 0$, $\deg(1,\cdots,\ell,m,\cdots,k+1)=d_2\geq 0$.

"Proof" Again by [12] Lemma 5.36, we find that $(v_1,\cdots,v_{k+1})\in\partial C(1,\cdots,k+1;c^{hol})$ if and only if (10.5) is nonempty. We take d_1 and d_2 as in "Lemma 10.8". In the case when $d_1,d_2\geq 0$ we find (10.9). Otherwise we have $d_1>d+1$ or $d_2>d+1$ by Lemma 10.7. Then the set of all $(v_1,\cdots,v_{k+1})$ such that (10.5) is nonempty is of codimension higher than d. Hence, by dimension counting, it is 0 as a codimension $d+1$ chain. The "proof" of "Lemma 10.8" is complete.

Now we take Lemmata 10.3 and 10.8 as axioms. Namely we consider :

Axiom III There exist locally finite chain $C^{(d)}(1,\cdots,k+1)$ on $\tilde{L}(1,\cdots,k+1)$ for $\deg(1,\cdots,k+1)=d$, with the following properties.

(10.10.1) The codimension of $C^{(d)}(1,\cdots,k+1)$ is d.
(10.10.2) $C^{(d)}(1,\cdots,k+1)$ is invariant of V action on $\tilde{L}(1,\cdots,k+1)$.
(10.10.3) $C^{(d)}(1,\cdots,k+1)$ is invariant of the map $[v_1,\cdots,v_k]\mapsto[cv_1,\cdots,cv_k]$.
(10.10.4) Let $\| \ \|$ be a norm on $L(1,\cdots,k+1)$ Then, there exists $\delta>0$ such that $(v_1,\cdots,v_{k+1})\in C^{(d)}(1,\cdots,k+1)$ implies

$$\operatorname{Re} Q(v_1,\cdots,v_{k+1})>\delta\|[v_1,\cdots,v_{k+1}]\|^2.$$

(10.10.5) $\partial C^{(d)}(1,\cdots,k+1)$ is a sum of

$$(-1)^\mu C^{(d_1)}(\ell,\cdots,m)\times C^{(d_2)}(1,\cdots,\ell,m,\cdots,k+1),$$

where the sum is taken over all ℓ,m *such that* $\deg(\ell,\cdots,m)=d_1\geq 0$, $\deg(1,\cdots,\ell,m,\cdots,k+1)=d_2\geq 0$.

We fix the sign in (10.10.5) later during the proof of Theorem 10.17. We can write (10.10.5) roughly as

$$\partial C^{(d)}+\sum\pm C^{(d')}\circ C^{(d+1-d')}=0. \tag{10.11}$$

We regard (10.11) as a Maurer-Cartan equation or Batalin-Vilkovisky master equation as we mentioned in the introduction. It is natural that we find it here, since we here are studying a family of A^∞ categories parametrized by $(v_1,\cdots,v_{k+1})$ and (10.11) describes a deformation of A^∞ structure (as is explained in the literatures quoted in the introduction.)

One advantage to restrict to such cycles is the following lemma.

Lemma 10.12 *Let c^s be a one parameter family of coefficient functions of degree 0 satisfying Axiom* I,II,III. *Then c^1 is homologous to c^0.*

Lemma 10.12 is an immediate consequence of Theorem 10.18, we prove later.

Note that $c_k \equiv 0$ satisfies Axioms I,II,III. We introduce an axiom which exclude such trivial c_k. We consider the case when $k+1=3$. Then $\partial C(1,2,3)=0$ by Axiom III. Axiom IV will determine the homology class of this cycle. We put

$$S(Q,1,2,3)=\left\{(v_1,v_2,v_3)\in L(1,2,3)\,\middle|\,Q(v_1,v_2,v_3)>0,\left\|[v_1,v_2,v_3]\right\|=1\right\}.$$

Lemma 10.13 *If* $\deg(1,2,3)=d$ *then the index of Q on $L(1,2,3)$ is d.*

Corollary 10.14 *If $L(1,2,3)$ then $S(Q,1,2,3)$ is homotopy equivalent to S^{n-d-1}.*

Corollary 10.14 is immediate from Lemma 10.13. Lemma 10.13 is immediate from definition. In fact, we have $Q[0,v_2,0]=\Omega\left(\pi_{L_1}v_2,\pi_{L_3}v_2\right)/2$, where $\pi_{L_i}:V/\tilde{L}_2\to\tilde{L}_i$ is an isomorphism.

Hereafter we omit (d) in $C^{(d)}(1,\cdots,k+1)$ in case no confusion can occur. Let $\bar{C}(1,\cdots,k+1;c^{hol})$, $\bar{C}(1,\cdots,k+1)$ be cycles on $L(1,\cdots,k+1)$ induced from $C(1,\cdots,k+1;c^{hol})$, $C(1,\cdots,k+1)$ respectively.

Theorem 10.15 $[\bar{C}(1,2,3;c^{hol})\cap S(Q,1,2,3)]\in H_{n-d-1}(S(Q,1,2,3);\boldsymbol{Z})=\boldsymbol{Z}$ *is the generator.*

Theorem 10.15 is a generalization of [12] Theorem 4.18 (which is the case when $d=0$) and is a motivation of Axiom IV below. Note that we do not put Theorem 10.15 in the quote while we put Lemmata 10.3 and 10.8 in the quote. The difference is that the statement of Theorem 10.15 is stable by the perturbation. So though we do not specify the perturbation, it is now standard to show that there exists a perturbation so that we can make sense of the left hand side of Theorem 10.15. On the other hand, it is not clear in what sense the statements of Lemmata 10.3 and 10.8 are stable by perturbation.

Axiom IV $[\bar{C}(1,2,3)\cap S(Q,1,2,3)]\in H_{n-d-1}(S(Q,1,2,3);\boldsymbol{Z})=\boldsymbol{Z}$ *is the generator.*

Note that we need to fix the sign of the generator of $H_{n-d-1}(S(Q,1,2,3);\boldsymbol{Z})$ for Axiom IV to make sense. The simplest way to do so is to use Theorem 10.15. Namely we assume

(10.16) $$[\bar{C}(1,2,3;c^{hol})\cap S(Q,1,2,3)]=[\bar{C}(1,2,3)\cap S(Q,1,2,3)].$$

The orientation of $\bar{C}(1,2,3;c^{hol})$ is determined by using its Morse homotopy limit. We discuss it later during the proof of Proposition 10.25. Our main results of this section are as follows :

Theorem 10.17 *There exists a coefficient function* c *satisfying Axioms* I,II,III,IV.

Theorem 10.18 *Let* c^1, c^2 *be two coefficient functions satisfying Axioms* I,II,III,IV. *Then* c^1 *is homologous to* c^2.

Proof of Theorem 10.15: We use Morse homotopy in a similar way as the proof of [12] Theorem 4.18. We choose a complex structure on V so that $J\tilde{L}_{j_1}$ is transversal to $\tilde{L}_{j_2}$ and $\tilde{L}_{j_3}$. Using it we regards $V = T^*\tilde{L}_{j_1}$. We regards $\tilde{L}_{j_2}$ and $\tilde{L}_{j_3}$ as graphs of closed one forms dV_2, dV_3, where V_i are quadratic functions on $\tilde{L}_1$. Let $\tilde{L}^{\varepsilon}_{j_2}$ and $\tilde{L}^{\varepsilon}_{j_3}$ be the graphs of εdV_2, εdV_3 respectively. We consider the isomorphism $V \to V$, $v_1 + v_2 \mapsto v_1 + \varepsilon v_2$ where $v_1 \in \tilde{L}_{j_1}$ $v_2 \in J\tilde{L}_{j_1}$. We then obtain an isomorphism $I_\varepsilon : V/\tilde{L}_1 \times V/\tilde{L}_2 \times V/\tilde{L}_3 \cong V/\tilde{L}_1 \times V/\tilde{L}^{\varepsilon}_2 \times V/\tilde{L}^{\varepsilon}_3$. We find $Q(I_\varepsilon(v_1,v_2,v_3)) = \varepsilon^2 Q(v_1,v_2,v_3)$. It follows that the homology class in Theorem 10.15 does not change if we replace $\tilde{L}_i$ by $\tilde{L}^{\varepsilon}_i$. So we may consider the limit where $\varepsilon \to 0$. By [18] this limit is described by Morse homotopy. Let us recall it here.

We remark that $L(j_1,j_2,j_3) = \{[0,v_2,0]\}$. Let $\tilde{L}^{\varepsilon}_{j_2}(v_2)$ be the graph of $\varepsilon dV_{2,q_2}$. It is easy to find that there exists a linear isomorphism $I : V/\tilde{L}_2 \to \tilde{L}^*_1$ such that $V_{2,q_2} = V_2 + I(v_2)$. Let $q_{31} = 0$ and let $q_{12}(v_2)$, $q_{23}(v_2)$ be the unique critical points of $V_{2,q_2} - V_1$ and $V_3 - V_{2,q_2}$ respectively. Let $U_{12}(v_2)$, $U_{23}(v_2)$, U_{31} be the unstable manifolds of $\mathrm{grad}(V_{2,q_2} - V_1)$, $\mathrm{grad}(V_3 - V_{2,q_2})$, $\mathrm{grad}(V_1 - V_3)$ respectively. Then for sufficiently small ε, $\bar{C}(j_1,j_2,j_3;c^{hol})$ is diffeomorphic to :

$$\left\{ v_2 \,\middle|\, U_{12}(v_2) \cap U_{23}(v_2) \cap U_{31} \neq \emptyset \right\}. \tag{10.19}$$

It is easy to see that (10.19) is a linear subspace of $L(j_1,j_2,j_3) = \{[0,v_2,0]\}$ and its codimension is $2n - \dim U_{12}(v_2) - \dim U_{23}(v_2) - \dim U_{31} = d$. This implies Theorem 10.15.

Proof of Theorem 10.17: We first generalize Lemma 10.13 as follows.

Lemma 10.20 *Let* $\deg(1,\cdots,k+1) = d$. *Then the index quadratic form* Q *on* $L(1,\cdots,k+1)$ *is* $d+k-2$.

Proof: We take ℓ, m with $\ell < \ell+2 \le m$. We remark

$$L(1,\cdots,k+1) = \{[v_1,\cdots,v_{\ell-1},0,v_{\ell+1},\cdots,v_{m-1},0,v_{m+1},\cdots,v_k]\}.$$

We have

$$(10.21)\qquad \begin{aligned} &Q(v_1,\cdots,v_{\ell-1},0,v_{\ell+1},\cdots,v_{m-1},0,v_{m+1},\cdots,v_{k+1}) \\ &\qquad = Q(v_{\ell+1},\cdots,v_{m-1}) + Q(v_1,\cdots,v_{\ell-1},v_{m+1},\cdots,v_{k+1}). \end{aligned}$$

Using (10.21) and Lemmata 10.7, 10.13, we can prove Lemma 10.20 by an induction on k.

We put

$$S(Q,1,\cdots,k+1) = \left\{[v_1,\cdots,v_{k+1}] \in L(1,\cdots,k+1) \,\middle|\, Q(v_1,\cdots,v_{k+1}) > 0, \|[v_1,\cdots,v_{k+1}]\| = 1\right\}.$$

Corollary 10.22 $H_*\left(S(Q,1,\cdots,k+1),\boldsymbol{Z}\right) \cong H_*(S^{n(k-1)-(d+k-1)},\boldsymbol{Z})$.

Corollary 10.22 is immediate from Lemma 10.20.

Now we start the proof of Theorem 10.17. We construct $C(1,\cdots,k+1)$ by induction on k. In case when $k+1=3$, we need to find $C(1,2,3)$ satisfying Axiom IV. Let $\deg(1,2,3)=d$, $d>0$. We take a codimension d linear subspace $\bar{C}(1,2,3)$ of $L(1,2,3)$ such that Q is positive on it. (We can choose such $\bar{C}(1,2,3)$ by Lemma 10.13 and $d>0$.) By perturbing it a bit, we may assume $\bar{C}(1,2,3) \cap \Gamma(1,2,3) \cong \boldsymbol{Z}^{\dim \bar{C}(1,2,3)}$. This $\bar{C}(1,2,3)$ satisfies Axioms IV.

We next consider the case $k=4$. Let $\deg(1,2,3,4)=d$. We consider the cycle :

$$(10.23)\qquad \begin{aligned} &(-1)^{\mu_1}\left[\left(C(1,2,3)\times C(1,3,4)\right)\cap S(Q;1,2,3,4)\right] \\ &\qquad +(-1)^{\mu_2}\left[\left(C(2,3,4)\times C(1,2,4)\right)\cap S(Q;1,2,3,4)\right] \end{aligned}$$

where

(10.24.1) $\mu_1 = \deg(1,3,4)+\deg(1,3)$,

(10.24.2) $\mu_2 = \deg(1,2,3)+\deg(2,3,4)+\deg(1,2,4)+\deg(1,2)\deg(2,3,4)+\deg(1,3)+1$.

(10.23) represents an element of $H_{2n-d-2}(S(Q;j_1,j_2,j_3,j_4);\boldsymbol{Z}) \cong \boldsymbol{Z}$.

Proposition 10.25 (10.23) *represents 0 in the homology group.*

Proof : We first show the following :

Lemma 10.26 *If* $\mathcal{J}(n,k,d) \neq \emptyset$ *then* $d \geq 2-k$.

Proof: Let $(j_1,\cdots,j_{k+1}) \in \mathcal{J}(n,k,d)$. We choose a complex structure on V so that $J\tilde{L}_{j_1}$ is transversal to $\tilde{L}_{j_\ell}$ $\ell = 2,\cdots,k+1$. Using it we regards $V = T^*\tilde{L}_{j_1}$. We regards $\tilde{L}_{j_\ell}$ as graphs of closed one forms dV_ℓ etc. where V_ℓ are quadratic functions on $\tilde{L}_1$. Let $f_{\ell\,\ell+1} = V_{\ell+1} - V_\ell$. Let $\mu\left(f_{\ell\,\ell+1}\right)$ be the number of *positive* eigenvalues of $f_{\ell\,\ell+1}$. By definition we have

(10.27) $$\mu\left(f_{12}\right)+\cdots+\mu\left(f_{k\,k+1}\right)-\mu\left(f_{1\,k+1}\right)=\eta\left(1,\cdots,k+1\right).$$

On the other hand, since $f_{12}+\cdots+f_{k\,k+1}=f_{1\,k+1}$, it is easy to see that the right hand side of (10.27) is nonnegative. Lemma 10.26 follows.

Now let $\deg(1,2,3,4)=d$. Using Lemma 10.26 and Axiom V, we find that $\left[\left(C(1,2,3)\times C(1,3,4)\right)\cap S(Q,1,2,3,4)\right]$ and $\left[\left(C(2,3,4)\times C(1,2,4)\right)\cap S(Q,1,2,3,4)\right]$ both represent the generator of $H_{2n-d-1}\left(S(Q,1,2,3,4);\mathbf{Z}\right)$. We are going to check the sign and show that (10.23) is 0 in the homology group. For this purpose we recall the definition of the orientation. We regards $\tilde L_i$ as the graph of df_i, where f_i is a quadratic form on some $\tilde L_0$ and we identify $V=T^*\tilde L_0$. Let $S(a,b)$ be the stable manifold of f_b-f_a. (Namely $S(a,b)$ is the eigen space of negative eigenvalues.) We remark that we may regards $C(1,2,3)$ as a linear subspace of $\tilde L_0$. Then we *define* the orientation on $C(1,2,3)$ etc. so that

(10.28) $$S(1,2)\oplus S(2,3)\cong C(1,2,3)\oplus S(1,3)$$

is an orientation preserving isomorphism. Then we have orientation preserving isomorphisms :

$$\begin{aligned} S(1,2)\oplus S(2,3)\oplus S(3,4) &\cong C(1,2,3)\oplus S(1,3)\oplus S(3,4)\\ &\cong C(1,2,3)\oplus C(1,3,4)\oplus S(1,4)\end{aligned}$$

$$\begin{aligned} S(1,2)\oplus S(2,3)\oplus S(3,4) &\cong S(1,2)\oplus C(2,3,4)\oplus S(2,4)\\ &\cong (-1)^{\deg(2,3,4)\deg(1,2)}\,C(2,3,4)\oplus S(1,2)\oplus S(2,4)\\ &\cong (-1)^{\deg(2,3,4)\deg(1,2)}\,C(2,3,4)\oplus C(1,2,4)\oplus S(1,4).\end{aligned}$$

Namely we have $C(1,2,3)\oplus C(1,3,4)\cong(-1)^{\deg(2,3,4)\deg(1,2)}C(2,3,4)\oplus C(1,2,4)$. Note $\deg(2,3,4)+\deg(1,2,4)=\deg(1,2,3)+\deg(1,3,4)$ by (9.1.2). Proposition 10.25 follows.

By Proposition 10.25, we can choose a chain $\bar C(1,2,3,4)$ which satisfies Axiom III.

Now the induction for the general k is as follows. We assume that we have constructed $C(1,\cdots,k')$ satisfying Axioms III for $k'\le k$. Let $(j_1,\cdots,j_{k+1})\in\mathcal{J}(n,k,d)$.

Lemma 10.29 $$\partial\sum_{\ell,m}\pm\left(C(\ell,\cdots,m)\times C(1,\cdots,\ell,m,\cdots,k+1)\right)=0.$$

Proof: We first prove the lemma up to sign. (Namely over $\mathbf{Z}_2$ coefficient.) The argument of the sign (together one in the statement) will be given later. We remark that the left hand side is a sum of the terms of the form

(10.30) $$C(a,\cdots,b)\times C(\ell,\cdots,a,b,\cdots,m)\times C(1,\cdots,\ell,m,\cdots,k+1)$$

for $1<\ell<a<b<m<k$. We put $\deg(a,\cdots,b)=d_1$, $\deg(\ell,\cdots,a,b,\cdots,m)=d_2$,

$(1,\cdots,\ell,m,\cdots,k+1)=d_3$. We may assume $d_1,d_2,d_3\geq 0$. We consider the following three cases.

Case 1: $d_1+d_2>0$, $d_2+d_3>0$. Since $\deg(\ell,\cdots,m)=d_1+d_2-1$, $\deg(1,\cdots,a,b,\cdots,k+1)=d_2+d_3-1$, it follows that both $\partial(C(\ell,\cdots,m)\times C(1,\cdots,\ell,m,\cdots,k+1))$ and $\partial(C(a,\cdots,b)\times C(1,\cdots,a,b,\cdots,k+1))$ appears in the left hand side of Lemma 10.29 and contains (10.30). Hence this case the term (10.30) cancels to each other (up to sign.)

Case 2: $d_1+d_2=0$. We apply induction hypothesis Axiom III to $(\ell,\cdots,m)$. We obtain

$$\sum_{a,b}\pm C(a,\cdots,b)\times C(\ell,\cdots,a,b,\cdots,m)=0\,.$$

Hence the sum of such terms in the left hand side of Lemma 10.29 vanishes.

Case 3: $d_2+d_3=0$. The same as Case 2.

Thus we proved Lemma 10.29 up to sign.

We now consider the homology class

$$\left[\sum\pm(C(\ell,\cdots,m)\times C(1,\cdots,\ell,m,\cdots,k+1))\cap S(Q;1,\cdots,k+1)\right]. \tag{10.31}$$

This homology class is in $H_{n(k-1)-d-2}\left(S(Q;1,\cdots,k+1),\boldsymbol{Z}\right)$. Corollary 10.22 implies that this group vanishes if $k+1\geq 5$. Therefore we can find $C(1,\cdots,k+1)$ which bounds (10.31). Namely this class satisfies Axiom III.

