LAGRANGIAN FLOER THEORY ON COMPACT TORIC MANIFOLDS II : BULK DEFORMATIONS.

KENJI FUKAYA, YONG-GEUN OH, HIROSHI OHTA, KAORU ONO

Abstract. This is a continuation of part I in the series (in progress) of the papers on Lagrangian Floer theory on toric manifolds. Using the deformations of Floer cohomology by the ambient cycles, which we call *bulk deformations*, we find a continuum of non-displaceable Lagrangian fibers on some compact toric manifolds. We also provide a method of finding all those fibers in arbitrary compact toric manifolds, which we call *bulk-balanced* Lagrangian fibers.

Contents

Date: November 1, 2008.

Key words and phrases. toric manifolds, Floer cohomology, weakly unobstructed Lagrangian submanifolds, potential function, Jacobian ring, bulk deformations, bulk-balanced Lagrangian submanifolds, open-closed Gromov-Wittten invariant.

KF is supported partially by JSPS Grant-in-Aid for Scientific Research No.18104001, YO by US NSF grant # 0503954, HO by JSPS Grant-in-Aid for Scientific Research No.19340017, and KO by JSPS Grant-in-Aid for Scientific Research, Nos. 17654009 and 18340014.

1. Introduction

This is the second of series of papers to study Lagrangian Floer theory on toric manifolds. The main purpose of this paper is to explore bulk deformations of Lagrangian Floer theory, which we introduced in section 13 [FOOO2] and draw its applications. In particular, we prove the following Theorems 1.1, 1.3. We call a Lagrangian submanifold L of a symplectic manifold X *non-displaceable* if $\psi(L) \cap L \neq \emptyset$ for any Hamiltonian diffeomorphism $\psi : X \to X$.

Theorem 1.1. *Let X_k be the k-points blow up of $\mathbb{C}P^2$ with $k \geq 2$. Then there exists a toric Kähler structure on X_k such that there exist a continuum of non-displaceable Lagrangian fibers $L(u)$.*

Moreover they have the following property : If $\psi : X \to X$ is a Hamiltonian isotopy such that $\psi(L(u))$ is transversal to $L(u)$ in addition, then

$$\#(\psi(L(u)) \cap L(u)) \geq 4.$$

Remark 1.2.
(1) We state Theorem 1.1 in the case of the blow up of $\mathbb{C}P^2$. We can construct many similar examples by the same method.
(2) We will prove Theorem 1.1 by proving the existence of $\mathfrak{b} \in H^2(X_k; \Lambda_+)$ and $\mathfrak{x} \in H(L(u); \Lambda_0)$ such that

$$HF((L(u), (\mathfrak{b}, \mathfrak{x})), (L(u), (\mathfrak{b}, \mathfrak{x})); \Lambda_0) \cong H(T^2; \Lambda_0). \tag{1.1}$$

Here

$$\Lambda_0 = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \in \Lambda \;\middle|\; \lambda_i \geq 0, \lim_{i \to \infty} \lambda_i = \infty, a_i \in R \right\}, \tag{1.2}$$

(R is a field of characteristic 0) and

$$\Lambda_+ = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \in \Lambda \;\middle|\; \lambda_i > 0 \right\} \tag{1.3}$$

are the universal Novikov ring and its maximal ideal. The left hand side of (1.1) is the Floer cohomology with bulk deformation. See section 13 [FOOO2] and section 2 of this paper for its definition.
(3) In Part IV of this series of papers, we will study this example further and prove that the universal cover $\widetilde{Ham}(X_k)$ of the group of Hamiltonian diffeomorphisms allows infinitely many continuous and homogeneous Calabi quasi-morphisms $\varphi_u : \widetilde{Ham}(X_k) \to \mathbb{R}$ (see [EP]) such that for any finitely many $u_1, \cdots, u_N$ there exists a subgroup $\cong \mathbb{Z}^N \subset \widetilde{Ham}(X_k)$ on which $(\varphi_{u_1}, \cdots, \varphi_{u_N}) : \mathbb{Z}^N \to \mathbb{R}^N$ is injective.

In sections 8 and 9 [FOOO3], we introduced the notion of leading term equation for each Lagrangian fiber $L(u)$ of a toric manifold X. See also section 4 of this paper. The leading term equation is a system consisting of n-elements of the Laurent polynomial ring $\mathbb{C}[y_1, \cdots, y_n, y_1^{-1}, \cdots, y_n^{-1}]$ of n variables. (Here $n = \dim L(u)$.) In section 9 [FOOO3], we proved that if the leading term equation has a solution in $(\mathbb{C} \setminus \{0\})^n$ then $L(u)$ has a nontrivial Floer cohomology for some bounding cochain $\mathfrak{x}$ in $H^1(L(u); \Lambda_0)$ *under certain nondegeneracy conditions.* The next theorem says that if we consider more general class of Floer cohomology integrating bulk deformations into its construction, we can remove this nondegeneracy condition.

Theorem 1.3. *Let X be a compact toric manifold and $L(u)$ its Lagrangian fiber. Suppose that the leading term equation of $L(u)$ has a solution in $(\mathbb{C}\setminus\{0\})^n$.*

Then there exists $\mathfrak{b}\in H^2(X;\Lambda_+)$ and $\mathfrak{x}\in H(L(u);\Lambda_0)$ satisfying

$$HF((L(u),(\mathfrak{b},\mathfrak{x})),(L(u),(\mathfrak{b},\mathfrak{x}));\Lambda_0)\cong H(T^n;\Lambda_0). \tag{1.4}$$

Corollary 1.4. *Let X be a compact toric manifold and $L(u)$ its Lagrangian fiber. Suppose that the leading term equation of $L(u)$ has a solution in $(\mathbb{C}\setminus\{0\})^n$. Then $L(u)$ is non-displaceable.*

Moreover $L(u)$ has the following property. If $\psi: X\to X$ is a Hamiltonian isotopy such that $\psi(L(u))$ is transversal to $L(u)$, then

$$\#(\psi(L(u))\cap L(u))\geq 2^n, \tag{1.5}$$

where $n=\dim L(u)$.

The converse to Theorem 1.3 also holds. (See Theorem 4.7.)

The leading term equation can be easily solved in practice for most of the compact toric manifolds, which are not necessarily Fano. Theorem 1.3 enables us to reduce the problem to locate all $L(u)$ such that there exists a pair $(\mathfrak{b},\mathfrak{x})\in H^2(X;\Lambda_+)\times H(L(u);\Lambda_0)$ satisfying (1.4) to the problem to decide existence of nonzero solution of explicitly calculable system of polynomial equations. In [FOOO3] we provided such a reduction for the case $\mathfrak{b}=0$. If all the solutions of the leading term equation are weakly nondegenerate (see Definition 9.2 [FOOO3]), Floer cohomology with $\mathfrak{b}=0$ seems to enough for the general study of non-displacement of Lagrangian fibers. The method employed in this paper works for arbitrary compact toric manifolds without nondegeneracy assumption, and the calculation is actually simpler. We believe that this method provides an optimal result on the non-displacement of Lagrangian fibers. (See Conjectures 3.16 & 3.20.)

Remark 1.5. In [Cho], Cho used Floer cohomology with 'B-field' to study non-displacement of Lagrangian fibers in toric manifolds. 'B-field' which Cho used is parameterized by $H^2(X;\sqrt{-1}\,\mathbb{R})$. The bulk deformation we use in this paper is parameterized by $\mathfrak{b}\in H^*(X;\Lambda_0)$. If we restrict to $\mathfrak{b}\in H^2(X;\sqrt{-1}\,\mathbb{R})$ our bulk deformation by $\mathfrak{b}$ in this paper coincides with the deformation by a 'B-field' in [Cho].

A brief outline of each section of the paper is now in order. In section 2, we review construction of the operator $\mathfrak{q}$ given in section 13 [FOOO2] and explain how we use $\mathfrak{q}$ to deform Floer cohomology. In section 3 we provide a more explicit description thereof for the case of compact toric manifolds and study its relation to the potential function with bulk, which is the generating function defined by the structure constants of $\mathfrak{q}$. This section also contains various results on the operator $\mathfrak{q}$ and on the potential function with bulk. These results will be used also in Parts III and IV of this series of papers. In section 4, we explain how we use the results of section 3 to study Floer cohomology of Lagrangian fibers of compact toric manifolds. Especially we prove Theorem 1.3 there. Section 5 is devoted to the proof of Theorem 1.1. In this section we discuss the case of two points blow up of $\mathbb{C}P^2$ in detail and illustrate the way to locate all the Lagrangian fibers that have nontrivial Floer cohomology (after bulk deformation). The calculation we perform in this section can be generalized to arbitrary compact toric manifolds. In section 6 we describe the results on the moduli space of pseudo-holomorphic discs with boundary on a Lagrangian fiber of a general toric manifold, which are basically due to [CO]. We use these results in the study of the operator $\mathfrak{q}$. In section 7 we carry out some calculation of the potential function with bulk deformation which is strong enough to locate all the Lagrangian fibers with nontrivial Floer cohomology (after bulk deformation).

In section 8 we use the Floer cohomology with bulk deformation in the study of non-displacement of Lagrangian submanifolds. For this purpose we define the cohomology between a pair of Lagrangian submanifolds L and $\psi(L)$ for a Hamiltonian diffeomorphism ψ. We also show that this Floer cohomology of the pair is isomorphic to the Floer cohomology of L itself. This is a standard process one takes to use Floer cohomology for the non-displacement problem dating back to Floer [Fl]. We include bulk deformations and deformations by bounding cochain there. These results were previously obtained in [FOOO2]. However we give rather detailed account of these constructions here in order to make this paper as self-contained as possible. To avoid too much overlap with that of [FOOO2] in this paper, we give a proof using the de Rham cohomology version here which is different from that of [FOOO2] in which we used the singular cohomology version. In section 9 we study the convergence property of potential functions. Namely we prove that the potential function is contained in the completion of the ring of Laurent polynomials over a Novikov ring with respect to an appropriate non-Archimedean norm. This choice of the norm depends on the Kähler structure (or equivalently to the moment polytope). We discuss the natural way to take completion and show that our potential function actually converges in that sense. In section 10, we discuss the relation of Euler vector fields and the potential function. In section 11, we slightly enlarge the parameter space of bulk deformations including $\mathfrak{b}$ from $H(X;\Lambda_0)$ not just from $H(X;\Lambda_+)$. In section 12, we review the construction of smooth correspondence in de Rham cohomology using continuous family of multisections and integration along fibers via its zero sets.

Notations and conventions

We take any field R containing $\mathbb{Q}$. The universal Novikov ring Λ_0 is defined as (1.2), where $a_i \in R$. Its ideal Λ_+ is defined as (1.3).

$$\Lambda = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \,\middle|\, a_i \in R, \lambda_i \in \mathbb{R},\ \lambda_i < \lambda_{i+1},\ \lim_{i\to\infty} \lambda_i = \infty \right\}$$

is the field of fraction of Λ_0.

In case we need to specify R we write $\Lambda_0(R)$, $\Lambda_+(R)$, $\Lambda(R)$. The (non-Archimedean) valuation $\mathfrak{v}_T$ on them are defined by

$$\mathfrak{v}_T\left(\sum_{i=1}^{\infty} a_i T^{\lambda_i}\right) = \inf\{\lambda_i \mid a_i \neq 0\}$$

It induces a non-Archimedean norm $\|x\| = e^{-\mathfrak{v}_T(x)}$ and defines a topology on them. Those rings are complete with respect to this norm.

If C is an R vector space, we denote by $C(\Lambda_0)$ the completion of $C \otimes \Lambda_0$ with respect to the non-Archimedean topology of Λ_0. In other words its elements are of the form

$$\sum a_i T^{\lambda_i}$$

such that $a_i \in C$, $\lambda_i < \lambda_{i+1}$, $\lambda_i \ge 0$, $\lim_{i\to\infty}\lambda_i = \infty$. $C(\Lambda_+)$, $C(\Lambda)$, $C(\Lambda_0(R))$, $C(\Lambda_+(R))$, $C(\Lambda(R))$ are defined in the same way.

We denote by (X, ω) the compact toric manifold, with Kähler form ω given. $\pi : X \to P$ is the moment map, where $P \subset \mathbb{R}^n$. We write the vector space $\mathbb{R}^n$ containing P by $M_{\mathbb{R}}$. Its dual space is denoted by $N_{\mathbb{R}}$. $L(u)$ is a fiber $\pi^{-1}(u)$ where $u \in \operatorname{Int} P$. We define an $\mathbb{R}$ linear isomorphism

$$H_1(L(u); \mathbb{R}) \to N_{\mathbb{R}}. \tag{1.6}$$

as follows. Let $\vec f \in H_1(L(u); \mathbb{Z})$. The moment map of the action of

$$S^1 = \vec f\,\mathbb{R} \mod H_1(L(u); \mathbb{Z}) \subset H_1(L(u); \mathbb{R})/H_1(L(u); \mathbb{Z}) = T^n$$

is denoted by $\mu_{\vec f}$. $\mu_{\vec f}$ factors thorough $P \subset M_{\mathbb{R}}$ so that $\mu_{\vec f} = \tilde\mu_{\vec f} \circ \pi$, where $\tilde\mu_{\vec f}$ is affine. We associate $d\tilde\mu_{\vec f} \in N_{\mathbb{R}}$ to $\vec f$. This induces (1.6).

The boundary ∂P is divided into m codimension 1 faces, which we denote by $\partial_i P$ $(i = 1, \cdots, m.)$ In [FOOO3], we defined affine maps $\ell_i : M_{\mathbb{R}} \to \mathbb{R}$ such that

$$\partial_i P = \{u \in M_{\mathbb{R}} \mid \ell_i(u) = 0\}, \qquad P = \{u \in M_{\mathbb{R}} \mid \ell_i(u) \ge 0, \quad i = 1, \cdots, m\}.$$

We put $\vec v_i = d\ell_i \in N_{\mathbb{R}} \cong H_1(L(u); \mathbb{R})$. In fact $\vec v_i \in H_1(L(u); \mathbb{Z})$, i.e., $\vec v_i$ is an integral vector.

We denote by x_i $(i = 1, \cdots n)$ the coordinates of $H^1(L(u); \Lambda_0)$ with respect to the basis $\mathbf{e}_i$ and put $y_i = e^{x_i}$.

2. Bulk deformations of Floer cohomology

In this section, we review the results of section 13 of [FOOO2].

Let (X, ω) be a compact symplectic manifold and L be its Lagrangian submanifold. We take a finite dimensional graded R-vector space H of smooth singular *cycles* of X. (Actually we may consider a subcomplex of the smooth singular chain complex of X and consider smooth singular *chains*. Since consideration of chain level arguments is not needed in this paper, we restrict ourselves to the case of *cycles*. See [FOOO2] and Part III of this series of papers for relevant explanations.)

We regard an element of H as a cochain (cocycle) by identifying a k-chain with a $(2n-k)$-cochain where $n = \dim L$.

In section 13 [FOOO2] we introduced a family of operators denoted by

$$\mathfrak{q}_{\beta;\ell,k} : E_\ell(H[2]) \otimes B_k(H^*(L; R)[1]) \to H^*(L; R)[1]. \tag{2.1}$$

Explanation of the various notations appearing in (2.1) is in order. β is an element of the image of $\pi_2(X, L) \to H_2(X, L; \mathbb{Z})$. $H[2]$ is the degree shift of H by 2 defined by $(H[2])^d = H^{d+2}$. $H^*(L; R)[1]$ is the degree shift of the cohomology group with R coefficient. The notations E_ℓ and B_k are defined as follows. Let C be a graded vector space. We put

$$B_k C = \underbrace{C \otimes \cdots \otimes C}_{k \text{ times}}.$$

The symmetric group $\mathfrak{S}_k$ of order $k!$ acts on $B_k C$ by

$$\sigma \cdot (x_1 \otimes \cdots \otimes x_k) = (-1)^* x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)},$$

where

$$* = \sum_{i<j;\sigma(i)>\sigma(j)} \deg x_i \deg x_j.$$

E_kC is the set of $\mathfrak{S}_k$-invariant elements of B_kC. The map (2.1) is a $\mathbb{Q}$-linear map of degree $1-\mu(\beta)$ here μ is the Maslov index.

We next describe the main properties of $\mathfrak{q}_{\beta;\ell,k}$. Let B_kC be as above and put

$$BC = \bigoplus_{k=0}^{\infty} B_kC.$$

(We remark $B_0C = R$.) BC has the structure of coassociative coalgebra with its coproduct $\Delta : BC \to BC \otimes BC$ defined by

$$\Delta(x_1 \otimes \cdots \otimes x_k) = \sum_{i=0}^{k} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_k).$$

This induces a coproduct $\Delta : EC \to EC \otimes EC$ with respect to which EC becomes a coassociative and graded cocommutative.

We also consider a map $\Delta^{n-1} : BC \to (BC)^n$ or $EC \to (EC)^n$ defined by

$$\Delta^{n-1} = (\Delta \otimes \underbrace{id \otimes \cdots \otimes id}_{n-2}) \circ (\Delta \otimes \underbrace{id \otimes \cdots \otimes id}_{n-3}) \circ \cdots \circ \Delta.$$

For an indecomposable element $\mathbf{x} \in BC$, it can be expressed as

$$\Delta^{n-1}(\mathbf{x}) = \sum_c \mathbf{x}_c^{n;1} \otimes \cdots \otimes \mathbf{x}_c^{n;n}$$

where c runs over all partitions of n. For an element

$$\mathbf{x} = x_1 \otimes \cdots \otimes x_k \in B_k(H(L;R)[1])$$

we put the shifted degree $\deg' x_i = \deg x_i + 1$ and

$$\deg' \mathbf{x} = \sum \deg' x_i = \deg \mathbf{x} + k.$$

(Recall $\deg x_i$ is the cohomological degree of x_i before shifted.)

Theorem 2.1. (Theorem 13.32 [FOOO2]) *The operators $\mathfrak{q}_{\beta;\ell,k}$ have the following properties.*

(1) *For each β and $\mathbf{x} \in B_k(H(L;R)[1])$, $\mathbf{y} \in E_k(H[2])$, we have the following :*

$$0 = \sum_{\beta_1+\beta_2=\beta} \sum_{c_1,c_2} (-1)^* \mathfrak{q}_{\beta_1}(\mathbf{y}_{c_1}^{2;1}, \mathbf{x}_{c_2}^{3;1} \otimes \mathfrak{q}_{\beta_2}(\mathbf{y}_{c_1}^{2;2}, \mathbf{x}_{c_2}^{3;2}) \otimes \mathbf{x}_{c_2}^{3;3}) \tag{2.2}$$

where

$$* = \deg' \mathbf{x}_{c_2}^{3;1} + \deg' \mathbf{x}_{c_2}^{3;1} \deg \mathbf{y}_{c_1}^{2;2} + \deg \mathbf{y}_{c_1}^{2;1}.$$

In (2.2) and hereafter, we write $\mathfrak{q}_\beta(\mathbf{y},\mathbf{x})$ in place of $\mathfrak{q}_{\beta;\ell,k}(\mathbf{y},\mathbf{x})$ if $\mathbf{y} \in E_\ell(H[2])$, $\mathbf{x} \in B_k(H(L;R)[1])$.

(2) *If $1 \in E_0(H[2])$ and $\mathbf{x} \in B_k(H(L;R)[1])$ then*

$$\mathfrak{q}_{\beta;0,k}(1,\mathbf{x}) = \mathfrak{m}_{\beta;k}(\mathbf{x}). \tag{2.3}$$

Here $\mathfrak{m}_{\beta;k}$ is the filtered A_∞ structure on $H(L;R)$.

(3) *Let $\mathbf{e} = PD([L])$ be the Poincaré dual to the fundamental class of L. Let $\mathbf{x}_i \in B(H(L;R)[1])$ and we put $\mathbf{x} = \mathbf{x}_1 \otimes \mathbf{e} \otimes \mathbf{x}_2 \in B(H(L;R)[1])$. Then*

$$\mathfrak{q}_\beta(\mathbf{y},\mathbf{x}) = 0 \tag{2.4}$$

except the following case.

$$\mathfrak{q}_{\beta_0}(1, \mathbf{e} \otimes x) = (-1)^{\deg x} \mathfrak{q}_{\beta_0}(1, x \otimes \mathbf{e}) = x, \tag{2.5}$$

where $\beta_0 = 0 \in H_2(X, L; \mathbb{Z})$ *and* $x \in H(L; R)[1] = B_1(H(L; R)[1])$.

Theorem 2.1 is proved in sections 13 and 32 [FOOO2]. We will recall its proof in section 7 in the case when X is a toric manifold, $R = \mathbb{R}$ and L is a Lagrangian fiber of X.

We next explain how we use the map $\mathfrak{q}$ to deform filtered A_∞ structure $\mathfrak{m}$ on L. In this section we use the universal Novikov ring

$$\Lambda_{0,nov} = \left\{ \sum c_i T^{\lambda_i} e^{n_i} \,\middle|\, c_i \in R, \lambda_i \geq 0, n_i \in \mathbb{Z}, \lim_{i\to\infty} \lambda_i = +\infty \right\}$$

which was introduced in [FOOO1]. We write $\Lambda_{0,nov}(R)$ in case we need to specify R. The ideal $\Lambda_{0,nov}^+$ of $\Lambda_{0,nov}$ is the set of all elements $\sum c_i T^{\lambda_i} e^{n_i}$ of $\Lambda_{0,nov}$ such that $\lambda_i > 0$. We put $F^\lambda \Lambda_{0,nov} = T^\lambda \Lambda_{0,nov}$. It defines a filtration on $\Lambda_{0,nov}$, under which $\Lambda_{0,nov}$ is complete. $\Lambda_{0,nov}$ becomes a graded ring by putting $\deg e = 2$, $\deg T = 0$.

We choose a basis $\mathbf{f}_a$ $(a = 1, \cdots, B)$ of H and consider an element

$$\mathfrak{b} = \sum_a \mathfrak{b}_a \mathbf{f}_a \in H(\Lambda_{0,nov}^+)$$

such that $\deg \mathfrak{b}_a + \deg \mathbf{f}_a = 2$ for each a. We then define

$$\mathfrak{m}_k^{\mathfrak{b}}(x_1, \cdots, x_k) = \sum_{\beta,\ell,k} e^{\mu(\beta)/2} T^{\omega\cap\beta/2\pi} \mathfrak{q}_{\beta;\ell,k}(\mathfrak{b}^{\otimes \ell}; x_1, \cdots, x_k). \tag{2.6}$$

Here $\mu : \pi_2(X, L) \to \mathbb{Z}$ is the Maslov index.

Lemma 2.2. *The family* $\{\mathfrak{m}_k^{\mathfrak{b}}\}_{k=0}^\infty$ *defines a filtered* A_∞ *structure on* $H(L; \Lambda_{0,nov})$.

Proof. We put

$$e^{\mathfrak{b}} = \sum_{\ell=0}^\infty \mathfrak{b}^{\otimes \ell}.$$

Then we have

$$\Delta(e^{\mathfrak{b}}) = e^{\mathfrak{b}} \otimes e^{\mathfrak{b}}.$$

Lemma 2.2 follows from this fact and Theorem 2.1. (See Lemma 13.39 [FOOO2] for detail.) $\square$

Let $b \in H^1(L; \Lambda_{0,nov}^+)$. We say b is a *weak bounding cochain* of the filtered A_∞ algebra $(H(L; \Lambda_{0,nov}), \{\mathfrak{m}_k^{\mathfrak{b}}\})$ if it satisfies

$$\sum_{k=0}^\infty \mathfrak{m}_k^{\mathfrak{b}}(b, \cdots, b) = cPD([L])$$

where $PD([L]) \in H^0(L; \mathbb{Q})$ is the Poincaré dual to the fundamental cycle and $c \in \Lambda_{0,nov}^+$. By a degree counting, we find that $\deg c = 2$.

We denote by $\widehat{\mathcal{M}}_{\text{weak,def}}(L; \Lambda_{0,nov}^+)$ the set of the pairs $(\mathfrak{b}, b)$ of elements $\mathfrak{b} \in H \otimes \Lambda_{0,nov}^+$ of degree 2 and weak bounding cochain b of $(H(L; \Lambda_{0,nov}), \{\mathfrak{m}_k^{\mathfrak{b}}\})$.

We define $\mathfrak{PO}(\mathfrak{b}, b)$ by the equation

$$\mathfrak{PO}(\mathfrak{b}, b) e = c.$$

By definition $\mathfrak{PO}(\mathfrak{b}, b)$ is an element of $\Lambda_{0,nov}^+$ of degree 0 i.e.,

$$\mathfrak{PO}(\mathfrak{b}, b) \in \Lambda_+$$

where we recall (1.3) for the definition of Λ_+.

We call the map

$$\mathfrak{PO} : \widehat{\mathcal{M}}_{\text{weak,def}}(L; \Lambda_{0,nov}^+) \to \Lambda_+$$

the potential function. We also define the projection

$$\pi : \widehat{\mathcal{M}}_{\text{weak,def}}(L; \Lambda_{0,nov}^+) \to H \otimes \Lambda_{0,nov}^+$$

by

$$\pi(\mathfrak{b}, b) = \mathfrak{b}.$$

Let $\mathbf{b}_1 = (\mathfrak{b}, b_1), \mathbf{b}_0 = (\mathfrak{b}, b_0) \in \widehat{\mathcal{M}}_{\text{weak,def}}(L; \Lambda_{0,nov}^+)$ with

$$\pi(\mathbf{b}_1) = \mathfrak{b} = \pi(\mathbf{b}_0).$$

We define an operator

$$\delta^{\mathbf{b}_1,\mathbf{b}_0} : H(L; \Lambda_0) \to H(L; \Lambda_0)$$

of degree $+1$ by

$$\delta^{\mathbf{b}_1,\mathbf{b}_0}(x) = \sum_{k_1,k_0} \mathfrak{m}^{\mathfrak{b}}_{k_1+k_0+1}(b_1^{\otimes k_1} \otimes x \otimes b_0^{\otimes k_0}).$$

Lemma 2.3.

$$(\delta^{\mathbf{b}_1,\mathbf{b}_0} \circ \delta^{\mathbf{b}_1,\mathbf{b}_0})(x) = (-\mathfrak{PO}(\mathbf{b}_1) + \mathfrak{PO}(\mathbf{b}_0))ex.$$

Proof. This is an easy consequence of Theorem 2.1. See [FOOO2] Proposition 12.17. □

Definition 2.4. ([FOOO2] Definition 13.61.) For a pair of elements $\mathbf{b}_1, \mathbf{b}_0 \in \widehat{\mathcal{M}}_{\text{weak,def}}(L; \Lambda_{0,nov}^+)$ with $\pi(\mathbf{b}_1) = \pi(\mathbf{b}_0)$, $\mathfrak{PO}(\mathbf{b}_1) = \mathfrak{PO}(\mathbf{b}_0)$, we define

$$HF((L, \mathbf{b}_1), (L, \mathbf{b}_0); \Lambda_{0,nov}) = \frac{\operatorname{Ker}(\delta^{\mathbf{b}_1,\mathbf{b}_0})}{\operatorname{Im}(\delta^{\mathbf{b}_1,\mathbf{b}_0})}.$$

This is well defined by Lemma 2.3.

By [FOOO2] Theorem 24.22, Floer cohomology is of the form

$$HF((L, \mathbf{b}_1), (L, \mathbf{b}_0); \Lambda_{0,nov}) \cong \Lambda_{0,nov}^a \oplus \bigoplus \frac{\Lambda_{0,nov}}{T^{\lambda_i}\Lambda_{0,nov}}.$$

We call a the Betti number and λ_i the torsion exponent of the Floer cohomology.

The following is a consequence of Theorems G and J [FOOO2] combined. (See also section 8.)

Theorem 2.5. *Let $\mathbf{b}_1, \mathbf{b}_0 \in \widehat{\mathcal{M}}_{\text{weak,def}}(L; \Lambda_{0,nov}^+)$ be as in Definition 2.4. Let $\psi : X \to X$ be a Hamiltonian diffeomorphism. We assume $\psi(L)$ is transversal to L.*

1. *The order of $\psi(L) \cap L$ is not smaller than the Betti number a of the Floer cohomology $HF((L, \mathbf{b}_1), (L, \mathbf{b}_0); \Lambda_{0,nov})$.*
2. *Let $\{\lambda_i\}$ be the torsion exponents of $HF((L, \mathbf{b}_1), (L, \mathbf{b}_0); \Lambda_{0,nov})$ and E be the Hofer distance of ψ from identity. Let b be the number of λ_i which is not smaller than E. Then the order of $\psi(L) \cap L$ is not smaller than $a + 2b$.*

3. Potential function with bulk

In this section, we specialize the story of the last section to the case of toric fibers, and make the construction of section 13 [FOOO2] explicit in this case. We also generalize the results from section 12 [FOOO3] and the story between Floer cohomology and the potential function to the case with bulk deformations.

Let X be a compact toric manifold and P its moment polytope. Let $\pi : X \to P$ be the moment map. For each face (of arbitrary codimension) P_a of P we have a complex submanifold $D_a = \pi^{-1}(P_a)$ for $a = 1, \cdots, B$. We enumerate P_a so that the first m P_a's correspond to the m codimension one faces of P. Here we note that the complex codimension of D_a is equal to the real codimension of P_a. Let $\mathcal{A}$ be the free abelian group generated by D_a. (In this paper we do not consider the case when $P_a = P$.) It is a graded abelian group $\mathcal{A} = \oplus_\ell \mathcal{A}_\ell$ with its grading given by the (real) dimension of D_a. We put $D = \pi^{-1}(\partial P) = \cup_a D_a$, that is, the toric divisor of X. We denote

$$\mathcal{A}^k(\mathbb{Z}) := \mathcal{A}_{2n-k}$$

We remark that $\mathcal{A}_\ell$ is nonzero only for even ℓ and so $\mathcal{A}^k$ is nonzero for even k. The homomorphism : $\mathcal{A}_{2n-k} \to H_{2n-k}(X;\mathbb{Z})$ and the Poincaré duality induce a surjective homomorphism

$$i_! : \mathcal{A}^k(\mathbb{Z}) \to H^k(X;\mathbb{Z})$$

for $k \neq 0$. We remark that $i_!$ is not injective. For example $\mathcal{A}^2(\mathbb{Z}) \cong \mathbb{Z}^m$ (where m is the number of irreducible components of D) and $H^2(X;\mathbb{Z}) = \mathbb{Z}^{m-n}$. In fact we have the exact sequence

$$0 \to H_2(X;\mathbb{Z}) \to H_2(X, X \setminus D;\mathbb{Z}) \to H_1(X \setminus D;\mathbb{Z}) \cong \mathbb{Z}^n \to 0.$$

On the other hand, since $H_2(N(D);\partial N(D)) \cong H^{2n-2}(N(D)) \cong H^{2n-2}(D)$, (where $N(D)$ is a regular neighborhood of D in X) we have $H_2(X, X \setminus D;\mathbb{Q}) \cong \mathbb{Q}^m \cong \mathcal{A}^2(\mathbb{Q})^*$.

We denote the set of Λ_+-cycles by $\mathcal{A}^k(\Lambda_+) = \mathcal{A}^k \otimes_{\mathbb{Z}} \Lambda_+$. The following is a generalization of Proposition 3.2 [FOOO3].

Proposition 3.1. *We have the canonical inclusion*

$$\mathcal{A}(\Lambda_+) \times H^1(L(u);\Lambda_+) \hookrightarrow \widehat{\mathcal{M}}_{\text{weak,def}}(L(u)).$$

Proposition 3.1 will be proved in section 7. We remark that the map $i_! : \mathcal{A}^k(\mathbb{Z}) \to H^k(X;\mathbb{Z})$ is not injective. Therefore, the gauge equivalence relation (See Definition 16.1 [FOOO2].) on the left hand side is nontrivial. So the right hand side is not $\mathcal{M}_{\text{weak,def}}(L(u))$, the set of gauge equivalence classes of the elements of $\widehat{\mathcal{M}}_{\text{weak,def}}(L(u))$.

For $\mathfrak{b} \in \bigoplus_k \mathcal{A}^k(\Lambda_+)$, $u \in \operatorname{Int} P$, we define

$$\mathfrak{PO}^u(\mathfrak{b}, b) : H^1(L(u);\Lambda_+) \to \Lambda_+$$

by

$$\mathfrak{PO}^u(\mathfrak{b}, b) = \sum_{\beta;\ell,k} T^{\omega\cap\beta/2\pi} \mathfrak{q}_{\beta;\ell,k}(\mathfrak{b}^{\otimes\ell}; b^{\otimes k}) \cap [L(u)]. \tag{3.1}$$

We remark that the summation on right hand side includes the term where $\ell = 0$. The term corresponding thereto is

$$\sum_{k,\beta} T^{\omega\cap\beta/2\pi} \mathfrak{m}_{\beta;k}(b^{\otimes k}) \cap [L(u)] = \mathfrak{PO}^u(b)$$

which is nothing but the potential function in the sense of section 3 [FOOO3]. Namely we have the identity

$$\mathfrak{PO}^u(0,b) = \mathfrak{PO}^u(b). \tag{3.2}$$

This function (3.1) is also a special case of the potential function we discussed in section 2. (We will not use the variable e in this section.)

We put

$$\mathfrak{b} = \sum_a w_a[D_a], \quad b = \sum x_i \mathbf{e}_i.$$

Here $\mathbf{e}_i$ $(i = 1, \cdots, n)$ is a basis of $H^1(L(u);\mathbb{Z})$. (See [FOOO3] Lemma 3.3.) We also put $u = (u_1, \cdots, u_n)$, and $y_i = e^{x_i}$. (See the end of section 1.) We next discuss a generalization of [FOOO3] Theorem 3.5.

To state it we need some notations.

Definition 3.2. A discrete submonoid of $\mathbb{R}_{\geq 0}$ is a subset $G \subset \mathbb{R}_{\geq 0}$ such that

(1) G is discrete.
(2) If g_1, g_2 then $g_1 + g_2 \in G$. $0 \in G$.

Hereafter we say *discrete submonoid* in place of discrete submonoid of $\mathbb{R}_{\geq 0}$ for simplicity.

For any discrete subset $\mathfrak{X}$ of $\mathbb{R}_{\geq 0}$ there exists a discrete submonoid containing it. The discrete submonoid G generated by $\mathfrak{X}$ is, by definition, the smallest one among them. We write $G = \langle \mathfrak{X} \rangle$.

Compare Condition 6.11 [FOOO2]. In [FOOO2] we considered $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$. Since we do not use the grading parameter e we consider $G \subset \mathbb{R}_{\geq 0}$ in this paper.

Definition 3.3. Let C_i be an R vector space. We denote by $C_i(\Lambda_0)$ the completion of $C_i \otimes \Lambda_0$. Let G be a discrete submonoid.

(1) An element x of $C_i(\Lambda_0)$ is said to be *G-gapped* if
$$x = \sum_{g \in G} x_g T^g$$
where $x_g \in C_i$.
(2) A filtered Λ_0 module homomorphism $f : C_1(\Lambda_0) \to C_2(\Lambda_0)$ is said to be G-gapped if there exists R linear maps $f_g : C_1 \to C_2$ for $g \in G$ such that
$$f(x) = \sum_{g \in G} T^g f_g(x).$$
Here we extend f_g to $f_g : C_1(\Lambda_0) \to C_2(\Lambda_0)$ in an obvious way.

The G-gappedness of potential functions, of filtered A_∞ structures, and etc. can be defined in a similar way.

We define

$$G(X) = \langle \{\omega \cap \beta \mid \beta \in \pi_2(X) \text{ is realized by a holomorphic sphere}\} \rangle. \tag{3.3}$$

Denote by G_{bulk} a discrete submonoid containing $G(X)$.

We put

$$\mathfrak{PO}^u_0(b) = \sum_{i=1}^m T^{\ell_i(u)} y_1^{v_{i,1}} \cdots y_n^{v_{i,n}} \tag{3.4}$$

and call it the *leading order potential function*.

Theorem 3.4. *Let X be an arbitrary compact toric manifold and $L(u)$ as above and $\mathfrak{b} \in \mathcal{A}(\Lambda_+)$ a $G_{\rm bulk}$ gapped element. Then there exist $c_\sigma \in \mathbb{Q}$, $e^i_\sigma \in \mathbb{Z}_{\geq 0}$, $\rho_\sigma \in G_{\rm bulk}$ and $\rho_\sigma > 0$, such that $\sum_{i=1}^m e^i_\sigma > 0$ and*

$$\mathfrak{PO}^u(\mathfrak{b}; b) - \mathfrak{PO}^u_0(b) = \sum_{\sigma=1}^{\infty} c_\sigma y_1^{v'_{\sigma,1}} \cdots y_n^{v'_{\sigma,n}} T^{\ell'_\sigma(u)+\rho_\sigma} \tag{3.5}$$

where

$$v'_{\sigma,k} = \sum_{i=1}^m e^i_\sigma v_{i,k}, \quad \ell'_\sigma = \sum_{i=1}^m e^i_\sigma \ell_i. \tag{3.6}$$

If there are infinitely many non-zero c_σ's, we have

$$\lim_{\sigma\to\infty} \rho_\sigma = \infty. \tag{3.7}$$

Theorem 3.4 is proved in section 7. (3.7) slightly improves corresponding statement in Theorem 3.5 [FOOO3].

We regard $\mathfrak{PO}^u$ as a function of w_i and y_i and write $\mathfrak{PO}^u(w_1, \cdots, w_B; y_1, \cdots, y_n)$. (Here $B = \sum_k \operatorname{rank} \mathcal{A}^k$.) Then Theorem 3.4, especially (3.7), implies the following

Lemma 3.5. *The potential function*

$$\mathfrak{PO}^u(w_1, \cdots, w_B; y_1, \cdots, y_n) : (\Lambda_+)^B \times (1+\Lambda_+)^n \to \Lambda_0$$

is extended to a function : $(\Lambda_+)^B \times (\Lambda_0)^n \to \Lambda_0$.

We remark that $1+\Lambda_+$ is the set of elements $1+x \in \Lambda_0$ with $x \in \Lambda_+$. It coincides with the image of $\exp : \Lambda_+ \to \Lambda_0$. We denote the extension by the same symbol $\mathfrak{PO}^u$.

Actually, we can extend the domain of the potential function to $(\Lambda_0)^B \times (\Lambda_0 \setminus \Lambda_+)^n$. Let $w_1, \cdots, w_m$ be the parameter corresponding to $\mathcal{A}^2$. (m is the number of irreducible components.) We put $\mathfrak{w}_i = e^{w_i}$ and consider the ring

$$\Lambda_0[\mathfrak{w}_1, \cdots, \mathfrak{w}_m, \mathfrak{w}_1^{-1}, \cdots, \mathfrak{w}_m^{-1}, w_{m+1}, \cdots, w_B, y_1, y_1^{-1}, \cdots, y_n, y_n^{-1}]. \tag{3.8}$$

We take its completion with respect to the norm induced by the non-Archimedean norm of Λ_0 and denote it by

$$\Lambda_0\{\mathfrak{w}, \mathfrak{w}^{-1}, w, y, y^{-1}\}.$$

In other words, its element is an infinite sum

$$\sum_k a_k \, \mathfrak{w}_1^{e_{k,1}} \cdots, \mathfrak{w}_m^{e_{k,m}} w_{m+1}^{e_{k,m+1}} \cdots w_B^{e_{k,B}} y_1^{f_{k,1}} \cdots y_n^{f_{k,n}}$$

where $e_{k,i} \in \mathbb{Z}$ $(i \leq m)$, $e_{k,i} \in \mathbb{Z}_{\geq 0}$ $(i > m)$, $f_{k,i} \in \mathbb{Z}$, $a_k \in \Lambda_0$ and

$$\lim_{k\to\infty} \mathfrak{v}_T(a_k) = \infty.$$

The ring (3.8) is called a *strictly convergent power series ring.* See [BGR]. We have

$$\mathfrak{PO}^u \in \Lambda_0\{\mathfrak{w}, \mathfrak{w}^{-1}, w, y, y^{-1}\}. \tag{3.9}$$

In particular, (3.9) implies that the potential function

$$\begin{aligned}&(\mathfrak{w}_1, \cdots, \mathfrak{w}_m, w_{m+1}, \cdots, w_B, y_1, \cdots, y_n)\\ &\mapsto \mathfrak{PO}(\mathfrak{w}_1, \cdots, \mathfrak{w}_m, w_{m+1}, \cdots, w_B, y_1, \cdots, y_n)\end{aligned}$$

is defined on $(\Lambda_0 \setminus \Lambda_+)^m \times (\Lambda_0)^{B-m} \times (\Lambda_0 \setminus \Lambda_+)^n$.

We can further improve (3.9) to Theorem 3.11. To make the precise statement on this we need some preparation.

First recall that P is convex and so $\mathrm{Int}P$ is contractible. Therefore we have a T^n-bundle isomorphism

$$\Psi : \pi^{-1}(\mathrm{Int}P) \cong T^n \times \mathrm{Int}P.$$

For example we can construct such an isomorphism by first picking a reference point u_{ref} and identifying a fiber $\pi^{-1}(u_{\mathrm{ref}}) = L(u_{\mathrm{ref}})$ with T^n and then using the parallel transport with respect to the natural affine connection associated the Lagrangian smooth fibration $\pi^{-1}(\mathrm{Int}P) \to \mathrm{Int}P$. (See [W], [Dui].) Then Ψ induces a natural isomorphism

$$\psi_u := (\Psi|_{\pi^{-1}(u)})^* : H^1(T^n;\mathbb{Z}) \to H^1(L(u);\mathbb{Z}).$$

Now we choose a basis $\{\mathbf{e}_i\}$ of $H^1(T^n;\mathbb{Z})$ and x_i for $i = 1,\cdots,n$ the associated coordinates. We then denote $y_i = e^{x_i}$. We note that $\{\mathbf{e}_i\}$ and x_i (and so y_i) depend only on T^n. Using the isomorphism ψ_u we can push-forward them to $H^1(L(u);\mathbb{Z})$ which are nothing but the coordinates associated to the basis

$$\{\psi_u(\mathbf{e}_i)\}_{1\le i\le n}$$

of $H^1(L(u);\mathbb{Z})$ mentioned in the end of section 1.

We denote the variable

$$y_i(u) = T^{-u_i} y_i \circ \psi_u^{-1} \tag{3.10}$$

and consider the ring

$$\Lambda[\mathfrak{w}_1,\cdots,\mathfrak{w}_m^{-1}, w_{m+1},\cdots,w_B, y_1(u),\cdots,y_n(u)^{-1}].$$

By definition we have a ring isomorphism, again denoted by ψ_u,

$$\psi_u : \Lambda[\mathfrak{w},\mathfrak{w}^{-1},w,y,y^{-1}] \to \Lambda[\mathfrak{w},\cdots,\mathfrak{w}^{-1},w,y(u),y(u)^{-1}]; \quad y_i \mapsto T^{u_i} y_i(u)\circ\psi_u.$$

Furthermore by definition, we have a ring isomorphism

$$\begin{aligned}\psi_{u',u} : \Lambda[&\mathfrak{w}_1,\cdots,\mathfrak{w}_m^{-1}, w_{m+1},\cdots,w_B, y_1(u),\cdots,y_n(u)^{-1}]\\ &\to \Lambda[\mathfrak{w}_1,\cdots,\mathfrak{w}_m^{-1}, w_{m+1},\cdots,w_B, y_1(u'),\cdots,y_n(u')^{-1}]\end{aligned}$$

given by $\psi_{u',u} = \psi_{u'}\circ\psi_u^{-1}$ or more explicitly by

$$\psi_{u',u}(y_i(u)) = T^{u'_i - u_i} y_i(u')$$

for any two u, $u' \in \mathrm{Int}P$. (Compare the discussion just below Remark 5.13 [FOOO3].) Clearly $\psi_{u'',u'}\circ\psi_{u',u} = \psi_{u'',u}$.

Now we define a family of valuations $\mathfrak{v}_T^u$ parameterized by $u \in \mathrm{Int}\,P$ on the ring $\Lambda[\mathfrak{w},\mathfrak{w}^{-1},w,y,y^{-1}]$ by the formula

$$\begin{aligned}&\mathfrak{v}_T^u\left(\sum_k a_k\,\mathfrak{w}_1^{e_{k,1}}\cdots,\mathfrak{w}_m^{e_{k,m}} w_{m+1}^{e_{k,m+1}}\cdots w_B^{e_{k,B}} y_1^{f_{k,1}}\cdots y_n^{f_{k,n}}\right)\\ &= \inf_k\{\mathfrak{v}_T(a_k) + \langle f_k, u\rangle \mid a_k \neq 0\}.\end{aligned} \tag{3.11}$$

We note that by definition we have

$$\mathfrak{v}_T^u(y_i) = u_i. \tag{3.12}$$

We remark that $\mathfrak{v}_T^u$ is independent of the choice of the basis $\{\mathbf{e}_i\}$ of $H^1(T^n;\mathbb{Z})$.

Definition 3.6. We define a function

$$\mathfrak{v}_P(x) = \inf\{\mathfrak{v}_T^u(x) \mid u \in \mathrm{Int}P\}$$

on the ring $\Lambda[\mathfrak{w}, \mathfrak{w}^{-1}, w, y, y^{-1}]$ which defines a non-Archimedean valuation. We denote its completion by $\Lambda^P\{\mathfrak{w}, \mathfrak{w}^{-1}, w, y, y^{-1}\}$. We put

$$\Lambda_0^P\{\mathfrak{w}, \mathfrak{w}^{-1}, w, y, y^{-1}\} = \{x \in \Lambda^P\{\mathfrak{w}, \mathfrak{w}^{-1}, w, y, y^{-1}\} \mid \mathfrak{v}_P(x) \geq 0\}.$$

If we denote

$$\begin{aligned} \Lambda_0^u\{\mathfrak{w}, \mathfrak{w}^{-1}, w, y, y^{-1}\} &= \{x \in \Lambda^P\{\mathfrak{w}, \mathfrak{w}^{-1}, w, y, y^{-1}\} \mid \mathfrak{v}_T^u(x) \geq 0\} \\ &\subset \Lambda^P\{\mathfrak{w}, \mathfrak{w}^{-1}, w, y, y^{-1}\}, \end{aligned}$$

we have

$$\Lambda_0^P\{\mathfrak{w}, \mathfrak{w}^{-1}, w, y, y^{-1}\} = \bigcap_{u \in P} \Lambda_0^u\{\mathfrak{w}, \mathfrak{w}^{-1}, w, y, y^{-1}\}. \tag{3.13}$$

Define the variable

$$z_j(u) = T^{\ell_j(u)} y_1(u)^{v_{j,1}} \cdots y_n(u)^{v_{j,n}}$$

for $j = 1, \cdots, m$.

The following lemma follows from the definition (3.10) of $y_i(u)$.

Lemma 3.7. *The expression*

$$z_j(u) \circ \psi_u \in \Lambda\{y, y^{-1}\}$$

is independent of $u \in \mathrm{Int}\, P$. We denote this common variable by z_j. Furthermore we have

$$\mathfrak{v}_T^u(z_j) = \ell_j(u). \tag{3.14}$$

In particular $z_j \in \Lambda_0^P\{y, y^{-1}\}$.

Proof. From (3.10), we have $y_i(u) \circ \psi_u = T^{-u_i} y_i$. Therefore we have

$$(y_1(u)^{v_{j,1}} \cdots y_n(u)^{v_{j,n}}) \circ \psi_u = T^{-\langle v_j, u\rangle} \prod_{i=1}^n y_i^{v_{j,i}}$$

for $j = 1, \cdots, m$. Recalling $\ell_j(u) = \langle v_j, u\rangle - \lambda_j$, we obtain

$$z_j(u) \circ \psi_u = T^{\ell_j(u)} (y_1(u)^{v_{j,1}} \cdots y_n(u)^{v_{j,n}}) \circ \psi_u = T^{-\lambda_j} \prod_{i=1}^n y_i^{v_{j,i}}$$

which shows independence of u.

We evaluate

$$\begin{aligned} \mathfrak{v}_T^u(z_j(u) \circ \psi_u) &= \mathfrak{v}_T^u(T^{-\lambda_j} \prod_{i=1}^n y_i^{v_{j,i}}) = -\lambda_j + \sum_{i=1}^m v_{j,i} \mathfrak{v}_T^u(y_i) \\ &= -\lambda_j + \sum_{i=1}^m v_{j,i} u_i = \langle v_j, u\rangle - \lambda_j = \ell_j(u) \end{aligned}$$

where we use (3.12) for the third equality and the definition of ℓ_j for the last. Finally since $\ell_j(u) > 0$ for $u \in \mathrm{Int}P$, the last statement follows. This finishes the proof. □

Remark 3.8. We note that the variables z_j, $j = 1, \cdots, m$ depend on Int$P$ and the vectors $\{v_j\}_{j=1,\cdots m}$. Recall the latter is the set of one dimensional generators of the fan Σ associated to the toric manifold $X = X_\Sigma$ which determines the *complex structure* on X. On the other hand the moment polytope P is determined by the *symplectic structure* of X. In other words, the variables depend on both complex structure and symplectic structure.

We consider formal power series of the form

$$\sum_{k=1}^{\infty} a_k \mathfrak{w}_1^{e_{k,1}} \cdots, \mathfrak{w}_m^{e_{k,m}} w_{m+1}^{e_{k,m+1}} \cdots w_B^{e_{k,B}} z_1^{f_{k,1}} \cdots z_m^{f_{k,m}}, \tag{3.15}$$

with the conditions

$$\begin{aligned} a_k &\in \Lambda_0, \qquad \lim_{k\to\infty} \mathfrak{v}_T(a_k) = \infty, \\ e_{k,i} &\in \begin{cases} \mathbb{Z} & i \le m, \\ \mathbb{Z}_{\ge 0} & i > m, \end{cases} \\ f_{k,j} &\in \mathbb{Z}_{\ge 0}. \end{aligned}$$

Lemma 3.9. *Any formal power series of the form* (3.15) *is an element of the ring* $\Lambda_0^P\{\mathfrak{w}, \mathfrak{w}^{-1}, w, y, y^{-1}\}$.

Conversely any element of $\Lambda_0^P\{\mathfrak{w}, \mathfrak{w}^{-1}, w, y, y^{-1}\}$ *can be written in the form of* (3.15).

Proof. Consider a monomial

$$\xi = a\, \mathfrak{w}_1^{e_1} \cdots, \mathfrak{w}_m^{e_m} w_{m+1}^{e_{m+1}} \cdots w_B^{e_B} y_1^{f_1} \cdots y_n^{f_n}.$$

Then we have the valuation

$$\mathfrak{v}_T^u(\xi) = \mathfrak{v}_T(a) + \langle f, u\rangle.$$

Put

$$c = \inf\{\langle f, u\rangle \mid u \in \operatorname{Int} P\}. \tag{3.16}$$

Since P is a convex polytope, we can take a vertex u^0 of P such that

$$c = \langle f, u^0\rangle.$$

There exist n faces $\partial_{j_i} P$, $i = 1, \cdots, n$ such that

$$\{u^0\} = \bigcap_{i=1}^{n} \partial_{j_i} P. \tag{3.17}$$

Since X is a smooth toric manifold, the corresponding fan is regular and so $\vec v_{i_j}$ $j = 1, \cdots, n$ forms a $\mathbb{Z}$-basis of M. (See section 2.1 [Ful], for example.) Therefore we have

$$\vec f = (f_1, \cdots, f_n) = \sum_{i=1}^{n} b_i \vec v_{j_i}$$

for some $b_i \in \mathbb{Z}$. By definition of c (3.16), we have

$$b_i \ge 0.$$

And we can express

$$\xi = aT^{\langle f, u^0\rangle} \mathfrak{w}_1^{e_1} \cdots, \mathfrak{w}_m^{e_m} w_{m+1}^{e_{m+1}} \cdots w_B^{e_B} z_{j_1}^{b_1} \cdots z_{j_n}^{b_n} :$$

Here we use the facts that $z_i(u) = T^{\ell_i(u)} y_1(u)^{v_{i,1}} \cdots y_n(u)^{v_{i,n}}$, $z_i(u) \circ \psi_u = z_i$ do not depend on u and $\ell_{j_i}(u^0) = 0$ by (3.17). We have

$$\mathfrak{v}_T(aT^{\langle f, u^0\rangle}) = \mathfrak{v}_T(a) + \langle f, u^0\rangle = \mathfrak{v}_T(a) + c.$$

If $\mathfrak{v}_T^u(\xi) \geq 0$ for all $u \in \mathrm{Int}P$, then

$$\mathfrak{v}_T(aT^{\langle f, u^0\rangle}) = \mathfrak{v}_T(a) + c = \inf_{u \in \mathrm{Int}P} \mathfrak{v}_T^u(\xi) \geq 0.$$

Therefore ξ is of the form (3.15).