In the final step, namely in the case $d=0$, we proceed in the same way. Axiom I,III,IV are satisfied. We finally verify Axiom II. Let $\deg(1,\cdots,k+1)=-1$. We consider the sum

$$\sum\pm C(\ell,\cdots,m)\times C(1,\cdots,\ell,m,\cdots,k+1). \tag{10.32}$$

where the summations is taken over $(\ell,\cdots,m)$ such that $\deg(\ell,\cdots,m)=\deg(1,\cdots,\ell,m,\cdots,k+1)=0$. (We discuss the sign later.) In the same way as Lemma 10.29 we can prove that (10.32) is a cycle. On the other hand, (10.32) is the top dimensional chain. Hence (10.32) is equal to the constant times the fundamental class. On the other hand, since Q is not positive definite on $L(1,\cdots,k+1)$, it follows that the cycle (10.32) is zero on some open set. Therefore (10.32) is zero everywhere. Axiom II follows immediately. We thus proved Theorem 10.17.

Remark 10.33 The proof of Theorem 10.17 as well as the proof of Theorem 10.18 is somewhat similar to the method of Acyclic model discovered by Eilenberg and MacLane [10] in the early days of homological algebra.

Before going further, we show how the wall $\mathcal{W}$ looks like combinatorially in the case

$k+1=4,5,6$. In the case $k+1=5$ and $\deg(1,2,3,4)=0$, we have

$$\mathcal{W}(1,2,3,4) = \pm\mathcal{C}(1,3,4)\times\mathcal{C}(1,2,3)\pm\mathcal{C}(1,2,4)\times\mathcal{C}(2,3,4).$$

Note $\deg(1,3,4)+\deg(1,2,3)=1$. In case $\deg(1,3,4)=1$, we may choose $\mathcal{C}(1,3,4)$ as a codimension 1 linear subspace of $\tilde{L}(1,3,4)$. Hence $\mathcal{C}(1,3,4)\times\mathcal{C}(1,2,3)$ is a codimension 1 linear subspace. The other case and other term $\mathcal{C}(1,2,4)\times\mathcal{C}(2,3,4)$ can be chosen to be a codimension 1 linear subspace. On the other hand, the index of Q on $L(1,2,3,4)$ also is 1 by Lemma 10.13. Hence we have the following Figure 10.

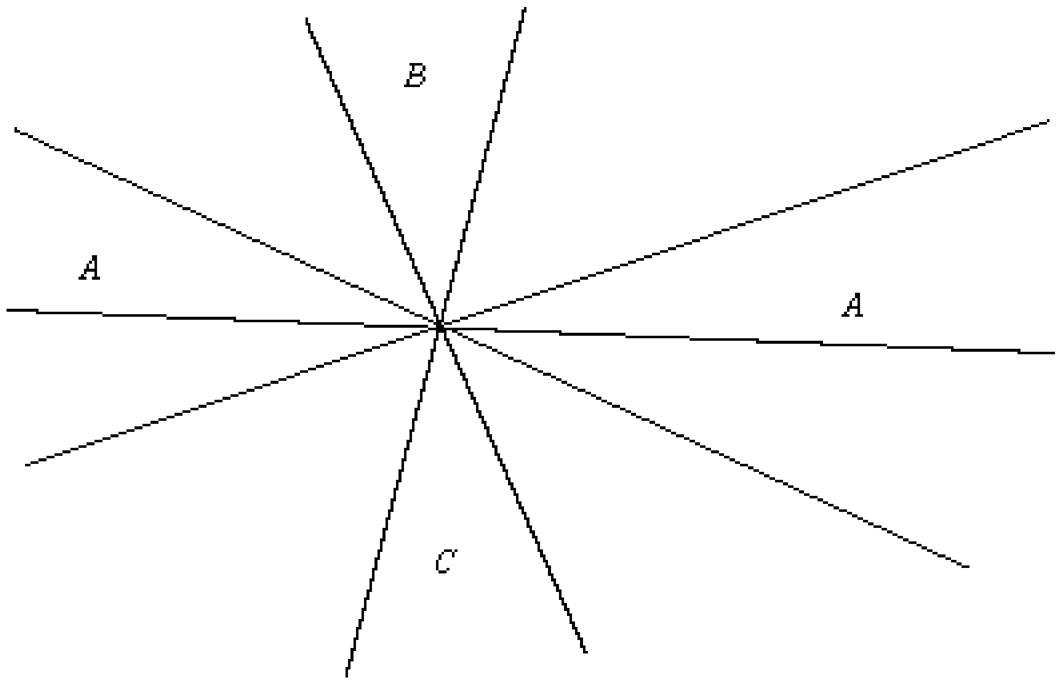

Figure 10

Here $Q<0$ on A. We have $c_3=\pm 1$ on B and $c_3=\mp 1$ on C, $c_3=0$ elsewhere. (Compare [21] (3.0.1).)

In the case when $k+1=5$, there are several possibilities according to the Maslov index. We first consider the case $\deg(1,2)=2$, $d(i,j)=0$ for other $i<j$. (Note then $\deg(1,2,3,4,5)=0$.) We find that $\mathcal{C}(i,j,k,\ell)$ is of negative virtual codimension, except $\mathcal{C}(1,2,3,4)$, $\mathcal{C}(1,2,3,5)$, $\mathcal{C}(1,2,4,5)$. Hence

$$\mathcal{W}(1,2,3,4,5) = \pm\mathcal{C}(1,2,3,4)\times\mathcal{C}(1,4,5)\pm\mathcal{C}(1,2,3,5)\times\mathcal{C}(3,4,5)\pm\mathcal{C}(1,2,4,5)\times\mathcal{C}(2,3,4).$$

Therefore combinatorially $\mathcal{W}(1,2,3,4,5)$ looks like as the following Figure 11. (We remark that index of $Q=2$ on $L(1,2,3,4,5)$ in this case.)

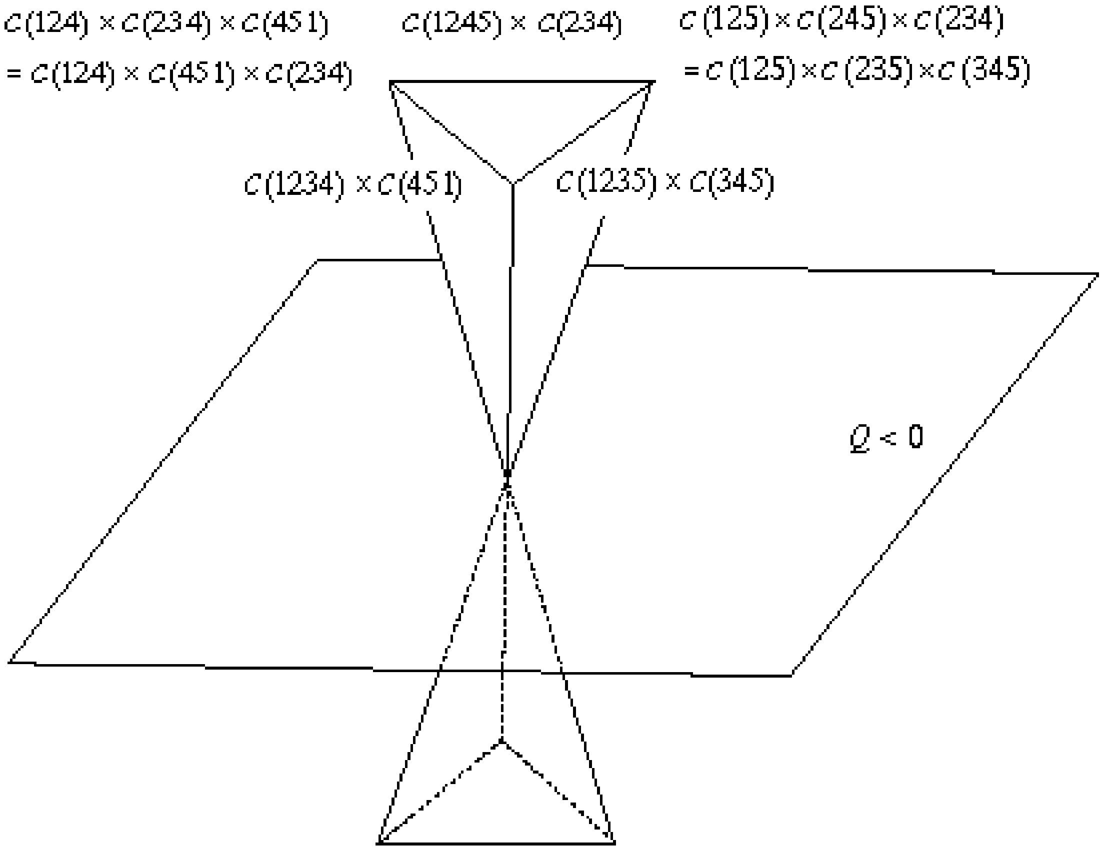

Figure 11

We remark however that Figure 11 is combinatorial or topological picture. Namely faces $C(1,2,3,4)\times C(1,4,5)$ etc. are not liner in this case. In fact, let us consider the case of $n=2$. Then $L(1,2,3,4)=\boldsymbol{R}^4$ and $C(1,3,4)\times C(1,2,3)\cong \boldsymbol{R}^2$. Hence $\partial C(1,2,3,4)$ is a union of two $\boldsymbol{R}^2$'s in $\boldsymbol{R}^4$. It is impossible to find a chain $C(1,2,3,4)$ contained in a single (flat) hyperplain. (We can take it as a union of two flat 3 dimensional sectors.) This is the reason why it is difficult to find a wall $\mathcal{W}$ such that $\overline{\mathcal{W}}\subset T^{2nk}$ is compact.

Let us consider other cases of $k+1=5$. Here we take the negative eigenspace $L(-)$ of Q and draw the figure of the intersection of $\mathcal{W}$ with a 2 dimensional plain parallel to $L(-)$.

If $\deg(1,2)=\deg(2,3)=1$ and $\deg(i,j)=0$ for other $i<j$, then 4 of $C(j_1,j_2,j_3,j_4)$'s can appear in $\mathcal{W}(1,2,3,4,5)$ and we find Figure 12-1. If $\deg(1,2)=\deg(3,4)=1$ and $\deg(1,2)=\deg(2,3)=1$ for other $i<j$, we have Figure 12-2.

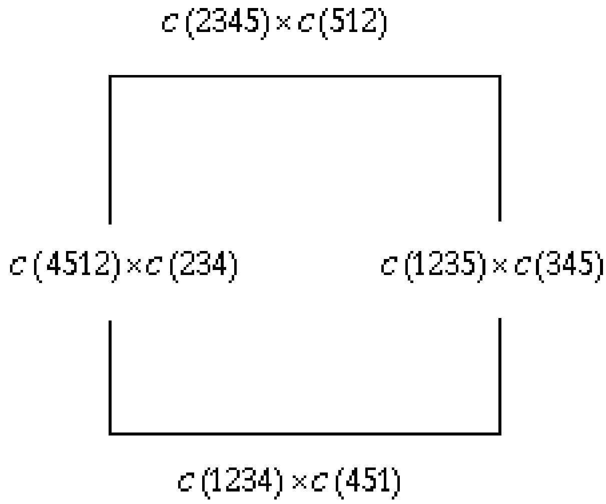

Figure 12-1

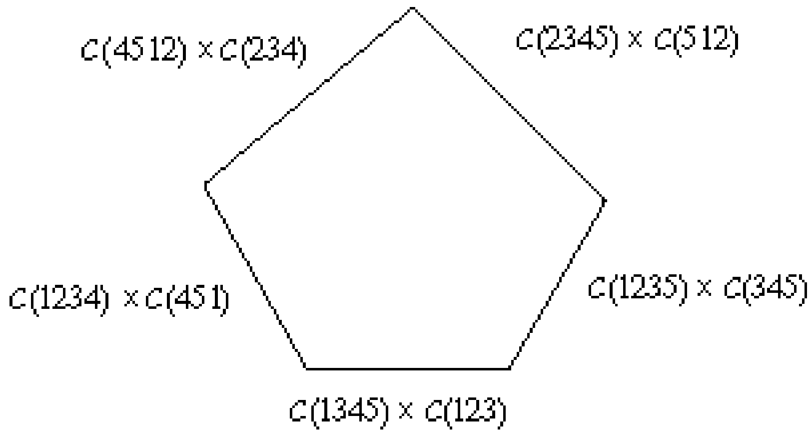

Figure 12-2

Let us consider the case when $k+1=6$. In this case, index of Q on $L(1,2,3,4,5,6)$ is 3. We take a 3 dimensional subspace $L(-)$ and are going to draw the figures of the intersection of $\mathcal{W}$ with a 3 dimensional plain parallel to $L(-)$. Let us consider first the case $\deg(1,2)=3$, $\deg(i,j)=0$ for other $i<j$. Then $\mathcal{W}(1,2,3,4,5,6)$ is a union of 4 faces and looks like

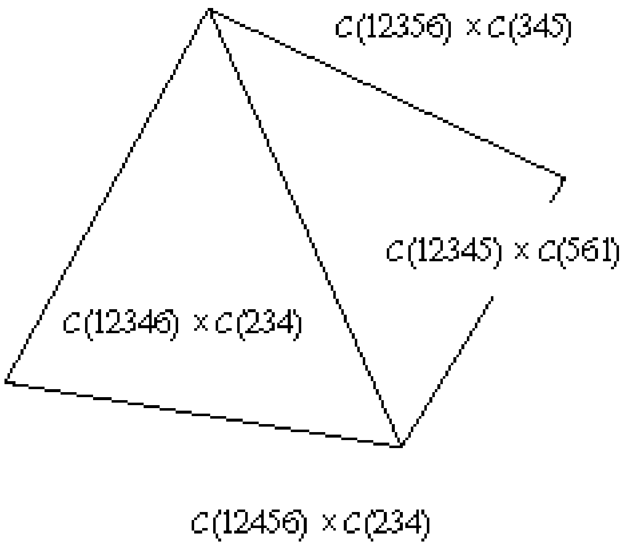

Figure 13.

Next we consider the case $\deg(1,2)=\deg(3,4)=\deg(5,6)=1$, $\deg(i,j)=0$ for other $i<j$. The we find that $\mathcal{W}(1,2,3,4,5,6)$ looks like

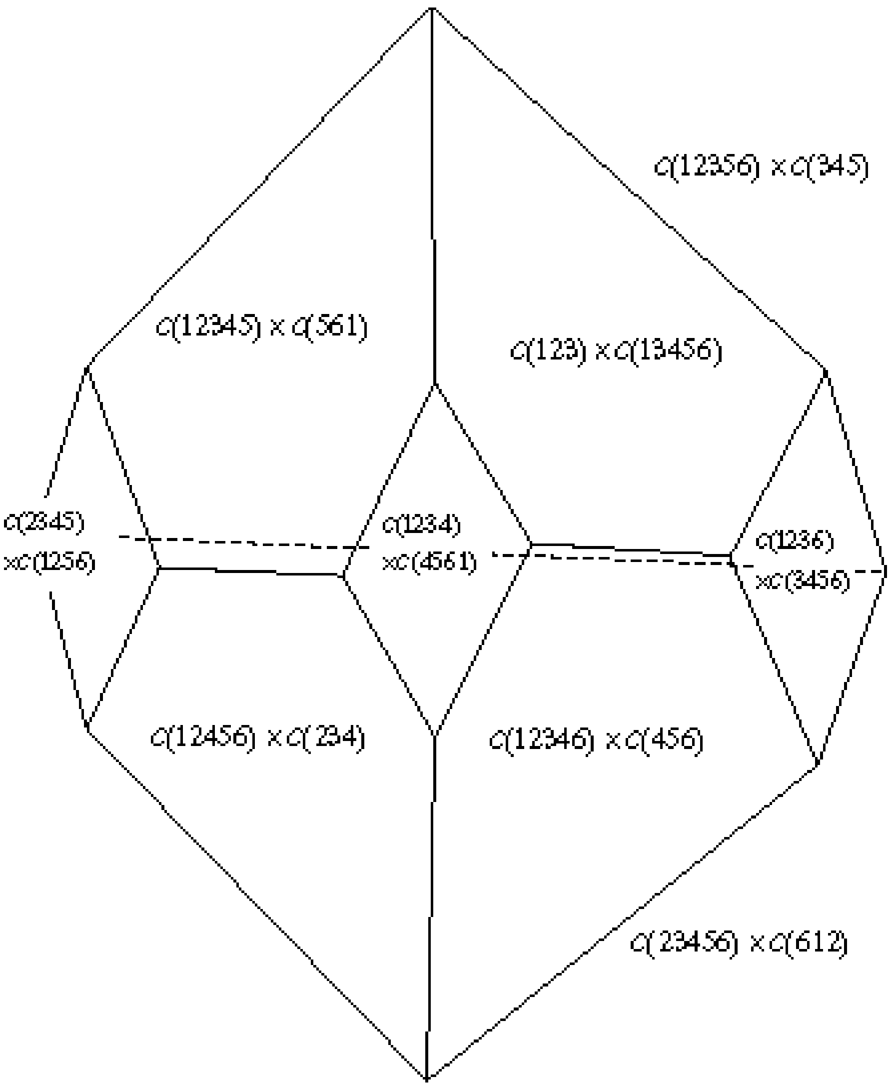

Figure 14.

This is a cell Stasheff introduced to study A^{∞}-structure in [45].

Finally we consider the case $\deg(1,2)=\deg(3,4)=\deg(4,5)=1$, $\deg(i,j)=0$ for other $i<j$. We then find that the following figure :

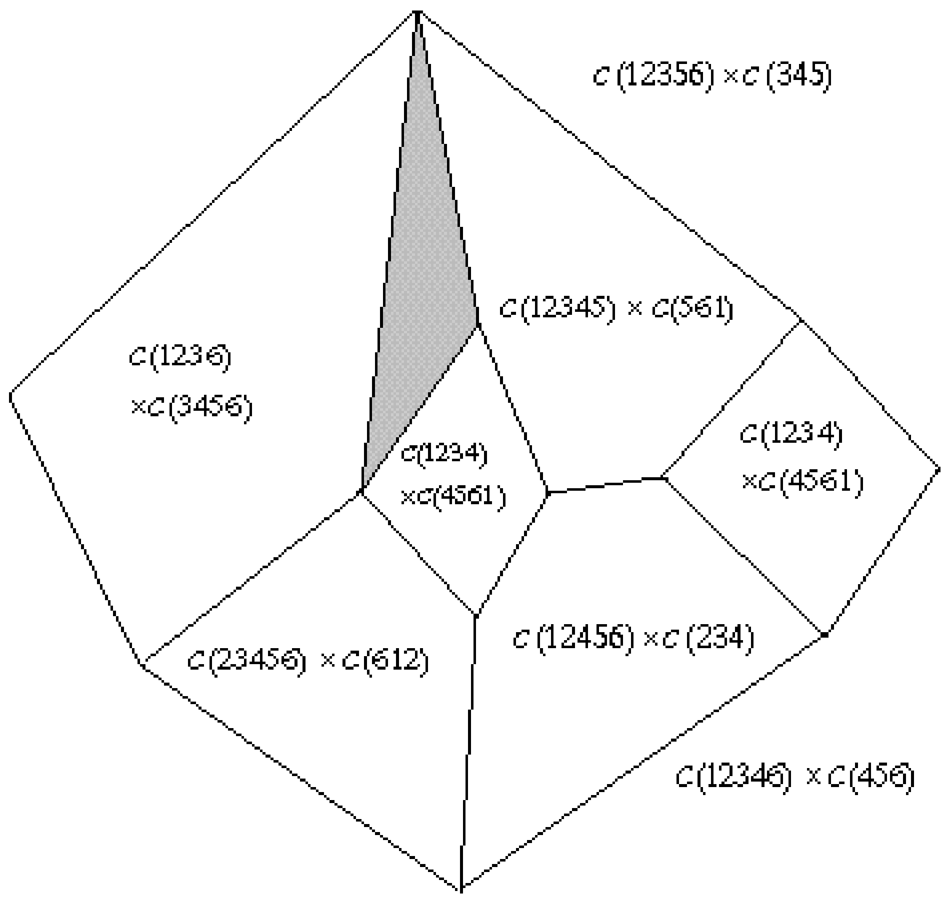

Figure 15

Here the shaded region can belong any one of $C(1,2,3,6)\times C(3,4,5,6)$, $C(1,2,3,4,5)\times C(5,6,1)$, $C(1,2,3,4)\times C(4,5,6,1)$. Note that we need to use the formula

$$\pm[C(1,5,6)\times C(1,2,3,5)]\pm[C(2,3,5)\times C(1,2,5,6)]\pm[C(3,5,6)\times C(1,2,3,6)]$$

to draw Figure 15.

Proof of Theorem 10.18: Let $C^{i}(1,\cdots,k+1)$ $i=1,2$ be the classes associated to m^{i}. We are going to prove the following by induction on k.

Lemma 10.34 *There exist codimension* $\deg(1,\cdots,k+1)-1$ *chains* $\mathcal{D}(1,\cdots,k+1)$ *in* $\tilde{L}(1,\cdots,k+1)$ *with the following properties.*

(10.35.1) $\mathcal{D}(1,2)$ *is the fundamental class.*

(10.35.2) $\mathcal{D}(1,\cdots,k+1)$ *is invariant of the translation by* $V\subseteq\tilde{L}(1,\cdots,k+1)$ *and by*

$[v_1,\cdots,v_{k+1}]\mapsto[cv_1,\cdots,cv_{k+1}]$.

(10.35.3) *There exists* $\delta>0$ *such that* $(v_1,\cdots,v_{k+1})\in\mathcal{D}(1,\cdots,k+1)$ *implies*

$$\operatorname{Re}Q(v_1,\cdots,v_{k+1})>\delta\left\|[v_1,\cdots,v_{k+1}]\right\|^2.$$

(10.35.4) *The boundary* $\partial\mathcal{D}(1,\cdots,k+1)$ *of* $\mathcal{D}(1,\cdots,k+1)$ *is a difference of two types of components. One of them is the sum of*

$$\pm\mathcal{D}(a(1),\cdots,a(2))\times\cdots\times\mathcal{D}(a(t-1),\cdots,a(t))\times C^2(a(1),a(2),\cdots,a(t)) \tag{10.36}$$

where the sum is taken over all t *and* $1=a(1)<\cdots<a(t)=k+1$ *with* $\deg(a(i),\cdots,a(i+1))\geq 0$, $a(i+1)\geq a(i)+1$, *and* $\deg\big(a(1),a(2),\cdots,a(t)\big)\geq 0$.

The other is the sum of

$$\pm C^1(\ell,\cdots,m)\times\mathcal{D}(1,\cdots,\ell,m,\cdots,k+1), \tag{10.37.1}$$

$$\pm\mathcal{D}(\ell,\cdots,m)\times C^1(1,\cdots,\ell,m,\cdots,k+1), \tag{10.37.2}$$

where the sum is taken over all ℓ,m *with* $\deg\big(1,\cdots,\ell,m,\cdots,k+1\big)\geq 0$, $\deg\big(\ell,\cdots,m\big)\geq 0$. *(*$k+1\geq m-\ell+1\geq 3$ *in* (10.37.1), $k\geq m-\ell+1\geq 2$ *in* (10.37.2).*)*

The sign will be fixed during the proof.

Remark 10.38 We can rewrite (10.35.3) as $\partial\mathcal{D}+C^2\circ\mathcal{D}-\mathcal{D}\circ C^1=0$.

The proof of Lemma 10.34 is similar to one of Theorem 10.17 and proceeds as follows. Let us first give a proof up to sign. (We discuss sign later.) The proof is by induction on k.

In the case $k+1=2$ we define $\mathcal{L}$ as in (10.35.1). In the case $k+1=3$, (10.36) – (10.37) is $C^2(1,2,3)-C^1(1,2,3)$, which is homologous to 0 by Axiom IV. Hence we have $\mathcal{D}(1,2,3)$.

Assume that $\mathcal{D}$ is constructed up to k. We consider the boundary of (10.36) – (10.37). It consists of three kinds of terms

$$\begin{aligned}&\pm\mathcal{D}\big(a(1),\cdots,a(2)\big)\times\cdots\times\mathcal{D}\big(a(t-1),\cdots,a(t)\big)\\ &\qquad C^2\big(a(1),\cdots,a(\alpha),a(\beta),\cdots,a(t)\big)\times C^2\big(a(\alpha),a(\alpha+1),\cdots,a(\beta)\big).\end{aligned} \tag{10.39.1}$$

$$\begin{aligned}&\pm C^1\big(a(\theta)+\ell,a(\theta)+\ell+1,\cdots,a(\theta)+m\big)\times\mathcal{D}\big(a(1),\cdots,a(2)\big)\times\cdots\\ &\quad\times\mathcal{D}\big(a(\theta),\cdots,a(\theta)+\ell,a(\theta)+m,\cdots,a(\theta+1)\big)\times\cdots\times\mathcal{D}\big(a(t-1),\cdots,a(t)\big)\\ &\quad\times C^2\big(a(1),\cdots,a(t)\big).\end{aligned} \tag{10.39.2.1}$$

$$\begin{aligned}&\pm\mathcal{D}\big(a(\theta)+\ell,a(\theta)+\ell+1,\cdots,a(\theta)+m\big)\times\mathcal{D}\big(a(1),\cdots,a(2)\big)\times\cdots\\ &\quad\times C^1\big(a(\theta),\cdots,a(\theta)+\ell,a(\theta)+m,\cdots,a(\theta+1)\big)\times\cdots\times\mathcal{D}\big(a(t-1),\cdots,a(t)\big)\\ &\quad\times C^2\big(a(1),\cdots,a(t)\big).\end{aligned} \tag{10.39.2.2}$$

$$\pm C^1(a,\cdots,b)\times C^1(\ell,\cdots,a,b,\cdots,m)\times\mathcal{D}(1,\cdots,\ell,m,\cdots,k+1). \tag{10.39.3.1}$$

(10.39.3.2) $\pm C^1(a,\cdots,b)\times \mathcal{D}(\ell,\cdots,a,b,\cdots,m)\times C^1(1,\cdots,\ell,m,\cdots,k+1)$.

(10.39.3.3) $\pm C^1(a,\cdots,b)\times C^1(\ell,\cdots,a,b,\cdots,m)\times \mathcal{D}(1,\cdots,\ell,m,\cdots,k+1)$.

For each term we can divide the case in the same way as the proof of Lemma 10.29. In one case, the term appears twice in the boundary of (10.36) − (10.37) and hence cancels. In the other case, terms cancel to each other by induction hypothesis or Axiom II.

Therefore (10.36) − (10.37) gives an element of $H_*(S(Q,1,\cdots,k+1),\boldsymbol{Z})$. We find the degree is $n(k-1)-d-1$. (This is $1+$ the degree in the case of the proof of Theorem 10.17.) Note $H_{n(k-1)-d-1}(S(Q,1,\cdots,k+1),\boldsymbol{Z})=0$ by Corollary 10.22. Hence we obtain $\mathcal{D}(1,\cdots,k+1)$. The proof of Lemma 10.34 is complete up to sign.

We can apply the same argument in the case when $\deg(1,\cdots,k+1)=0$. Then we obtain (10.36) = (10.37), since in this case they are top dimensional cycle which is zero somewhere.

We now use $\mathcal{D}(1,\cdots,k+1)$ in case $\deg(1,\cdots,k+1)=1$ in the same way as $\mathcal{C}(1,\cdots,k+1)$ to obtain f. Then (10.36) = (10.37) for $\deg(1,\cdots,k+1)=0$ implies that $b^2\circ f = f\circ b^1$ in the sense of Definition 9.32. Therefore c^1 is homologous to c^2. This completes the proof of Theorem 10.18 up to sign.

Now we are going to discuss the sign in Axiom III and check the sign in the proofs of Theorem 10.17 and Lemma 10.34. For this purpose we continue the discussion of § 9 on the A^∞ structure. In § 9, we consider only c_k of degree 0 (or b_k of degree -1) we generalize it to other degree. We discuss using b_k to simplify the sign. Let $\deg(1,\cdots,k+1)=d$. We consider integral current $b^{(d)}_{1,\cdots,k+1}[v_1,\cdots,v_{k+1}]$ of degree d on $L(1,\cdots,k+1)$. Let $\Lambda^{(\lambda)}$ [resp. $\Lambda^{(\lambda)}_{smooth}$] denote the vector space of all degree λ current [resp. smooth differential forms] on $L(1,\cdots,k+1)$. We define $b^{(d)}: Ts(J)\otimes\Lambda^{(\lambda)}_{smooth}\to Ts(J)\otimes\Lambda^{(\lambda+d)}$ by

(10.40)
$$b^{(d)}\Big(\big[e_{12}\big|\cdots\big|e_{k\,k+1}\big]\otimes u\Big) = \sum_{\ell<m}(-1)^{(d+1)\sum_{i=1}^{\ell-1}(\deg(i,i+1)+1)}\big[e_{12}\big|\cdots\big|e_{\ell-1\,\ell}\big|e_{\ell\,m}\big|e_{m\,m+1}\big|\cdots\big|e_{k\,k+1}\big]\otimes\Big(u\wedge b^{(d)}_{1,\cdots,k+1}\Big).$$

(Note that $Ts(J)$ degree of $b^{(d)}$ is $-d-1$ and $Ts(J)$ degree of e_{ij} is $\deg(i,j)$.) Since the current degree of $b^{(d)}$ is d, the total degree of $b^{(d)}$ is -1 and is odd. We consider the equation

(10.41)
$$db^{(d)}+\sum_{d_1+d_2=d+1}(-1)^{d_2}b^{(d_2)}\circ b^{(d_1)}=0.$$

Here d is the exterior derivative on $L(1,\cdots,k+1)$.

Lemma 10.42 (10.41) *is equivalent to*

$$db^{(d)}_{1,\cdots,k+1} + \sum_{\substack{\ell<m \\ d_1+d_2=d+1}} (-1)^{d_2+(d_2+1)\sum_{i=1}^{\ell-1}(\deg(i,i+1)+1)} b^{(d_1)}_{\ell,\cdots,m} \wedge b^{(d_2)}_{1,\cdots,\ell,m,\cdots,k+1} = 0. \tag{10.43}$$

Proof: We apply $db^{(d)} + \sum_{d_1+d_2=d+1}(-1)^{d_2} b^{(d_2)} \circ b^{(d_1)}$ to $\left[e_{12}\middle|\cdots\middle|e_{kk+1}\right] \otimes 1$ and obtain (10.43). On the contrary, if we apply $db^{(d)} + \sum_{d_1+d_2=d+1}(-1)^{d_2} b^{(d_2)} \circ b^{(d_1)}$ to general $\left[e_{\bullet\bullet}\middle|\cdots\middle|e_{\bullet\bullet}\right] \otimes u$ we obtain the terms

$$\begin{aligned} &\left[e_{\bullet\bullet}\middle|\cdots\middle|e_{j_1 j_{k+1}}\middle|\cdots\middle|e_{\bullet\bullet}\right] \\ &\otimes\left(u \wedge \left(db^{(d)}_{1,\cdots,k+1} + \sum_{\ell<m} (-1)^{d_2+(d_2+1)\sum_{i=1}^{\ell-1}(\deg(i,i+1)+1)} b^{(d_1)}_{\ell,\cdots,m} \wedge b^{(d_2)}_{1,\cdots,\ell,m,\cdots,k+1} \right)\right), \end{aligned} \tag{10.44}$$

and

$$\pm\left[e_{\bullet\bullet}\middle|\cdots\middle|b^{(d_2)}(e_{\bullet\bullet})\middle|\cdots\middle|b^{(d_1)}(e_{\bullet\bullet})\middle|\cdots\middle|e_{\bullet\bullet}\right] \pm \left[e_{\bullet\bullet}\middle|\cdots\middle|b^{(d_1)}(e_{\bullet\bullet})\middle|\cdots\middle|b^{(d_2)}(e_{\bullet\bullet})\middle|\cdots\middle|e_{\bullet\bullet}\right]. \tag{10.45}$$

(10.44) vanishes by (10.43). (10.45) cancels to each other since the total degree of $b^{(d)}$ is odd. The proof of Lemma 10.42 is complete.

We now put

$$c^{(d)}_k = s \circ b^{(d)}_k \circ \left(s^{-1} \otimes \cdots \otimes s^{-1}\right). \tag{10.46}$$

We regards our chain $C^{(d)}_k[v_1,\cdots,v_k]$ in Axiom III as a degree d integral current. Let $b^{(d)}_k$ correspond to it by (10.46). We choose the sign in (10.10.5) so that it is equivalent to (10.41) or (10.43). (We will check that this choice coincides with (10.23) and (10.24) in the case $k+1=4$ later.)

To check that the sign in the proof of Theorem 10.17 is correct, we proceed as follows. We construct $b^{(d)}_k$ by induction on k such that

$$db^{(d)}_k + \sum_{\substack{d_1+d_2=\ell+1 \\ k_1+k_2=k+1}} (-1)^{d_2} b^{(d_2)}_{k_2} \circ b^{(d_1)}_{k_1} = 0. \tag{10.47}$$

The proof of Theorem 10.7 for $k \le 3$ gives $b^{(d)}_2$, $b^{(d)}_3$. We calculate

$$
\begin{aligned}
d & \sum_{\substack{d_1+d_2=\ell+1\\ k_1+k_2=k+1}} (-1)^{d_2} b_{k_2}^{(d_2)} \circ b_{k_1}^{(d_1)} \\
&= \sum_{\substack{d_1+d_2=\ell+1\\ k_1+k_2=k+1}} (-1)^{d_2} db_{k_2}^{(d_2)} \circ b_{k_1}^{(d_1)} + \sum_{\substack{d_1+d_2=\ell+1\\ k_1+k_2=k+1}} b_{k_2}^{(d_2)} \circ db_{k_1}^{(d_1)} \\
&= -\sum_{\substack{d_1+d_2=\ell+1,\, d_3\\ k_1+k_2+k_3=k+2}} (-1)^{d_2+d_3} b_{k_3}^{(d_3)} \circ b_{k_2}^{(d_2-d_3+1)} \circ b_{k_1}^{(d_1)} \\
&\qquad - \sum_{\substack{d_1+d_2=\ell+1,\, d_3\\ k_1+k_2+k_3=k+2}} (-1)^{d_3} b_{k_2}^{(d_2)} \circ b_{k_3}^{(d_3)} \circ b_{k_1}^{(d_1-d_3+1)} \\
&= 0.
\end{aligned}
$$

Thus induction works.

We next check that the choice of sign above coincides with (10,23), (10.24). We calculate using

$$
m_2^{(\deg(j_1,j_2,j_3))}(x_{j_1j_2}, x_{j_2j_3}) = (-1)^{\deg(j_1j_2)+1} sb_2[x_{j_1j_2}, x_{j_2j_3}],
$$

and obtain

$$
\begin{aligned}
&\sum (-1)^{d_2} b_2^{(d_2)} \circ b_2^{(d_1)} \left[x_{12} \middle| x_{23} \middle| x_{34}\right] \\
&= \sum (-1)^{\deg(1,3,4)} b_2^{(\deg(1,3,4))} \left[b_2^{(\deg(1,2,3))} \left[x_{12} \middle| x_{23}\right] \middle| x_{34}\right] \\
&\qquad + (-1)^{(\deg(2,3,4)+1)(\deg(1,2)+1)+\deg(1,2,4)} b_2^{(\deg(1,2,4))} \left[x_{12} \middle| b_2^{(d_1)} \left[x_{23} \middle| x_{34}\right]\right] \\
&= (-1)^{\mu_1} sm_2^{(\deg(1,3,4))}(m_2^{(\deg(1,2,3))}(x_{12}, x_{23}), x_{34}) \\
&\qquad + (-1)^{\mu_2} sm_2^{(\deg(1,2,4))}(x_{12}, m_2^{(\deg(2,3,4))}(x_{23}, x_{34})).
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_1 &= \deg(1,3,4) + \deg(1,2) + \deg(1,3), \\
\mu_2 &= (\deg(2,3,4)+1)(\deg(1,2)+1) + \deg(1,2,4) + \deg(1,2) + \deg(2,3) \\
&= \deg(1,2)\deg(2,3,4) + \deg(1,2,3) \\
&\qquad + \deg(1,2,4) + \deg(2,3,4) + \deg(1,3) + 1.
\end{aligned}
$$

Thus the sign coincide with one of (10.23) and (10.24). The proof of Theorem 10.17 is complete.

We use $c_k^{(d)}$ in place of c_k in Definitions 9.13 and 9.17 and obtain a map

(10.48)
$$
\begin{aligned}
m_k^{(d)} : \pi_{12}^* \mathcal{P}(\tilde L_{j_1}, \tilde L_{j_2}; \tilde M_{j_1}, \tilde M_{j_2}) \otimes \cdots \otimes \pi_{kk+1}^* \mathcal{P}(\tilde L_{j_k}, \tilde L_{j_{k+1}}; \tilde M_{j_k}, \tilde M_{j_{k+1}}) \\
\to \pi_{1k+1}^* \mathcal{P}(\tilde L_{j_1}, \tilde L_{j_{k+1}}; \tilde M_{j_1}, \tilde M_{j_{k+1}}) \otimes \Lambda^d\left(\prod \mathcal{M}(\tilde L_{j_i}, \tilde M_{j_i})\right).
\end{aligned}
$$

where $\Lambda^d\left(\prod \mathcal{M}(\tilde L_{j_i}, \tilde M_{j_i})\right)$ is the totality of degree d currents on $\prod \mathcal{M}(\tilde L_{j_i}, \tilde M_{j_i})$. Note that the Floer degree of $m_k^{(d)}$ is $2-k-d$. Its current degree (degree as differential form) is d. Using complex structure of $\mathcal{M}(\tilde{L}_{j_i}, \tilde{M}_{j_i})$ we decompose

$$\Lambda^d\Big(\prod \mathcal{M}(\tilde{L}_{j_i}, \tilde{M}_{j_i})\Big) = \bigoplus_{d_1+d_2=d} \Lambda^{(d_1,d_2)}\Big(\prod \mathcal{M}(\tilde{L}_{j_i}, \tilde{M}_{j_i})\Big).$$

Let $m_k^{(d)} = \sum_{d_1+d_2=d} m_k^{(d_1,d_2)}$ be the decomposition of $m_k^{(d)}$ to $(d_1 d_2)$ forms. We generalize Proposition 9.20 as follows.