For the converse, we first obtain

$$\mathfrak{v}_T^u(a\mathfrak{w}_1^{e_1} \cdots, \mathfrak{w}_m^{e_m} w_{m+1}^{e_{m+1}} \cdots w_B^{e_B} z_1^{f_1} \cdots z_m^{f_m}) = \mathfrak{v}_T(a) + \sum_{j=1}^m \ell_j(u) f_j$$

from (3.14). Since $a \in \Lambda_0$, $f_j \geq 0$ and $\ell_j(u) > 0$ for all $u \in \mathrm{Int}P$, $\mathfrak{v}_T(a) + \sum_{j=1}^m \ell_j(u) f_j \geq 0$ for all $u \in \mathrm{Int}P$ and so

$$\mathfrak{v}_P(a\mathfrak{w}_1^{e_1} \cdots, \mathfrak{w}_m^{e_m} w_{m+1}^{e_{m+1}} \cdots w_B^{e_B} z_1^{f_1} \cdots z_m^{f_m}) \geq 0.$$

This prove the converse and hence the proof of the lemma. □

Remark 3.10. (1) We remark that the representation (3.15) of an element $x \in \Lambda_0^P\{\mathfrak{w}, \mathfrak{w}^{-1}, w, y, y^{-1}\}$ is *not* unique. The non-uniqueness is due to the fact that z_i's in $\Lambda_0^P\{\mathfrak{w}, \mathfrak{w}^{-1}, w, y, y^{-1}\}$ satisfy the quantum Stanley-Reisner relation. (See Proposition 5.5 [FOOO3].)

(2) The proof of Lemma 3.9 implies the following : A *monomial* of the form (3.15) is a monomial in $\Lambda_0^P\{\mathfrak{w}, \mathfrak{w}^{-1}, w, y, y^{-1}\}$ and vice versa.

(3) The discussion above shows that the moment polytope P is closely related to the Berkovich spectrum [Ber], [KS] of $\Lambda_0^P\{\mathfrak{w}, \mathfrak{w}^{-1}, w, y, y^{-1}\}$.

Now we can state the following theorem whose proof we will postpone until section 9.

Theorem 3.11. *The function* $\mathfrak{PO}^u \circ \psi_u$ *lies in*

$$\Lambda_0^P\{\mathfrak{w}, \mathfrak{w}^{-1}, w, y, y^{-1}\}$$

and is independent of u.

We denote the common function by $\mathfrak{PO}$.

Now we can generalize the result of section 3 [FOOO3] as follows. Using Lemma 3.5 and the idea of Cho (see section 11 [FOOO3]) we can define Floer cohomology

$$HF((L(u), \mathfrak{b}, \mathfrak{x}), (L(u), \mathfrak{b}, \mathfrak{x}); \Lambda_0)$$

for any $(\mathfrak{b}, \mathfrak{x}) \in H(X; \Lambda_0) \times H^1(L(u); \Lambda_0) \cong (\Lambda_0)^B \times (\Lambda_0)^n$. See sections 8 and 11. The following is the generalization of Theorem 3.9 [FOOO3].

Theorem 3.12. *Let* $(\mathfrak{b}, \mathfrak{x}) \in (\Lambda_0)^B \times (\Lambda_0)^n$. *We assume*

$$\frac{\partial \mathfrak{PO}}{\partial x_i}(\mathfrak{b}, \mathfrak{x}) = 0 \tag{3.18}$$

for $i = 1, \cdots, n$. *Then we have*

$$HF((L(u_0), \mathfrak{b}, \mathfrak{x}), (L(u_0), \mathfrak{b}, \mathfrak{x}); \Lambda_0) \cong H(T^n; \Lambda_0). \tag{3.19}$$

If we assume

$$\frac{\partial \mathfrak{PO}}{\partial x_i}(\mathfrak{b}, \mathfrak{x}) \equiv 0 \mod T^{\mathcal{N}} \tag{3.20}$$

then we have

$$HF((L(u_0),\mathfrak{b},\mathfrak{x}),(L(u_0),\mathfrak{b},\mathfrak{x});\Lambda_0/T^{\mathcal{N}}) \cong H(T^n;\Lambda_0/T^{\mathcal{N}}). \tag{3.21}$$

Theorem 3.12 will be proved in section 7. We next define :

Definition 3.13. Let $L(u)$ be a Lagrangian fiber of a compact toric manifold (X,ω). We say that $L(u)$ is *bulk-balanced* if there exist sequences ω_i, P_i, $\mathfrak{b}_i$, $\mathfrak{x}_i$, $\mathcal{N}_i$, and u_i with the following properties.

(1) (X,ω_i) is a sequence toric manifolds such that $\lim_{i\to\infty}\omega_i = \omega$.
(2) P_i is a moment polytope of (X,ω_i). It converges to the moment polytope P of (X,ω).
(3) $u_i \in P_i$ and $\lim_{i\to\infty} u_i = u$.
(4) $\mathfrak{b}_i \in \mathcal{A}(\Lambda_+(\mathbb{C}))$, $\mathfrak{x}_i \in H^1(L(u_i);\Lambda_0(\mathbb{C}))$, $\mathcal{N}_i \in \mathbb{R}_+$.
(5)
$$HF((L(u_i),\mathfrak{b}_i,\mathfrak{x}_i),((L(u_i),\mathfrak{b}_i,\mathfrak{x}_i);\Lambda_0(\mathbb{C})/T^{\mathcal{N}_i}) \cong H(T^n;\Lambda_0(\mathbb{C})/T^{\mathcal{N}_i}).$$
(6) $\lim_{i\to\infty}\mathcal{N}_i = \infty$.

Remark 3.14. (1) Definition 3.13 is related to Definitions 3.10 [FOOO3]. Namely it is easy to see that

"Strongly balanced" $\Rightarrow$ "balanced"$\Rightarrow$ "bulk-balanced"

On the other hand the three notions are all different. (See Example 9.17 [FOOO3] and section 5 of the present paper.)

(2) We can generalize Theorem 3.12 to the case $\mathfrak{b} \in \mathcal{A}(\Lambda_0(\mathbb{C}))$ in place of $\mathfrak{b} \in \mathcal{A}(\Lambda_+(\mathbb{C}))$. See section 11.

The next result is a generalization of Proposition 3.11 [FOOO3] which will be proved in section 8.

Proposition 3.15. *Suppose that $L(u) \subset X$ is bulk-balanced. Then $L(u)$ is non-displaceable.*

Moreover if $\psi : X \to X$ is a Hamiltonian diffeomorphism such that $\psi(L(u))$ is transversal to $L(u)$, then

$$\#(\psi(L(u)) \cap L(u)) \geq 2^n. \tag{3.22}$$

It seems reasonable to expect the following converse to this proposition.

Conjecture 3.16. *If $L(u)$ is a non-displaceable fiber of a compact toric manifold then $L(u)$ is bulk-balanced.*

The next definition is related to Definition 4.10 [FOOO3].

Definition 3.17. Let $L(u)$ be a Lagrangian fiber of a compact toric manifold (X,ω). The *bulk $\mathfrak{PD}$-threshold*, $\overline{\mathfrak{E}}^{\text{bulk}}(L(u))$ is $2\pi\mathcal{N}$ where $\mathcal{N}$ is the supremum of the numbers $\mathcal{N}_i$ such that there exist ω_i, P_i, $\mathfrak{b}_i$, $\mathfrak{x}_i$, and u_i satisfying Definition 3.13 (1) - (5).

Remark 3.18. In Definition 4.10 [FOOO3] we defined two closely related numbers $\overline{\mathfrak{E}}(L(u))$, $\mathfrak{E}(L(u))$. It is easy to see

$$\mathfrak{E}(L(u)) \leq \overline{\mathfrak{E}}(L(u)) \leq \overline{\mathfrak{E}}^{\text{bulk}}(L(u)). \tag{3.23}$$

The equalities in (3.23) do *not* hold in general. (See Example 9.17 [FOOO3] and section 5.)

We recall that the displacement energy $e(L)$ of a Lagrangian submanifold $L \subset X$ is the infimum of the Hofer distance $dist(\psi, id)$ ([H]) between identity and a Hamiltonian isotopy $\psi : X \to X$ such that $\psi(L) \cap L = \emptyset$. (See Definition 4.9 [FOOO3].)

We will prove the following in section 8.

Theorem 3.19.
$$e(L) \geq \overline{\mathfrak{E}}^{\rm bulk}(L(u)). \tag{3.24}$$

It would be interesting to see if the following holds :

Conjecture 3.20. *The equality holds in* (3.24).

4. Elimination of higher order terms by bulk deformations

The purpose of this section is to apply the result of the last section to locate bulk-balanced Lagrangian fibers. We first recall the notion of leading term equation. We denote by $\Lambda_0\{y, y^{-1}\}$ the completion of the Laurent polynomial ring $\Lambda_0[y_1, y_1^{-1}, \cdots, y_n, y_n^{-1}]$ with respect to the non-Archimedean norm. For each fixed $\mathfrak{b} \in \rho A(\Lambda_+)$ and u, we have
$$\mathfrak{PO}^u(\mathfrak{b}; y_1, \cdots, y_n) \in \Lambda_0\{y, y^{-1}\}.$$
We also put
$$\mathfrak{PO}^u_{\mathfrak{b}}(y_1, \cdots, y_n) = \mathfrak{PO}^u(\mathfrak{b}; y_1, \cdots, y_n)$$
and regard $\mathfrak{PO}^u_{\mathfrak{b}}$ as an element of $\Lambda_0\{y, y^{-1}\}$.

Henceforth we write $y^{\vec v}$ for $y_1^{v_1} \cdots y_n^{v_n}$ with $\vec v = (v_1, \cdots, v_n)$.

Let $\vec v_i = d\ell_i = (v_{i,1}, \cdots, v_{i,n}) \in H_1(L(u); \mathbb{Z}) \cong \mathbb{Z}^n \cong N_{\mathbb{Z}}$ $(i = 1, \cdots, m)$ as in the end of section 1. We define $S_l \in \mathbb{R}_+$ by $S_l < S_{l+1}$ and
$$\{S_l \mid l = 1, 2, \cdots, \mathcal{L}\} = \{\ell_i(u) \mid i = 1, 2, \cdots, m\}. \tag{4.1}$$
We re-enumerate the set $\{\vec v_k \mid \lambda_k = S_l\}$ as
$$\{\vec v_{l,1}, \cdots, \vec v_{l,a(l)}\}. \tag{4.2}$$

Let $A_l^{\perp} \subset N_{\mathbb{R}} \cong \mathbb{R}^n$ be the $\mathbb{R}$-vector space generated by $\vec v_{l',r}$ for $l' \leq l$, $r = 1, \cdots, a(l')$. We remark that $A_l^{\perp}$ is defined over $\mathbb{Q}$. Namely $A_l^{\perp} \cap \mathbb{Q}^n$ generates $A_l^{\perp}$ as an $\mathbb{R}$ vector space. Denote by K the smallest integer l such that $A_l^{\perp} = N_{\mathbb{R}}$. We put $d(l) = \dim A_l^{\perp} - \dim A_{l-1}^{\perp}$, $d(1) = \dim A_1^{\perp}$.

We remark
$$\{\vec v_{l,r} \mid l = 1, \cdots, K, r = 1, \cdots, a(l)\} \subset \{\vec v_i \mid i = 1, \cdots, m\}.$$
Henceforth we assume $l \leq K$ whenever we write $\vec v_{l,r}$. For each (l, r) we define the integer $i(l, r) \in \{1, \cdots, m\}$ by
$$\vec v_{l,r} = \vec v_{i(l,r)}. \tag{4.3}$$
Renumbering $\vec v_i$, if necessary, we can enumerate them so that
$$\begin{aligned} &\{\vec v_i \mid i = 1, \cdots, m\} \\ &= \{\vec v_{l,r} \mid l = 1, \cdots, K, r = 1, \cdots, a(l)\} \setminus \{\vec v_i \mid i = \mathcal{K} + 1, \cdots, m\} \end{aligned} \tag{4.4}$$
for some $1 \leq \mathcal{K} \leq m - 1$.

Recall we have chosen a basis $\mathbf{e}_i$ of $H^1(L(u); \mathbb{Z})$ in the end of section 1. It can be identified with a basis of $M_{\mathbb{R}} \cong H^1(L(u); \mathbb{R})$. Denote its dual basis on $N_{\mathbb{R}}$ by $\mathbf{e}_i^*$.

We choose a basis $\mathbf{e}^*_{l,s}$ of $N_{\mathbb{R}}$ such that $\mathbf{e}^*_{1,1}, \cdots, \mathbf{e}^*_{l,d(l)}$ forms a $\mathbb{Q}$-basis of $A_l^{\perp}$ and that each of $\vec{v}_i$ lies in $\bigoplus_{l,s} \mathbb{Z}\mathbf{e}^*_{l,s}$.

We put

$$\mathbf{e}^*_i = \sum_{l=1}^{K} \sum_{s=1}^{d(l)} a_{i;(l,s)} \mathbf{e}^*_{l,s},$$

$(a_{(l,s);i} \in \mathbb{Q})$. Regarding $\mathbf{e}^*_i$ and $\mathbf{e}^*_{l,s}$ as functions on $M_{\mathbb{R}}$, this equation can be written as

$$x_i = \sum_{l=1}^{K} \sum_{s=1}^{d(l)} a_{i;(l,s)} x_{l,s}$$

with $x_i = \mathbf{e}^*_i$ and $x_{l,s} = \mathbf{e}^*_{l,s}$. If we associate $y_{l,s} = e^{x_{l,s}}$, it is contained in a finite field extension of $\mathbb{Q}[y_1, y_1^{-1}, \cdots, y_n, y_n^{-1}]$ and satisfies

$$y_i = \prod_{l=1}^{K} \prod_{s=1}^{d(l)} y_{l,s}^{a_{i;(l,s)}}. \tag{4.5}$$

We put $\vec{v}_{l,r} = (v_{l,r;1}, \cdots, v_{l,r;n}) \in \mathbb{Z}^n$.

Lemma 4.1. *The product*

$$y^{\vec{v}_{l,r}} = y_1^{v_{l,r;1}} \cdots y_n^{v_{l,r;n}}$$

is a monomial of $y_{l',s}$ for $l' \le l$, $s \le d(l')$.

Proof. By the definition of $A_\ell^{\perp}$, $\vec{v}_{l,r}$ is an element of $A_l^{\perp}$ and so

$$\vec{v}_{l,r} = \sum_{l' \le l, s \le d(l')} c_{l,r;l',s} \mathbf{e}^*_{l',s}$$

for some integer $c_{l,r;l',s}$. Therefore

$$y^{\vec{v}_{l,r}} = \prod_{l' \le l, s \le d(l')} y_{l',s}^{c_{l,r;l',s}}$$

and the lemma follows. $\square$

We put

$$(\mathfrak{PO}^u_{\mathfrak{b}})_l = \sum_{r=1}^{a(l)} y^{\vec{v}_{l,r}}. \tag{4.6}$$

By Lemma 4.1, $(\mathfrak{PO}^u_{\mathfrak{b}})_l$ can be written as a Laurent polynomial of $y_{l',s}$ for $l' \le l$, $s \le d(l')$ with its coefficients are scalers i.e., elements of R.

Now we consider the equation

$$y_k \frac{\partial \mathfrak{PO}^u_{\mathfrak{b}}}{\partial y_k} = 0 \tag{4.7}$$

with $k = 1, \cdots, n$ for y_k from Λ_0 . By changing the coordinates to $y_{l,s}$ $(l = 1, \cdots, K, s = 1, \cdots, d(l))$, (4.7) becomes

$$y_{l,s} \frac{\partial \mathfrak{PO}^u_{\mathfrak{b}}}{\partial y_{l,s}} = 0. \tag{4.8}$$

Lemma 4.2. *The equation* (4.8) *has a solution with* $y_{l,s}$ *from* $\Lambda_0(R) \setminus \Lambda_+(R)$ *if and only if* (4.7) *has a solution with* $y_k \in \Lambda_0(R) \setminus \Lambda_+(R)$.

If R *is algebraically closed, then the ratio between the numbers of solutions counted with multiplicity is equal to the degree of field extension*

$$\Big[\mathbb{Q}[y_{1,1}, y_{1,1}^{-1}, \cdots, y_{K,d(K)}, y_{K,d(K)}^{-1}] : \mathbb{Q}[y_1, y_1^{-1}, \cdots, y_n, y_n^{-1}]\Big].$$

Proof. This is obvious from the form of the change of coordinate (4.5). □

Definition 4.3. The *leading term equation* of (4.7) or of (4.8) is the system of equations

$$\frac{\partial(\mathfrak{PO}^u_{\mathfrak b})_l}{\partial y_{l,s}} = 0 \tag{4.9}$$

with $y_{l,s}$ from $R \setminus \{0\}$ for $l = 1, \cdots, K$, $s = 1, \cdots, d(l)$.

We remark that (4.7) is an equation for $y_1, \cdots, y_n \in \Lambda_0$. On the other hand, the equation (4.9) is one for $y_{l,s} \in R \setminus \{0\}$. The following lemma describes the relation between these two equations.

Lemma 4.4. *Let* $y_{l,s} \in \Lambda_0(R) \setminus \Lambda_+(R)$ *be a solution of* (4.8). *We define* $\overline{y}_{l,s} \in \mathbb{C} \setminus \{0\}$ *by* $y_{l,s} \equiv \overline{y}_{l,s} \mod \Lambda_+(R)$. *Then* $\overline{y}_{l,s}$ *solves the leading term equation* (4.9).

The proof is easy. (See sections 8,9 [FOOO3].)

We remark that the discussion above applies to the leading order potential function $\mathfrak{PO}^u_0$ (See (3.4)) without changes. See sections 8,9 [FOOO3].

Lemma 4.5. *The leading term equation of* $\mathfrak{PO}^u(\mathfrak b, y)$ *is independent of* $\mathfrak b \in \mathcal{A}(\Lambda_+)$. *Moreover it coincides with the leading term equation of* $\mathfrak{PO}^u_0$.

Proof. The first half follows from Theorem 3.4. The second half follows from Theorem 3.5 [FOOO3]. □

We denote by $\Lambda_0\{y_{**}, y_{**}^{-1}\}$ the completion of $\Lambda_0[y_{1,1}, y_{1,1}^{-1}, \cdots, y_{K,d(K)}, y_{K,d(K)}^{-1}]$ with respect to the non-Archimedean norm. It is a finite field extension of $\Lambda_0\{y, y^{-1}\}$.

Definition 4.6. We say that (X, ω) is rational if $c[\omega] \in H^2(X;\mathbb{Q})$ for some $c \in \mathbb{R} \setminus \{0\}$. We say that a Lagrangian submanifold $L \subset X$ is rational if $\{\omega \cap \beta \mid \beta \in H_2(X, L; \mathbb{Z})\} \subset \mathbb{R}$ is isomorphic to $\mathbb{Z}$ or $\{0\}$.

We remark that only rational symplectic manifold (X, ω) carries a rational Lagrangian submanifold L. (In the general situation $\pi_2(X, L)$ is used sometimes in the definition of rationality of L. In our case of toric fibers, they are equivalent.)

Now we state the main result of this section.

Theorem 4.7. *The following two conditions on* u *are equivalent to each other :*

(1) *The leading term equation of* $\mathfrak{PO}^u_0$ *has a solution* $y_{l,s} \in R \setminus \{0\}$.
(2) *There exists* $\mathfrak b \in \mathcal{A}(\Lambda_+(R))$ *such that* $\mathfrak{PO}^u_{\mathfrak b}$ *has a critical point on* $(\Lambda_0(R) \setminus \Lambda_+(R))^n$.

Corollary 4.8. *If the leading term equation of* $\mathfrak{PO}^u_0$ *has a solution then* $L(u)$ *is bulk-balanced.*

Proof of Theorem 4.7. The proof of (2) ⇒ (1) is a consequence of Lemmata 4.2, 4.4 and 4.5. The rest of this section is devoted to the proof of the converse.

Let $\mathfrak{y}_{1,1}, \cdots, \mathfrak{y}_{K,d(K)}$ be a solution of the leading term equation. We remark $\mathfrak{y}_{l,s} \in R \setminus \{0\} \subset \Lambda_0(R) \setminus \Lambda_+(R)$. We will fix $\mathfrak{y}_{l,s}$ during the proof of Theorem 4.7 and find $\mathfrak{b}$ such that $\mathfrak{y}_{l,s}$ is a critical point of $\mathfrak{PO}^u_{\mathfrak{b}}$. We also require $\mathfrak{b}$ to have the form

$$\mathfrak{b} = \sum_{l=1}^{K} \sum_{r=1}^{a(l)} \mathfrak{b}_{l,r} D_{i(l,r)} \tag{4.10}$$

where $\mathfrak{b}_{l,r} \in \Lambda_+$. (Here and hereafter in this section we omit R in $\Lambda_+(R)$ and etc.)

Note $i(l,r) \le m$ and so $\deg D_{i(l,r)} = 2$. In other words, we use $\mathfrak{b}$ in the second cohomology $H^2(X;\Lambda_+)$ (more precisely $\mathfrak{b} \in \mathcal{A}^2(\Lambda_+)$) only to prove Theorem 4.7.

We first consider the case when X is Fano. In this case we can calculate $\mathfrak{PO}^u(\mathfrak{b}; y)$ explicitly as follows.

Proposition 4.9. *Suppose X is Fano and $\mathfrak{b}$ is as in* (4.10)*. Then*

$$\mathfrak{PO}^u(\mathfrak{b}, y) = \sum_{l=1}^{K} \sum_{r=1}^{a(l)} \exp(\mathfrak{b}_{l,r}) T^{S_l} y^{\vec{v}_{i(l,r)}} + \sum_{i=\mathcal{K}+1}^{m} T^{\ell_i(u)} y^{\vec{v}_i}. \tag{4.11}$$

We will prove Proposition 4.9 in section 7.

We put

$$\vec{v}_{i(l,r)} = \sum_{l'=1}^{l} \sum_{s=1}^{d(l')} v_{l,r;l',s} \mathbf{e}^*_{l',s}, \qquad \vec{v}_i = \sum_{l=1}^{K} \sum_{s=1}^{d(l)} v_{i;l,s} \mathbf{e}^*_{l,s}.$$

Lemma 4.10. *If $\mathfrak{y}_{l,s} \in R \setminus \{0\}$ is a solution of the leading term equation, then*

$$\begin{aligned} \mathfrak{y}_{l',s} \frac{\partial \mathfrak{PO}^u}{\partial y_{l',s}}(\mathfrak{b}; \mathfrak{x}) = & \sum_{l=l'}^{K} \sum_{r=1}^{a(l)} \left(\mathfrak{b}_{l,r} + \sum_{h=2}^{\infty} \frac{1}{h!} \mathfrak{b}_{l,r}^h \right) T^{S_l} v_{l,r;l',s} \mathfrak{y}^{\vec{v}_{i(l,r)}} \\ & + \sum_{i=\mathcal{K}+1}^{m} v_{i;l',s} T^{\ell_i(u)} \mathfrak{y}^{\vec{v}_i}. \end{aligned} \tag{4.12}$$

Here $\mathfrak{x} = \sum (\log \mathfrak{y}_i) \mathbf{e}_i$ and $\mathfrak{y}_i$ is determined from $\mathfrak{y}_{l,s}$ by (4.5)*.*

Proof. Differentiating (4.11), we obtain

$$\begin{aligned} y_{l',s} \frac{\partial \mathfrak{PO}^u_{\mathfrak{b}}}{\partial y_{l',s}} = & \sum_{l=l'}^{K} \sum_{r=1}^{a(l)} \left(1 + \mathfrak{b}_{l,r} + \sum_{h=2}^{\infty} \frac{1}{h!} \mathfrak{b}_{l,r}^h \right) T^{S_l} v_{l,r;l',s} y^{\vec{v}_{i(l,r)}} \\ & + \sum_{i=\mathcal{K}+1}^{m} v_{i;l',s} T^{\ell_i(u)} y^{\vec{v}_i}. \end{aligned}$$

On the other hand, the leading term equation is

$$0 = \sum_{r=1}^{a(l)} v_{l,r;l',s} \mathfrak{y}^{\vec{v}_{i(l,r)}}$$

Therefore (4.12) follows. □

To highlight the idea of the proof, we first consider the *rational* case. In this case, by rescaling the symplectic form ω to $c\omega$ by some $c \in \mathbb{R}_+$, we may assume that ω is integral, i.e.,

$$\{\omega \cap \beta / 2\pi \mid \beta \in H_2(X, L(u); \mathbb{Z})\} \in \mathbb{Z}.$$

It follows that $S_l, \ell_i(u) \in \mathbb{Z}$. Thus, we can reduce the coefficient rings from the universal Novikov rings Λ_0, Λ_+, Λ to the following rings respectively :

$$\Lambda_0^{\rm int} := R[[T]], \quad \Lambda_+^{\rm int} := TR[[T]], \quad \Lambda_0^{\rm int} := R[[T]][T^{-1}].$$

Here $R[[T]]$ is the formal power series ring.

We also consider pairs $(\mathfrak b, b)$ only from $\mathcal A^2(\Lambda_+^{\rm int}) \times H^1(L(u);\Lambda_0^{\rm int})$. Under these restrictions, the exponents of T appearing in our discussion following always become integers.