Theorem 10.49 $\bar{\partial} m_k^{(0\,d)} + \sum_{\substack{d_1+d_2=\ell+1 \\ k_1+k_2=k+1}} \pm m_{k_2}^{(0\,d_2)} \circ m_{k_1}^{(0\,d_1)} = 0.$

The sign is so that it is equivalent to

$$\bar{\partial} B_k^{(0\,d)} + \sum_{\substack{d_1+d_2=\ell+1 \\ k_1+k_2=k+1}} (-1)^{d_2} B_{k_1}^{(0\,d_2)} \circ B_{k_2}^{(0\,d_1)} = 0$$

here $B_k^{(0\,d)}$ is the operator obtained from $b_k^{(d)}$.

Proof of Theorem 10.49. As in the proof of Proposition 9.20, $m_k^{(0\,d)}$ fails to be holomorphic only because of its discontinuity. Hence (10.47) implies Theorem 10.49.

We turn to the proof of Lemma 10.34. Consider degree d integral current $f^{(d)}_{1,\cdots,k+1}[v_1,\cdots,v_{k+1}]$ for $\deg(1,\cdots,k+1)=d+1$. We use it to define $f^{(d)}: Ts(J)\otimes \Lambda^{(\lambda)}_{smooth} \to Ts(J)\otimes \Lambda^{(\lambda+d)}$ by

(10.50)
$$\begin{aligned} &f^{(d)}\Big(\big[e_{12}\big|\cdots\big|e_{kk+1}\big]\otimes u\Big) \\ &= \sum_{d_1+\cdots+d_{i-1}=d} (-1)^{\mu}\big[e_{a(1)a(2)}\big|\cdots\big|e_{a(i-1)a(i)}\big]\otimes\Big(u\wedge f^{(d_{i-1})}_{a(i-1)\cdots a(i)}\wedge\cdots\wedge f^{(d_1)}_{a(1)\cdots a(2)}\Big) \end{aligned}$$

where

$$\begin{aligned} \mu = d_{i-1}&\big((\deg(1,2)+1)+\cdots+(\deg(a(i-1)-1,a(i-1))+1)\big) \\ &+\cdots\cdots+ d_2\big((\deg(1,2)+1)+\cdots+(\deg(a(2)-1,a(2))+1)\big). \end{aligned}$$

We note that $Ts(J)$ degree of $f^{(d)}$ is $-d$ and current degree is d. Hence its total degree is 0 and is even. Now we consider the equation

(10.51)
$$df_k^{(d)} + \sum_{\substack{d_1+d_2=d+1 \\ k_1+k_2=k+1}} (-1)^{d_2} f_{k_2}^{(d_2)} \circ b_{k_1}^{1(d_1)} - \sum_{\substack{d_1+d_2=d+1 \\ k_1+k_2=k+1}} (-1)^{d_2} b_{k_2}^{2(d_2)} \circ f_{k_1}^{(d_1)} = 0.$$

We solve it by induction on k in the same way as the proof of Theorem 10.17. We can check using induction hypothesis that

$$d\left(\sum_{d_1+d_2=\ell+1}(-1)^{d_2} f^{(d_2)} \circ b^{1(d_1)} - \sum_{d_1+d_2=\ell+1}(-1)^{d_2} b^{2(d_2)} \circ f^{(d_1)}\right) = 0 .$$

The proof of Lemma 10.34 and Theorem 10.18 is complete.

In a similar way as Theorem 10.49, we can define $n_k^{(0d)}$ and show :

Lemma 10.52 $$\bar{\partial} n_k^{(0d)} + \sum_{\substack{d_1+d_2=d+1 \\ k_1+k_2=k+1}} \pm n_{k_2}^{(0d_2)} \circ m_{k_1}^{1(0d_1)} - \sum_{\substack{d_1+d_2=d+1 \\ k_1+k_2=k+1}} \pm m_{k_2}^{2(0d_2)} \circ n_{k_1}^{(0d_1)} = 0 .$$

Where the sign is so that it is equivalent to

$$\bar{\partial} F_k^{(0d)} + \sum_{\substack{d_1+d_2=d+1 \\ k_1+k_2=k+1}} (-1)^{d_2} F_{k_2}^{(0d_2)} \circ B_{k_1}^{1(0d_1)} - \sum_{\substack{d_1+d_2=d+1 \\ k_1+k_2=k+1}} (-1)^{d_2} B_{k_2}^{2(0d_2)} \circ F_{k_1}^{(0d_1)} = 0$$

where $F_k^{(0d)}$ is the operator corresponding to $f_k^{(d)}$.

Remark 10.53 We can prove one more step. Namely the homotopy equivalence n_k is unique up to chain homotopy. The method of proof is similar. (See Remark 12.42.) We can further continue and will arrive the notion of A^∞ category consisting of all A^∞ functors. (See [15].) We do not discuss it here.

§ 11 Extension and Floer cohomology 2 (Higher cohomology)

In this section we describe the isomorphisms in Theorems 3.1 and 6.1 in terms of m_k and prove the commutativity of the higher cohomology analogue of Diagrams 1,2 in §6. We first consider the case of Abelian variety. Let $(L,\mathcal{L})$, $(L',\mathcal{L}')$ be pairs of affine Lagrangian submanifolds and flat line bundles on it. For simplicity, we assume that L, L' are transversal to L_{pt}. Let k be the number such that

(11.1) $$HF^k((L',\mathcal{L}'),(L,\mathcal{L})) \neq 0.$$

Since $(T^{2n},\Omega)^\vee$ is an Abelian variety, it follows from Corollary 5.27 that there exist line bundles $\mathcal{E}(L_i(w_i),\beta_i)$ and an exact sequence

(11.2) $$0 \to \mathcal{E}(L,\mathcal{L}) \to \mathcal{E}(L_1(w_1),\beta_1)^{\oplus N_1} \to \cdots \to \mathcal{E}(L_{k+2}(w_{k+2}),\beta_{k+2})^{\oplus N_{k+2}}$$

such that

(11.3.1) $$Ext^m\big(\mathcal{E}(L_i(w_i),\beta_i),\mathcal{E}(L_j(w_j),\beta_j)\big) = 0 \quad \text{for } m \neq 0 \text{ and } i < j,$$

(11.3.2) $$Ext^m\big(\mathcal{E}(L',\mathcal{L}'),\mathcal{E}(L_j(w_j),\beta_j)\big) = 0 \quad \text{for } m \neq 0.$$

Hereafter we write $\mathcal{E}(L_i)$ etc. in place of $\mathcal{E}(L_i(w_i),\beta_i)$ etc. We put

$$\mathcal{F}_i \cong Ker\left(\mathcal{E}(L_i)^{\oplus N_i} \to \mathcal{E}(L_{i+1})^{\oplus N_{i+1}}\right).$$

Hence $0 \to \mathcal{F}_i \to \mathcal{E}(L_i)^{\oplus N_i} \to \mathcal{F}_{i+1} \to 0$ is exact and $\mathcal{F}_1 \cong \mathcal{E}(L,\mathcal{L})$. It follows from Assumption (11.1), (11.3) and Theorem 6.1 that $Ext^k(\mathcal{E}(L',\mathcal{L}'),\mathcal{E}(L,\mathcal{L})) \cong Ext^k(\mathcal{E}(L',\mathcal{L}'),\mathcal{F}_1) \cong \cdots \cong Ext^1(\mathcal{E}(L',\mathcal{L}'),\mathcal{F}_k)$. Therefore, we obtain an exact sequence :

(11.4) $$\begin{aligned} 0 \to Hom\big(\mathcal{E}(L',\mathcal{L}'),\mathcal{F}_k\big) &\to Hom\big(\mathcal{E}(L',\mathcal{L}'),\mathcal{E}(L_k)\big)^{\oplus N_k} \\ &\to Hom\big(\mathcal{E}(L',\mathcal{L}'),\mathcal{F}_{k+1}\big) \to Ext^k\big((L',\mathcal{L}'),(L,\mathcal{L})\big) \to 0. \end{aligned}$$

We are going to construct a map $\Phi : HF^k((L',\mathcal{L}'),(L,\mathcal{L})) \cong Ext^k(\mathcal{E}(L',\mathcal{L}'),\mathcal{E}(L,\mathcal{L}))$ by imitating the above constructions in its mirror. We first remark that the morphisms in (11.2) are elements of $Hom(\mathcal{E}(L,\mathcal{L}),\mathcal{E}(L_1(w_1),\beta_1))^{\oplus N_1}$ and $Hom\big(\mathcal{E}(L_{i-1}(w_{i-1}),\beta_{i-1}),\mathcal{E}(L_i(w_i),\beta_i)\big) \otimes M(N_{i-1},N_i)$, (where $M(N_{i-1},N_i)$ is the totality of $N_{i-1} \times N_i$ matrices.) Since we have a canonical isomorphism between 0-th Floer cohomology and Hom by Theorem 6.4, we obtain elements

(11.5.1) $$x_1 \in HF^0\big(\mathcal{E}(L,\mathcal{L}),\mathcal{E}(L_1(w_1),\beta_1)\big)^{\oplus N_1},$$

(11.5.2) $$x_i \in HF^0\Big((L_{i-1}(w_{i-1}),\beta_{i-1}),(L_i(w_i),\beta_i)\Big)\otimes M(N_{i-1},N_i).$$

Since (11.2) is exact, it follows from Theorem 6.5 that

(11.5.3) $$m_2(x_i, x_{i+1}) = 0.$$

Definition 11.6 Let $s \in HF^k\big((L',\mathcal{L}'),(L,\mathcal{L})\big)$. We define $\tilde{\Phi}(s) \in HF^0((L,\mathcal{L}),L_{k+1})^{\oplus N_{k+1}} \cong Hom(\mathcal{E}(L,\mathcal{L}),\mathcal{E}(L_{k+1}))^{\oplus N_{k+1}}$ by

(11.7) $$\tilde{\Phi}(s) = m_{k+2}\big(s, x_1, \cdots, x_{k+1}\big).$$

Lemma 11.8 $\tilde{\Phi}(s) \in Hom\big(\mathcal{E}(L',\mathcal{L}'),\ \mathcal{F}_{k+1}\big).$

Proof: $m_2\Big(m_{k+2}\big(s, x_1, \cdots, x_{k+1}\big), x_{k+2}\Big) = 0$ by (11.5.3), (11.3) and A^∞ formulae (Lemma 9.28.) The lemma follows.

Definition 11.9 $\Phi(s)$ is the image in $Ext^k((L',\mathcal{L}'),(L,\mathcal{L}))$ of $\tilde{\Phi}(s) \in Hom\big(\mathcal{E}(L',\mathcal{L}'),\ \mathcal{F}_{k+1}\big).$

Lemma 11.10 $\Phi(s)$ *is independent of the coefficient function defining* m_k.

Proof: Let m_k^1, m_k^2 be the higher multiplications obtained by two choices of coefficient functions. Then, by using Theorem 10.18, we obtain n. Using (9.33) and Assumption (11.5.3), we find that

$$m_{k+2}^2\big(s, x_1, \cdots, x_{k+1}\big) - m_{k+2}^1\big(s, x_1, \cdots, x_{k+1}\big) = \pm\, m_2\Big(n_{k+1}\big(s, x_1, \cdots, x_k\big), x_{k+1}\Big).$$

The proof of Lemma 11.10 is complete.

Proposition 11.11 Φ *is independent of the choice of the resolution* (11.2).

Proof: We first consider the "dual" way to construct the map Φ. Let

(11.12) $$\to \mathcal{E}(L'_{k+2}(w'_{k+2}),\beta'_{k+2})^{\oplus N'_{k+2}} \to \cdots \to \mathcal{E}(L'_1(w'_1),\beta'_1)^{\oplus N'_1} \to \mathcal{E}(L',\mathcal{L}') \to 0$$

be an exact sequence of sheaves such that

(11.13.1) $Ext^m\Big(\mathcal{E}(L'_i(w'_i),\beta'_i),\mathcal{E}(L'_j(w'_j),\beta'_j)\Big) = 0$ for $m \neq 0$ and $i > j$.

(11.13.2) $Ext^m\Big(\mathcal{E}(L'_j(w'_j),\beta'_j),\mathcal{E}(L,\mathcal{L})\Big) = 0$ for $m \neq 0$.

(11.13.3) $Ext^m\Big(\mathcal{E}(L'_j(w'_j),\beta'_j),\mathcal{E}(L_i(w_i),\beta_i)\Big) = 0$ for $m \neq 0$.

Hereafter, we write $\mathcal{E}(L_i')$ in place of $\mathcal{E}(L_i'(w_i'),\beta_i')$. We put

$$\mathcal{G}_i \cong coker\left(\mathcal{E}(L_{i+1}')^{\oplus N_{i+1}} \to \mathcal{E}(L_i')^{\oplus N_i}\right).$$

Hence $0 \to \mathcal{G}_{i+1} \to \mathcal{E}(L_i') \to \mathcal{G}_i \to 0$ is exact and $\mathcal{G}_1 \cong \mathcal{E}(L',\mathcal{L}')$. We then have an exact sequence

$$(11.14)\qquad \begin{aligned} 0 \to Hom\left(\mathcal{G}_k, \mathcal{E}(L,\mathcal{L})\right) &\to Hom\left(\mathcal{E}(L_k'), \mathcal{E}(L,\mathcal{L})\right)^{\oplus N_k'} \\ &\to Hom\left(\mathcal{G}_{k+1}, \mathcal{E}(L,\mathcal{L})\right) \to Ext^k\left((L',\mathcal{L}'),(L,\mathcal{L})\right) \to 0. \end{aligned}$$

Let $y_1 \in HF^0\left(L_1',(L',\mathcal{L}')\right)^{\oplus N_1'}$, and $y_i \in HF^0\left(L_i',L_{i-1}'\right) \otimes M(N_i',N_{i-1}')$ be the elements corresponding to the maps in (11.12). We have $m_2(y_i,y_{i-1}) = 0$. We put

$$(11.15)\qquad \tilde{\Phi}'(s) = m_{k+2}(y_{k+1},\cdots,y_1,s) \in HF^0\left(L_{k+1}',(L,\mathcal{L})\right) \cong Hom\left(\mathcal{E}(L_{k+1}'),\mathcal{E}(L,\mathcal{L})\right).$$

We can prove that $\tilde{\Phi}'(s) = Hom\left(\mathcal{G}_{k+1},\mathcal{E}(L,\mathcal{L})\right)$ in the same way as Lemma 11.8 and that the class $\Phi'(s) \in Ext^k\left((L',\mathcal{L}'),(L,\mathcal{L})\right)$ induced from $\tilde{\Phi}'(s)$ is independent of the coefficient function.

Lemma 11.16 $\Phi(s) = \pm\Phi'(s)$, where the sign depends only on the degree.

Proof: By A^∞ formula and the fact the Floer cohomology of nonzero degree appears only in $HF\left((L',\mathcal{L}'),(L,\mathcal{L})\right)$, we have

$$(11.17)\qquad m_2\left(m_{k+2}(y_{k+1},\cdots,y_1,s),x_1\right) = \pm\, m_2\left(y_{k+1}, m_{k+2}(y_k,\cdots,y_1,s,x_1)\right),$$

We compare (11.17) to the standard argument of double complex. Let $C_{ab} = HF(L_a',L_b) \otimes M(N_a',N_b)$. We define $\delta^1_{ab} : C_{ab} \to C_{a+1\,b}$, $\delta^2_{ab} : C_{ab} \to C_{a\,b+1}$ by

$$\delta^1_{ab}(z) = m_2(y_a,z),\ \ \delta^2_{ab}(z) = m_2(z,x_b).$$

(11.2) and (11.14) implies

$$Ker\left(\delta^1_{k+1\,0}\right)\big/ Im\left(\delta^1_{k\,0}\right) \cong Ext^k((L',\mathcal{L}'),(L,\mathcal{L})) \cong Ker\left(\delta^2_{0\,k+1}\right)\big/ Im\left(\delta^2_{0\,k}\right).$$

The isomorphism $Ker\left(\delta^1_{k+1\,0}\right)\big/ Im\left(\delta^1_{k\,0}\right) \cong Ker\left(\delta^2_{0\,k+1}\right)\big/ Im\left(\delta^2_{0\,k}\right)$ is constructed in standard homological algebra as follows. Let $z \in Ker\left(\delta^1_{k+1\,0}\right)$. We obtain $z_i \in C_{k+1-i\,i}$ such that $z = z_0$ and $\delta^2_{k+1-i\,i}(z_i) = \delta^1_{k-i\,i+1}(z_{i+1})$. Then $z_{k+1} \in Ker\left(\delta^2_{0\,k+1}\right)\big/ Im\left(\delta^2_{0\,k}\right)$ is the element corresponding to z. We consider the case $z = z_{k+1\,0} = m_{k+2}(y_{k+1},\cdots,y_1,s)$. (11.17) implies that we can take $z_1 = m_{k+2}(y_k,\cdots,y_1,s,x_1)$. In a similar way, we can take $z_i = m_{k+2}(y_{k-i+1},\cdots,y_1,s,x_1,\cdots,x_i)$. Thus we obtain $z_{k+1} = m_{k+2}(s,x_1,\cdots,x_{k+1})$. Namely $\pm\Phi(s) = \Phi'(s)$ as required.

It is easy to see that Proposition 11.11 follows from Lemma 11.16.

Theorem 11.18 *The following diagram commutes up to sign.*

$$\begin{array}{ccc} HF^{k'}((L'',\mathcal{L}''),(L',\mathcal{L}'))\otimes HF^{k}((L',\mathcal{L}'),(L,\mathcal{L})) & \xrightarrow{m_2} & HF^{k+k'}((L'',\mathcal{L}''),(L,\mathcal{L})) \\ \downarrow \Phi\otimes\Phi & & \downarrow \Phi \\ Ext^{k'}(\mathcal{E}(L'',\mathcal{L}''),\mathcal{E}(L',\mathcal{L}'))\otimes Ext^{k}(\mathcal{E}(L',\mathcal{L}'),\mathcal{E}(L,\mathcal{L})) & \longrightarrow & Ext^{k+k'}(\mathcal{E}(L'',\mathcal{L}''),(L,\mathcal{L})) \end{array}$$

Diagram 3

Here the map in the second horizontal line is Yoneda product.

To prove it we need another results, Propositions 11.20 and 11.22. We recall $\tilde\Phi(s)\in Hom\left(\mathcal{E}(L',\mathcal{L}'),\ \mathcal{F}_{k+1}\right)$. We first find a discontinuous section of $\mathcal{E}(L_i)^{\oplus N_i}$ which projects to $\tilde\Phi(s)$ by $Hom\left(\mathcal{E}(L',\mathcal{L}'),\mathcal{E}(L_k)\right)^{\oplus N_{k-1}} \to Hom\left(\mathcal{E}(L',\mathcal{L}'),\ \mathcal{F}_{k+1}\right)\to 0$. Let $(v,\sigma)\in(T^{2n},\Omega)^{\vee}$ and $z\in \mathcal{E}(L',\mathcal{L}')_{(v,\sigma)}\cong HF^{n}((L_{pt}(v),\sigma),(L',\mathcal{L}'))$. We put

$$(11.19)\quad \tilde\Phi_{k-1}(s)(v,\sigma)(z) = m_{k+2}(z,s,x_1,\cdots,x_k)\in HF^{n}((L_{pt}(v),\sigma),L_k)^{\oplus N_k}\cong \mathcal{E}(L_{k-1})^{\oplus N_k}_{(v,\sigma)}.$$

We remark that $m_{k+2}(z,s,x_1,\cdots,x_k)$ is ill-defined if $(v,\sigma)\in(T^{2n},\Omega)^{\vee}$ is on a (Hausdorff) codimension k subset that is the wall. $\tilde\Phi_{k-1}(s)$ is discontinuous there.