Lemma 4.11. *Suppose X is Fano and $L(u)$ is rational. For each k, l, r, there exists $\mathfrak b_{l,r}(k) \in \Lambda_+^{\rm int}$ such that*

$$\mathfrak b(k) = \sum_{l=1}^{K}\sum_{r=1}^{a(l)} \mathfrak b_{l,r}(k) D_{i(l,r)}$$

has the following property.

$$\mathfrak y_{l,s}\frac{\partial \mathfrak{PO}^u}{\partial y_{l,s}}(\mathfrak b(k);\mathfrak x) \equiv 0 \mod T^k. \tag{4.13}$$

for $l = 1,\cdots,K$, $s = 1,\cdots,d(l)$. Here $\mathfrak x = \sum(\log \mathfrak y_i)\mathbf e_i$. We also have

$$\mathfrak b_{l,r}(k+1) \equiv \mathfrak b_{l,r}(k) \mod T^{k-S_l}. \tag{4.14}$$

Proof. The proof is by an induction over k. If $k \le S_1$, we apply Lemma 4.10 to $\mathfrak b = \mathfrak b(k) = 0$ and obtain

$$\mathfrak y_{l',s}\frac{\partial \mathfrak{PO}^u}{\partial y_{l',s}}(\mathfrak b(k);\mathfrak x) \equiv 0 \mod T^{S_1}.$$

Hence (4.13) holds for $k \le S_1$.

Now suppose $k > S_1$ and assume $\mathfrak b(k-1)$ with the required property. By the induction hypothesis we may put

$$\mathfrak y_{l',s}\frac{\partial \mathfrak{PO}^u}{\partial y_{l',s}}(\mathfrak b(k-1);\mathfrak x) \equiv T^k E_{l',s} \mod T^{k+1}, \tag{4.15}$$

for some $E_{l',s} \in R$. Let $\vec E = \sum E_{l',s}\mathbf e^*_{l',s} \in N_R = N\otimes_{\mathbb Z} R$.

Sublemma 4.12. *$\vec E$ is contained in the vector space generated by $\{e^*_{l,s} \mid S_l < k,\ s = 1,\cdots,d(l)\}$.*

Proof. This is a consequence of (4.12). □

By Sublemma 4.12, we can express $\vec E$ as

$$-\vec E = \sum_{S_l<k} c_{l,r}\mathfrak y^{\vec v_{l,r}}\vec v_{l,r} \tag{4.16}$$

for some $c_{l,r} \in R$. Note $\vec v_{l,r}$, $l \le l_0$, $r \le a(l)$ span the vector space

$$A_{l_0}^{\perp} = \operatorname{span}_R\{e^*_{l,s} \mid l \le l_0,\ s = 1,\cdots,d(l)\}.$$

We define $\mathfrak b_{l,r}(k)$ by

$$\mathfrak b_{l,r}(k) = \mathfrak b_{l,r}(k-1) + c_{l,r}(\mathfrak y^{\vec v_{l,r}})^{-1}T^{k-S_l}D_{i(l,r)}.$$

Since $k - S_l > 0$ it follows $k - S_l \in \mathbb Z_+$ by the integrality hypothesis of ω. Namely $\mathfrak b_{l,r}(k) \in \Lambda_+^{\rm int}$. Lemma 4.10, (4.15) and (4.16) imply (4.13). This finishes the induction steps and so the proof of Lemma 4.11 is complete. □

Now we consider

$$\mathfrak{b} = \lim_{k\to\infty} \mathfrak{b}(k).$$

The right hand side converges by (4.14) and so $\mathfrak{b}$ is well-defined as an element of $\rho A(\Lambda_+^{\rm int})$ and satisfies

$$\mathfrak{y}_{l',s}\frac{\partial \mathfrak{PO}^u}{\partial y_{l',s}}(\mathfrak{b};\mathfrak{x}) = 0$$

as required. Thus Theorem 4.7 is proved for the case where X is Fano and $L(u)$ is rational.

We now turn to the case where X is not necessarily Fano or $L(u)$ not necessarily rational. We will still use an induction argument but we need to choose the discrete submonoids of $\mathbb{R}$ that we work with carefully to carry out the induction.

Let $G(X)$ be as in (3.3). We define :

$$G(L(u)) = \langle\{\omega[\beta]/2\pi \mid \beta\in\pi_2(X,L(u)) \text{ is realized by a holomorphic disc}\}\rangle. \tag{4.17}$$

Definition 4.13. Let $G(X)$ be as in (3.3). We define $G_{\rm bulk}$ to be the discrete submonoid of $\mathbb{R}$ generated by $G(X)$ and the subset

$$\{\lambda - S_l \mid \lambda\in G(L(u)),\quad l=1,\cdots,K, \lambda > S_l\} \subset \mathbb{R}_+\subset\mathbb{R}.$$

Condition 4.14. We consider

$$\mathfrak{b} = \sum_{l=1}^{K}\sum_{r=1}^{a(l)}\mathfrak{b}_{l,r}D_{i(l,r)} \in \mathcal{A}^2(\Lambda_+(R)) \tag{4.18}$$

such that all $\mathfrak{b}_{l,r}$ are $G_{\rm bulk}$-gapped.

The main geometric input to the proof of the non-Fano case of Theorem 4.7 is the following.

Proposition 4.15. *We assume* $\mathfrak{b}$ *satisfies Condition* 4.14 *and consider*

$$\mathfrak{b}' = \mathfrak{b} + cT^{\lambda}D_{i(l,r)}, \tag{4.19}$$

with $c\in R$, $\lambda\in G_{\rm bulk}$, $l\le K$. *Then we have*

$$\begin{aligned}
&\mathfrak{PO}^u(\mathfrak{b}';y) - \mathfrak{PO}^u(\mathfrak{b};y)\\
&= cT^{\lambda+\ell_{i(l,r)}(u)}y^{\vec v_{i(l,r)}} + \sum_{h=2}^{\infty}c_hT^{h\lambda+\ell_{i(l,r)}(u)}y^{\vec v_{i(l,r)}}\\
&\qquad\qquad + \sum_{h=1}^{\infty}\sum_{\sigma}c_{h,\sigma}T^{h\lambda+\ell'_\sigma(u)+\rho_\sigma}y^{\vec v_\sigma}.
\end{aligned} \tag{4.20}$$

Here $c_h, c_{h,\sigma}\in R$, $\rho_\sigma\in G_{\rm bulk}$. *Moreover there exists* $e_\sigma^i\in\mathbb{Z}_{\ge 0}$ *such that* $\vec v_\sigma = \sum e_\sigma^i\vec v_i$, $\ell'_\sigma = \sum e^i_\sigma\ell_i$ *and* $\sum_i e^i_\sigma > 0$.

We prove Proposition 4.15 in section 7.

Definition 4.16. We enumerate elements of $G_{\rm bulk}$ so that

$$G_{\rm bulk} = \{\lambda_j^b \mid j=0,1,2,\cdots\}$$

where $0=\lambda_0^b<\lambda_1^b<\lambda_2^b<\cdots$.

(1) For $k \geq 1$, we define $\Lambda_0^{G_{\rm bulk}}\{y_{**}, y_{**}^{-1}\}_k$ to be a subspace of $\Lambda_0\{y_{**}, y_{**}^{-1}\}$ consisting of elements of the form
$$\sum_{l=1}^{K}\sum_{j=k}^{\infty} T^{S_l+\lambda_j^b} P_{j,l}(y_{1,1}, y_{1,1}^{-1}, \cdots, y_{l,d(l)}, y_{l,d(l)}^{-1}) \tag{4.21}$$
where each $P_{j,l}$ is a Laurent polynomial of $y_{1,1}, \cdots, y_{l,d(l)}$ with R coefficients, i.e.,
$$P_{j,l} \in R[y_{1,1}, y_{1,1}^{-1}, \cdots, y_{l,d(l)}, y_{l,d(l)}^{-1}].$$
We put $\Lambda_0^{G_{\rm bulk}}\{y_{**}, y_{**}^{-1}\}_0 := \Lambda_0^{G_{\rm bulk}}\{y_{**}, y_{**}^{-1}\}$.

(2) We define $N_R^{G_{\rm bulk}}(k)$ to be the set of elements of the form
$$\sum_{l=1}^{K}\sum_{s=1}^{d(l)}\sum_{j=k}^{\infty} c_{l,s,j} T^{S_l+\lambda_j^b}\mathbf{e}_{l,s}^*$$
from $N_R \otimes_R \Lambda_0$ with $c_{l,s,j} \in R$.

Lemma 4.17. *If $\mathfrak b$ satisfies Condition 4.14 then $\mathfrak{PO}_{\mathfrak b}^u \in \Lambda_0^{G_{\rm bulk}}\{y_{**}, y_{**}^{-1}\}$.*

Proof. This will follow from Theorem 3.4. It is easy to see $\mathfrak{PO}_0^u \in \Lambda_0^{G_{\rm bulk}}\{y_{**}, y_{**}^{-1}\}$ from the definitions of $\mathfrak{PO}_0^u$ and $G_{\rm bulk}$. So it suffices to show that the right hand side of (3.5) in Theorem 3.4 lies in $\Lambda_0^{G_{\rm bulk}}\{y_{**}, y_{**}^{-1}\}$. We consider a term $c_\sigma y^{\vec v'_\sigma} T^{\ell'_\sigma(u)+\rho_\sigma}$ thereof. Let $\vec v'_\sigma = \sum_{i=1}^m e_\sigma^i \vec v_i$ as in (3.6). We put
$$l_0 = \sup\{l \mid \exists r\ e_\sigma^{i(l,r)} \neq 0\}.$$
Then
$$c_\sigma y^{\vec v'_\sigma} \in R[y_{1,1}, y_{1,1}^{-1}, \cdots, y_{l_0,d(l_0)}, y_{l_0,d(l_0)}^{-1}].$$
On the other hand
$$\ell'_\sigma(u) = \sum e_\sigma^i \ell_i(u) \geq \ell_{i(l_0,r)}(u) = S_{l_0}$$
because $e_\sigma^i \geq 0$ and $\sum_i e_\sigma^i > 0$ and $e_\sigma^{i(l_0,r)} \neq 0$ for some r. Therefore
$$\ell'_\sigma(u) - S_{l_0} \in G_{\rm bulk}.$$
It follows that
$$c_\sigma y^{\vec v'_\sigma} T^{\ell'_\sigma(u)+\rho_\sigma} \in \Lambda_0^{G_{\rm bulk}}\{y_{**}, y_{**}^{-1}\}$$
as required. □

We now state the following lemma

Lemma 4.18. *If $\mathfrak P$ lies in $\Lambda_0^{G_{\rm bulk}}\{y_{**}, y_{**}^{-1}\}_k$ for some $k \in \mathbb Z_{\geq 0}$, so does $\frac{\partial \mathfrak P}{\partial y_{l',s}}$ for the same k and so*
$$\sum_{l'=1}^{K}\sum_{s=1}^{d(l')} \mathfrak c_{l',s}\frac{\partial \mathfrak P}{\partial y_{l',s}}(\mathfrak c)\mathbf e_{l',s}^* \in N_R^{G_{\rm bulk}}(k) \tag{4.22}$$
for $\mathfrak c = (\mathfrak c_{1,1}, \cdots, \mathfrak c_{K,d(K)}) \in (R \setminus \{0\})^n$.

Proof. By the form (4.21) of the elements from $\Lambda_0^{G_{\rm bulk}}\{y_{**}, y_{**}^{-1}\}_k$, the first statement immediately follows. Then the last statement follows from the definition of $N_R^{G_{\rm bulk}}(k)$. □

Proposition 4.19. *There exists a sequence*

$$\mathfrak{b}(k) = \sum_{l=1}^{K} \sum_{r=1}^{a(l)} \mathfrak{b}_{l,r}(k) D_{i(l,r)} \tag{4.23}$$

that satisfies Condition 4.14 *and*

$$\sum_{l'=1}^{K} \sum_{s=1}^{d(l')} \mathfrak{y}_{l',s} \frac{\partial \mathfrak{PO}^u_{\mathfrak{b}(k)}}{\partial y_{l',s}}(\mathfrak{y}) \mathbf{e}^*_{l',s} \in N_R^{G_{\rm bulk}}(k). \tag{4.24}$$

Moreover

$$\mathfrak{b}(k+1) - \mathfrak{b}(k) \equiv 0 \mod T^{\lambda_k^b} \Lambda_0. \tag{4.25}$$

Proof. We prove this by induction over k. The case $k = 0$ follows from Lemma 4.17. ($\mathfrak{b}(0) = 0$.)

Suppose we have found $\mathfrak{b}(k)$ as in the proposition. Then we have

$$\sum_{l'=1}^{K} \sum_{s=1}^{d(l')} \mathfrak{y}_{l',s} \frac{\partial \mathfrak{PO}^u_{\mathfrak{b}(k)}}{\partial y_{l',s}}(\mathfrak{y}) \mathbf{e}^*_{l',s} \equiv \sum_{l=1}^{K} \sum_{s=1}^{d(l)} c_{l,s,k} T^{S_l + \lambda_k^b} \mathbf{e}^*_{l,s} \mod N_R^{G_{\rm bulk}}(k+1)$$

with $c_{l,s,k} \in R$.

Since $\{\vec{v}_{i(l',r)} \mid l' \le l\}$ spans $A_l^{\perp}$ for all $l \le K$ by definition, we can find $a_{l,r,k} \in R$ such that

$$\sum_{s=1}^{d(l)} c_{l,s,k} \mathbf{e}^*_{l,s} - \sum_{r=1}^{a(l)} a_{l,r,k} \vec{v}_{i(l,r)} \in A_{l-1}^{\perp}.$$

Therefore by definition of $N_R^{G_{\rm bulk}}(k)$ we have

$$\sum_{l=1}^{K} \sum_{s=1}^{d(l)} c_{l,s,k} T^{S_l + \lambda_k^b} \mathbf{e}^*_{l,s} - \sum_{l=1}^{K} \sum_{r=1}^{a(l)} a_{l,r,k} T^{S_l + \lambda_k^b} \vec{v}_{i(l,r)} \in N_R^{G_{\rm bulk}}(k+1).$$

Thus

$$\begin{aligned} &\sum_{l'=1}^{K} \sum_{s=1}^{d(l')} \mathfrak{y}_{l',s} \frac{\partial \mathfrak{PO}^u_{\mathfrak{b}(k)}}{\partial y_{l',s}}(\mathfrak{y}) \mathbf{e}^*_{l',s} \\ &\equiv \sum_{l=1}^{K} \sum_{r=1}^{a(l)} a_{l,r,k} T^{S_l + \lambda_k^b} \vec{v}_{i(l,r)} \mod N_R^{G_{\rm bulk}}(k+1) \end{aligned} \tag{4.26}$$

We now put

$$b_{l,r}(k+1) = b_{l,r}(k) - T^{\lambda_k^b} a_{l,r,k} (\mathfrak{y}^{\vec{v}_{i(l,r)}})^{-1}.$$

Lemma 4.20. *Let $\mathfrak{b}(k)$ be as in the induction hypothesis above. If $\lambda = \lambda_k^b$, then the second and the third terms of* (4.20) *are contained in $\Lambda_0^{G_{\rm bulk}}\{y_{**}, y_{**}^{-1}\}_{k+1}$.*

Proof. We first consider

$$c_h T^{h\lambda_k^b + \ell_{i(l,r)}(u)} y^{\vec{v}_{i(l,r)}} \tag{4.27}$$

which is in the second term of (4.20). ($h \ge 2$.) We remark that

$$c_h y^{\vec{v}_{i(l,r)}} \in R[y_{1,1}, y_{1,1}^{-1}, \cdots, y_{l,d(l)}, y_{l,d(l)}^{-1}].$$

On the other hand,

$$h\lambda_k^b + \ell_{i(l,r)}(u) - S_l = h\lambda_k^b$$

is contained in $G_{\rm bulk}$ and so must be equal to $\lambda^b_{k'}$ for some $k' > k$ since $h \ge 2$. Therefore (4.27) is contained in $\Lambda_0^{G_{\rm bulk}}\{y_{**}, y_{**}^{-1}\}_{k+1}$, as required.

We next consider

$$c_{h,\sigma} T^{h\lambda^b_k + \ell'_\sigma(u) + \rho_\sigma} y^{\vec v_\sigma} \tag{4.28}$$

which is in the third term of (4.20). ($h \ge 1$.) We have $\vec v_\sigma = \sum e^i_\sigma \vec v_i$, $\ell'_\sigma = \sum e^i_\sigma \ell_i$. We put

$$l_0 = \sup\{l \mid \exists r\ e_\sigma^{i(l,r)} \ne 0\}.$$

Then

$$c_{h,\sigma} y^{\vec v_\sigma} \in R[y_{1,1}, y_{1,1}^{-1}, \cdots, y_{l_0,d(l_0)}, y_{l_0,d(l_0)}^{-1}].$$

On the other hand, since

$$\ell'_\sigma(u) = \sum_i e^i_\sigma \ell_i(u) \ge \ell_{i(l_0,r)}(u) = S_{l_0}$$

it follows that

$$\ell'_\sigma(u) + \rho_\sigma - S_{l_0} \in G_{\rm bulk} \setminus \{0\}.$$

Therefore

$$h\lambda^b_k + \ell'_\sigma(u) + \rho_\sigma - S_{l_0} > \lambda^b_k$$

and so equal to $\lambda^b_{k'}$ for some $k' > k$. Hence (4.28) is contained in $\Lambda_0^{G_{\rm bulk}}\{y_{**}, y_{**}^{-1}\}_{k+1}$, as required.

The proof of Lemma 4.20 is complete. □

Then Proposition 4.15, (4.26), Lemma 4.18 and Lemma 4.20 imply that (4.24) is satisfied for $k+1$. The proof of Proposition 4.19 is complete. □

Now we are ready to complete the proof of Theorem 4.7. By (4.25)

$$\lim_{k\to\infty} \mathfrak b(k) = \mathfrak b$$

converges. Then (4.24) implies

$$\mathfrak y_{l,s} \frac{\partial \mathfrak{PO}^u_{\mathfrak b}}{\partial y_{l,s}}(\mathfrak y) = 0,$$

as required. □

We next show that the proof of Theorem 4.7 also provides a way to calculate bulk $\mathfrak{PO}$-threshold $\overline{\mathfrak E}^{\rm bulk}(L(u))$ from the leading term equation.

Theorem 4.21. *The following two conditions for $\mathcal N$ are equivalent to each other.*

1. *There exists $(\mathfrak b, b) \in \mathcal A(\Lambda_+) \times H^1(L(u);\Lambda_0)$ such that*
$$HF((L(u_0),\mathfrak b,\mathfrak x),(L(u_0),\mathfrak b,\mathfrak x);\Lambda_0/T^{\mathcal N}) \cong H(T^n;\Lambda_0/T^{\mathcal N}). \tag{4.29}$$
2. *We put $l_0 = \max\{l \mid S_l \le \mathcal N\}$. Then there exist $\mathfrak y_{l,j} \in R \setminus \{0\}$ for $l \le l_0$, $j = 1,\cdots,d(l)$ which solve the leading term equation (4.9) for $l \le l_0$.*

Corollary 4.22. *If the statement (2) of Theorem 4.21 holds then $\overline{\mathfrak E}^{\rm bulk}(L(u)) \ge 2\pi\mathcal N$.*

Proof of Theorem 4.21. The proof of (1) ⇒ (2) is similar to one in Theorem 4.7.

If (2) is satisfied, we can repeat the proof of Theorem 4.7 up to the order $\mathcal N$ to find $\mathfrak b$ such that (3.20) is satisfied. Then, (1) follows from Theorem 3.12. □

5. Two points blow up of $\mathbb{C}P^2$: an example

Our main example is the two-points blow up X_2 of $\mathbb{C}P^2$. We take its Kähler form $\omega_{\alpha,\beta}$ such that the moment polytope is

$$P_{\alpha,\beta} = \{(u_1,u_2) \mid 0 \le u_1 \le 1, 0 \le u_2 \le 1-\alpha, \beta \le u_1+u_2 \le 1\}. \tag{5.1}$$

Here

$$(\alpha,\beta) \in \Delta = \{(\alpha,\beta) \mid 0 \le \alpha, \beta, \quad \alpha+\beta \le 1\}. \tag{5.2}$$

We remark that $\mathbb{R}_+\Delta$ is the Kähler cone of X_2.

In Example 9.17 [FOOO3] we studied this example in the case

$$\beta = \frac{1-\alpha}{2}, \quad \frac{1}{3} < \alpha. \tag{5.3}$$

We continue the study this time involving bulk deformations.

We consider the point

$$\mathbf{u} = (u,\beta)\,, \qquad u \in \left(\beta, \frac{1-\beta}{2}\right) \tag{5.4}$$

and compute

$$\mathfrak{PO}^{\mathbf{u}}(0;y_1,y_2) = T^{\beta}(y_2+y_2^{-1}) + T^{u}(y_1+y_1y_2) + T^{1-\beta-u}y_1^{-1}y_2^{-1}. \tag{5.5}$$

We note that (5.4) implies

$$\beta < u < 1-\beta-u. \tag{5.6}$$

Therefore the leading term equation is

$$1-y_2^{-2} = 0, \qquad 1+y_2 = 0. \tag{5.7}$$

Namely $(y_1,-1)$ is its solution for any y_1. Therefore Theorem 4.7 implies :

Proposition 5.1. *$L(\mathbf{u}) \subset (X_2,\omega_{\alpha,\beta})$ is bulk-balanced if (5.3) and (5.4) are satisfied.*

Theorem 1.1 for $k=2$ will then follow from Proposition 3.15.

Proof of Theorem 1.1. The case $k=2$ (the two points blow up) is already proved. We consider $k=3$. We blow up $(X_2,\omega_{\alpha,\beta})$ at the fixed point corresponding to $(1,0) \in P_{\alpha,\beta}$. Then we have a toric Kähler structure on X_3 whose moment polytope is

$$\{(u_1,u_2) \in P_{\alpha,\beta} \mid u_1 \le 1-\epsilon\}.$$

We have

$$\begin{aligned}\mathfrak{PO}_0^{\mathbf{u}}(y_1,y_2) = T^{\beta}(y_2+y_2^{-1}) &+ T^{u}(y_1+y_1y_2)\\ &+ T^{1-\beta-u}y_1^{-1}y_2^{-1} + T^{1-\epsilon-u}y_1^{-1}.\end{aligned} \tag{5.8}$$

We remark that

$$1-\beta-u < 1-\epsilon-u$$

if ϵ is sufficiently small. Therefore the leading term equation at (5.4) is again (5.7). Therefore we can apply Theorem 4.7 to show that all $L(\mathbf{u})$ satisfying (5.4) are bulk-balanced. Thus Theorem 1.1 is proved for $k=3$.

We can blow up again at the fixed point corresponding to $(1-\epsilon,0)$. We can then prove the case $k=4$. (We remark that this time our toric manifold is not Fano. We never used the property X to be Fano in the above discussion.) We can continue arbitrary many times to complete the proof of Theorem 1.1. □

Below we will examine the effect of bulk deformations more explicitly for the example of two points blow up. We consider the divisor

$$D_1 = \pi^{-1}(\{(u_1,u_2) \in P \mid u_2 = 0\})$$

and let

$$\mathfrak{b}_{w,\kappa} = wT^{\kappa}[D_1] \in \mathcal{A}^2(\Lambda_+). \tag{5.9}$$

By Proposition 4.9, we have :

$$\begin{aligned}\mathfrak{PO}^{\mathbf{u}}(\mathfrak{b}_{w,\kappa}; y_1, y_2) =& T^{\beta}(\exp(\mathfrak{b}_{w,\kappa})y_2 + y_2^{-1}) \\ &+ T^{u}(y_1 + y_1y_2) + T^{1-\beta-u}y_1^{-1}y_2^{-1}.\end{aligned} \tag{5.10}$$

We study the equation

$$\frac{\partial \mathfrak{PO}^{\mathbf{u}}}{\partial y_1}(\mathfrak{b}_{w,\kappa}; y_1, y_2) = \frac{\partial \mathfrak{PO}^{\mathbf{u}}}{\partial y_2}(\mathfrak{b}_{w,\kappa}; y_1, y_2) = 0. \tag{5.11}$$

We put $y_2 = -1 + cT^{\mu}$, $y_1 = d$, with $c, d \in \Lambda_0 \setminus \Lambda_+$. Taking the inequality (5.6) into account, we obtain

$$\begin{aligned} cT^{\mu} + d^{-2}T^{1-\beta-2u} &\equiv 0 \mod T^{\max\{\mu, 1-\beta-2u\}} \\ -2cT^{\mu} + wT^{\kappa} + dT^{u-\beta} &\equiv 0 \mod T^{\max\{\mu,\kappa,u-\beta\}}. \end{aligned} \tag{5.12}$$

(Case 1) $\mu = \kappa < u - \beta$.

We have $c = w/2$, $\mu = 1 - \beta - 2u$. $d = \pm\sqrt{-2/w}$. $u = (1-\beta)/2 - \kappa/2 = (1+\alpha)/4 - \kappa/2$. It implies $1/3 < u < (1+\alpha)/4$. The equation for (c,d) has 2 solutions. They are both simple. Hence in the same way as the proof of Theorem 9.4 [FOOO3] (the strongly non-degenerate case) we can show that these two solutions correspond to the solutions of (5.11).

(Case 2) $\mu = u - \beta < \kappa$.

We have $d = 2c$, $1 - \beta - 2u = \mu$. Hence $u = 1/3$. We can show that there are 3 solutions of (5.11) in the same way.

(Case 3) $\kappa = u - \beta < \mu$.

We have $d = -w$. Then $\mu = 1 - \beta - 2u$. $c = -w^{-2}$. Hence $u < 1/3$. We can show that there is 1 solution of (5.11) in the same way.

(Case 4) $\kappa = u - \beta = \mu$.