Lemma 11.20 $\tilde\Phi_{k-1}(s)(v,\sigma)$ *projects to* $\pm\tilde\Phi(s)(v,\sigma)\in Hom\left(\mathcal{E}(L',\mathcal{L}'),\ \mathcal{F}_{k+1}\right)$ *by the sheaf homomorphism* $\mathcal{H}om\left(\mathcal{E}(L',\mathcal{L}'),\mathcal{E}(L_k)\right)^{\oplus N_{k-1}} \to \mathcal{H}om\left(\mathcal{E}(L',\mathcal{L}'),\ \mathcal{F}_{k+1}\right)\to 0$.

Proof: By A^{∞} formulae we have

$$m_2\left(m_{k+2}(z,s,x_0,\cdots,x_k),x_{k+1}\right)=\pm m_2\left(z,m_{k+2}(s,x_0,\cdots,x_k,x_{k+1})\right).$$

Lemma 11.20 follows.

Lemma 11.20 implies that in particular that $\tilde\Phi_{k-1}(s)(v,\sigma)$ determines a smooth element of $\mathcal{H}om\left(\mathcal{E}(L',\mathcal{L}'),\ \mathcal{F}_{k+1}\right)$. Namely the singularity is contained in the kernel of $\mathcal{E}(L_k)^{\oplus N_k}\to\ \mathcal{F}_{k+1}$.

Proof of Theorem 11.18: We consider resolutions :

$$(11.21.1)\qquad 0\to\mathcal{E}(L,\mathcal{L})\to\mathcal{E}(L_1)^{\oplus N_1}\to\cdots\to\mathcal{E}(L_{k+k'+2})^{\oplus N_{k+k'+2}}$$

$$(11.21.2)\qquad \to\mathcal{E}(L'_{k+k'+2})^{\oplus N'_{k+k'+2}}\to\cdots\to\mathcal{E}(L'_1)^{\oplus N'_1}\to\mathcal{E}(L'',\mathcal{L}'')\to 0.$$

Suppose the maps in (11.21.1) is represented by $x_i \in HF^0\left(L_{i-1}, L_i\right) \otimes M(N_{i-1}, N_i)$ and the maps in (11.21.2) is represented by $y_i \in HF^0\left(L'_i, L'_{i-1}\right) \otimes M(N'_i, N'_{i-1})$. Let $s' \in HF^k\left((L'', \mathcal{L}''), (L', \mathcal{L}')\right)$, $s \in HF^k\left((L', \mathcal{L}'), (L, \mathcal{L})\right)$, $(v, \sigma) \in (T^{2n}, \Omega)^\vee$ and $z \in \mathcal{E}(L'_{k'+1})_{(v,\sigma)} \cong HF^n((L_{pt}(v), \sigma), L'_{k'+1})$. By definition

$$\tilde{\Phi}'(s')(v, \sigma)(z) = m_2\left(z, m_{k'+2}(y_{k'+1}, \cdots, y_1, s')\right).$$

Hence by Proposition 11.20 the Yoneda Product $\Phi(s') \circ \Phi(s)$ is represented by a map sending z to

$$m_{k+2}\left(m_2\left(z, m_{k'+2}(y_{k'+1}, \cdots, y_1, s')\right), s, x_1, \cdots, x_k\right). \tag{11.22}$$

By A^∞ formula, we find that (11.22) is equal to

$$\begin{aligned} &\pm m_2\left(z, m_{k+2}\left(m_{k'+2}(y_{k'+1}, \cdots, y_1, s'), s, x_1, \cdots, x_k\right)\right) \\ = \quad &\pm m_2\left(z, m_{k+k'+2}\left(y_{k'+1}, \cdots, y_1, m_2(s', s), x_1, \cdots, x_k\right)\right) \end{aligned}$$

By the proof of Lemma 11.16, we find that this element is $\pm\Phi(m_2(s', s))$. The proof of Theorem 11.18 is complete.

Theorem 11.23 *Let* $s_{ii+1} \in HF((L(i), \mathcal{L}(i)), (L(i+1), \mathcal{L}(i+1)))$, $i = 1, 2, 3$. *Suppose* $m_2(s_{12}, s_{23}) = m_2(s_{23}, s_{34}) = 0$. *Then* $\Phi\left(m_3(s_{12}, s_{23}, s_{34})\right)$ *coincides with the triple Massey-Yoneda product of* $\Phi(s_{12})$, $\Phi(s_{23})$, $\Phi(s_{34})$ *up to sign.*

Proof: Let k_i be the degree of s_{ii+1}. We take resolutions :

$$\to \mathcal{E}(L'_{k+2})^{\oplus N'_{k+2}} \to \cdots \to \mathcal{E}(L'_1)^{\oplus N'_1} \to \mathcal{E}(L(1), \mathcal{L}(1)) \to 0, \tag{11.24}$$

$$0 \to \mathcal{E}(L(4), \mathcal{L}(4)) \to \mathcal{E}(L_1)^{\oplus N_1} \to \cdots \to \mathcal{E}(L_{k+2})^{\oplus N_{k+2}} \to, \tag{11.25}$$

satisfying (11.3), (11.13). Let $y_i \in HF^0(L'_i, L'_{i-1})$, $x_i \in HF^0(L_{i-1}, L_i)$ be elements corresponding to the boundary operators of (11.24), (11.25). Using assumptions we calculate

$$\begin{aligned} &m_{k_1+k_2+k_3+1}\left(y_{k_1+k_2+k_3}, \cdots, y_1, m_3(s_{12}, s_{23}, s_{34})\right) \\ = \quad &\pm m_{k_2+k_3+2}\left(y_{k_1+k_2+k_3}, \cdots, y_{k_1+2}, m_{k_1+2}(y_{k_1+1}, \cdots, y_1, s_{12}), s_{23}, s_{34}\right) \\ &\pm m_{k_3+2}\left(y_{k_1+k_2+k_3}, \cdots, y_{k_1+k_2+1}, m_{k_1+k_2+2}(y_{k_1+k_2}, \cdots, y_1, s_{12}, s_{23}), s_{34}\right). \end{aligned} \tag{11.26}$$

On the other hand, by using $m_2(s_{12}, s_{23}) = 0$ we have :

$$\begin{aligned} &m_2\left(y_{k_1+k_2+1}, m_{k_1+k_2+2}(y_{k_1+k_2}, \cdots, y_1, s_{12}, s_{23})\right) \\ &= \pm m_{k_2+2}\left(y_{k_1+k_2+1}, \cdots, y_{k_1+1}, m_{k_1+2}(y_{k_1+1}, \cdots, y_1, s_{12}), s_{23}\right). \end{aligned} \tag{11.27}$$

The right hand side of (11.27) belongs to $Hom(\mathcal{E}(L'_{k_1+k_2+1}),\mathcal{E}(L(3)))$ and goes to zero by $Hom(\mathcal{E}(L'_{k_1+k_2+1}),\mathcal{E}(L(3))) \to Ext^{k_1+k_2}(\mathcal{E}(L(1)),\mathcal{E}(L(3)))$, since $m_2(s_{12},s_{23})=0$. Hence $m_{k_1+k_2+2}(y_{k_1+k_2},\cdots,y_1,z_{12},z_{23})$ is a chain which bounds the cycle representing the Yoneda product $m_2(z_{12},z_{23})$. Thus the second term in (11.26) is one of the terms defining Massey-Yoneda product. In a similar way we find that the first term gives another term of Massey Yoneda product. The proof of Theorem 11.23 is complete.

Theorem 11.28 $\Phi : HF^k\big((L',\mathcal{L}'),(L,\mathcal{L})\big) \to Ext^k\big((L',\mathcal{L}'),(L,\mathcal{L})\big)$ *is an isomorphism.*

Proof: We need to prove injectivity only since we know the groups are isomorphic to each other by Theorem 6.1. To show injectivity we study the map

$$\begin{aligned} m_{k+2} : HF^n((L_{pt}(v,\sigma),(L',\mathcal{L}')) \otimes HF^k((L',\mathcal{L}'),(L,\mathcal{L})) \\ \otimes HF^0((L,\mathcal{L}),L_1) \otimes \cdots \otimes HF^0(L_{k-1},L_k) \to HF^n((L_{pt}(v,\sigma),L_k). \end{aligned}$$

First we determine the combinatorial structure of its wall. To save notation, we put $L_0=(L,\mathcal{L})$ and $L_{-1}=(L',\mathcal{L}')$, $\tilde L_{-2}=\tilde L_{pt}$ and $\eta(L_{j_1},\cdots,L_{j_m})=\eta(j_1,\cdots,j_m)$. We consider $\eta(j_1,\cdots,j_m)$ only in the case $j_1<j_2<\cdots<j_m$. By Assumption (11.3.1) and Lemma 2.25, we have

(11.29) $\eta(-2,-1,0)=\eta(-1,0,i)=k$ and all other $\eta(j_1,j_2,j_3)=0$.

We can study the wall in the same way as the examples in § 10 (especially Figure 13) and obtain the following :

Lemma 11.30 $\mathcal{C}(-2,-1,0,j_1,\cdots,j_{m-1})$, *(resp.* $\mathcal{C}(-1,0,j_1,\cdots,j_m)$*) is homeomorphic to a product of the* $nm-k-m$ *dimensional vector space and a cone of two* $m-1$ *- dimensional simplexes.*

We recall $L_i=(L_i(w_i),\beta_i)$. We write $(L,\mathcal{L})=(L_0(w_0),\beta_0)$, $(L',\mathcal{L}')=(L_{-1}(w_{-1}),\beta_{-1})$ $\tilde L_{pt}=\tilde L_{-2}$. Note that here we move only (v,σ). Namely the other variables w_i, β_i are fixed. By Bair's category theorem, we can choose the coefficient function c_k such that the following (11.31) holds.

(11.31) If $j(i)\neq -2,-1,0$ then $\mathcal{C}(-1,0,j(1),\cdots,j(m))$ does not contain $[w_{-1},w_0,w_{j(1)},\cdots,w_{j(m)}]$.

We next use the Maurer-Cartan equation (Theorem 10.53) and obtain

$$\bar\partial m_k^{(0d)} + \sum_{\substack{d_1+d_2=\ell+1\\ k_1+k_2=k+1}} \pm m_{k_2}^{(0d_2)} \circ m_{k_1}^{(0d_1)} = 0. \tag{11.32}$$

We remark we are moving only v. (11.32) makes sense and holds in this situation because of (11.31). We then have

(11.33) $$\bar\partial\left(\tilde\Phi_{k-1}(s)\right)(v,\sigma)(z) = \pm\, m_2(z, m_k^{(01)}(s, x_1, \cdots, x_k)).$$

Now we recall the exact sequence

$$\begin{aligned} Hom\left(\mathcal{E}(L',\mathcal{L}),\ \mathcal{F}_k\right) \quad &\to Hom\left(\mathcal{E}(L',\mathcal{L}), \mathcal{E}(L_k)\right)^{\oplus N_{k-1}} \\ &\to Hom\left(\mathcal{E}(L',\mathcal{L}),\ \mathcal{F}_{k+1}\right) \xrightarrow{\ \delta\ } Ext^1(\mathcal{E}(L',\mathcal{L}),\ \mathcal{F}_k). \end{aligned}$$

The standard construction of the coboundary operator $Hom\left(\mathcal{E}(L',\mathcal{L}),\ \mathcal{F}_{k+1}\right) \to Ext^1(\mathcal{E}(L',\mathcal{L}),\ \mathcal{F}_k)$, in Dolbeault cohomology is as follows. We start from $u \in Hom\left(\mathcal{E}(L',\mathcal{L}),\ \mathcal{F}_{k+1}\right)$. We lift it to a section $\tilde u$ of $\mathcal{H}om\left(\mathcal{E}(L',\mathcal{L}), \mathcal{E}(L_k)\right)^{\oplus N_{k-1}}$ which is, in general, not holomorphic. Then, since u is holomorphic, $\bar\partial \tilde u$ is a section of $\mathcal{H}om\left(\mathcal{E}(L',\mathcal{L}), \mathcal{F}_k\right) \otimes \Lambda^{(01)}$ and represent the class $\delta u \in Ext^1(\mathcal{E}(L',\mathcal{L}),\ \mathcal{F}_k)$. We consider $\tilde\Phi(s) \in Hom\left(\mathcal{E}(L',\mathcal{L}),\ \mathcal{F}_{k+1}\right)$, the holomorphic section in Lemma 11.20. By Lemma 11.20, we can lift it to $\tilde\Phi_{k-1}(s)$, which is a (discontinuous) section of $\mathcal{H}om\left(\mathcal{E}(L',\mathcal{L}), \mathcal{E}(L_k)\right)^{\oplus N_{k-1}}$. Hence the 01 current $\bar\partial\tilde\Phi_{k-1}(s)$ represents $\delta\tilde\Phi(s) \in Ext^1(\mathcal{E}(L',\mathcal{L}),\ \mathcal{F}_k)$. To be more explicit we put

(11.34) $$\Phi_{k-2}(s) = m_{k+1}^{(01)}\left(s, x_1, \cdots, x_k\right).$$

$\Phi_{k-2}(s)$ is a $\mathcal{H}om\left(\mathcal{E}(L',\mathcal{L}), \mathcal{E}(L_k)\right)^{\oplus N_k}$ valued 01 current.

Lemma 11.35 $\Phi_{k-2}(s)$ *is a* $\mathcal{F}_k$ *valued* 01 *current.*

Proof: $m_2\left(m_{k+1}^{(01)}\left(s, x_1, \cdots, x_k\right), x_{k+1}\right) = \pm\, m_{k+1}^2\left(s, x_1, \cdots, x_{k-1}, m_2(x_k, x_{k+1})\right) = 0$. The lemma follows. (In fact this lemma is also a consequence of the construction of boundary operator summarized above.)

Thus, by (11.33), we have

Lemma 11.36 $\delta\tilde\Phi(s) = \pm\Phi_{k-2}(s)$.

We next find a 01 current with coefficient in $\mathcal{H}om\left(\mathcal{E}(L',\mathcal{L}), \mathcal{E}(L_{k-1})\right)^{\oplus N_{k-1}}$ which goes to $\Phi_{k-2}(s)$. We put

(11.37.1) $\tilde\Phi_{k-2}(s)(v,\sigma)(z) = m_{k+1}^{(01)}\left(z, s, x_1, \cdots, x_{k-1}\right),$

(11.37.2) $\Phi_{k-3}(s) = m_k^{(02)}\left(s, x_1, \cdots, x_{k-1}\right).$

(11.37) implies $m_2\left(\tilde{\Phi}_{k-2}(s), x_k\right) = \pm\Phi_{k-2}(s)$. Using Theorem 10.53 we can show $\bar{\partial}\tilde{\Phi}_{k-2}(s) = \pm\Phi_{k-3}(s)$. Therefore $\delta\delta\tilde{\Phi}(s) = \pm\Phi_{k-3}(s)$. We continue in the same way and conclude :

Proposition 11.38 $(\delta\cdots\delta)\left(\tilde{\Phi}(s)\right) \in Ext^k(\mathcal{E}(L',\mathcal{L}'), \mathcal{F}_1) = Ext^k(\mathcal{E}(L',\mathcal{L}'), \mathcal{E}(L,\mathcal{L}))$ *is represented by Dolbeault cycle* $\Psi(s)$ *where*

(11.39) $$\Psi(s)(v,\sigma)(z) = m_2^{(0\,k)}(z,s).$$

We thus describe our element $\Phi(s)$ using Dolbeault cohomology. To show that it is nontrivial, we use the above representative and Serre duality. Namely we prove the following :

Theorem 11.40 *The following diagram commutes up to nonzero constant.*

$$\begin{array}{ccccc} HF^k((L',\mathcal{L}'),(L,\mathcal{L})) & \otimes & HF^{n-k}((L,\mathcal{L}),(L',\mathcal{L}')) & \to & \mathbf{C} \\ & \downarrow \Phi\otimes\Phi & & & \| \\ Ext^k(\mathcal{E}(L',\mathcal{L}'),\mathcal{E}(L,\mathcal{L})) & \otimes & Ext^{n-k}(\mathcal{E}(L,\mathcal{L}),\mathcal{E}(L',\mathcal{L}')) & \to & \mathbf{C} \end{array}$$

Diagram 4

Here the inner products are defined in Remark 3.3.

Proof: We apply Proposition 11.38 also to $t \in HF^{n-k}((L,\mathcal{L}),(L',\mathcal{L}'))$ and obtain $\Psi(t)$. To prove Theorems 11.40 and 11.26, we calculate $\Psi(s)$ and $\Psi(t)$ more explicitly. We may regard $L(-2,-1,0) \cong V/\tilde{L}_{pt}$. By definition, we may choose $C(-2,-1,0) \subseteq L(-2,-1,0)$ as a codimension k linear subspace such that Q is positive definite on it. Furthermore, we may assume that $C(-2,-1,0) \cap \Gamma \cong \mathbf{Z}^{n-k}$. We put $C(-2,-1,0) \cap \Gamma \cong \Gamma_1$, $\bar{C}(-2,-1,0) = C(-2,-1,0)/\Gamma_1$. We consider the case $s = [p]$, where $p \in L \cap L'$. Let $\tilde{p} \in V$ be a lift of it and $v(p) \in V/\tilde{L}_{pt}$ be its equivalence class. Then the support of $\Psi([p])$ is

(11.41) $$T = \left\{[[v]] \,\middle|\, \tilde{p} \in \hat{L}(v) + C(-2,-1,0)\right\} = \left\{[[v]] \,\middle|\, v - v(p) \in \bar{C}(-2,-1,0)\right\} \cong T^k \subset \left(V/\tilde{L}_{pt}\right)/\left(\Gamma/\Gamma\cap\tilde{L}_{pt}\right).$$

We consider $[[v]] \in T$. Let $q_1 \in L(v) \cap L$, $q_2 \in L(v) \cap L'$. We are going to calculate the $[q_2]$ component of $\Phi_0([p])([q_1])$. It is zero unless

(11.42) $$q_1, q_2 \in v(p) + C(-2,-1,0) + \tilde{L}_{pt} \mod \Gamma.$$

In case (11.42) holds, we choose lifts $\tilde{q}_1, \tilde{q}_2$ in V such that

(11.43) $$\tilde q_1, \tilde q_2 \in v(p) + C(-2,-1,0) + \tilde L_{pt}.$$

We may choose $\tilde q_1 \in \hat L(v(p))$, $\tilde q_2 \in \hat L'(v(p))$. We put :

(11.44)
$$\begin{aligned}&\Theta_0^{(k)}(v,\sigma)_{p,q_1,q_2}\\ &= \sum_{\gamma\in\Gamma_1} \exp\left(-2\pi Q(q_1(\gamma), v(p), q_2(\gamma)) + 2\pi\sqrt{-1}H(q_1(\gamma), v(p), q_2(\gamma); \beta_1, \sigma, \beta_2)\right).\end{aligned}$$

Here $\{q_1(\gamma)\} = \hat L(\tilde q_1) \cap L(v(p)+\gamma)$, $\{q_2(\gamma)\} = \hat L'(\tilde q_2) \cap L(v(p)+\gamma)$.

By definition the $[q_2]$ coefficient of $\tilde\Phi_0([p])([q_1])$ around (v,σ) is $(0,k)$ component of the delta current times $\Theta_0^{(k)}(v,\sigma)_{p,q_1,q_2}$. We remark that

(11.45) $$\Theta_0^{(k)}(v,\sigma)_{p,q_1,q_2} = \sum_{\gamma\in\Gamma_1} C_\gamma \exp\left(2\pi\sqrt{-1}\sigma(q_2(\gamma) - q_1(\gamma))\right).$$

Here C_γ is independent of σ.