We have $-2c + w + d = 0$ and $1 - \beta - 2u = \mu$. Hence $u = 1/3$. $\kappa = \alpha/2 - 1/6$.

$$d^2(d+w) + 2 = 0. \tag{5.13}$$

This has three simple roots unless

$$\frac{4}{27}w^3 + 2 = 0. \tag{5.14}$$

When κ is small Case 1 and Case 3 occur. There are two fibers with nontrivial Floer cohomology (on (5.4)), that is $((\beta+\kappa,\beta)$ and $((1+\alpha)/4 - \kappa/2, \beta))$. They move from (β,β), $(1+\alpha)/4, \beta)$ to $(1/3,\beta)$. Then, when $\kappa = \alpha/2 - 1/6$, Case 4 occurs. If $\kappa > \alpha/2 - 1/6$ then Case 2 occurs and bulk deformation does not change the 'secondary' leading term equation (5.13).

It might be interesting to observe that it actually occurs that the 'secondary' leading term equation (5.13) has multiple roots. That is the case where (5.14) is satisfied. (We remark that the example where there is a multiple root for the leading term equation was found in [OsTy].)

6. Operator $\mathfrak{q}$ in the toric case

In this section and the next, we study the moduli space of holomorphic discs and its effects on the operator $\mathfrak{q}$ and on the potential function $\mathfrak{PO}^u(\mathfrak{b}; y_1, \cdots, y_n)$.

Let $u \in \operatorname{Int} P$ and $\beta \in H_2(X, L(u); \mathbb{Z})$. We denote by $\mathcal{M}^{\text{main}}_{k+1;\ell}(L(u), \beta)$ the moduli space of stable maps from bordered Riemann surfaces of genus zero with $k+1$ boundary marked points and ℓ interior marked points, in homology class β. (See section 3 [FOOO1]. We require the boundary marked points to respect the cyclic order of $S^1 = \partial D^2$. (In other words, we consider the *main component* in the sense of section 3 [FOOO1].)) We assume $k \geq 0$. Then $\mathcal{M}^{\text{main}}_{k+1;\ell}(L(u), \beta)$ is compact. (See sections 13.2 and 32.1 [FOOO2], for the reason why we need to assume $k \geq 0$ for compactness.)

We denote an element of $\mathcal{M}^{\text{main}}_{k+1;\ell}(L(u), \beta)$ by

$$(\Sigma, \varphi, \{z_i^+ \mid 1 = 1, \cdots, \ell\}, \{z_i \mid i = 0, 1, \cdots, k\})$$

where Σ is a connected genus zero bordered semi-stable curve, $\varphi : (\Sigma, \partial\Sigma) \to (X, L(u))$ is a holomorphic map and $z_i^+ \in \operatorname{Int}\Sigma$ and $z_i \in \partial\Sigma$. Let $\mathcal{M}^{\text{main,reg}}_{k+1;\ell}(L(u), \beta)$ be its subset consisting of all maps from a *smooth* disc. (Namely the stable map without disc or sphere bubble.)

We have the following proposition. Let $\beta_i \in H_2(X, L(u); \mathbb{Z})$ $(i = 1, \cdots, m)$ be the classes with $\mu(\beta_i) = 2$ and

$$\beta_i \cap D_j = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

We recall from [CO] that the spin structure of $L(u)$ induced from the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ as its orbit is called the *standard* spin structure.

Proposition 6.1. (1) *If $\mu(\beta) < 0$, or $\mu(\beta) = 0$, $\beta \neq 0$, then $\mathcal{M}^{\text{main,reg}}_{k+1;\ell}(L(u), \beta)$ is empty.*

(2) *If $\mu(\beta) = 2$, $\beta \neq \beta_1, \cdots, \beta_m$, then $\mathcal{M}^{\text{main,reg}}_{k+1;\ell}(L(u), \beta)$ is empty.*

(3) *For $i = 1, \cdots, m$, we have*

$$\begin{aligned} \mathcal{M}^{\text{main,reg}}_{1;0}(L(u), \beta_i) &= \mathcal{M}^{\text{main}}_{1;0}(L(u), \beta_i), \\ \mathcal{M}^{\text{main}}_{1;\ell}(L(u), \beta_i) &= \mathcal{M}^{\text{main,reg}}_{1;0}(L(u), \beta_i) \times \operatorname{Conf}(\ell; D^2). \end{aligned} \tag{6.1}$$

Here $\operatorname{Conf}(\ell; D^2)$ is a compactification of

$$\{(z_1^+, \cdots, z_\ell^+) \mid z_i^+ \in \operatorname{Int}D^2, z_i^+ \neq z_j^+ \text{ for } i \neq j.\}$$

(See Remark 6.2.) Moreover $\mathcal{M}^{\text{main}}_{1;0}(L(u), \beta_i)$ is Fredholm regular. Furthermore the evaluation map

$$ev : \mathcal{M}^{\text{main}}_{1;0}(L(u), \beta_i) \to L(u)$$

is an orientation preserving diffeomorphism if we equip $L(u)$ with the standard spin structure.

(4) *For any β, the moduli space $\mathcal{M}^{\text{main,reg}}_{1;\ell}(L(u), \beta)$ is Fredholm regular. Moreover*

$$ev : \mathcal{M}^{\text{main,reg}}_{1;\ell}(L(u), \beta) \to L(u)$$

is a submersion.

(5) *If $\mathcal{M}^{\rm main}_{1;\ell}(L(u),\beta)$ is not empty then there exist $k_i \in \mathbb{Z}_{\ge 0}$ and $\alpha_j \in H_2(X;\mathbb{Z})$ such that*
$$\beta = \sum_i k_i\beta_i + \sum_j \alpha_j$$
and α_j is realized by a holomorphic sphere. There is at least one nonzero k_i.

Remark 6.2. We define the compactification of $\mathrm{Conf}(\ell;D^2)$ as follows. We consider $X=\mathbb{C}$, $L=S^1$. Let β_1 be the generator of $H_2(X;L)$ which is represented by a holomorphic disc. Then, clearly $\mathcal{M}^{\rm reg}_{0;\ell}(L;\beta_1)$ is identified with $\mathrm{Conf}(\ell;D^2)$. Hence $\mathcal{M}_{0;\ell}(L;\beta_1)$ is a compactification of it. We use this compactification.

Proposition 6.1 follows easily from Theorem 10.1 [FOOO3] which in turn follows from [CO] as we explained in section 10 [FOOO3].

We next discuss Kuranishi structure of $\mathcal{M}^{\rm main}_{k+1;\ell}(L(u),\beta)$. In section 17, 18 [FOOO1] or section 29 [FOOO2], we defined a Kuranishi structure on $\mathcal{M}^{\rm main}_{k+1;\ell}(L(u),\beta)$. In our toric case, this structure can be chosen to be T^n equivariant in the following sense. Let (V,E,Γ,ψ,s) be a Kuranishi chart (see section 5 [FO] and section A1 [FOOO2]). Here $V\subset\mathbb{R}^N$ is an open set with a linear action of a finite group Γ, $E\to V$ is a Γ equivariant vector bundle, s its Γ-equivariant section and $\psi: s^{-1}(0)/\Gamma \to \mathcal{M}^{\rm main}_{k+1;\ell}(L(u),\beta)$ is a homeomorphism onto an open set. Then we have a T^n action on V,E which commutes with Γ action, such that s is T^n equivariant. Moreover ψ is T^n equivariant. Here the T^n action is induced by one on X. (We recall that $L(u)$ is T^n invariant.) The construction of such Kuranishi structure is obvious from construction given in section 29 [FOOO2]. We use the T^n equivariance of ψ and the fact that T^n action is free on $L(u)$ to conclude that the T^n action on V is free.

Let $\{D_a \mid a=1,\cdots,B\}$ be the basis of $\mathcal{A}(\mathbb{Z})$. (Each D_a corresponds to a face of P.) We note that each of D_a is a T^n invariant submanifold. Let
$$ev_i^{\rm int} : \mathcal{M}^{\rm main}_{k+1;\ell}(L(u),\beta)\to X$$
be the evaluation map at the i-th interior marked point. ($i=1,\cdots,\ell$.) Namely
$$ev_i^{\rm int}((\Sigma,\varphi,\{z_i^+\},\{z_i\})) = \varphi(z_i^+).$$
We put $\underline{B}=\{1,\cdots,B\}$. We denote by $Map(\ell,\underline{B})$ the set of all maps $\mathbf{p}:\{1,\cdots,\ell\}\to\underline{B}$. We write $|\mathbf{p}|=\ell$ if $\mathbf{p}\in Map(\ell,\underline{B})$.

We define a fiber product
$$\mathcal{M}^{\rm main}_{k+1;\ell}(L(u),\beta;\mathbf{p}) = \mathcal{M}^{\rm main}_{k+1;\ell}(L(u),\beta)\,{}_{(ev_1^{\rm int},\cdots,ev_\ell^{\rm int})}\times_{X^\ell} \prod_{i=1}^{\ell} D_{\mathbf{p}(i)}. \tag{6.2}$$
Here the right hand side is the set of all $((\Sigma,\varphi,\{z_i^+\},\{z_i\}),(p_1,\cdots,p_\ell))$ such that $(\Sigma,\varphi,\{z_i^+\},\{z_i\})\in\mathcal{M}^{\rm main}_{k+1;\ell}(L(u),\beta)$, $p_i\in D_{\mathbf{p}(i)}$, and that $\varphi(z_i^+)=p_i$.

We define
$$ev_i : \mathcal{M}^{\rm main}_{k+1;\ell}(L(u),\beta)\to L(u)$$
by
$$ev_i((\Sigma,\varphi,\{z_i^+\},\{z_i\})) = \varphi(z_i).$$
It induces
$$ev_i : \mathcal{M}^{\rm main}_{k+1;\ell}(L(u),\beta;\mathbf{p})\to L(u)$$
in an obvious way.

Lemma 6.3. *$\mathcal{M}^{\text{main}}_{k+1;\ell}(L(u),\beta;\mathbf{p})$ has a Kuranishi structure such that each Kuranishi chart is T^n-equivariant and the coordinate change preserves the T^n action. Moreover the evaluation map*

$$ev = (ev_0, ev_1, \cdots, ev_k) : \mathcal{M}^{\text{main}}_{k+1;\ell}(L(u),\beta;\mathbf{p}) \to L(u)^{k+1}$$

is weakly submersive and T^n-equivariant. Our Kuranishi structure has a tangent bundle and is oriented.

Proof. The fiber product of Kuranishi structures is defined in section A1.2 [FOOO2]. Since the maps we used here to define the fiber product are all T^n-equivariant it follows that the Kuranishi structure on the fiber product is T^n-equivariant. The orientability is proved in Chapter 9 [FOOO2]. The fact that ev is well defined and is weakly submersive is proved in section 29 [FOOO2] also. □

We next describe the boundary of our Kuranishi structure. For the description, we need to prepare some notations. We denote the set of shuffles of ℓ elements by

$$\text{Shuff}(\ell) = \{(\mathbb{L}_1, \mathbb{L}_2) \mid \mathbb{L}_1 \cup \mathbb{L}_2 = \{1, \cdots, \ell\},\ \mathbb{L}_1 \cap \mathbb{L}_2 = \emptyset\}. \tag{6.3}$$

We will define a map

$$\text{Split} : \text{Shuff}(\ell) \times Map(\ell, \underline{B}) \longrightarrow \bigcup_{\ell_1+\ell_2=\ell} Map(\ell_1, \underline{B}) \times Map(\ell_2, \underline{B}), \tag{6.4}$$

as follows : Let $\mathbf{p} \in Map(\ell, \underline{B})$ and $(\mathbb{L}_1, \mathbb{L}_2) \in \text{Shuff}(\ell)$. We put $\ell_j = \#(\mathbb{L}_j)$ and let $\mathfrak{i}_j : \{1, \cdots, \ell_j\} \cong \mathbb{L}_j$ be the order preserving bijection. We consider the map $\mathbf{p}_j : \{1, \cdots, \ell_j\} \to \underline{B}$ defined by $\mathbf{p}_j(i) = \mathbf{p}(\mathfrak{i}_j(i))$, and set

$$\text{Split}((\mathbb{L}_1, \mathbb{L}_2), \mathbf{p}) := (\mathbf{p}_1, \mathbf{p}_2).$$

We now define a gluing map, associated to

$$\begin{aligned}\text{Glue}^{(\mathbb{L}_1,\mathbb{L}_2),\mathbf{p}}_{\ell_1,\ell_2;k_1,k_2;i;\beta_1,\beta_2} :& \mathcal{M}^{\text{main}}_{k_1+1;\ell_1}(L(u),\beta_1;\mathbf{p}_1)\,{}_{ev_0}\times_{ev_i} \mathcal{M}^{\text{main}}_{k_2+1;\ell_2}(L(u),\beta_2;\mathbf{p}_2) \\ &\to \mathcal{M}^{\text{main}}_{k+1;\ell}(L(u),\beta;\mathbf{p})\end{aligned} \tag{6.5}$$

below. Here $k = k_1 + k_2 - 1$, $\ell = \ell_1 + \ell_2$, $\beta = \beta_1 + \beta_2$, and $i = 1, \cdots, k_2$. Let

$$\mathbb{S}_j = ((\Sigma_{(j)}, \varphi_{(j)}, \{z^+_{i,(j)}\}, \{z_{i,(j)}\}) \in \mathcal{M}^{\text{main}}_{k_j+1;\ell_j}(L(u),\beta_j;\mathbf{p}_j)$$

$j = 1, 2$. We glue $z_{0,(1)} \in \partial\Sigma_1$ with $z_{i,(2)} \in \partial\Sigma_2$ to obtain

$$\Sigma = \Sigma_1 \#_i \Sigma_2.$$

Suppose $(\mathbb{S}_1, \mathbb{S}_2)$ is an element of the fiber product in the left hand side of (6.5). Namely we assume

$$\varphi_{(1)}(z_{0,(1)}) = \varphi_{(2)}(z_{i,(2)}).$$

This defines a holomorphic map

$$\varphi = \varphi_{(1)} \#_i \varphi_{(2)} : \Sigma \to X$$

by putting $\varphi = \varphi_{(j)}$ on Σ_j.

Let $m \in \mathbb{L}_j$. Then $\mathfrak{i}_j(c) = m$, $\mathfrak{i}_j : \{1, \cdots, \ell_j\} \cong \mathbb{L}_j$ be the order preserving bijection. We define the m-th interior marked point z^{int}_m of φ as $z^{\text{int}}_{c;(j)} \in \Sigma_j \subset \Sigma$. We define the boundary marked points $(z_0, z_1, \cdots, z_k)$ by

$$(z_0, z_1, \cdots, z_k) = (z_{0,(2)}, \cdots, z_{i-1,(2)}, z_{1,(1)}, \cdots, z_{k_1,(1)}, z_{i+1,(2)}, \cdots, z_{k_2,(2)}).$$

Now we put

$$\mathbb{S} = ((\Sigma, \varphi, \{z^+_i\}, \{z_i\})$$

and

$$\mathrm{Glue}^{(\mathbb{L}_1,\mathbb{L}_2),\mathbf{p}}_{\ell_1,\ell_2;k_1,k_2;i;\beta_1,\beta_2}(\mathbb{S}_1,\mathbb{S}_2)=\mathbb{S}.$$

Lemma 6.4. *The boundary of $\mathcal{M}^{\mathrm{main}}_{k+1;\ell}(L(u),\beta;\mathbf{p})$ is isomorphic to the union of the images of $\mathrm{Glue}^{(\mathbb{L}_1,\mathbb{L}_2),\mathbf{p}}_{\ell_1,\ell_2;k_1,k_2;i;\beta_1,\beta_2}$ for $k=k_1+k_2-1$, $\ell=\ell_1+\ell_2$, $\beta=\beta_1+\beta_2$, and $i=1,\cdots,k_2$ as a space with Kuranishi structure. The isomorphism preserves the T^n action.*

The isomorphism commutes with the evaluation maps at the boundary marked points.

The lemma directly follows from our construction of the Kuranishi structure we gave in section 29 [FOOO2].

Let $\mathfrak{S}_\ell$ be the symmetric group of ℓ elements. It acts on $\mathcal{M}^{\mathrm{main}}_{k+1;\ell}(L(u),\beta)$ by changing the indices of interior marked points. It also acts on $Map(\ell,\underline{B})$ by $\sigma\cdot\mathbf{p}=\mathbf{p}\circ\sigma^{-1}$. Then for $\sigma\in\mathfrak{S}_\ell$ we have

$$\sigma_*:\mathcal{M}^{\mathrm{main}}_{k+1;\ell}(L(u),\beta;\mathbf{p})\to\mathcal{M}^{\mathrm{main}}_{k+1;\ell}(L(u),\beta;\sigma\cdot\mathbf{p}). \tag{6.6}$$

We next generalize Lemma 10.2 [FOOO3] to our situation. Let

$$\mathfrak{forget}_0:\mathcal{M}^{\mathrm{main}}_{k+1;\ell}(L(u),\beta;\mathbf{p})\to\mathcal{M}^{\mathrm{main}}_{1;\ell}(L(u),\beta;\mathbf{p}) \tag{6.7}$$

be the forgetful map which forgets all the boundary marked points except the 0-th one. We may choose our Kuranishi structures so that (6.7) is compatible with $\mathfrak{forget}_0$ in the same sense as Lemma 31.8 [FOOO2].

Lemma 6.5. *For each given $E>0$ and ℓ_0, there exists a system of multisections $\mathfrak{s}_{\beta,k+1,\ell,\mathbf{p}}$ on $\mathcal{M}^{\mathrm{main}}_{k+1;\ell}(L(u),\beta;\mathbf{p})$ for $\beta\cap\omega<E$, $\ell\le\ell_0$, $\mathbf{p}\in Map(\ell,\underline{B})$. They have the following properties :*

1. *They are transversal to* 0.
2. *They are invariant under the T^n action.*
3. *The multisection $\mathfrak{s}_{\beta,k+1,\ell,\mathbf{p}}$ is the pull-back of the multisection $\mathfrak{s}_{\beta,1,\ell,\mathbf{p}}$ by the forgetful map* (6.7).
4. *The restriction of $\mathfrak{s}_{\beta,k+1,\ell,\mathbf{p}}$ to the image of $\mathrm{Glue}^{(\mathbb{L}_1,\mathbb{L}_2),\mathbf{p}}_{\ell_1,\ell_2;k_1,k_2;i;\beta_1,\beta_2}$ is the fiber product of the multisections $\mathfrak{s}_{\beta_j,k_j+1,\ell_j,\mathbf{p}_j}$ $j=1,2$ with respect to the identification of the boundary given in Lemma* 6.4.
5. *For $\ell=0$ the multisection $\mathfrak{s}_{\beta,k+1,0,\emptyset}$ coincides with one defined in Lemma* 10.2 [FOOO3].
6. *The map* (6.6) *preserves our system of multisections.*

Proof. The proof is similar to the proof of Lemma 10.2 [FOOO3]. We define $\mathfrak{s}_{\beta,k+1,\ell,\mathbf{p}}$ for $\mathbf{p}\in Map(\ell,\underline{B})$ by a double induction over ℓ and $\omega\cap\beta$. The case $\ell=0$ is proved in Lemma 10.2 [FOOO3]. Condition (4) above determines the multisection on the boundary. T^n equivariance implies that $ev_0:\mathcal{M}^{\mathrm{main,reg}}_{k+1;\ell}(L(u),\beta;\mathbf{p})^{\mathfrak{s}_{\beta,k+1,\ell,\mathbf{p}}}\to L(u)$ is a submersion. Here

$$\mathcal{M}^{\mathrm{main,reg}}_{k+1;\ell}(L(u),\beta;\mathbf{p})^{\mathfrak{s}_{\beta,k+1,\ell,\mathbf{p}}}=(\mathfrak{s}_{\beta,k+1,\ell,\mathbf{p}})^{-1}(0).$$

This fact and the induction hypothesis imply that the multisection we defined by (4) on the boundary of our moduli space is automatically transversal. (This is the important point that makes the proof of Lemma 6.5 easier than corresponding general discussion given in section 30 [FOOO2]. See section 10 [FOOO3] for more discussion about this point.)

Thus we have defined a multisection on a neighborhood of the boundary. We can extend it to the interior so that it satisfies (1) and (2) in the following way : We first take the quotient $(V/T^n, E/T^n)$ of our Kuranishi chart. Since the T^n action is free on V the quotient space is a manifold on which Γ acts. Thus we can use the standard result of the theory of Kuranishi structure to define a transversal multisection on this chart where the multisection is already defined. We lift it to V and obtain a required multisection there. In this way we can construct the multisection inductively on the Kuranishi charts using the good coordinate system. (See section A1 [FOOO2].)

To show (6) it suffices to take the quotient by the action of symmetric group and work out the induction on the quotient spaces. The proof of Lemma 6.5 is now complete. □

Corollary 6.6. $\mathcal{M}^{\text{main}}_{k+1;\ell}(L(u),\beta;\mathbf{p})^{\mathfrak{s}_{\beta,k+1,\ell,\mathbf{p}}}$ *is empty, if one of the following conditions are satisfied.*

(1) $\mu(\beta)-\sum_i(2n-\dim D_{\mathbf{p}_i}-2)<0$.
(2) $\mu(\beta)-\sum(2n-\dim D_{\mathbf{p}_i}-2)=0$ *and* $\beta\neq 0$.

Proof. We may assume $k=0$, by Lemma 6.5 (3).

We first consider the case of $\beta=0$. All the holomorphic curves in this homotopy class are constant maps. Then our moduli space is empty for $\ell>0$, since $L(u)\cap D=\emptyset$. This implies the lemma for the case $\beta=0$.

We next consider the case $\beta\neq 0$. The virtual dimension of $\mathcal{M}^{\text{main}}_{1;\ell}(L(u),\beta;\mathbf{p})$ (which is, by definition, its dimension as a space with Kuranishi structure) is

$$n+\mu(\beta)-\sum(2n-\dim D_{\mathbf{p}_i}-2)-2. \tag{6.8}$$

By the transversality (Lemma 6.5 (1)) and T^n equivariance (Lemma 6.5 (2)), we find that (6.8) is not smaller than $\dim L(u)=n$ if the perturbed moduli space is nonempty. (We use $\beta\neq 0$ here : If $\beta=0$ the virtual dimension of $\mathcal{M}^{\text{main}}_{1;0}(L(u),\beta_0)$ is $n-2$ but it is nonempty.) This finishes the proof of the lemma for the case $\beta\neq 0$. □

We now assume

$$\mu(\beta)-\sum(2n-\dim D_{\mathbf{p}_i}-2)=2, \tag{6.9}$$

and $\beta\neq 0$. Then

$$\mathcal{M}^{\text{main}}_{1;\ell}(L(u),\beta;\mathbf{p})^{\mathfrak{s}_{\beta,1,\ell,\mathbf{p}}}$$

has a virtual fundamental *cycle*, by Corollary 6.6. We introduce the following invariant

Definition 6.7. We define $c(\beta;\mathbf{p})\in\mathbb{Q}$ by

$$c(\beta;\mathbf{p})[L(u)]=ev_{0*}([\mathcal{M}^{\text{main}}_{1;\ell}(L(u),\beta;\mathbf{p})^{\mathfrak{s}_{\beta,1,\ell,\mathbf{p}}}]).$$

Lemma 6.8. *The number* $c(\beta;\mathbf{p})$ *is independent of the choice of the system of multisections* $\mathfrak{s}_{\beta,k+1}$ *satisfying* (1) - (6) *of Proposition* 6.5.

The proof is the same as the proof of Lemma 10.7 [FOOO3] and so is omitted.

Remark 6.9. The independence of open Gromov-Witten invariant such as $c(\beta;\mathbf{p})$ was proved in [KL] by taking equivariant perturbations in the situation where an appropriate S^1-action exists.

We use the above moduli spaces to define the operators $\mathfrak{q}_{\beta;k,\ell}$ as follows. Let $\mathbf{p} \in Map(\ell, \underline{B})$. We put

$$D(\mathbf{p}) = D_{\mathbf{p}(1)} \otimes \cdots \otimes D_{\mathbf{p}(\ell)}.$$

Let $h_1, \cdots, h_k$ be differential forms on $L(u)$. We put

$$\sum(\deg h_i + 1) - \mu(\beta) + \sum(2n - \dim D_{\mathbf{p}_i} - 2) + 2 = d$$

where we note that

$$\begin{aligned}\deg[\mathcal{M}^{\rm main}_{1;\ell}(L(u), \beta; \mathbf{p})^{\mathfrak{s}_{\beta,1,\ell,\mathbf{p}}}] &= \operatorname{codim}[\mathcal{M}^{\rm main}_{1;\ell}(L(u), \beta; \mathbf{p})^{\mathfrak{s}_{\beta,1,\ell,\mathbf{p}}}] \\ &= -\mu(\beta) + \sum(2n - \dim D_{\mathbf{p}_i} - 2) + 2.\end{aligned}$$

(See (6.8).) We then define a differential form of degree d on $L(u)$ by

$$\mathfrak{q}^{dR}_{\beta;\ell,k}(D(\mathbf{p}); h_1, \cdots, h_k) = \frac{1}{\ell!}(ev_0)_!(ev_1, \cdots, ev_k)^*(h_1 \wedge \cdots \wedge h_k), \tag{6.10}$$

here ev_0, ev_i are the maps

$$(ev_0, \cdots, ev_k) : \mathcal{M}^{\rm main}_{k+1;\ell}(L(u), \beta; \mathbf{p})^{\mathfrak{s}_{\beta,1,\ell,\mathbf{p}}} \longrightarrow L(u)^{k+1}$$

and $(ev_0)_!$ is the integration along the fiber. More precisely we use (6.10) for $(\beta, \ell, k) \neq (0,0,0), (0,0,1)$ and we put

$$\mathfrak{q}_{0;0,1}(h) = (-1)^{n+\deg h+1} dh, \quad \mathfrak{q}_{0;0,2}(h_1, h_2) = (-1)^{\deg h_1(\deg h_2+1)} h_1 \wedge h_2.$$

We use T^n-equivariance to show that

$$ev_0 : \mathcal{M}^{\rm main}_{k+1;\ell}(L(u), \beta; \mathbf{p})^{\mathfrak{s}_{\beta,1,\ell,\mathbf{p}}} \to L(u)$$

is a proper submersion. Hence the integration along the fiber is well-defined and gives rise to smooth forms. (It is fairly obvious that the integration along the fiber on the zero set of a transversal multisection is well defined and that it satisfies Stokes' theorem. See section 12 [Fu3], section 33 [FOOO2] or section 12 of present paper.) Let $\Omega(L(u))$ be the de Rham complex of $L(u)$.