We next consider $\tilde\Phi_0(t)$. Note $L(-2,0,-1) \cong L(-2,-1,0) \cong V/\tilde L_{pt}$, and $Q|_{L(-2,0,-1)} = -Q|_{L(-2,-1,0)}$ by this isomorphism. Hence $C(-2,0,-1)$ is transversal to $C(-2,-1,0)$.

Let $p' \in L \cap L'$. We consider $t = [p']$. We define $v(p') \in V/\tilde L_{pt}$ in a similar way as $v(p)$. We put

(11.46) $$T' = \left\{[[v]] \mid v - v(p') \in C(-2,0,-1)\right\} \cong T^{n-k} \subset \left(V/\tilde L_{pt}\right)/\left(\Gamma/\Gamma\cap\tilde L_{pt}\right).$$

Let $[[v]] \in T'$, $q_1 \in L(v) \cap L$, $q_2 \in L(v) \cap L'$. We choose $\tilde q_1'$, $\tilde q_2'$ such that $\tilde q_1, \tilde q_2 \in v(p) + C(-2,-1,0) + \tilde L_{pt}$, $\tilde q_1' \in \hat L(v(p'))$, $\tilde q_2' \in \hat L'(v(p'))$. We define

(11.47)
$$\begin{aligned}&\Theta_0^{(n-k)}(t)(v,\sigma)_{p',q_1',q_2'}\\ &= \sum_{\gamma\in\Gamma_1} \exp\left(2\pi Q(q_1'(\gamma), v(p'), q_2'(\gamma)) - 2\pi\sqrt{-1}H(q_1'(\gamma), v(p'), q_2'(\gamma); \beta_1, \sigma, \beta_2)\right).\end{aligned}$$

Here $\{q_1'(\gamma)\} = \hat L'(\tilde q_1') \cap L(v(p')+\gamma)$, $\{q_2'(\gamma)\} = \hat L'(\tilde q_2') \cap L(v(p')+\gamma)$. We have

(11.48) $$\Theta_0^{(n-k)}(t)(v,\sigma)_{p,q_1,q_2} = \sum_{\gamma\in\Gamma_2} C_\gamma' \exp\left(-2\pi\sqrt{-1}\sigma(q_2'(\gamma) - q_1'(\gamma))\right).$$

Let ω^n be the nontrivial holomorphic n form on $(T^{2n},\Omega)^\vee$. By definition, we have

(11.49)
$$\begin{aligned}&\int_{(T^{2n},\Omega)^\vee} \tilde\Phi_0([p]) \wedge \tilde\Phi_0([p']) \wedge \omega^n\\ &= C \sum_{v\in T\cap T'} \int_{\sigma\in(T^{2n},\Omega)^\vee} \Theta^{(k)}([p])(v,\sigma) \wedge \Theta^{(n-k)}([p'])(v,\sigma)\, d\sigma.\end{aligned}$$

By (4.45), (4.45), we find that (4.49) is equal to

$$(11.50) \qquad C\sum_{\substack{\gamma_1\in\Gamma_1,\\ \gamma_2\in\Gamma_2}} C_{\gamma_1}C_{\gamma_2}\int_{\sigma\in(T^{2n},\Omega)^\vee}\exp\Big(2\pi\sqrt{-1}\sigma\big(q_2(\gamma)-q_1(\gamma)-q_2'(\gamma)+q_1'(\gamma)\big)\Big)d\sigma\,.$$

We remark that $q_1(\gamma)-q_2(\gamma)\in C(-2,-1,0)$, $q_1(\gamma)-q_2'(\gamma)\in C(-2,0,-1)$. Since $C(-2,-1,0)\cap C(-2,0,-1)=\{0\}$, it follows that the integral in (11.50) is 0 unless $q_1(\gamma)-q_2(\gamma)=q_1'(\gamma)-q_2'(\gamma)=0$. Therefore (11.50) is zero unless $v(p)=v(p')=v$. In that case, (11.50) is $CC_0C_0\mathrm{Vol}(T^n)$ and is a constant. The proofs of Theorems 11.40 and 11.28 are complete.

We finally consider the case when $(T^{2n},\Omega)^\vee$ is not necessary an Abelian variety. In that case we *define* $\Phi: HF^k\big((L',\mathcal{L}'),(L,\mathcal{L})\big)\to Ext^k\big((L',\mathcal{L}'),(L,\mathcal{L})\big)$ by (10.39). Then by the argument above, Φ is an isomorphism. We are going to prove Theorems 11.18 and 11.23 by using this definition of Φ. (Then they are generalized to the case when $(T^{2n},\Omega)^\vee$ is not necessary an Abelian variety.)

Alternative proof of Theorem 11.18: Let s,s' be as in the proof of Theorem 11.18. By Theorem 10.49, we have

$$(11.51) \qquad \begin{aligned}\Psi(m_2(s,s'))(v,\sigma)(z) &= m_2^{(0\,k+k')}(z,m_2(s,s'))\\ &= \pm m_2^{(0\,k')}\Big(m_2^{(0\,k)}(z,s),s'\Big)\pm\Big(\bar\partial m_3^{(0\,k+k'-1)}\Big)(z,s,s').\end{aligned}$$

Since $\bar\partial m_3^{(0k+k'-1)}$ is zero in Dolbeault cohomology Theorem 11.18 follows.

Alternative proof of Theorem 11.23: Let s_{12},s_{23},s_{34} be as in the proof of Theorem 11.23. By Theorem 10.49, we have

$$(11.52) \qquad \begin{aligned}&\Psi(m_3(s_{12},s_{23},s_{34}))(v,\sigma)(z)\\ &= m_2^{(0\,k_1+k_2+k_3-1)}(z,m_3(s_{12},s_{23},s_{34}))\\ &= \pm m_3^{(0\,k_2+k_3-1)}(m_2^{(0\,k_1)}(z,s_{12}),s_{23},s_{34}))\pm m_2^{(0\,k_3)}(m_3^{(0\,k_1+k_2-1)}(z,s_{12},s_{23}),s_{34}))\\ &\qquad\pm\Big(\bar\partial m_4^{(0\,k_1+k_2+k_3-2)}\Big)(z,s_{12},s_{23},s_{34}).\end{aligned}$$

By (11.51) $z\mapsto\pm m_3^{(0\,k_1+k_2-1)}(z,s_{12},s_{23})$ is a Dolbeault chain which bounds $z\mapsto m_2^{(0\,k')}\big(m_2^{(0\,k)}(z,s_{12}),s_{23}\big)$. Therefore the right hand side of (11.52) is the Massey-Yoneda product of $\Phi(s_{12}),\Phi(s_{23}),\Phi(s_{34})$. The proof of Theorem 11.23 is complete.

§ 12 Resolution and Lagrangian surgery

The purpose of this section is to show that multi theta function m_k describes various important properties of the sheaves on complex tori. In fact, in this section, we do not use so much the fact that our complex manifold is a torus. Many of the arguments of this section may be generalized if we can construct m_k satisfying Theorem 10.49 etc. on more general complex manifolds. We study the derived category $\mathbf{D}\big((T^{2n},\Omega)^\vee\big)$ of coherent sheaves of complex torus. Note that the derived category we study in this section is the usual one (see [23], [22]) and not one we introduced in § 2. For $\mathcal{F} \in Ob\big(\mathbf{D}((T^{2n},\Omega)^\vee)\big)$, $u \in \boldsymbol{Z}$ let $\mathcal{F}[u] \in Ob\big(\mathbf{D}((T^{2n},\Omega)^\vee)\big)$ be the object obtained by shifting degree. Namely $H^k\big((T^{2n},\Omega)^\vee, \mathcal{F}(u)\big) \cong H^{k+u}\big((T^{2n},\Omega)^\vee, \mathcal{F}\big)$. Roughly speaking, we construct objects such as

$$\bigoplus_a \mathcal{E}\big(L_{0,a}(w_{0,a}),\alpha_{0,a}\big)[u(0,a)] \to \cdots \to \bigoplus_a \mathcal{E}\big(L_{I,a}(w_{I,a}),\alpha_{I,a}\big)[u(I,a)]\,.$$

We consider

$$x_{i,j;a,b} \in HF^{i-j+u(j,b)-u(i,a)+1}\Big((L_{i,a}(w_{i,a}),\alpha_{i,a}),(L_{j,b}(w_{j,b}),\alpha_{j,b})\Big). \tag{12.1}$$

For each $0 \le i < j \le k$, a,b we consider an equation

$$\sum_k \sum_{\substack{i=\ell(1)<\cdots<\ell(k+1)=j\\ a=c(1),\cdots,c(k+1)=b}} (-1)^\mu\, m_k\Big(x_{\ell(1),\,\ell(2);\,c(1),\,c(2)},\cdots,x_{\ell(k),\,\ell(k+1);\,c(k),c(k+1)}\Big)=0\,. \tag{12.2}$$

Here the sign is so that it is equivalent to :

$$\sum_k \sum_{\substack{i=\ell(1)<\cdots<\ell(k+1)=j\\ a=c(1),\cdots,c(k+1)=b}} b_k\Big[x_{\ell(1),\,\ell(2);\,c(1),\,c(2)}\Big|\cdots\Big|x_{\ell(k),\,\ell(k+1);\,c(k),\,c(k+1)}\Big]=0\,. \tag{12.3}$$

Definition 12.4 We say a system $\mathcal{L}=\Big(\big((L_{i,a}(w_{i,a}),\alpha_{i,a})\big),(u(i,a)),\big(x_{i,j;a,b}\big)\Big)$ a *Lagrangian resolution*, if (12.2) is satisfied.

Theorem 12.5 *For any Lagrangian resolution* $\mathcal{L}$, *we have an object* $\mathbf{E}(\mathcal{L}) \in Ob\Big(\mathbf{D}\big((T^{2n},\Omega)^\vee\big)\Big)$.

Remark 12.6 Any Lagrangian resolution determines an A^∞ functor $\mathcal{L}ag((T^{2n},\Omega)) \to Ch$. (See [15] for definition and notation.) Hence Theorem 12.5 associates an object of a derived category of sheaves of the mirror to certain A^∞ functors $\mathcal{L}ag((T^{2n},\Omega)) \to Ch$. The conjecture we mentioned in the introduction of [12] is $\mathbf{D}\big((T^{2n},\Omega)^\vee\big) \cong \mathcal{F}unc\big(\mathcal{L}ag((T^{2n},\Omega)),Ch\big)^{op}$, (where $\mathcal{F}unc\big(\mathcal{L}ag((T^{2n},\Omega)),Ch\big)^{op}$ is the opposite category of the A^∞ category of A^∞ functors $\mathcal{Lag}(T^{2n},\Omega)\to Ch$. See [15].) Thus Lagrangian resolution naturally appears in homological mirror conjecture.

Conjecture 12.7 Any object of $\mathbf{D}\left((T^{2n},\Omega)^\vee\right)$ is obtained as $\mathbf{E}(\mathcal{L})$ from some Lagrangian resolution $\mathcal{L}$, if $(T^{2n},\Omega)^\vee$ is an Abelian variety.

Remark 12.8 Mukai mentioned to the author that he proposed a conjecture (10 years ago) that any coherent sheaf on abelian variety has a resolution by semi-homogeneous sheaves. Conjecture 12.7 will follow from this conjecture of Mukai.

Proof of Theorem 12.5: We first give an idea of the proof of Theorem 12.5. The detail will be given later in this section. We consider a direct sum of holomorphic vector bundles :

$$C(\mathcal{L}) = \bigoplus_{d,i,a} \mathcal{E}(L_{i,a}(w_{i,a}),\alpha_{i,a})\otimes\Lambda^{(0,d)}, \tag{12.9}$$

where the degree of an element of $\Lambda^{(0,d)}\left(\mathcal{E}(L_{i,a}(w_{i,a}),\alpha_{i,a}\right)$ is $d+i-u(i,a)$. We will define a boundary operator on (12.9) and will regard it as a complex of $\mathcal{O}_{(T^{2n},\Omega)^\vee}$ module sheaves. Let $i=\ell(1)<\cdots<\ell(k)=j$, $a=c(1),c(2),\cdots,c(k)=b$. We put $\vec{\ell}=(\ell(1),\cdots,\ell(k))$, $\vec{c}=(c(1),\cdots,c(k))$. Let

$$\begin{aligned} d(\vec{\ell},\vec{c}) &= \sum_{s=1}^{k-1}\Big(\ell(s)-\ell(s+1)+u(\ell(s+1),c(s+1))-u(\ell(s),c(s))+1\Big)+2-k \\ &= \ell(1)-\ell(k)+c(k)-c(1)+1. \end{aligned} \tag{12.10}$$

If $\ell_1(k_1)=\ell_2(1)$ and $c_1(k_1)=c_2(1)$, we put

$$(\vec{\ell}_1\cup\vec{\ell}_2)(i)=\begin{cases}\ell_1(i) & i\le k_1\\ \ell_2(i-k_1+1) & i>k_1,\end{cases}\quad (\vec{c}_1\cup\vec{c}_2)(i)=\begin{cases}c_1(i) & i\le k_1\\ c_2(i-k_1+1) & i>k_1.\end{cases}$$

Definition 12.11 We define a distribution valued homomorphism $m_{(\vec{\ell},\vec{c})}:\mathcal{E}(L_{i,a}(w_{i,a}),\alpha_{i,a})\to\mathcal{E}(L_{j,b}(w_{j,b}),\alpha_{j,b})\otimes\Lambda^{(0,d(\vec{\ell},\vec{c}))}$, by

$$m_{(\vec{\ell},\vec{c})}(z)=m_k^{(0\ d(\vec{\ell},\vec{c}))}\Big(z,x_{\ell(1),\ell(2);c(1),c(2)},\cdots,x_{\ell(k-1),\ell(k);c(k-1),c(k)}\Big). \tag{12.12}$$

Here $z\in\mathcal{E}(L_{i,a}(w_{i,a}),\alpha_{i,a})_{(v,\sigma)}\cong HF^n(L_{pt}(v),\sigma),(L_{i,a}(w_{i,a}),\alpha_{i,a}))$.

(12.10) implies that the right hand side of (12.12) is in $\mathcal{E}(L_{j,b}(w_{j,b}),\alpha_{j,b})_{(v,\sigma)}\cong HF^n(L_{pt}(v),\sigma),(L_{j,b}(w_{j,b}),\alpha_{j,b}))\otimes\Lambda^{(0,d)}$.

Definition 12.13 Let $S=(s_{i,a})$ is a smooth section of $C(\mathcal{L})$. We define a distribution section $\hat{\hat{\partial}}S$ of $\hat{\hat{\partial}}S$ by

(12.14) $$\left(\hat{\bar{\partial}}S\right)_{j,b} = \bar{\partial}s_{j,b} + \sum_{\vec{\ell},\vec{c}}(-1)^{\mu(\vec{\ell},\vec{c})+\deg S+d(\vec{\ell},\vec{c})}m_{(\vec{\ell},\vec{c})}(s_{i,a}).$$

Here the sum is taken over all $(\vec{\ell},\vec{c})$ such that $i=\ell(1)<\cdots<\ell(k)=j$, $a=c(1),c(2),\cdots,c(k)=b$. $\mu(\vec{\ell},\vec{c})$ in (12.14) is determined so that $(-1)^{\mu(\vec{\ell},\vec{c})}B^{(d)}_{(\vec{\ell},\vec{c})}=s\circ m_{(\vec{\ell},\vec{c})}\circ\left(s^{-1}\otimes\cdots\otimes s^{-1}\right)$ Here s in this formula is the suspension in §9 and $B^{(d)}_{(\vec{\ell},\vec{c})}$ is defined from $b^{(d)}_k$ in the same way as we defined $m^{(d)}_{(\vec{\ell},\vec{c})}$ from $c^{(d)}_k$.

Lemma 12.15 $\hat{\bar{\partial}}\circ\hat{\bar{\partial}}$ is well-defined and $\hat{\bar{\partial}}\circ\hat{\bar{\partial}}=0$.

Proof: Note that $\hat{\bar{\partial}}S$ is in general not well-defined for a distribution section. However the definition of $m^{(d)}_k$ implies that $\hat{\bar{\partial}}\circ\hat{\bar{\partial}}(S)$ is well-defined for smooth S. We calculate

(12.16) $$\begin{aligned}\pm\left(\left(\hat{\bar{\partial}}\circ\hat{\bar{\partial}}\right)S\right)_{j,b} &= \bar{\partial}\bar{\partial}s_{j,b}-\sum_{\vec{\ell},\vec{c}}(-1)^{\deg S+d(\vec{\ell},\vec{c})}\bar{\partial}\circ B_{(\vec{\ell},\vec{c})}(s_{i,a})\\ &\quad-\sum_{\vec{\ell},\vec{c}}(-1)^{\deg S+1+d(\vec{\ell},\vec{c})}B_{(\vec{\ell},\vec{c})}\circ\bar{\partial}(s_{i,a})\\ &\quad-\sum_{\substack{\vec{\ell}=\vec{\ell}_1\cup\vec{\ell}_2\\ \vec{c}=\vec{c}_1\cup\vec{c}_2}}(-1)^{d(\vec{\ell}_1,\vec{c}_1)}\left(B_{(\vec{\ell}_1,\vec{c}_1)}\circ B_{(\vec{\ell}_2,\vec{c}_2)}\right)(s_{i,a})\\ &=-\sum_{\vec{\ell},\vec{c}}\left(\bar{\partial}B_{(\vec{\ell},\vec{c})}\right)(s_{i,a})-\sum_{\substack{\vec{\ell}=\vec{\ell}_1\cup\vec{\ell}_2\\ \vec{c}=\vec{c}_1\cup\vec{c}_2}}(-1)^{\mu(\vec{\ell}_1,\vec{c}_1)}\left(B_{(\vec{\ell}_1,\vec{c}_1)}\circ B_{(\vec{\ell}_2,\vec{c}_2)}\right)(s_{i,a}).\end{aligned}$$

(Note that our convention in (10.40) is that we take wedge with b from the right.) On the other hand, (12.2) implies

(12.17) $$\begin{aligned}&\sum_{\vec{\ell},\vec{c}}(-1)^{(d_2)}\left(B^{(d_2)}\circ B^{(d_1)}\right)\left(s_{i,a},x_{\ell(1),\ell(2);c(1),c(2)},\cdots,x_{\ell(k-1),\ell(k);c(k-1),c(k)}\right)\\ &\qquad=\sum_{\substack{\vec{\ell}=\vec{\ell}_1\cup\vec{\ell}_2\\ \vec{c}=\vec{c}_1\cup\vec{c}_2}}(-1)^{\mu(\vec{\ell}_1,\vec{c}_1)}\left(B_{(\vec{\ell}_1,\vec{c}_1)}\circ B_{(\vec{\ell}_2,\vec{c}_2)}\right)(s_{i,a}).\end{aligned}$$

Theorem 10.49 and (12.17) implies that (12.16) vanishes. Lemma 12.15 is proved.

We remark that $\hat{\bar{\partial}}$ is an $\mathcal{O}_{(T^{2n},\Omega)^\vee}$ module homomorphism. ($\bar{\partial}$ is an $\mathcal{O}_{(T^{2n},\Omega)^\vee}$ module homomorphism and $m_{(\vec{\ell},\vec{c})}$ does not contain derivative and so is an $\mathcal{O}_{(T^{2n},\Omega)^\vee}$ module homomorphism.) Thus we are almost done. Namely we obtain a “complex of $\mathcal{O}_{(T^{2n},\Omega)^\vee}$ module sheaf”, which gives an element of $Ob\left(\mathbf{D}\left((T^{2n},\Omega)^\vee\right)\right)$.

However there is one trouble. Namely $m_{(\vec{\ell},\vec{c})}$ is singular. Hence, we need to be careful to choose the regularity we assume to associate $\mathcal{O}_{(T^{2n},\Omega)^\vee}$ module sheaf to the holomorphic bundle $C(\mathcal{L})$. Namely if we consider the sheaf of smooth sections S, then $m_{(\vec{\ell},\vec{c})}(S)$ is not smooth hence $\hat{\bar{\partial}}$ does not give a sheaf homomorphism. On the other hand, if we consider the sheaf of distribution valued sections S, then $m_{(\vec{\ell},\vec{c})}(S)$ is not well defined in general. We go back to this point later in this section.

The next result calculates the cohomology of $\mathbf{E}(\mathcal{L})$. We define a chain complex $C(\mathcal{O},\mathcal{L})$. We put :

(12.18) $$C^{\ell}(\mathcal{O},\mathcal{L}) = \oplus HF^{\ell-i+u(i,a)}\left((L_{st},0),\left(L_{i,a}(w_{i,a}),\alpha_{i,a}\right)\right).$$

We next define $\delta : C^{\ell}(\mathcal{O},\mathcal{L}) \to C^{\ell+1}(\mathcal{O},\mathcal{L})$. Let $S = (s_{i,a}) \in C^{\ell}(\mathcal{O},\mathcal{L})$. We put

(12.19) $$(\delta S)_{j,b} = \sum_{\vec{\ell},\vec{c}} (-1)^{\mu(\vec{\ell},\vec{c})} m_k\left(s_{i,a}, x_{\ell(1),\ell(2);c(1),c(2)};\cdots, x_{\ell(k-1),\ell(k);c(k-1),c(k)}\right).$$

Here $\mu(\vec{\ell},\vec{c})$ is as in (12.14).

Lemma 12.20 $\delta\delta = 0$.

The proof is a straight forward calculation using A^{∞} formulae and (12.2). We omit it.

Theorem 12.21 $H^k\left((T^{2n},\Omega)^{\vee},\mathbf{E}(\mathcal{L})\right) \cong H^k\left(C^*(\mathcal{O},\mathcal{L}),\delta\right)$.

We can generalize Theorem 12.21 as follows. Let $\mathcal{L},\mathcal{L}'$ be Lagrangian resolutions. We put

(12.22.1) $$C^{\ell}(\mathcal{L}',\mathcal{L}) = \oplus HF^{\ell+i-j-u'(i,a)+u(j,b)}\left(\left(L'_{i,a}(w'_{i,a}),\alpha'_{i,a}\right),\left(L_{j,b}(w_{j,b}),\alpha_{j,b}\right)\right)$$

(12.22.2) $$\begin{aligned}(\delta S)_{i',i;a',a} = \sum_{\vec{\ell}',\vec{c}',\vec{\ell},\vec{c}} (-1)^{\mu(\vec{\ell}',\vec{c}';\vec{\ell},\vec{c})} m_{k+k'-1}\Big(& x'_{\ell'(1),\ell'(2);c'(1),c'(2)},\cdots,\\ & x'_{\ell'(k'-1),\ell'(k');c'(k'-1),c'(k')}, s_{\ell'(k'),\ell(1);c'(k'),c(1)}, x_{\ell(1),\ell(2);c(1),c(2)},\\ & \cdots, x_{\ell(k-1),\ell(k);c(k-1),c(k)}\Big).\end{aligned}$$

Again by using A^{∞} formulae and (12.2) we can check $\delta\circ\delta = 0$.

Theorem 12.23 $Ext^k(\mathbf{E}(\mathcal{L}'),\mathbf{E}(\mathcal{L})) \cong H^k\left(C^*(\mathcal{L}',\mathcal{L}),\delta\right)$.

Remark 12.24 We can also describe the Yoneda and Massey-Yoneda products among elements of $Ext^k(\mathbf{E}(\mathcal{L}'),\mathbf{E}(\mathcal{L}))$ in terms of m_k. We leave it to the reader. (Compare the definition of (higher) compositions among A^{∞} functors in [15].)

Next we slightly rewrite Theorem 12.21. We recall $HF\left((L_{st},0),\left(L_{i,a}(w_{i,a}),\alpha_{i,a}\right)\right) \cong HF\left(\left(L_{i,a}(w_{i,a}),\alpha_{i,a}\right),(L_{st},0)\right)^*$. We use the (canonical) basis of them to associate element $s^*_{i,a} \in HF\left(\left(L_{i,a}(w_{i,a}),\alpha_{i,a}\right),(L_{st},0)\right)^*$ to each element of

$s_{i,a} \in HF\Big(\big(L_{st},0\big),\big(L_{i,a}(w_{i,a}),\alpha_{i,a}\big)\Big)$.

Corollary 12.25 $H\Big((T^{2n},\Omega)^{\vee},\mathbf{E}(\mathcal{L})\Big)$ *is isomorphic to the linear space of the solutions of the linear equations*

$$\sum_{\vec{\ell},\vec{c}}(-1)^{\mu(\vec{\ell},\vec{c})} m_k\big(s_{i,a}, x_{\ell(1),\ell(1);c(2),c(2)},\cdots,x_{\ell(k-1),\ell(k);c(k-1),c(k)}\big)=0,$$
$$\sum_{\vec{\ell}',\vec{c}'}(-1)^{\mu(\vec{\ell}',\vec{c}')} m_{k'}\big(x_{\ell'(1),\ell'(2);c'(1),c'(2)},\cdots,x_{\ell'(k'-1),\ell'(k');c'(k'-1),c'(k')},s^{*}_{j,b}\big)=0,$$

for $S=\big(s_{i,a}\big)$.

Corollary 12.25 follows immediately from Theorem 12.11 by using the cyclicity :

$$\begin{aligned}&\Big(m_{k'}\big(t_{i,a},x_{\ell'(1),\ell'(2);c'(1),c'(2)},\cdots,x_{\ell'(k'-1),\ell'(k');c'(k'-1),c'(k')}\big),s_{j,b}\Big)\\ &=\Big\langle m_{k'}\big(t_{i,a},x_{\ell'(1),\ell'(2);c'(1),c'(2)},\cdots,x_{\ell'(k'-1),\ell'(k');c'(k'-1),c'(k')}\big),s^{*}_{j,b}\Big\rangle\\ &=\pm\Big\langle m_{k'}\big(x_{\ell'(1),\ell'(2);c'(1),c'(2)},\cdots,x_{\ell'(k'-1),\ell'(k');c'(k'-1),c'(k')},s^{*}_{j,b}\big),t_{i,a}\Big\rangle.\end{aligned}$$

Before proving theorems, we give some examples.

Example 12.26 Suppose $\eta(\tilde{L}_1,\tilde{L}_2)=0$. We choose

(12.27) $$x=\sum_{p\in L_1(w_1)\cap L_2(w_2)} c_p[p]\in HF^0((L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2)),$$

$(L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2),x$ determine a Lagrangian resolution $\mathcal{L}$. Then $\mathbf{E}(\mathcal{L})$ is an element of $\mathbf{E}(\mathcal{L})\in Ob\big(\mathbf{D}\big((T^{2n},\Omega)^{\vee}\big)\big)$ determined by the complex $\mathcal{E}(L_1(w_1),\alpha_1)\xrightarrow{\Phi(x)}\mathcal{E}(L_2(w_2),\alpha_2)$.

Example 12.28 (Compare [41].) Suppose $\eta(\tilde{L}_1,\tilde{L}_2)=1$. We choose $x\in HF^1((L_1(v_1),\alpha_1),(L_2(v_2),\alpha_2))$. $(L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2),x$ determine a Lagrangian resolution $\mathcal{L}$. ($u(2)=1$.) We have an exact sequence

(12.29) $$0\to\mathcal{E}(L_2(w_2),\alpha_2)\to\mathbf{E}(\mathcal{L})[1]\to\mathcal{E}(L_1(w_1),\alpha_1)\to 0.$$

which corresponds to $x\in Ext^1(\mathcal{E}(L_1(w_1),\alpha_1),\mathcal{E}(L_2(w_2),\alpha_2))$. To show (12.29) we consider the operator $\hat{\bar{\partial}}$ we used in the proof of the Theorem 12.5. In our case $C^d(\mathcal{L})=\big(\mathcal{E}(L_1(w_1),\alpha_1)\otimes\Lambda^{(0,d)}\big)\oplus\big(\mathcal{E}(L_2(w_2),\alpha_2)\otimes\Lambda^{(0,d)}\big)$, and $\hat{\bar{\partial}}:C^d(\mathcal{L})\to C^{d+1}(\mathcal{L})$ is

$$\hat{\bar{\partial}}(s_1,s_2)=\big(\bar{\partial}s_1,\bar{\partial}s_2+m_2^{(1)}(s_1,x)\big).$$

Therefore the Dolbeault resolution $\underset{d}{\oplus}\left(\mathcal{E}(L_2(w_2),\alpha_2)\otimes\Lambda^{(0,d)}\right)$ of $\mathcal{E}(L_2(w_2),\alpha_2)$ is a subcomplex of the complex $C^*(\mathcal{L})$ defining $\mathbf{E}(\mathcal{L})[1]$ and we have an exact sequence

$$0\to\underset{d}{\oplus}\left(\mathcal{E}(L_2(w_2),\alpha_2)\otimes\Lambda^{(0,d)}\right)\to C^*(\mathcal{L})\to\underset{d}{\oplus}\left(\mathcal{E}(L_1(w_2),\alpha_1)\otimes\Lambda^{(0,d)}\right)\to 0\,.$$

compatible with differentials. (12.29) follows.

Example 12.30 Let $\eta^*(\tilde{L}_1,\tilde{L}_2)=\eta^*(\tilde{L}_3,\tilde{L}_4)=1$ and $\eta^*(\tilde{L}_i,\tilde{L}_j)=0$ for other $i<j$. We put $u(1)=0$, $u(2)=u(3)=1$, $u(4)=u(5)=2$ and consider $x_{ij}\in HF^{d_{ij}}\left(\left(L_i(w_i),\alpha_i\right),\left(L_j(w_j),\alpha_j\right)\right)$, where $d_{12}=d_{34}=1$, $d_{23}=d_{45}=d_{13}=d_{14}=d_{35}=0$. Our equation (12.2) is

(12.31.1) $$m_3(x_{23},x_{34},x_{45})\pm m_2(x_{24},x_{45})\pm m_2(x_{23},x_{35})=0$$

(12.31.2) $$m_4(x_{12},x_{23},x_{34},x_{45})\pm m_3(x_{13},x_{34},x_{45})\pm m_3(x_{12},x_{24},x_{45})\\ \pm m_2(x_{13},x_{35})\pm m_2(x_{14},x_{45})=0.$$

They are third and fourth order equations of $L_1\bullet L_2+L_2\bullet L_3+L_3\bullet L_4+L_4\bullet L_5+L_1\bullet L_3+L_1\bullet L_4+L_2\bullet L_4+L_3\bullet L_5$ variables. (The number of equations is $L_2\bullet L_5+L_1\bullet L_5$.) In case when $\mathbf{E}(\mathcal{L})[2]$ is represented by a vector bundle, we have a diagram of exact sequences :

$$\begin{array}{ccccccccccc}
 & & & & & & 0 & & & & \\
 & & & & & & \downarrow & & & & \\
 & & 0 & \to & \mathcal{E}_2 & \to & \mathcal{F} & \to & \mathcal{E}_1 & \to & 0\\
 & & & & 0 & & & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & \mathbf{E}(\mathcal{L})[2] & & \mathcal{E}_3 & \to & 0 & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{E}_4 & \to & \mathcal{H} & \to & \mathcal{G} & \to & 0 & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & \mathcal{E}_5 & & 0 & & & & \\
 & & & & \downarrow & & & & & & \\
 & & & & 0 & & & & & &
\end{array}$$

Diagram 5

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathcal{F} & & \\
 & & & & & & \downarrow & & \\
0 & \to & \mathcal{E}_4 & \to & \vartheta & \to & \mathcal{E}_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{E}_4 & \to & \mathcal{H} & \to & \mathcal{G} & \to & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

Diagram 6

Here $\mathcal{E}_i = \mathcal{E}(L_i(w_i),\alpha_i)$. The extension of the first line in Diagram 5 is given by x_{12}. The composition $\mathcal{E}_2 \to \mathcal{F} \to \mathcal{E}_3$ is x_{23}. There exists a lift $\mathcal{F} \to \mathcal{E}_3$ of x_{23} since $Ext^1(\mathcal{E}_1,\mathcal{E}_3) = 0$. The lift is not unique. The ambiguity is controlled by $x_{13} \in Hom(\mathcal{E}_1,\mathcal{E}_3)$. The extension of the first horizontal line in Diagram 6 is given by $x_{34} \in Ext^1(\mathcal{E}_3,\mathcal{E}_4)$. Note that we can find $\mathcal{H}$ as in the second horizontal line of Diagram 6, since $Ext^1(\mathcal{F},\mathcal{E}_4) = 0$. (The extension $\mathcal{H}$ such that Diagram 6 commutes is not unique. The ambiguity is controlled by x_{14} and x_{24}.) The equations (12.31) give a condition for the map $x_{45} \in Hom(\mathcal{E}_4,\mathcal{E}_5)$ to extend to $\mathcal{H} \to \mathcal{E}_5$. It seems possible but complicated to identify this obstruction as an element of $Hom(\mathcal{E}_1,\mathcal{E}_5) \oplus Hom(\mathcal{E}_2,\mathcal{E}_5)$. We do not need to do so since we can construct Diagram 5 directly from

$$\hat{\bar\partial} = \begin{pmatrix}
\bar\partial & & & & \mathbf{0} \\
m_2^{(1)}(\bullet,x_{12}) & \bar\partial & & & \\
m_3(\bullet,x_{12},x_{23}) \pm m_2(\bullet,x_{13}) & m_2(\bullet,x_{23}) & \bar\partial & & \\
m_4(\bullet,x_{12},x_{23},x_{34}) \pm m_3(\bullet,x_{13},x_{34}) \pm m_2(\bullet,x_{14}) & m_3(\bullet,x_{23},x_{34}) \pm m_2(\bullet,x_{24}) & m_2^{(1)}(\bullet,x_{34}) & \bar\partial & \\
0 & 0 & m_3(\bullet,x_{34},x_{45}) \pm m_2(\bullet,x_{35}) & m_2(\bullet,x_4) & \bar\partial
\end{pmatrix}.$$

We next explain how Theorems 12.5 12.21 and 12.23 are related to the study of Lagrangian submanifolds in tori. We will show what Examples 12.26 and 28 correspond in the mirror. We regard $L_1(w_1) \cup L_2(w_2)$ as a singular Lagrangian submanifold in (T^{2n},Ω). We suppose $B = 0$ for simplicity. We can perform Lagrangian surgery at each $p \in L_1(w_1) \cap L_2(w_2)$ and obtain a smooth Lagrangian submanifold $L \subseteq \left(T^{2n},\omega\right)$. In case $n = 2$, $L_1(w_1) \bullet L_2(w_2) = 1$, we obtain a genus two Lagrangian surface in T^4.

Conjecture 12.32 The bundle $\mathcal{E}(L,\mathcal{L})$ constructed in § 2 is equal to $\mathbf{E}(\measuredangle)$ with $c_p \neq 0$.

We explain an argument to justify the conjecture. Let $(v,\sigma)\in(T^{2n},\Omega)^\vee$. The fiber $\mathbf{E}(\mathcal{L})_{(v,\sigma)}$ is a cohomology of the complex

$$HF((L_{pt}(v),\sigma),(L_1(w_1),\alpha_1))\xrightarrow{m_2(\bullet,x)}HF((L_{pt}(v),\sigma),(L_2(w_2),\alpha_2)). \tag{12.33}$$

We remark that the isomorphism class of the complex (12.33) is independent of c_p as far as it is nonzero. In fact if $c'_p=\kappa c_p$, we have an isomorphism which is $\times\kappa$ on $HF((L_{pt}(v),\sigma),(L_2(w_2),\alpha_2))$.

On the other hand, the fiber $\mathcal{E}(L,\mathcal{L})_{(v,\sigma)}$ of $\mathcal{E}(L,\mathcal{L})$ is the Floer cohomology $HF^n((L_{pt}(v),\sigma),(L,\mathcal{L}))$ by definition. Floer's chain complex to calculate it is (as graded Abelian group) :

$$\begin{aligned} CF((L_{pt}(v),\sigma),(L,\mathcal{L})) &= \sum_{p\in L_{pt}(\sigma)\cap L} Hom\Big(\mathcal{L}(\sigma)_p,\mathcal{L}_p\Big) \\ &\cong \sum_{p\in L_{pt}(\sigma)\cap L_1(w_1)} Hom\Big(\mathcal{L}(\sigma)_p,\mathcal{L}(\alpha_1)\Big)\oplus\sum_{p\in L_{pt}(\sigma)\cap L_2(w_2)} Hom\Big(\mathcal{L}(\sigma)_p,\mathcal{L}(\alpha_2)\Big) \\ &\cong HF\Big(\mathcal{L}(\sigma)_p,\mathcal{L}(\alpha_1)\Big)\oplus HF\Big(\mathcal{L}(\sigma)_p,\mathcal{L}(\alpha_2)\Big). \end{aligned} \tag{12.34}$$

Floer's boundary operator of $CF((L_{pt}(v),\sigma),(L,\mathcal{L}))$ is obtained by counting the number of holomorphic 2 gons bounding $L_{pt}(v)$ and L. As can be seen in Figure 16, such 2 gon will become a holomorphic triangle used in the definition of m_2 in (12.33).

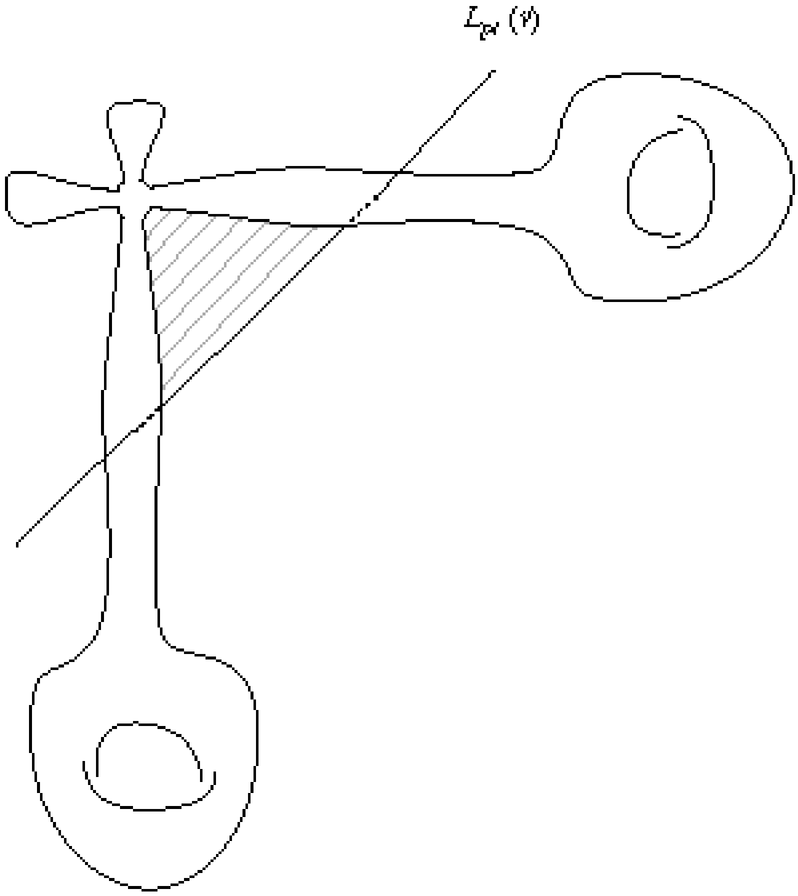

Figure 16

Thus we find that the boundary operator in (12.33) and the map in (12.32) coincides. Hence $\mathbf{E}(\mathcal{L})_{(v,\sigma)} \cong \mathcal{E}(L,\mathcal{L})_{(v,\sigma)}$.