Definition 6.10. We put

$$\mathfrak{q}^{dR}_{\ell,k} = \sum_{\beta} T^{\omega\cap\beta/2\pi} \mathfrak{q}^{dR}_{\beta;\ell,k}.$$

By restricting $\mathfrak{q}^{dR}_{\ell,k}$ to $E_\ell\mathcal{A} \subset B_\ell\mathcal{A}$ we obtain

$$\mathfrak{q}^{dR}_{\beta;\ell,k} : E_\ell(\mathcal{A}[2]) \otimes B_k(\Omega(L(u))[1]) \to \Omega(L(u))[1]$$

of degree $1 - \mu(\beta)$ and

$$\mathfrak{q}^{dR}_{\ell,k} : E_\ell(\mathcal{A}(\Lambda_+)(\mathbb{R})) \otimes B_k((\Omega(L(u)) \widehat{\otimes} \Lambda_0(\mathbb{R}))[1]) \to (\Omega(L(u)) \widehat{\otimes} \Lambda_0(\mathbb{R}))[1].$$

Proposition 6.11. *$\mathfrak{q}^{dR}_{\beta;\ell,k}$ satisfies* (2.2).

Proof. For $\mathbf{p} \in Map(\ell, \underline{B})$, $(\mathbb{L}_1, \mathbb{L}_2) \in \mathrm{Shuff}(\ell)$ we put

$$\mathrm{Split}((\mathbb{L}_1, \mathbb{L}_2), \mathbf{p}) = (\mathrm{Split}((\mathbb{L}_1, \mathbb{L}_2), \mathbf{p})_1, \mathrm{Split}((\mathbb{L}_1, \mathbb{L}_2), \mathbf{p})_2)$$

It is easy to see that the coproduct $\Delta(D(\mathbf{p}))$ is given by the formula

$$\Delta(D(\mathbf{p})) = \sum_{(\mathbb{L}_1,\mathbb{L}_2)\in\mathrm{Shuff}(\ell)} \frac{\ell!}{\#|\mathbb{L}_1|!\#|\mathbb{L}_2|!} D(\mathrm{Split}((\mathbb{L}_1, \mathbb{L}_2), \mathbf{p})_1) \otimes D(\mathrm{Split}((\mathbb{L}_1, \mathbb{L}_2), \mathbf{p})_2).$$

Then (6.4) and (6.5) imply (2.2) in the same way as section 13 [FOOO2]. □

Now for $\mathfrak{b} \in \mathcal{A}^2(\Lambda_+)$, we define

$$\mathfrak{m}_k^{dR,\mathfrak{b}}(h_1, \cdots, h_k) = \mathfrak{q}(e^{\mathfrak{b}}; h_1, \cdots, h_k). \tag{6.11}$$

Here

$$e^{\mathfrak{b}} = 1 + \mathfrak{b} + \mathfrak{b} \otimes \mathfrak{b} + \cdots .$$

Proposition 6.11 implies that $\mathfrak{m}_k^{Dr,\mathfrak{b}}$ defines a structure of filtered A_∞ algebra on $\Omega(L(u))$.

Lemma 6.12. $(\Omega(L(u)) \widehat{\otimes} \Lambda_0(\mathbb{R}), \{\mathfrak{m}_k^{dR,\mathfrak{b}}\}_{k=0}^{\infty})$ *is homotopy equivalent to the filtered A_∞ algebra defined by* (2.6)*.*

The proof of the this lemma is the same as that of Lemma 37.55 [FOOO2] and is omitted here. We refer readers thereto for the details. In fact we do not need to use Lemma 6.12 for our applications. We can just use the de Rham version without involving the singular homology version.

We take a canonical model of $(\Omega(L(u)), \{\mathfrak{m}_k^{dR,\mathfrak{b}}\}_{k=0}^{\infty})$ to obtain a filtered A_∞ algebra $(H(L(u); \Lambda_0(\mathbb{R})), \{\mathfrak{m}_k^{dR,\mathfrak{b},can}\}_{k=0}^{\infty})$. The canonical model in the situation where we include bulk deformations can be defined also by using section 32 [FOOO2] as follows. In Corollary 32.40 [FOOO2] we reinterpreted the operator $\mathfrak{q}$ as follows : We define

$$\mathfrak{q}^o(D(\mathrm{op}(\mathbf{p})); x_1, \cdots, x_k) = (-1)^* \mathfrak{q}(D(\mathbf{p}); x_1, \cdots, x_k). \tag{6.12}$$

Here

$$(\mathrm{op}(\mathbf{p}))(i) = \mathbf{p}(\ell - i)$$

if $\mathbf{p} \in Map(\ell, \underline{B})$. We do not discuss sign $*$ here. In our case where $\deg D_i$ is even and $\deg x_i$ is odd, there is no such a sign factor, i.e., $(-1)^* = 1$. We regard $\mathfrak{q}^o$ as a homomorphism

$$E\mathcal{A}(\Lambda_+(\mathbb{R}))[2] \longrightarrow \bigoplus_k Hom(B_k(\Omega(L(u))[1]), \Omega(L(u))[1]) \widehat{\otimes} \Lambda_0(\mathbb{R}). \tag{6.13}$$

We regard $E\mathcal{A}(\Lambda_+(\mathbb{R}))[2]$ as a filtered L_∞ algebra with trivial operations. The right hand side of (6.13) is identified with Hochschild complex of differential graded algebra $\Omega(L(u))$. So it is a differential graded Lie algebra. Then the formula (2.6) which we proved in Lemma 6.11 is equivalent to saying that (6.13) is a filtered L_∞ homomorphism. This is Proposition 32.34 [FOOO2].

Let $H(L(u); \mathbb{R})$ be the de Rham cohomology of $L(u)$. Then it has the structure of A_∞ algebra.

Remark 6.13. We remark that in our case where $L(u)$ is a torus, this A_∞ structure is formal. Namely there is no higher operations $\mathfrak{m}_k$ for $k > 2$. In other words, The cohomology *ring*, $H(L(u); \mathbb{R})$, is homotopy equivalent to the de Rham complex as an A_∞ algebra defined by (6.10).

We identify

$$\bigoplus_k Hom(B_k(H(L(u); \mathbb{R})[1]), H(L(u); \mathbb{R})[1])$$

with the Hochschild complex of $H(L(u);\mathbb{R})$. In Theorem 32.41 [FOOO2] we defined a homotopy equivalence of L_∞ algebras

$$\begin{aligned}&\bigoplus_k Hom(B_k(\Omega(L(u))[1]),\Omega(L(u))[1])\\ &\longrightarrow \bigoplus_k Hom(B_k(H(L(u);\mathbb{R})[1]),H(L(u);\mathbb{R})[1]).\end{aligned} \tag{6.14}$$

(We will review some part of the construction of (6.14) at the beginning of the next section.)

Remark 6.14. The domain and the target of (6.14) are both differential graded Lie algebras. However the homotopy equivalence is one as an L_∞ algebra and is not as a differential graded Lie algebra homomorphism.

We compose two (filtered) L_∞ homomorphisms (6.13) and (6.14)$\otimes\Lambda_0(\mathbb{R})$ to obtain

$$\mathfrak{q}^{can,o} : E(\mathcal{A}(\Lambda_+(\mathbb{R}))[2]) \longrightarrow \bigoplus_k Hom(B_k(H(L(u);\Lambda_0(\mathbb{R}))[1]),H(L(u);\Lambda_0(\mathbb{R}))[1]).$$

We reinterpret $\mathfrak{q}^{can,o}$ in the opposite direction as (6.12) to obtain

$$\mathfrak{q}^{can}_{\ell,k} : E_\ell(\mathcal{A}(\Lambda_+(\mathbb{R}))[2]) \otimes B_k(H(L(u);\Lambda_0(\mathbb{R}))[1]) \to H(L(u);\Lambda_0(\mathbb{R}))[1].$$

It can be decomposed as

$$\mathfrak{q}^{can}_{\ell,k} = \sum_\beta T^{\beta\cap\omega/2\pi}\mathfrak{q}^{can}_{\beta;\ell,k}.$$

We use this to obtain a filtered A_∞ structure $\mathfrak{m}^{\mathfrak{b},can}_k = \mathfrak{q}^{can}_{*,k}(e^{\mathfrak{b}},\cdots)$ as in (6.11) on $H(L(u);\Lambda(\mathbb{R}))$ for $\mathfrak{b}\in\Lambda_+$.

7. Calculation of potential function with bulk

We define a potential function $\mathfrak{PO}^u$ as we discussed in section 3. In this section we study and will partially calculate it. The next lemma is used for this purpose.

Lemma 7.1. *Let $\mathfrak{x}\in H^1(L(u);\Lambda_+)$, $\beta\in H_2(X,L;\mathbb{Z})$, and $\mathbf{p}\in Map(\ell,\underline{B})$. We assume* (6.9)*. Then we have*

$$\mathfrak{q}^{can}_{\beta;\ell,k}(D(\mathbf{p});\mathfrak{x},\cdots,\mathfrak{x}) = \frac{c(\beta;\mathbf{p})}{\ell!k!}(\partial\beta\cap\mathfrak{x})^k\cdot PD([L(u)]).$$

where $PD([L(u)])$ is the Poincaré dual to the fundamental class $[L(u)]\in H_n(L(u);\mathbb{Z})$.

Proof. The proof is similar to that of Lemma 10.8 [FOOO3] and proceed as follows.

Let h be a harmonic representative of the class $\mathfrak{x}$. We first prove

$$\int_{L(u)}\mathfrak{q}^{dR}_{\beta;\ell,k}(D(\mathbf{p});h,\cdots,h) = \frac{c(\beta;\mathbf{p})}{\ell!k!}(\partial\beta\cap\mathfrak{x})^k. \tag{7.1}$$

The proof is the same as that of Formula (10.7) [FOOO3], using Definition 6.7.

We next use (7.1) to calculate operations in the canonical model. According to the construction of section 32.4 [FOOO2] and at the end of this section of this paper, the operator $q^{can}_{\beta;\ell,k}$ induced on the canonical model is a sum :

$$q^{can}_{\beta;\ell,k}(D(\mathbf{p});\mathfrak{x},\cdots,\mathfrak{x}) = \sum_\Gamma \mathfrak{q}_\Gamma(h,\cdots,h). \tag{7.2}$$

Explanation of the formula (7.2) is in order. The right hand side is a sum over Γ. Γ consists of the following data :

(1) $|\Gamma|$ is a tree together with an isotopy type of its embedding to $\mathbb{R}^2$. (It determines cyclic order to each of the sets of edges containing a vertex. In other words we fix its ribbon structure.)
(2) Each of the vertex $|\Gamma|$ is either exterior or interior.
(3) Each of the exterior vertices has only one edge. The set of the exterior vertices is numbered from 0 to k which respect the counter-clockwise cyclic order of $\mathbb{R}^2$, that is the order induced by the orientation of $\mathbb{R}^2$.
(4) Each of the interior vertices is either of the type Q or C. Let $C_0^Q(\Gamma)$, $C_0^C(\Gamma)$ be the set of interior vertices of type Q (= quantum) or C (= classical), respectively.
(5) The set $\{1, \cdots, \ell\}$ is divided into a *disjoint* union $\bigcup_{v \in C_0^Q(\Gamma)} I_v$.
(6) To the vertex v of type Q, an element $\beta_v \in H_2(X; L(u); \mathbb{Z})$ such that $\mathcal{M}_1^{\text{main}}(L(u), \beta_v) \neq \emptyset$, $\beta_v \neq 0$ is chosen.
(7) The vertex v of type Q with $k_v + 1$ edges is assigned the A_∞ operation $\mathfrak{q}^{dR}_{\beta_v;\ell_v,k_v}(D(\mathbf{p}_{I_v}); \cdots)$, where $k_v \geq 0$ and $\mathbf{p}_{I_v} = (\mathbf{p}(I_v(1)), \cdots, \mathbf{p}(I_v(\#I_v))$. Here we identify $I_v \subset \{1, \cdots, \ell\}$ with an order preserving injective map $I_v : \{1, \cdots, \#I_v\} \to \{1, \cdots, \ell\}$.
(8) The vertex v of type C has exactly 3 edges. We assign the wedge product $\mathfrak{m}_{2,\beta_0} = \pm\wedge$ to it. (We remark $\beta_0 = 0$. So the operations $\mathfrak{m}_{k,\beta_0}$ are operations of $\Omega(L(u))$, that is the differential graded algebra regarded as an A_∞ algebra. In particular $\mathfrak{m}_{k,\beta_0} = 0$ for $k \geq 3$.)

We take a T^n-invariant Riemannian metric on $L(u)$ and hence the Green operator (or the propagator)

$$G : \Omega^k(L(u)) \to \Omega^{k+1}(L(u))$$

is also T^n-invariant. We identify $H(L(u); \mathbb{R})$ to the space of harmonic forms and let

$$\Pi : \Omega^k(L(u)) \to H^k(L(u); \mathbb{R}) \subset \Omega^k(L(u))$$

be the harmonic projection. They satisfies the relation :

$$-(d \circ G + G \circ d) = id - \Pi.$$

We assign the Green operator G to each of the interior edges, that is the edges which do not contain exterior vertex. We assign Π to the edge which contains the zero-th exterior vertex. We define $\mathfrak{q}_\Gamma(h, \cdots, h)$ by composing operations assigned to the vertices and edges according to the way they are connected. More precisely, we first define

$$\mathfrak{f}_\Gamma(h, \cdots, h) \in \Omega(L(u))$$

by the induction of the number of vertices as follows.

If there is a unique interior vertex v then

$$\mathfrak{f}_\Gamma(h, \cdots, h) = \mathfrak{q}^{dR}_{\beta_v;\ell_v,k_v}(D(\mathbf{p}_{I_v}); h, \cdots, h),$$

when v is of type Q and

$$\mathfrak{f}_\Gamma(h, h) = h \wedge h,$$

when v is of type C.

We next assume that Γ has more than one interior vertices. Let v_{last} be the unique edge which is joined with the zero's exterior vertex. We remove v_{last}, zero's exterior vertex, and the edge joining them from $|\Gamma|$. Let $|\Gamma_1|, \cdots, |\Gamma_{k_{v_{\rm last}}}|$ be the closure of the connected component of it. (The number of the connected components is $k_{v_{\rm last}}$, that is the number of edges of $v_{\rm last}$ minus 1.) The other data consisting Γ induces ones on Γ_i in an obvious way. We then put

$$\begin{aligned}\mathfrak f_\Gamma(h,\cdots,h) = \mathfrak q^{dR}_{\beta_{v_{\rm last}};\ell_{v_{\rm last}},k_{v_{\rm last}}}(D(\mathbf p_{I_{v_{\rm last}}});&G(\mathfrak f_{\Gamma_1}(h,\cdots,h)),\\ &\cdots, G(\mathfrak f_{\Gamma_{k_{v_{\rm last}}}}(h,\cdots,h))),\end{aligned}$$

when $v_{\rm last}$ is of type Q and

$$\mathfrak f_\Gamma(h,\cdots,h) = G(\mathfrak f_{\Gamma_1}(h,\cdots,h)) \wedge G(\mathfrak f_{\Gamma_2}(h,\cdots,h),$$

when $v_{\rm last}$ is of type C.

We then define

$$\mathfrak q_\Gamma(h,\cdots,h) = \Pi\circ \mathfrak f_\Gamma(h,\cdots,h).$$

We have thus defined $\mathfrak q_\Gamma(h,\cdots,h)$. (7.2) is its sum over Γ such that

$$\sum_{v\in C^Q_0(\Gamma)} \beta_v = \beta.$$

This finishes the description of (7.2). Let us go back to the proof of Lemma 7.1.

Sublemma 7.2. *In our situation the nonzero* $\mathfrak q_\Gamma$ *appears only in the case of* Γ *which has only one interior vertex.*

Proof. We remark that since $L(u)$ is a torus the wedge product between harmonic forms is again harmonic. Therefore application of the Green operator to the wedge product gets zero. Namely if there is an interior vertex of type C which is not the vertex $v_{\rm last}$ (that is the vertex which is joined by an edge to the zero's exterior vertex), then $\mathfrak q_\Gamma$ is zero.

We next consider the vertices of type Q. Consider

$$\mu(\beta_v) - \sum_i (2n - \dim D_{\mathbf p_{I_v}} - 2) \tag{7.3}$$

for each $v \in C^Q_0(\Gamma)$. We remark that $\beta_v \ne 0$. Hence (7.3) is not smaller than 2 for each v.

On the other hand the sum of (7.3) over vertices of type Q is 2. This is a consequence of (6.9) which we assumed. It follows that there is only one vertex of type Q.

To complete the proof of the sublemma it suffices to consider the case where $v_{\rm last}$ is of type C and there is another interior vertex of type Q. In such a case we have

$$\mathfrak q_\Gamma(h,\cdots,h) = \pm\Pi(G(\mathfrak q^{dR}_{\beta;\ell,k-1}(D(\mathbf p);h,\cdots,h))\wedge h) \tag{7.4}$$

Using β and the above argument we have

$$\mathfrak q^{dR}_{\beta;\ell,k-1}(D(\mathbf p);h,\cdots,h) \in \Omega^n(L(u)).$$

Hence $G\mathfrak q^{dR}_{\beta;\ell,k-1}(D(\mathbf p);h,\cdots,h) \in \Omega^{n+1}(L(u)) = 0$. Therefore (7.4) is zero. The proof of Sublemma 7.2 is now complete. $\square$

The above discussion implies that the only nonzero term in (7.2) is (7.1). The proof of Lemma 7.1 is complete. $\square$

Proof of Proposition 3.1. This is an immediate consequence of Corollary 6.6. In fact it implies that $\mathfrak{m}_k^{can,\mathfrak{b}}(b,\cdots,b)$ can be only degree 0, that is proportional to $PD[L(u)]$. □

Proof of Theorem 3.4. Let $\mathfrak{b} = \sum_{a=1}^B \mathfrak{b}_a D_a$, $\mathfrak{b}_a \in \Lambda_+$. We assume $\mathfrak{b}_a$ is $G_{\rm bulk}$-gapped. We have

$$e^{\mathfrak{b}} = \sum_{\ell} \sum_{\mathbf{p}\in Map(\ell,\underline{B})} \mathfrak{b}^{\mathbf{p}} D(\mathbf{p}).$$

Here

$$\mathfrak{b}^{\mathbf{p}} = \prod_j \mathfrak{b}_{\mathbf{p}(j)}.$$

We have :

$$\mathfrak{PO}^u(\mathfrak{b};b) = \sum_{\beta,\mathbf{p},k} \mathfrak{b}^{\mathbf{p}} T^{\beta\cap\omega/2\pi} \mathfrak{q}^{can}_{\beta;|\mathbf{p}|,k}(D(\mathbf{p});b,\cdots,b).$$

By the degree counting the sum is nonzero only when (6.9) is satisfied. Therefore by Lemma 7.1 we have

$$\begin{aligned}\mathfrak{PO}^u(\mathfrak{b};b) &= \sum_{\beta,\mathbf{p},k} \mathfrak{b}^{\mathbf{p}} T^{\beta\cap\omega/2\pi} \frac{c(\beta;\mathbf{p})}{k!|\mathbf{p}|!}(b\cap\partial\beta)^k \\ &= \sum_{\beta,\mathbf{p}} \frac{1}{|\mathbf{p}|!} \mathfrak{b}^{\mathbf{p}} T^{\beta\cap\omega/2\pi} c(\beta;\mathbf{p}) \exp(b\cap\partial\beta).\end{aligned} \tag{7.5}$$

The sum of the cases $\beta = \beta_i$ $(i = 1,\cdots,m)$ and $|\mathbf{p}| = 0$ is $\mathfrak{PO}^u_0(b)$.

We next study other terms for $|\mathbf{p}| \ne 0$. We first consider the case $\beta = \beta_i$, $\ell \ne 0$. Then the corresponding term is a sum of the terms written as

$$cT^{\ell_i(u)+\rho} y^{\vec v_i}. \tag{7.6}$$

Here $c \in \mathbb{Q}$ and ρ is a sum of the numbers which appears as an exponents of $\mathfrak{b}_a$ for various a. It is nonzero since $\ell \ne 0$ and $\mathfrak{b}_a \in \Lambda_+$. Therefore $\rho \in G_{\rm bulk} \setminus \{0\}$. Therefore (7.6) is of the form appearing in the right hand side of (3.5).

We next consider the case $\beta \ne \beta_i$ $(i = 1,\cdots,m)$. We assume $c(\beta;\mathbf{p}) \ne 0$ in addition. Then by Proposition 6.1 (5) we have e^i and ρ such that

$$\frac{\beta\cap\omega}{2\pi} = \sum_i e^i \ell_i(u) + \rho.$$

Here $e^i \in \mathbb{Z}_{\ge 0}$ and $\sum e^i > 0$ and ρ is a sum of symplectic areas of holomorphic spheres divided by 2π. Thus the corresponding term is a sum of the terms

$$cT^{\sum_i e^i\ell_i(u)+\rho+\rho'} y^{\sum e^i \vec v_i}.$$

Here $c \in \mathbb{Q}$ and ρ' is a sum of the numbers that appear as the exponents of $\mathfrak{b}_a$ for various a. This is exactly of the form in the right hand side of (3.5).

Finally we prove (3.7). We first fix β. There can be infinitely many terms contributing to $\mathfrak{q}^{can}_{\beta;\ell,k}$. Namely it is possible that $\ell \to \infty$. The exponent of any such term is not smaller than

$$\frac{\beta\cap\omega}{2\pi} + \ell\rho_0 \tag{7.7}$$

where

$$\rho_0 = \inf(G_{\rm bulk} \setminus \{0\}).$$

(7.7) goes to infinity as $\ell \to \infty$.

We next consider the case where infinitely many different β's contribute to $\mathfrak{q}^{can}_{\beta;\ell,k}$. We denote the β's by β_γ. Suppose $\mathfrak{q}^{can}_{\beta_\gamma;\ell_\gamma,k_\gamma}$ is nonzero. The term corresponding thereto in (3.5) is of the form :

$$c_\gamma T^{\ell'_\gamma(u)+\rho_\gamma} y_1^{v'_{\gamma,1}} \cdots y_n^{v'_{\gamma,n}} \tag{7.8}$$

such that $d\ell'_\gamma = (v'_{\gamma,1}, \cdots, v'_{\gamma,n})$. We study $\ell'_\gamma(u)$ and ρ_γ and prove that ρ_γ goes to infinity.

We apply Proposition 6.1 (5) and obtain

$$\beta_\gamma = \sum_{i=1}^m k_{i,\gamma}\beta_i + \sum_j \alpha_{\gamma,j}.$$

We have

$$\ell'_\gamma(u) = \sum_{i=1}^m k_{i,\gamma}\ell_i(u)$$

and

$$\rho_\gamma = \sum_j \frac{\alpha_{\gamma,j} \cap [\omega]}{2\pi} + (\text{a sum of exponents appearing in } \mathfrak{b}).$$

If $(k_{1,\gamma}, \cdots, k_{n,\gamma}) \in \mathbb{Z}^n$ is bounded as $\gamma \to \infty$, then $\sum_j \alpha_{\gamma,j} \in H_2(X;\mathbb{Z})$ is necessarily unbounded. Therefore

$$\rho_\gamma \ge \sum_j \frac{\alpha_{\gamma,j} \cap [\omega]}{2\pi}$$

goes to infinity, as required.

We next assume that $(k_{1,\gamma}, \cdots, k_{n,\gamma}) \in \mathbb{Z}^n$ is unbounded. Then the sum of its Maslov indices

$$\sum_{i=1}^n k_{i,\gamma}\mu(\beta_i) = 2\sum_{i=1}^n k_{i,\gamma}$$

is unbounded. (We remark $k_{i,\gamma} \ge 0$.) Therefore one of the following occurs.

(a) $|\sum_j c_1(X) \cap \alpha_{\gamma,j}|$ is unbounded.
(b) $\mu(\beta_\gamma)$ is unbounded.

In case (a), $\sum_j \alpha_{\gamma,j} \in H_2(X;\mathbb{Z})$ is unbounded. Therefore

$$\rho_\gamma \ge \sum_j \alpha_{\gamma,j} \cap [\omega]$$

goes to infinity, as required.

For the case of (b), we have

$$\dim \mathcal{M}^{\text{main}}_{1;\ell_\gamma}(L(u);\beta_\gamma) = 2\ell_\gamma + n + \mu(\beta_\gamma) - 2.$$

On the other hand,

$$\dim \mathcal{M}^{\text{main}}_{1;\ell_\gamma}(L(u);\beta_\gamma : \mathbf{p}) = n,$$

since $\mathfrak{q}_{\beta_\gamma;\ell_\gamma,k_\gamma}(\mathbf{p};b)$ is nonzero. Therefore

$$\sum_{j=1}^{\ell_\gamma}(\deg \mathbf{p}_j - 2) = \mu(\beta_\gamma) - 2$$

goes to infinity. Hence ℓ_γ goes to infinity. It follows that

$$\rho_\gamma \geq \ell_\gamma \rho_0$$

goes to infinity, as required.

The proof of Theorem 3.4 is now complete. $\square$

Proof of Proposition 4.9. We assume that $\mathfrak{b}$ is as (4.10). We remark that $D_{i(l,r)} \in H^2(D;\mathbb{Z})$. Therefore a dimension counting argument shows that only β with $\mu(\beta) = 2$ contributes to $\mathfrak{PO}^u(\mathfrak{b};b)$. Then by the assumption that X is Fano we derive that only β_i's for $(i = 1, \cdots, m)$ contribute among those β's.

Thus we have obtained

$$\mathfrak{PO}^u(\mathfrak{b};b) = \sum_{i=1}^{m} \sum_{\mathbf{p}} \frac{1}{|\mathbf{p}|!} \mathfrak{b}^{\mathbf{p}} T^{\ell_i(u)} c(\beta_i;\mathbf{p}) y^{\vec{v}_i}. \tag{7.9}$$

We next calculate $c(\beta_i;\mathbf{p})$. By definition we have

$$c(\beta_i;\mathbf{p})[L(u)] = ev_{0*}(\mathcal{M}^{\text{main}}_{1;|\mathbf{p}|}(L(u),\beta;\mathbf{p})^{\mathfrak{s}_{\beta,1,\ell,\mathbf{p}}})$$

and

$$\mathcal{M}^{\text{main}}_{1;|\mathbf{p}|}(L(u),\beta;\mathbf{p}) = \mathcal{M}^{\text{main}}_{1;|\mathbf{p}|}(L(u),\beta) \times_{X^{|\mathbf{p}|}} \prod_{j=1}^{|\mathbf{p}|} D_{\mathbf{p}(j)}.$$

We consider

$$ev_0 : \mathcal{M}^{\text{main}}_{1;0}(L(u),\beta) \to L(u).$$

It is a diffeomorphism by Proposition 6.1. We fix $p_0 \in L(u)$ and let $\{\varphi\}$ is $ev_0^{-1}(p_0)$. Since $[\varphi] = \beta_i$ it follows that

$$[\varphi] \cap D(\mathbf{p}(j)) = \begin{cases} 1 & j = i, \\ 0 & j \neq i. \end{cases}$$

We remark that the number $c(\beta_i;\mathbf{p})$ is well defined, that is, independent of the perturbation. So we can perform the calculation in the homology level to find that

$$c(\beta_i;\mathbf{p}) = \begin{cases} 1 & \mathbf{p}(j) = i \text{ for all } j, \\ 0 & \text{otherwise.} \end{cases} \tag{7.10}$$

Thus (7.9) is equal to

$$\sum_{i=1}^{m} \exp(\mathfrak{b}_i) T^{\ell_i(u)} y^{\vec{v}_i}.$$

By using the decomposition of $\mathfrak{b}_i$ in (4.10), the proof of Proposition 4.9 is complete. $\square$

Proof of Proposition 4.15. We assume that $\mathfrak{b}$ satisfies Condition 4.14. Again by dimension counting only β with $\mu(\beta) = 2$ contributes to $\mathfrak{PO}^u(\mathfrak{b};b)$. In Proposition 4.15 we do not assume that X is Fano. So the homology classes β other than β_i $(i = 1, \cdots, m)$ may contribute.

We first study the contribution of β_i. We put

$$\mathfrak{P}_0(\mathfrak{b};b) = \sum_{i=1}^{m} \sum_{\mathbf{p}} \frac{1}{|\mathbf{p}|!} \mathfrak{b}^{\mathbf{p}} T^{\ell_i(u)} c(\beta_i;\mathbf{p}) y^{\vec{v}_i}.$$

(7.10) holds also in our case. Therefore we have

$$\begin{aligned}\mathfrak{P}_0(\mathfrak{b}';b)-\mathfrak{P}_0(\mathfrak{b};b) &= \sum_{i=1}^{m}(\exp(\mathfrak{b}'_i)-\exp(\mathfrak{b}_i))T^{\ell_i(u)}y^{\vec v_i}\\ &= (\exp(\mathfrak{b}_{i(l,r)}+cT^{\lambda})-\exp(\mathfrak{b}_{i(l,r)}))T^{S_l}y^{\vec v_{i(l,r)}}.\end{aligned}$$

This is of the form of the sum of the first 2 terms of the right hand side of (4.20).

We next study the contribution of $\beta \ne \beta_i$. We assume $\mu(\beta)=2$ and $\mathcal{M}^{\rm main}(L(u);\beta)\ne\emptyset$. We put

$$\mathfrak{P}_\beta(\mathfrak{b};b)=\sum_{\mathbf{p}}\frac{1}{|\mathbf{p}|!}\mathfrak{b}^{\mathbf{p}}T^{\beta\cap\omega/2\pi}c(\beta;\mathbf{p})\exp(b\cap\partial\beta).$$

We write

$$\beta=\sum_{i=1}^m e^i_\beta\beta_i+\sum_j\alpha_{\beta,j}$$

as in Proposition 6.1 (5). Then

$$\begin{aligned}\beta\cap[\omega] &= \sum_{i=1}^m e^i_\beta\ell_i(u)+\sum_j\alpha_{\beta,j}\cap[\omega]\\ \exp(b\cap\partial\beta) &= y^{\sum_{i=1}^m e^i_\beta\vec v_i}.\end{aligned}$$

We have $e^i_\beta\ge 0$ and $\sum_i e^i_\beta>0$. Moreover, since $\beta\ne\beta_i$ $(i=1,\cdots,m)$ it follows that $\sum_j\alpha_{\beta,j}\ne 0$. (We use $\mu(\beta)=2$ to prove this.) Therefore

$$\rho_\beta=\sum_j\alpha_{\beta,j}\cap[\omega]/2\pi>0.$$

Hence

$$\mathfrak{P}_\beta(\mathfrak{b}';b)-\mathfrak{P}_\beta(\mathfrak{b};b)=\sum_\sigma\sum_{h=1}^\infty c_{\sigma,h}T^{\sum_i e^i_\beta\ell_i(u)+\rho_\beta+h\lambda+\rho'_\sigma}y^{\sum_i e^i_\beta\vec v_i},$$

where $c_{\sigma,h}\in R$ and ρ'_σ is a sum of exponents of T in $\mathfrak{b}$. This corresponds to the third term of (4.20). In fact $\ell'_\sigma=\sum_i e^i_\beta\ell_i$, $\rho_\sigma=\rho'_\sigma+\rho_\beta$.

Now Proposition 4.15 follows if we rewrite

$$\mathfrak{PO}(\mathfrak{b}';b)-\mathfrak{PO}(\mathfrak{b};b)=(\mathfrak{P}_0(\mathfrak{b}';b)-\mathfrak{P}_0(\mathfrak{b};b))+\sum_\beta(\mathfrak{P}_\beta(\mathfrak{b}';b)-\mathfrak{P}_\beta(\mathfrak{b};b)).$$

□

8. Floer cohomology and non-displacement of Lagrangian submanifolds

In this section we discuss how we apply Floer cohomology and the potential function to the study of non-displacement property of Lagrangian submanifolds. Especially we will prove Proposition 3.15 and Theorem 3.19. The argument of this section is a minor modification and combination of the one given in [FOOO3] except that we integrate *bulk deformations* into the argument therein. (The way to use bulk deformation in the study of non-displacement of Lagrangian submanifold is described in section 13 [FOOO2].) This generalization is quite straightforward. We however gives details in order to make this paper as self-contained as possible for readers' convenience. To avoid too much repetition of the materials from [FOOO2] or [FOOO3], we will use the de Rham cohomology version instead of the singular cohomology version of filtered A_∞ algebra associated to a Lagrangian submanifolds in some part of this section. De Rham version is suitable for the purpose of present paper since we can easily realize *exact* unit in de Rham theory. We are using weak bounding cochain which is easier to handle in case exact unit (rather than homotopy unit) exits.

In this section we put $R = \mathbb{C}$. We write Λ_0, Λ_+, Λ in place of $\Lambda_0(\mathbb{C})$, $\Lambda_+(\mathbb{C})$, $\Lambda(\mathbb{C})$ respectively, in this section.

We first explain how we enlarge the deformation parameters $(\mathfrak{b}, \mathfrak{x})$ of Floer cohomology to

$$\mathcal{A}(\Lambda_+) \times H^1(L(u); \Lambda_0) \supset \mathcal{A}(\Lambda_+) \times H^1(L(u); \Lambda_+),$$

by including $b \in H^1(L(u); \Lambda_0) \supset H^1(L(u); \Lambda_+)$ as in [FOOO3] where we borrowed the idea of Cho [Cho] of considering Floer cohomology twisted with flat *non-unitary* line bundles in the study of displacement problem of Lagrangian submanifolds.

Definition 8.1. Let

$$\mathfrak{x} = \sum_i \mathfrak{x}_i \mathbf{e}_i \in H^1(L(u_0); \Lambda_0) \tag{8.1}$$

and

$$\mathfrak{x}_i = \mathfrak{x}_{i,0} + \mathfrak{x}_{i,+} \tag{8.2}$$

where $\mathfrak{x}_{i,0} \in \mathbb{C}$ and $\mathfrak{x}_{i,+} \in \Lambda_+$. We put

$$\mathfrak{y}_{i,0} = \exp(\mathfrak{x}_{i,0}) = \sum_{n=0}^{\infty} \frac{\mathfrak{x}_{i,0}^n}{n!} \in \mathbb{C}.$$

Let $\rho : H_1(L(u); \mathbb{Z}) \to \mathbb{C} \setminus \{0\}$ be the representation defined by $\rho(\mathbf{e}_i) = \mathfrak{y}_{i,0}$.

Definition 8.2. We define

$$\mathfrak{q}_{\ell,k}^{can,\rho} : E_\ell \mathcal{A}(\Lambda_+)[2] \otimes B_k(H(L(u); \Lambda_0)[1] \to H(L(u); \Lambda_0)[1].$$

by

$$\mathfrak{q}_{\ell,k}^{can,\rho} = \sum_\beta \mathfrak{y}_{1,0}^{\partial\beta\cap\mathbf{e}_1^*} \cdots \mathfrak{y}_{n,0}^{\partial\beta\cap\mathbf{e}_n^*} T^{\beta\cap\omega/2\pi} \mathfrak{q}_{\beta;\ell,k}^{can}.$$

We then define :

$$\mathfrak{m}_k^{\mathfrak{b},can,\mathfrak{x}}(x_1, \cdots, x_k) = \sum_\ell \mathfrak{q}_\ell^{can,\rho}(\mathfrak{b}^\ell; e^{\mathfrak{x}_+} x_1 e^{\mathfrak{x}_+} \cdots e^{\mathfrak{x}_+} x_k e^{\mathfrak{x}_+}), \tag{8.3}$$

and

$$\mathfrak{PO}_\rho^u(\mathfrak{b}, \mathfrak{x}_+) = \sum_{\ell,k} \mathfrak{q}_{\ell,k}^{can,\rho}(\mathfrak{b}^\ell; \mathfrak{x}_+^k). \tag{8.4}$$

We define

$$\mathfrak{q}_{\ell,k}^\rho : E_\ell(\mathcal{A}(\Lambda_+)[2]) \otimes B_k((\Omega(L(u) \widehat{\otimes} \Lambda_0)[1]) \to (\Omega(L(u)) \widehat{\otimes} \Lambda_0)[1].$$

and $\mathfrak{m}_k^{\mathfrak{b},\mathfrak{x}}$ in the same way.

Lemma 8.3. (1) $\mathfrak{m}_k^{\mathfrak{b},\mathfrak{x}}$, $\mathfrak{m}_k^{\mathfrak{b},can,\mathfrak{x}}$ *define structures of a filtered A_∞ algebra on $\Omega(L(u))\widehat{\otimes}\Lambda_0$ and on $H(L(u); \Lambda_0)$, respectively.*

(2) *Let* $\mathfrak{PO}^u : H^1(L(u);\Lambda_0) \to \Lambda_0$ *be the extended potential function as in Lemma* 3.5. *Then we have*

$$\mathfrak{PO}^u_\rho(\mathfrak{b};\mathfrak{x}_+) = \mathfrak{PO}^u(\mathfrak{b};\mathfrak{x})$$

if (8.2) *holds.*

Proof. The proof of (1) is the same as that of Proposition 11.2 [FOOO3]. The proof of (2) is the same as the proof of Lemma 3.8 [FOOO3]. □

Definition 8.4.

$$HF((L(u),\mathfrak{b},\mathfrak{x}),(L(u),\mathfrak{b},\mathfrak{x});\Lambda_0) := \frac{\text{Ker } \mathfrak{m}_1^{\mathfrak{b},can,\mathfrak{x}}}{\text{Im } \mathfrak{m}_1^{\mathfrak{b},can,\mathfrak{x}}}.$$

We remark

$$\frac{\text{Ker } \mathfrak{m}_1^{\mathfrak{b},can,\mathfrak{x}}}{\text{Im } \mathfrak{m}_1^{\mathfrak{b},can,\mathfrak{x}}} \cong \frac{\text{Ker } \mathfrak{m}_1^{\mathfrak{b},\mathfrak{x}}}{\text{Im } \mathfrak{m}_1^{\mathfrak{b},\mathfrak{x}}}.$$

Proof of Theorem 3.12. Based on the above definition the proof goes in the same way as the proof of Theorem 3.9 [FOOO3]. □

We next prove Proposition 3.15 and Theorem 3.19. Again the proofs will be similar to the proofs of Proposition 3.11 and Theorem 4.11 [FOOO3] in which we use a variant of Theorem 2.5 that also employs Floer cohomology twisted by non-unitary flat bundles (whose holonomy is ρ as above).

Now we provide the details of the above mentioned proofs.

Let $\psi_t : X \to X$ be a Hamiltonian isotopy with $\psi_0 =$ identity. We put $\psi_1 = \psi$. We consider the pair

$$L^{(0)} = L(u), \quad L^{(1)} = \psi(L(u))$$

such that $L^{(1)}$ is transversal to $L^{(0)}$. By perturbing ψ_t a bit, we may assume the following :

Condition 8.5. If $p \in L(u) \cap \psi(L(u))$ then

$$\psi_t(p) \notin \pi^{-1}(\partial P) \tag{8.5}$$

for any $t \in [0,1]$.

We put $\psi_t^* J = J_t$ where J is the standard complex structure of X. Then $J_0 = J$ and $J_1 = \psi_*(J)$.

Let $p, q \in L^{(0)} \cap L^{(1)}$. We consider the homotopy class of maps

$$\varphi : \mathbb{R} \times [0,1] \to X \tag{8.6}$$

such that

(1) $\lim_{\tau\to-\infty} \varphi(\tau,t) = p$, $\lim_{\tau\to+\infty} \varphi(\tau,t) = q$.
(2) $\varphi(\tau,0) \in L^{(0)}$, $\varphi(\tau,1) \in L^{(1)}$.

We denote by $\pi_2(L^{(1)},L^{(0)};p,q)$ the set of all such homotopy classes. We then define maps

$$\begin{aligned}
&\pi_2(L^{(1)},L^{(0)};p,r) \times \pi_2(L^{(1)},L^{(0)};r,q) \to \pi_2(L^{(1)},L^{(0)};p,q),\\
&\pi_2(X;L^{(1)}) \times \pi_2(L^{(1)},L^{(0)};p,q) \to \pi_2(L^{(1)},L^{(0)};p,q),\\
&\pi_2(L^{(1)},L^{(0)};p,q) \times \pi_2(X;L^{(0)}) \to \pi_2(L^{(1)},L^{(0)};p,q),
\end{aligned} \tag{8.7}$$

as follows. The map in the first line is an obvious concatenation. To define the map in the second line we first fix a base point $p_0 \in L^{(1)}$. Let $\varphi : \mathbb{R} \times [0,1] \to X$ represent an element of $\pi_2(L^{(1)}, L^{(0)}; p, q)$ and $\phi : D^2 \to X$ an element of $\pi_2(X; L^{(1)})$, respectively. ($\phi(1) = p_0$ and $\phi(\partial D^2) \subset L^{(1)}$.) We take a path γ joining p_0 and $\varphi(0,1)$ in $L^{(1)}$. We take the boundary connected sum $(\mathbb{R} \times [0,1]) \# D^2$ of $\mathbb{R} \times [0,1]$ and D^2 along $(0,1)$ and 1, which is nothing but $\mathbb{R} \times [0,1]$. We use γ to obtain the map $\varphi \#_\gamma \phi : \mathbb{R} \times [0,1] \cong (\mathbb{R} \times [0,1]) \# D^2 \to X$ joining φ and ϕ. The homotopy class of $\varphi \#_\gamma \phi$ is independent of γ since $\pi_1(L^{(1)})$ acts trivially on $\pi_2(X; L^{(1)})$. (We use the fact that $L^{(1)}$ is a torus here.) We thus defined the map in the second line. The map in the third line is defined in the same way.

We denote the maps in (8.7) by $\#$.

Remark 8.6. (1) We here use the set $\pi_2(L^{(1)}, L^{(0)}; p, q)$ of *homotopy* classes. In the last two sections we use homology group $H_2(X, L(u); \mathbb{Z})$. In fact $H_2(X, L(u); \mathbb{Z}) \cong \pi_2(X, L(u))$ in our situation and so we can instead use the latter.

(2) The definition of $\#$ above is rather ad hoc since we use the fact that $L^{(1)}$ is a torus. In the general case we use the set of Γ equivalence classes of the elements of $\pi_2(L^{(1)}, L^{(0)}; p, q)$ in place of $\pi_2(L^{(1)}, L^{(0)}; p, q)$ itself. (See Definition-Proposition 4.9 [FOOO2].)

Definition 8.7. We consider the moduli space of maps (8.6) satisfying (1), (2) above, in homotopy class $B \in \pi_2(L^{(1)}, L^{(0)}; p, q)$, and satisfying the equation :

$$\frac{\partial \varphi}{\partial \tau} + J_t \left(\frac{\partial \varphi}{\partial t} \right) = 0. \tag{8.8}$$

We denote it by

$$\widetilde{\mathcal{M}}^{\text{reg}}(L^{(1)}, L^{(0)}; p, q; B).$$

We put k_1 marked points $(\tau_i^{(1)}, 1)$ on $\{(\tau, 1) \mid \tau \in \mathbb{R}\}$, k_0 marked points $(\tau_i^{(0)}, 0)$ on $\{(\tau, 0) \mid \tau \in \mathbb{R}\}$, and ℓ marked points (τ_i, t_i) on $\mathbb{R} \times (0,1)$. We number the $k_1 + k_0$ marked points so that it respects to the counter-clockwise cyclic order. The totality of such $(\varphi, \{(\tau_i^{(1)}, 1)\}, \{(\tau_i^{(0)}, 0)\}, \{(\tau_i, t_i)\})$ is denoted by

$$\widetilde{\mathcal{M}}^{\text{reg}}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; p, q; B).$$

We divide this space by the $\mathbb{R}$ action induced by the translation of τ direction to obtain $\mathcal{M}^{\text{reg}}(L^{(1)}, L^{(0)}; p, q; B)$, and $\mathcal{M}^{\text{reg}}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; p, q; B)$. Finally we compactify them to obtain $\mathcal{M}(L^{(1)}, L^{(0)}; p, q; B)$, and $\mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; p, q; B)$.

See Definition 12.24 [FOOO2] (the case $\ell = 0$) and section 13.8 [FOOO2] for the detail.

Remark 8.8. In [FOOO2] we defined $\mathcal{M}_{k_1,k_0}(L^{(1)}, L^{(0)}; [\ell_p, w_1], [\ell_q, w_2])$. The choice of $[w_1]$ and B uniquely determines $[w_2]$ by the relation $[w_1]\#B = [w_2]$, but there could be more than one element $B \in \pi_2(L^{(1)}, L^{(0)}; p, q)$ satisfying $[w_1]\#B = [w_2]$. This is because the equivalence class $[\ell_p, w]$ is not the homotopy class but the equivalence class of a weaker relation. But the number of such classes B for which $\mathcal{M}(L^{(1)}, L^{(0)}; p, q; B) \neq \emptyset$ is finite by Gromov's compactness. Therefore $\mathcal{M}_{k_1,k_0}(L^{(1)}, L^{(0)}; [\ell_p, w_1], [\ell_q, w_2])$ is a finite union of $\mathcal{M}(L^{(1)}, L^{(0)}; p, q; B)$ with B satisfying $[w_1]\#B = [w_2]$.

We define the evaluation map

$$ev = (ev^{\rm int}, ev^{(1)}, ev^{(0)}) : \mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; p, q; B) \to X^\ell \times (L(u))^{k_1} \times (L(u))^{k_0},$$

as follows.

$$\begin{aligned} ev_i^{(0)}(\varphi, \{(\tau_i^{(1)}, 1)\}, \{(\tau_i^{(1)}, 0)\}, \{(\tau_i, t_i)\}) &= \varphi((\tau_i^{(0)}, 0)), \\ ev_i^{(1)}(\varphi, \{(\tau_i^{(1)}, 1)\}, \{(\tau_i^{(0)}, 0)\}, \{(\tau_i, t_i)\}) &= \psi^{-1}(\varphi((\tau_i^{(1)}, 1))), \\ ev_i^{\rm int}(\varphi, \{(\tau_i^{(1)}, 1)\}, \{(\tau_i^{(0)}, 0)\}, \{(\tau_i, t_i)\}) &= \psi_{t_i}^{-1}(\varphi((\tau_i, t_i))). \end{aligned} \tag{8.9}$$

We have diffeomorphisms $L(u) \cong L^{(0)}$ and $L(u) \cong L^{(1)}$. (The former is the identity and the latter is ψ.)

Lemma 8.9. *$\mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; p, q; B)$ has an oriented Kuranishi structure with corners. Its boundary is isomorphic to the union of the follows three kinds of fiber products as spaces with Kuranishi structure.*

(1)
$$\mathcal{M}_{k'_1,k'_0;\ell'}(L^{(1)}, L^{(0)}; p, r; B') \times \mathcal{M}_{k''_1,k''_0;\ell''}(L^{(1)}, L^{(0)}; r, q; B'')$$
where $k'_j + k''_j = k_j$, $\ell' + \ell'' = \ell$, $B' \# B'' = B$. The product is the direct product.

(2)
$$\mathcal{M}_{k'_1+1;\ell'}(L(u); \beta') \ {}_{ev_0}\times_{ev_i^{(1)}} \ \mathcal{M}_{k''_1,k_0;\ell''}(L^{(1)}, L^{(0)}; p, q; B'').$$
Here $\beta' \in \pi_2(X; L^{(1)}) \cong \pi_2(X; L(u))$, $k'_1 + k''_1 = k_1 + 1$, $\ell' + \ell'' = \ell$, $\beta' \# B'' = B$. The fiber product is taken over $L^{(1)} \cong L(u)$ by using $ev_0 : \mathcal{M}_{\ell';k'_1+1}(L(u); \beta') \to L(u)$ and $ev_i^{(1)} : \mathcal{M}_{k''_1,k_0;\ell''}(L^{(1)}, L^{(0)}; p, q; B'') \to L^{(1)}$. Here $i = 1, \cdots, k''_1$.

(3)
$$\mathcal{M}_{k_1,k'_0;\ell'}(L^{(1)}, L^{(0)}; p, q; B') \ {}_{ev_i^{(0)}}\times_{ev_0} \ \mathcal{M}_{k''_0+1;\ell''}(L(u); \beta'').$$
Here $\beta'' \in \pi_2(X; L^{(0)}) \cong \pi_2(X; L(u))$, $k'_0 + k''_0 = k_0 + 1$, $\ell' + \ell'' = \ell$, $B' \# \beta'' = B$. The fiber product is taken over $L^{(0)} \cong L(u)$ by using $ev_0 : \mathcal{M}_{k''_0+1;\ell''}(L(u); \beta'') \to L(u)$ and $ev_i^{(0)} : \mathcal{M}_{k_1,k'_0;\ell'}(L^{(1)}, L^{(0)}; p, q; B') \to L^{(0)}$.

Lemma 8.9 is proved in section 29.4 [FOOO2].

Definition 8.10. We next take $\mathbf{p} \in Map(\ell, \underline{B})$ and define

$$\mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; p, q; B; \mathbf{p}) = \mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; p, q; B) \ {}_{ev^+}\times \prod_{i=1}^{\ell} D_{\mathbf{p}(i)}. \tag{8.10}$$

It is a space with oriented Kuranishi structure with corners.

We remark that Condition 8.5 implies that if $p = q$, $B = B_0 = 0$ then the set $\mathcal{M}(L^{(1)}, L^{(0)}; p, p; B_0)$ is empty.

Lemma 8.11. *The boundary of $\mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; p, q; B; \mathbf{p})$ is a union of the following three types of fiber product as a space with Kuranishi structure.*

(1)

$$\mathcal{M}_{k'_1,k'_0;\ell'}(L^{(1)},L^{(0)};p,r;B';\mathbf{p}_1) \times \mathcal{M}_{k''_1,k''_0;\ell''}(L^{(1)},L^{(0)};r,q;B'';\mathbf{p}_2).$$

Here the notations are the same as Lemma 8.9 (1) *and*

$$(\mathbf{p}_1,\mathbf{p}_2) = \mathrm{Split}((\mathbb{L}_1,\mathbb{L}_2),\mathbf{p}) \tag{8.11}$$

for some $(\mathbb{L}_1,\mathbb{L}_2) \in \mathrm{Shuff}(\ell)$.

(2)

$$\mathcal{M}_{k'_1+1;\ell'}(L(u);\beta';\mathbf{p}_1)\ {}_{ev_0}\times_{ev_i^{(1)}}\ \mathcal{M}_{k''_1,k_0;\ell''}(L^{(1)},L^{(0)};p,q;B'';\mathbf{p}_2).$$

Here the notations are the same as Lemma 8.9 (2) *and* (8.11).

(3)

$$\mathcal{M}_{k_1,k'_0;\ell'}(L^{(1)},L^{(0)};p,q;B';\mathbf{p}_1)\ {}_{ev_i^{(0)}}\times_{ev_0}\ \mathcal{M}_{k''_0+1;\ell''}(L(u);\beta'';\mathbf{p}_2).$$

Here the notations are the same as Lemma 8.9 (3) *and* (8.11).

The proof is immediate from Lemma 8.9. We remark that by our definition of evaluation map $ev_i^{\rm int}$ the homology class β', β'' in (2), (3) above are nonzero.

We now construct a virtual fundamental chains on the moduli space (8.10). We remark that we already defined a system of multisections on $\mathcal{M}_{k+1;\ell}(L(u);\beta;\mathbf{p})$ in Lemma 6.5.

Lemma 8.12. *There exists a system of multisections* (8.10) *which are compatible to one another and to the multisections provided in Lemma* 6.5 *under the identification of the boundaries given in Lemma* 8.11.

Proof. We construct multisections on the moduli space (8.10) by induction over k and $\int_\beta \omega$.