From this argument, the reader finds that calculating m_k gives a way to calculate Floer homology of various Lagrangian submanifolds (which are not affine) in tori. We do not pursue this line in this paper and leave it to future research.

We now go back to the proof of Theorem 12.5. To overcome the difficulty mentioned before, we are going to replace $m_k^{(d)}$, (which are singular), by smooth one. In § 9, we constructed a family $c_k^{(d)}$ of integral currents by solving the equation :

$$dc_k^{(d)} + \sum \pm c_{k_1}^{(d_1)} \circ c_{k-k_1+1}^{(d-d_1+1)} = 0 \tag{12.35}$$

inductively. We do the same process but using smooth forms in place of integral currents. First we take a smooth d form $c_{smooth}(1,2,3)$ for each $\deg(1,2,3) = d$. More precisely, we choose $c'_{smooth}(1,2,3)$ first so that the following is satisfied.

$$\begin{aligned} &\operatorname{supp}\left(c'_{smooth}(1,2,3)\right) \\ &\qquad \subseteq \left\{[v_1,v_2,v_3] \in L(1,2,3) \,\middle|\, Q[v_1,v_2,v_3] \geq \delta \left\|[v_1,v_2,v_3]\right\|^2 \right\}. \end{aligned} \tag{12.36.1}$$

(12.36.2) $c'_{smooth}(1,2,3)$ is invariant of $[v_1,v_2,v_3] \mapsto [cv_1,cv_2,cv_3]$.
(12.36.3) $c'_{smooth}(1,2,3)$ is smooth outside origin.
(12.36.4) $dc'_{smooth}(1,2,3) = 0$. And $c'_{smooth}(1,2,3)$ represents a generator in $H^d_{DR}(S(Q,1,2,3),\boldsymbol{R})$.

To remove the singularity at the origin we replace it by $c_{smooth}(1,2,3)$ such that

$$\begin{aligned}&\mathrm{supp}\left(c_{smooth}(1,2,3)\right)\\ &\quad\subseteq \left\{[v_1,v_2,v_3] \in L(1,2,3) \mid Q[v_1,v_2,v_3] > \delta\left\|[v_1,v_2,v_3]\right\|^2 - C\right\}\end{aligned} \tag{12.37.1}$$

for some constant C.
(12.37.2) $c_{smooth}(1,2,3)$ is invariant of $[v_1,v_2,v_3] \mapsto [cv_1,cv_2,cv_3]$ outside a compact set $\mathcal{K}(1,2,3)$.
(12.37.3) $c_{smooth}(1,2,3)$ is smooth.
(12.37.4) $c_{smooth}(1,2,3) - c'_{smooth}(1,2,3) = d\left(\Delta c(1,2,3)\right)$ where $\Delta c(1,2,3)$ is of compact support in $\mathcal{K}(1,2,3)$.

We next construct $c_{smooth}(1,\cdots,k+1)$ inductively. We can solve (12.35) inductively since appropriate De-Rham cohomology vanishes. We then obtain $c_{smooth}(1,\cdots,k+1)$ which is smooth. Also we may choose it so that a condition similar to (12.37.1) , (12.37.2) hold outside the set $\mathcal{K}(1,\cdots,k+1)$, where $\mathcal{K}(1,\cdots,k+1)$ is defined inductively as :

$$\begin{aligned}\mathcal{K}(1,\cdots,k+1) &= \mathcal{K}_0(1,\cdots,k+1) \cup \bigcup \mathcal{K}(1,\cdots,\ell,m,\cdots,k+1) \times C(\ell,\cdots,m)\\ &\qquad \cup \bigcup C(1,\cdots,\ell,m,\cdots,k+1) \times \mathcal{K}(\ell,\cdots,m).\end{aligned} \tag{12.38}$$

Here $\mathcal{K}_0(1,\cdots,k+1)$ is a small compact neighborhood of the origin.

We now use $c_{k,smooth}$ in place of c_k in Definitions 9.13 and 9.17 to obtain $m^{(d)}_{k,smooth}$. We remark that the properties (9.7.1) and (9.7.2) are used to show that $m^{(d)}_k$ converges. However we can easily find that it is enough if (12.37.1) , (12.37.2) are satisfied and $\mathcal{K}(1,\cdots,k+1)$ is of the form (12.38). We then can use exponential decay estimate to prove that $m^{(d)}_{k,smooth}$ is a (homomorphism bundle valued) smooth d form.

We want to use $m^{(d)}_{k,smooth}$ in place of $m^{(d)}_k$ to construct $\left(C(\mathcal{L}),\hat{\bar{\partial}}_{smooth}\right)$. However $x_{i,j;a,b}$ satisfies (12.2) for m_k but not for $m_{k,smooth}$. So we need to replace $x_{i,j;a,b}$ by $x'_{i,j;a,b}$ as follows.

We use the method of the proof of Theorem 12.18 to find $n^{(d)}_k$ such that

$$\bar{\partial} n_k^{(0\,d)} + \sum_{\substack{d_1+d_2=d+1\\ k_1+k_2=k+1}} \pm\, n_{k_2}^{(0\,d_2)} \circ m_k^{(0\,d_1)} + \sum_{\substack{d_1+d_2=d+1\\ k_1+k_2=k+1}} \pm \circ m_{k,smooth}^{(0 d_1)} \circ n_{k_1}^{(0 d_2)} = 0. \tag{12.39}$$

(See Lemma 10.52. Sign is as in there.) Now we put

(12.40) $$x'_{i,j;a,b} = \sum_{\vec{\ell},\vec{c}} (-1)^{\mu(\vec{\ell},\vec{c})} n_k\Big(x_{\ell(1),\ell(1);c(2),\,c(2)},\cdots,x_{\ell(k),\,\ell(k+1);c(k),\,c(k+1)}\Big).$$

Then (12.2) and the $d = 0$ case of (12.39) imply

(12.41) $$\sum_{\vec{\ell},\vec{c}} (-1)^{\mu(\vec{\ell},\vec{c})} m_{k,smooth}\Big(x'_{\ell(1),\ell(2);\,c(1),c(2)},\cdots,x'_{\ell(k),\ell(k+1);\,c(k),c(k+1)}\Big) = 0\,.$$

Hence we can use $x'_{i,j;a,b}$ and $m^{(0d)}_{k,smooth}$ to construct $\Big(C(\mathcal{L}),\hat{\bar{\partial}}_{smoot}\Big)$. (We remark that $m^{(0d)}_{k,smooth}$ satisfies the conclusion of Theorem 10.49.) We then take the sheaf of smooth sections and regard $\hat{\bar{\partial}}_{smoot}$ as a chain complex of $\mathcal{O}_{(T^{2n},\Omega)^\vee}$ module sheaf. Note that the difference of $\Big(C(\mathcal{L}),\hat{\bar{\partial}}_{smoot}\Big)$ from the direct sum of Dolbeault complex is degree zero term with smooth coefficient. Hence by usual Fredholm theory (elliptic estimate) we find that the cohomology sheaf of $\Big(C(\mathcal{L}),\hat{\bar{\partial}}_{smoot}\Big)$ is coherent. We now define

Definition 12.42 $\mathbf{E}(\mathcal{L}) = \Big(C(\mathcal{L}),\hat{\bar{\partial}}_{smoot}\Big).$

The proof of Theorem 12.5 is complete.

We remark that we can start with $m^{(0d)}_{k,smooth}$ and can avoid using singular $m^{(0d)}_k$. However it seems that $m^{(0d)}_k$ is more canonical than $m^{(0d)}_{k,smooth}$. In fact $m^{(0d)}_k$ has a theta series expansion whose coefficients are integers. While the coefficients of the expansion of $m^{(0d)}_{k,smooth}$ are not integer.

Remark 12.43 We can prove that $\mathbf{E}(\mathcal{L})$ in Definition 12.42 is independent of the smoothing $c^{(d)}_{k,smooth}$ as follows. Let $c^{1(d)}_{k,smooth}$, $c^{2(d)}_{k,smooth}$ etc. be two choices. Let $n^{1(0d)}_k$ and $n^{2(0d)}_k$ be as in (12.38) and let $x^1_{i,j;a,b}$, $x^2_{i,j;a,b}$ be as in (12.39). We also have $n^{12(0d)}_k$ such that

(12.44) $$\bar{\partial} n^{12(0d)}_k + \sum_{\substack{d_1+d_2=d+1\\ k_1+k_2=k+1}} \pm\, n^{12(0d_2)}_{k_2}\circ m^{1(0d_1)}_{k_1,smooth} + \sum_{\substack{d_1+d_2=d+1\\ k_1+k_2=k+1}} \pm \circ m^{2(0d_2)}_{k_2,smooth}\circ n^{(0d_1)}_{k_1,smooth} = 0\,.$$

We put

(12.45) $$x'^2_{i,j;a,b} = \sum_{\vec{\ell},\vec{c}} (-1)^{\mu(\vec{\ell},\vec{c})} n^{12}_k\Big(x^1_{\ell(1),\,\ell(1);c(2),\,c(2)},\cdots,x^1_{\ell(k),\ell(k+1);c(k),c(k+1)}\Big).$$

We cam easily check

(12.46) $$\sum_{\vec{\ell},\vec{c}} (-1)^{\mu(\vec{\ell},\vec{m})} m^2_{k,smooth}\Big(x'^2_{\ell(1),\ell(2);c(1),c(2)},\cdots,x'^2_{\ell(k),\,\ell(k+1);\,c(k),c(k+1)}\Big) = 0\,.$$

Namely we can use either $x^2_{i,j;a,b}$ or $x'^2_{i,j;a,b}$ (together with $m^{2(0d)}_{k,smooth}$) to construct $\left(C(\mathcal{L}),\hat{\bar{\partial}}^2_{smooth}\right)$. Let $\left(C(\mathcal{L}),\hat{\bar{\partial}}^2_{smooth}\right)$ be one obtained from $x^2_{i,j;a,b}$ and $\left(C(\mathcal{L}),\hat{\bar{\partial}}'^2_{smooth}\right)$ be one obtained from $x'^2_{i,j;a,b}$. We can use $n^{12(0d)}_k$ to construct a chain map $n^{12}:\left(C(\mathcal{L}),\hat{\bar{\partial}}^1_{smooth}\right)\to\left(C(\mathcal{L}),\hat{\bar{\partial}}'^2_{smooth}\right)$. (We use Formula (12.45) to do so.) Using the fact that $n^{(00)}_1$ is isomorphism (identity) we can show that n^{12} is an isomorphism.

So it suffices to show that $\left(C(\mathcal{L}),\hat{\bar{\partial}}'^2_{smooth}\right)=\left(C(\mathcal{L}),\hat{\bar{\partial}}'^2_{smooth}\right)$ as an element of $\mathcal{F}\in Ob\left(\mathbf{D}\left((T^{2n},\Omega)^\vee\right)\right)$. For this purpose we need to proceed as follows. We use the terminology of [15].) Let us consider the composition of A^∞ functors $n^{12}\circ n^1$ and another A^∞ functor n^2. (Here n^i is an A^∞ functor constructed from $n^{i(0d)}_k$.) We can prove that they are homotopic. The proof is similar to the proof is Theorem 10.18 and is by Acyclic model. Using the homotopy we can construct a chain homotopy equivalence $\left(C(\mathcal{L}),\hat{\bar{\partial}}'^2_{smooth}\right)\cong\left(C(\mathcal{L}),\hat{\bar{\partial}}'^2_{smooth}\right)$. We omit the detail of the proof.

Proof of Theorem 12.20: We construct $\left(C^*(\mathcal{O},\mathcal{L}),\delta_{smooth}\right)$ in the same way as $\left(C^*(\mathcal{O},\mathcal{L}),\delta\right)$ and by using $m^{(0d)}_{k,smooth}$ and $x'_{i,j;a,b}$ in place of $m^{(0d)}_k$ and $x_{i,j;a,b}$.

Lemma 12.47 *$\left(C^*(\mathcal{O},\mathcal{L}),\delta_{smooth}\right)$ is chain homotopy equivalent to $\left(C^*(\mathcal{O},\mathcal{L}),\delta\right)$.*

Proof: Let $n^{(d)}_k$ satisfy (12.39). We define $N:\left(C^*(\mathcal{O},\mathcal{L}),\delta\right)\to\left(C^*(\mathcal{O},\mathcal{L}),\delta_{smooth}\right)$ by

$$N(S)_{i,a}=\sum_{\vec{\ell},\vec{c}}(-1)^{\mu(\vec{\ell},\vec{c})}n_k\left(s_{i,a},x_{\ell(1),\ell(2);c(1),c(2)},\cdots,x_{\ell(k-1),\ell(k);c(k-1),c(k)}\right),$$

where $S=(s_{i,a})$. By using (12.2), (12.39), (12.40) and (12.41) we find that N is a chain map. Since $n^{(0)}_1=identity$, we find that N is an isomorphism. The proof of Lemma 12.47 is complete.

We next define a chain map

$$\Psi:\left(C^*(\mathcal{O},\mathcal{L}),\delta_{smooth}\right)\to\left(\Gamma(C(\mathcal{L})),\hat{\bar{\partial}}'^2_{smooth}\right). \tag{12.48}$$

(Here $\Gamma\left(C(\mathcal{L})\right)$ is the vector space of smooth sections of $C(\mathcal{L})$.) We put

$$(\Psi S)_{j,b}(v,\sigma)=\sum_{\vec{\ell},\vec{c}}(-1)^{\mu(\vec{\ell},\vec{c})+d(\vec{\ell},\vec{c})+\deg S}m^{(0d(\vec{\ell},\vec{c}))}_{k+1,smooth}\Big(x_0(v,\sigma),s_{i,a},x'_{\ell(1),\ell(1);c(2),c(2)},\\ \cdots,x'_{\ell(k-1),\ell(k);c(k-1),c(k)}\Big). \tag{12.49}$$

where $\deg S$ is the degree as differential form.

Lemma 12.50 Ψ *is a chain map.*

Proof: This is an easy calculation using A^∞ formulae and (12.39), (12.40) and (12.41). We omit it.

We now prove that Ψ induces an isomorphism in cohomologies. The proof is by induction on I (the number of i's). In case when $I=1$, Ψ coincides with the composition of the direct sum of the map Ψ in Proposition 11.38 and the isomorphism N. Hence Ψ induce an isomorphism by Theorem 11.28. Let us assume that Ψ induces an isomorphism for $I-1$. We consider $I = I_1 + I_2 + 1$. Let us consider $\mathcal{L}_{i\le I_1}$ and $\mathcal{L}_{i\ge I_1+1}$, where $\mathcal{L}_{i\le I_1}$ is the part of $\mathcal{L}$ for $i \le I_1$ and $\mathcal{L}_{i\ge I_1+1}$ is a part for $i \ge I_1+1$. There is a chain homomorphism $\left(C(\mathcal{L}),\hat{\hat{\partial}}_{smooth}\right) \to \left(C\left(\mathcal{L}_{i\le I_1}\right),\hat{\hat{\partial}}_{smooth}\right)$ whose kernel is $\left(C\left(\mathcal{L}_{i\ge I_1+1}\right),\hat{\hat{\partial}}_{smooth}\right)$.

Lemma 12.51 *There exists an exact triangle*

$$
\begin{array}{ccc}
\left(C\left(\mathcal{L}_{i\ge I_1+1}\right),\hat{\hat{\partial}}_{smooth}\right) & \longrightarrow & \left(C(\mathcal{L}),\hat{\hat{\partial}}_{smooth}\right) \\
\nwarrow & & \swarrow \\
& \left(C\left(\mathcal{L}_{i\le I_1}\right),\hat{\hat{\partial}}_{smooth}\right) &
\end{array}
$$

Proof: We can use a part of $\hat{\hat{\partial}}$ to define $\left(C\left(\mathcal{L}_{i\le I_1}\right),\hat{\hat{\partial}}_{smooth}\right) \to \left(C\left(\mathcal{L}_{i\ge I_1+1}\right),\hat{\hat{\partial}}_{smooth}\right)[1]$. The lemma follows.

One the other hand we have an exact sequence

$$0 \to \left(C^*\left(\mathcal{O},\mathcal{L}_{i\ge I_1+1}\right),\delta_{smooth}\right) \to \left(C^*\left(\mathcal{O},\mathcal{L}\right),\delta_{smooth}\right) \to \left(C^*\left(\mathcal{O},\mathcal{L}_{i\le I_1}\right),\delta_{smooth}\right) \to 0$$

of chain complex. Therefore we obtain a diagram of long exact sequences :

$$
\begin{array}{ccccccc}
\to H^k\left(C^*\left(\mathcal{O},\mathcal{L}_{i\ge I_1+1}\right),\delta_{smooth}\right) & \to & H^k\left(C^*\left(\mathcal{O},\mathcal{L}\right),\delta_{smooth}\right) & \to & H^k\left(C^*\left(\mathcal{O},\mathcal{L}_{i\le I_1}\right),\delta_{smooth}\right) & \to \\
\downarrow & & \downarrow & & \downarrow & \\
\to H^k((T^{2n},\Omega)^\vee,\mathbf{E}(\mathcal{L}_{i\ge I_1+1})) & \to & H^k((T^{2n},\Omega)^\vee,\mathbf{E}(\mathcal{L})) & \to & H^k((T^{2n},\Omega)^\vee,\mathbf{E}(\mathcal{L}_{i\le I_1})) & \to
\end{array}
$$

Diagram 7

The diagram commutes by definition. Hence the induction hypothesis and five lemma imply that Ψ induces an isomorphism for I. The proof of Theorem 12.20 is complete.

The proof of Theorem 12.23 is similar to one of Theorem 12.20 and is omitted.

In this section, we consider x_{ij} for $i < j$ only. Hence in the mirror we have a “tree” of Lagrangian submanifolds. It seems possible to study more general “graph” of Lagrangian submanifolds. Then we remove the restriction $i < j$ in the sheaf theory sides. The construction in that case seems to become more complicated.

We finally remark that there is one important point of view which is not studied in this paper. That is, in this paper we fix (T^{2n},Ω) and regard m_k as a function on Abelian variety. In the theory of theta function, it is more important to regard it as a function of Ω (the moduli parameter of Abelian variety). This point of view is important also for Mirror symmetry. Note that we can generalize the equation $\bar\partial m^{(d)} + \sum \pm m^{(d')} \circ m^{(d-d'+1)} = 0$ so that $\bar\partial$ include derivative with respect to Ω, under certain circumstances. In the case when $d = 0$ and the case when the image of the wall is compact in T^{2n}, this equation can be regarded as one to control wall crossing of a function m_k. (Here we regard it as a function of Ω.) The wall crossing studied in [6], [21] seems to be more directly related to it.

We leave systematic study of multi theta function as a function of Ω, as a target of the future research.

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