We remark that the boundary condition for (8.8) is not T^n equivariant anymore : while the boundary $L^{(0)} = L(u)$ is T^n invariant, $L^{(1)} = \psi(L(u))$ is not. So there is no way to define a T^n-action on our moduli space (8.10).

We however remark that ev_0 in (2) and (3) of Lemma 8.11 is a submersion after perturbation. This is a consequence of (2) of Lemma 6.5. Moreover the fiber product in (1) of Lemma 8.11 is actually a direct product. Therefore the perturbation near the boundary at each step of the induction is automatically transversal by the induction hypothesis. Therefore we can extend the perturbation by the standard theory of Kuranishi structure and multisection. This implies Lemma 8.12. □

We are now ready to define Floer cohomology with bulk deformation denoted by

$$HF((L^{(1)},\mathfrak{b},\psi_*(\mathfrak{x})),(L^{(0)},\mathfrak{b},\mathfrak{x});\Lambda_0).$$

Let us use the notation of Definition 8.1. We have a representation $\rho : \pi_1(L(u)) \to \mathbb{C} \setminus \{0\}$. We choose a flat $\mathbb{C}$-bundle $(\mathcal{L},\nabla)$ whose holonomy representation is ρ. It determines flat $\mathbb{C}$ bundles on $L^{(0)}$, $L^{(1)}$, which we denote by $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(1)}$, respectively. The fiber of $\mathcal{L}^{(j)}$ at p is denoted by $\mathcal{L}^{(j)}_p$.

Definition 8.13. We define

$$CF((L^{(1)},\rho),(L^{(0)},\rho);\Lambda_0) = \bigoplus_{p\in L^{(1)}\cap L^{(0)}} Hom(\mathcal{L}^{(0)}_p,\mathcal{L}^{(1)}_p)\otimes_{\mathbb{C}}\Lambda_0.$$

With elements $p \in L^{(1)} \cap L^{(0)}$ equipped with the degree 0 or 1 according to the parity of Maslov index, it becomes a $\mathbb{Z}_2$-graded free Λ_0-module.

We are now ready to define an operator $\mathfrak{r}$, following section 13.8 [FOOO2]. We first define

$$\mathrm{Comp} : \pi_2(L^{(1)}, L^{(0)}; p, q) \times Hom(\mathcal{L}_p^{(0)}, \mathcal{L}_p^{(1)}) \to Hom(\mathcal{L}_q^{(0)}, \mathcal{L}_q^{(1)})$$

Let $B = [\varphi] \in \pi_2(L^{(1)}, L^{(0)}; p, q)$, $\sigma \in Hom(\mathcal{L}_p^{(0)}, \mathcal{L}_p^{(1)})$.

$\tau \mapsto \varphi(\tau, j)$ defines a path joining p to q in $L^{(j)}$. Let

$$\mathrm{Pal}_{\partial_j B} : \mathcal{L}_p^{(j)} \to \mathcal{L}_q^{(j)} \tag{8.12}$$

be the parallel transport along this path with respect to the flat connection ∇. Since ∇ is flat this is independent of the choice of the representative φ but depends only on B. We define

$$\mathrm{Comp}(B, \sigma) = \mathrm{Pal}_{\partial_1 B} \circ \sigma \circ \mathrm{Pal}_{\partial_0 B}^{-1}. \tag{8.13}$$

Lemma 8.14. *Let $B \in \pi_2(L^{(1)}, L^{(0)}; p, q)$, $B' \in \pi_2(L^{(1)}, L^{(0)}; q, r)$ and $\beta_j \in \pi_2(X, L^{(j)})$, $\sigma \in Hom(\mathcal{L}_p^{(0)}, \mathcal{L}_p^{(1)})$. Then we have*

$$\begin{aligned}
\mathrm{Comp}(B \# B', \sigma) &= \mathrm{Comp}(B', \mathrm{Comp}(B, \sigma)), \\
\mathrm{Comp}(\beta_0 \# B, \sigma) &= \rho(\beta_0)\mathrm{Comp}(B, \sigma), \\
\mathrm{Comp}(B \# \beta_1, \sigma) &= \rho(\beta_1)\mathrm{Comp}(B, \sigma).
\end{aligned}$$

The proof is easy and so omitted.

Definition 8.15. Let $B \in \pi_2(L^{(1)}, L^{(0)}; p, q)$, $\mathbf{p} \in Map(\ell, \underline{B})$ and $h_i^{(j)}$ $(i = 1, \cdots, k_j)$ be differential forms on $L^{(j)}$. We define

$$\begin{aligned}
&\mathfrak{r}_{\rho,k_1,k_0;\ell;B}(D(\mathbf{p}); h_1^{(1)}, \cdots, h_{k_1}^{(1)}; \sigma; h_1^{(0)}, \cdots, h_{k_0}^{(0)}) \\
&= \frac{1}{\ell!} T^{\omega\cap B/2\pi} \mathrm{Comp}(B, \sigma) \int_{\mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; p, q; B; \mathbf{p})} ev^{(1)*}h^{(1)} \wedge ev^{(0)*}h^{(0)} \\
&\in Hom(\mathcal{L}_q^{(0)}, \mathcal{L}_q^{(1)}) \otimes_{\mathbb{C}} \Lambda_0.
\end{aligned} \tag{8.14}$$

Here

$$h^{(j)} = h_1^{(j)} \times \cdots \times h_{k_j}^{(j)}$$

is a differential form on $(L^{(j)})^{k_j}$.

$$\mathfrak{r}_{\rho,k_1,k_0;\ell} = \sum_B \mathfrak{r}_{\rho,k_1,k_0;\ell;B}$$

converges in non-Archimedean topology by the energy estimate (see section 22.5 [FOOO2]) and defines

$$\begin{aligned}
\mathfrak{r}_{\rho,k_1,k_0;\ell} : E_\ell\mathcal{A}(\Lambda_+)[2] &\otimes B_{k_1}((\Omega(L^{(1)}) \widehat{\otimes} \Lambda_0)[1]) \\
&\otimes CF((L^{(1)}, \rho), (L^{(0)}, \rho); \Lambda_0) \otimes B_{k_0}((\Omega(L^{(0)}) \widehat{\otimes} \Lambda_0)[1]) \\
&\longrightarrow CF((L^{(1)}, \rho), (L^{(0)}, \rho); \Lambda_0).
\end{aligned}$$

The following is a slight modification of Theorem 13.71 [FOOO2].

Proposition 8.16. *Let* $\mathbf{y} \in \mathcal{A}(\Lambda_+)[2]$, $\mathbf{x} \in B_{k_1}((\Omega(L^{(1)}) \widehat{\otimes} \Lambda_0)[1])$, *and let* $\mathbf{z} \in B_{k_0}((\Omega(L^{(0)}) \widehat{\otimes} \Lambda_0)[1])$, $v \in CF((L^{(1)}, \rho), (L^{(0)}, \rho); \Lambda_0)$. *Then, we have*

$$\begin{aligned}
0 = &\sum_{c_1,c_2} (-1)^{\deg \mathbf{y}^{(2;2)}_{c_1} \deg' \mathbf{x}^{(3;1)}_{c_2} + \deg' \mathbf{x}^{(3;1)}_{c_2} + \deg \mathbf{y}^{(2;1)}_{c_1}} \\
&\qquad \mathfrak{r}_\rho(\mathbf{y}^{(2;1)}_{c_1} \otimes (\mathbf{x}^{(3;1)}_{c_2} \otimes \mathfrak{q}_\rho(\mathbf{y}^{(2;2)}_{c_1} \otimes \mathbf{x}^{(3;2)}_{c_2}) \otimes \mathbf{x}^{(3;3)}_{c_2}) \otimes v \otimes \mathbf{z}) \\
&+ \sum_{c_1,c_2,c_3} (-1)^{\deg \mathbf{y}^{(2;2)}_{c_1} \deg' \mathbf{x}^{(2;1)}_{c_2} + \deg' \mathbf{x}^{(2;1)}_{c_2} + \deg \mathbf{y}^{(2;1)}_{c_1}} \\
&\qquad \mathfrak{r}_\rho(\mathbf{y}^{(2;1)}_{c_1} \otimes \mathbf{x}^{(2;1)}_{c_2} \otimes \mathfrak{r}_\rho(\mathbf{y}^{(2;2)}_{c_1} \otimes \mathbf{x}^{(2;2)}_{c_2} \otimes v \otimes \mathbf{z}^{(2;1)}_{c_3})) \otimes \mathbf{z}^{(2;2)}_{c_3}) \\
&+ \sum_{c_1,c_3} (-1)^{(\deg \mathbf{y}^{(2;2)}_{c_1} + 1)(\deg' \mathbf{x} + \deg' v + \deg' \mathbf{z}^{(3;1)}_{c_3}) + \deg \mathbf{y}^{(2;1)}_{c_1}} \\
&\qquad \mathfrak{r}_\rho(\mathbf{y}^{(2;1)}_{c_1} \otimes (\mathbf{x} \otimes v \otimes (\mathbf{z}^{(3;1)}_{c_3} \otimes \mathfrak{q}_\rho(\mathbf{y}^{(2;2)}_{c_1} \otimes \mathbf{z}^{(3;2)}_{c_3}) \otimes \mathbf{z}^{(3;3)}_{c_3}))).
\end{aligned} \tag{8.15}$$

Proof. The 1st, 2nd and 3rd terms correspond to (2), (1) and (3) of Lemma 8.11 respectively. The associated weights of symplectic area behave correctly under the composition rules in Lemma 8.14. The proposition follows from Stokes' formula. (We do not discuss sign here, since the sign will be trivial for the case of our interest where the degrees of ambient cohomology classes are even and the degrees of the cohomology classes of Lagrangian submanifold are odd.) □

Lemma 8.17. *If* $\mathbf{x} = \mathbf{x}_1 \otimes 1 \otimes \mathbf{x}_2$ *where* 1 *is the degree* 0 *form* 1, *then*

$$\mathfrak{r}_\rho(\mathbf{y} \otimes \mathbf{x} \otimes v \otimes \mathbf{z}) = 0. \tag{8.16}$$

The same holds if $\mathbf{z} = \mathbf{z}_1 \otimes 1 \otimes \mathbf{z}_2$.

Proof. This is an immediate consequence of the definition. □

Using the algebraic formalism developed in section 32.7 [FOOO2] we can define

$$\begin{aligned}
\mathfrak{r}^{can}_{\rho,k_1,k_0;\ell} : E_\ell\mathcal{A}(\Lambda_+)[2] \otimes B_{k_1}(H(L^{(1)}; \Lambda_0)[1]) \\
\otimes CF((L^{(1)}, \rho), (L^{(0)}, \rho); \Lambda_0) \otimes B_{k_0}(H(L^{(1)}; \Lambda_0)[1]) \\
\longrightarrow CF((L^{(1)}, \rho), (L^{(0)}, \rho); \Lambda_0),
\end{aligned}$$

such that (8.15), (8.16) hold when $\mathfrak{r}$ and $\mathfrak{q}$ is replaced by $\mathfrak{r}^{can}$ and $\mathfrak{q}^{can}$.

Definition 8.18. Let $\mathfrak{b} \in \mathcal{A}(\Lambda_+)$, $\mathfrak{x} \in H^1(L(u), \Lambda_0)$. We use the notations of Definition 8.1 and define

$$\delta^{\mathfrak{b},\mathfrak{x}} : CF((L^{(1)}, \rho), (L^{(0)}, \rho); \Lambda_0) \to CF((L^{(1)}, \rho), (L^{(0)}, \rho); \Lambda_0)$$

by

$$\delta^{\mathfrak{b},\mathfrak{x}}(v) = \mathfrak{r}^{can}_\rho(e^{\mathfrak{b}} \otimes e^{\mathfrak{x}_+} \otimes v \otimes e^{\mathfrak{x}_+}).$$

By taking a harmonic representative of $\mathfrak{x}_+$ we also have

$$\delta^{\mathfrak{b},\mathfrak{x}}(v) = \mathfrak{r}_\rho(e^{\mathfrak{b}} \otimes e^{\mathfrak{x}_+} \otimes v \otimes e^{\mathfrak{x}_+}).$$

Lemma 8.19.

$$\delta^{\mathfrak{b},\mathfrak{x}} \circ \delta^{\mathfrak{b},\mathfrak{x}} = 0.$$

Proof. We remark that $\Delta e^{\mathfrak b} = e^{\mathfrak b} \otimes e^{\mathfrak b}$ and $\Delta e^{\mathfrak x_+} = e^{\mathfrak x_+} \otimes e^{\mathfrak x_+}$. Therefore Proposition 8.16 implies

$$\begin{aligned} 0 =& \mathfrak r_\rho^{can}(e^{\mathfrak b} \otimes e^{\mathfrak x_+} \otimes \mathfrak r_\rho^{can}(e^{\mathfrak b} \otimes e^{\mathfrak x_+} \otimes v \otimes e^{\mathfrak x_+}) \otimes e^{\mathfrak x_+}) \\ &+ \mathfrak r_\rho^{can}(e^{\mathfrak b} \otimes e^{\mathfrak x_+} \otimes \mathfrak q_\rho^{can}(e^{\mathfrak b} \otimes e^{\mathfrak x_+}) \otimes e^{\mathfrak x_+} \otimes v \otimes e^{\mathfrak x_+}) \\ &+ (-1)^{\deg v+1}\mathfrak r_\rho^{can}(e^{\mathfrak b} \otimes e^{\mathfrak x_+} \otimes v \otimes e^{\mathfrak x_+} \otimes \mathfrak q_\rho^{can}(e^{\mathfrak b} \otimes e^{\mathfrak x_+}) \otimes e^{\mathfrak x_+}). \end{aligned}$$

Since $\mathfrak q_\rho^{can}(e^{\mathfrak b} \otimes e^{\mathfrak x_+})$ is a (harmonic) 0 form, Lemma 8.17 implies that the second and the third terms vanish. This proves the lemma. □

Definition 8.20.

$$HF((L^{(1)}, \mathfrak b, \psi_*(\mathfrak x)), (L^{(0)}, \mathfrak b, \mathfrak x); \Lambda_0) = \frac{\text{Ker } \delta^{\mathfrak b,\mathfrak x}}{\text{Im } \delta^{\mathfrak b,\mathfrak x}}.$$

We recall we are considering the Hamiltonian isotopic pair

$$L^{(0)} = L(u), \quad L^{(1)} = \psi(L(u)).$$

For this case, we prove

Proposition 8.21. *We have*

$$HF((L^{(1)}, \mathfrak b, \psi_*(\mathfrak x)), (L^{(0)}, \mathfrak b, \mathfrak x); \Lambda) \cong HF((L(u), \mathfrak b, \mathfrak x), (L(u), \mathfrak b, \mathfrak x); \Lambda).$$

Remark 8.22. We use Λ coefficients instead of Λ_0 coefficients in Proposition 8.21.

Proof. We can prove Proposition 8.21 in the same way as sections 13, 22, 32 of [FOOO2]. We will give an alternative proof here using de Rham theory. Let ψ_t be the Hamiltonian isotopy such that ψ_0 is the identity and ψ_1 is ψ. We put $L^{(t)} = \psi_t(L(u))$.

Let $\chi : \mathbb R \to [0, 1]$ be a smooth function such that $\chi(\tau) = 0$ for τ sufficiently small and $\chi(\tau) = 1$ for τ sufficiently large. We choose a two-parameter family of compatible almost complex structures $\{J_{\tau,t}\}_{\tau,t}$ by

$$J_{\tau,t} = \psi^*_{t\chi(\tau)} J.$$

Then it satisfies the following :

1. $J_{\tau,t} = J_t$ for sufficiently large τ.
2. $J_{\tau,t} = J$ for sufficiently small τ.
3. $J_{\tau,1} = \psi_{\chi(\tau)*} J$.
4. $J_{\tau,0} = J$.

Let $p \in L^{(0)} \cap L^{(1)}$. We consider maps $\varphi : \mathbb R \times [0, 1] \to X$ such that

1. $\lim_{\tau\to+\infty} \varphi(\tau, t) = p$.
2. $\lim_{\tau\to-\infty} \varphi(\tau, t)$ converges to a point in $L^{(0)}$ independent of t.
3. $\varphi(\tau, 0) \in L^{(0)}$, $\varphi(\tau, 1) \in L^{(\chi(\tau))}$.

We denote by $\pi_2(L^{(1)}, L^{(0)}; *, p)$ the set of homotopy classes of such maps. There are obvious maps

$$\begin{aligned} &\pi_2(L^{(1)}, L^{(0)}; *, p) \times \pi_2(L^{(1)}, L^{(0)}; p, q) \to \pi_2(L^{(1)}, L^{(0)}; *, q), \\ &\pi_2(X; L^{(1)}) \times \pi_2(L^{(1)}, L^{(0)}; *, p) \to \pi_2(L^{(1)}, L^{(0)}; *, p), \\ &\pi_2(L^{(1)}, L^{(0)}; *, p) \times \pi_2(X; L^{(0)}) \to \pi_2(L^{(1)}, L^{(0)}; *, p). \end{aligned} \tag{8.17}$$

(We here use the fact that the action of $\pi_1(L^{(i)})$ on $\pi_2(X; L^{(i)})$ is trivial.) We denote (8.17) by #.

Definition 8.23. We consider the moduli space of maps satisfying (1) - (3) above and of homotopy class $C_+ \in \pi_2(L^{(0)}, L^{(1)}; *, p)$ and satisfying the following equation :

$$\frac{\partial \varphi}{\partial \tau} + J_{\tau,t}\left(\frac{\partial \varphi}{\partial t}\right) = 0. \tag{8.18}$$

We denote it by

$$\mathcal{M}^{\rm reg}(L^{(1)}, L^{(0)}; *, p; C_+).$$

We also consider the moduli spaces with maps with interior and boundary marked points and their compactifications. We then get the moduli space

$$\mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; *, p; C_+).$$

We remark that we do not divide by $\mathbb{R}$ action since (8.18) is not invariant under the translation. We can define an evaluation map

$$ev = (ev^{\rm int}, ev^{(1)}, ev^{(0)}) : \mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; *, p; C_+) \to X^{\ell} \times (L^{(1)})^{k_1} \times (L^{(0)})^{k_0},$$

in a similar way as (8.9) as follows :

$$\begin{aligned}
ev_i^{(0)}(\varphi, \{(\tau_i^{(1)}, 1)\}, \{(\tau_i^{(1)}, 0)\}, \{(\tau_i, t_i)\}) &= \varphi((\tau_i^{(0)}, 0)),\\
ev_i^{(1)}(\varphi, \{(\tau_i^{(1)}, 1)\}, \{(\tau_i^{(0)}, 0)\}, \{(\tau_i, t_i)\}) &= \psi^{-1}_{\chi(\tau_i)}(\varphi((\tau_i^{(1)}, 1))),\\
ev_i^{\rm int}(\varphi, \{(\tau_i^{(1)}, 1)\}, \{(\tau_i^{(0)}, 0)\}, \{(\tau_i, t_i)\}) &= \psi^{-1}_{t_i\chi(\tau_i)}(\varphi((\tau_i, t_i))).
\end{aligned} \tag{8.19}$$

Moreover there is another evaluation map

$$ev_{-\infty} : \mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; *, p; C_+) \to L(u)$$

defined by

$$ev_{-\infty}(\varphi) = \lim_{\tau \to -\infty} \varphi(\tau, t).$$

Using fiber product with the cycle $D(\mathbf{p})$ we define $\mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; *, p; C_+; \mathbf{p})$ in the same way as above.

Lemma 8.24. $\mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; *, p; C_+; \mathbf{p})$ *has an oriented Kuranishi structure with boundary. Its boundary is a union of the following four types of fiber products as the space with Kuranishi structure.*

(1)
$$\mathcal{M}_{k_1',k_0';\ell'}(L^{(1)}, L^{(0)}; *, q; C'_+; \mathbf{p}_1) \times \mathcal{M}_{k_1'',k_0'';\ell''}(L^{(1)}, L^{(0)}; q, p; B''; \mathbf{p}_2).$$
Here the notations are the same as Lemma 8.9 (1) *and* (8.11).

(2)
$$\mathcal{M}_{k_1'+1;\ell'}(L(u); \beta'; \mathbf{p}_1) \;{}_{ev_0}\times_{ev_i^{(1)}}\; \mathcal{M}_{k_1'',k_0;\ell''}(L^{(1)}, L^{(0)}; *, p; C''_+; \mathbf{p}_2).$$
Here the notations are the same as Lemma 8.9 (2) *and* (8.11).

(3)
$$\mathcal{M}_{k_1,k_0';\ell'}(L^{(1)}, L^{(0)}; *, p; C'_+; \mathbf{p}_1) \;{}_{ev_i^{(0)}}\times_{ev_0}\; \mathcal{M}_{k_0''+1;\ell''}(L(u); \beta''; \mathbf{p}_2).$$
Here the notations are the same as Lemma 8.9 (3) *and* (8.11).

(4)
$$\mathcal{M}_{k_1'+k_0'+1;\ell'}(L(u); \beta'; \mathbf{p}_1)_{ev_0} \times_{ev_{-\infty}} \mathcal{M}_{k_1'',k_0'';\ell'}(L^{(1)}, L^{(0)}; *, p; C''_+; \mathbf{p}_2),$$
where $k_j' + k_j'' = k_j$, $\ell' + \ell'' = \ell$, $\beta' \# C''_+ = C_+$ *and* (8.11).

The proof is the same as one in section 29.4 [FOOO2].

Lemma 8.25. *There exists a system of multisections on $\mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; *, p; C_+; \mathbf{p})$ so that it is compatible with one constructed before at the boundaries described in Lemma 8.24.*

Proof. We can still use the fact ev_0 is a submersion on the perturbed moduli space to perform the inductive construction of multisection in the same way as the proof of Lemma 8.12. □

For $C_+ \in \pi_2(L^{(1)}, L^{(0)}; *, p)$, we define $\rho(C_+) \in Hom(\mathcal{L}_p^{(0)}, \mathcal{L}_p^{(1)})$ by

$$\rho(C_+) = \mathrm{Pal}_{\partial_1 C_+} \circ \mathrm{Pal}^{-1}_{\partial_0 C_+}. \tag{8.20}$$

Here we use the notation of (8.12).

Lemma 8.26. *Let $C_+ \in \pi_2(L^{(1)}, L^{(0)}; *, p)$, $B' \in \pi_2(L^{(1)}, L^{(0)}; p, q)$ and $\beta_j \in \pi_2(X, L^{(j)})$. Then we have*

$$\mathrm{Comp}(B', \rho(C_+)) = \rho(C_+ \# B'),$$
$$\rho(\beta_0 \# C_+) = \rho(\beta_0)\rho(C_+), \quad \rho(C_+ \# \beta_1) = \rho(\beta_1)\rho(C_+).$$

The proof is easy and is left to the reader.

Now let $C_+ \in \pi_2(L^{(1)}, L^{(0)}; *, p)$, $\mathbf{p} \in Map(\ell, \underline{B})$ and let $h_i^{(j)}$ $(i = 1, \cdots, k_j)$ be differential forms on $L^{(j)}$ and h also a differential form on $L(u)$. We define

$$\begin{aligned}
&\mathfrak{f}_{k_1,k_0;\ell;C_+}(D(\mathbf{p}); h_1^{(1)}, \cdots, h_{k_1}^{(1)}; h; h_1^{(0)}, \cdots, h_{k_0}^{(0)}) \\
&= \frac{1}{\ell!}\rho(C_+) \int_{\mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; *, p; C_+; \mathbf{p})} ev^{(1)*} h^{(1)} \wedge ev_{-\infty}^* h \wedge ev^{(0)*} h^{(0)} \\
&\in Hom(\mathcal{L}_p^{(0)}, \mathcal{L}_p^{(1)}) \otimes \Lambda.
\end{aligned} \tag{8.21}$$

Here

$$h^{(j)} = h_1^{(j)} \times \cdots \times h_{k_j}^{(j)}$$

is a differential form on $(L^{(j)})^{k_j}$. It induces

$$\begin{aligned}
\mathfrak{f}_{C_+} : B((\Omega(L^{(1)}) \widehat{\otimes} \Lambda_0)[1]) \otimes (\Omega(L(u)) \widehat{\otimes} \Lambda)[1] \otimes B((\Omega(L^{(0)}) \widehat{\otimes} \Lambda_0)[1]) \\
\to \bigoplus_{p \in L^{(1)} \cap L^{(0)}} Hom(\mathcal{L}_p^{(0)}, \mathcal{L}_p^{(1)}) \otimes \Lambda.
\end{aligned}$$

Now we define

$$\mathfrak{f} : \Omega(L(u)) \otimes \Lambda \to CF((L^{(1)}, \rho), (L^{(0)}, \rho); \Lambda)$$

by

$$\mathfrak{f}(h) = \sum_{C_+} T^{\omega \cap C_+/2\pi} \mathfrak{f}_{C_+}(e^{\mathfrak{b}} \otimes e^{\mathfrak{x}_+} \otimes h \otimes e^{\mathfrak{x}_+}). \tag{8.22}$$

We remark that $\omega \cap C_+/2\pi$ may not be positive in this case since (8.18) is τ-dependent.

The fact that the right hand side converges in non-Archimedean topology follows from the energy estimate. See section 22.5 [FOOO2].

Lemma 8.27. *$\mathfrak{f}$ is a chain map.*

Proof. With Lemmata 8.24, 8.25, 8.26, the proof is similar to the proof of Lemmata 8.15, 8.17 and 8.19. □

We next define the chain map of the opposite direction. Let $p \in L^{(0)} \cap L^{(1)}$. We consider maps $\varphi : \mathbb{R} \times [0,1] \to X$ such that

(1) $\lim_{\tau \to -\infty} \varphi(\tau, t) = p$.
(2) $\lim_{\tau \to +\infty} \varphi(\tau, t)$ converges to a point in $L^{(0)}$ and is independent of t.
(3) $\varphi(\tau, 0) \in L^{(0)}$, $\varphi(\tau, 1) \in L^{(\chi(-\tau))}$.

We denote by $\pi_2(L^{(1)}, L^{(0)}; p, *)$ the set of homotopy classes of such maps. There are obvious maps

$$\begin{aligned} &\pi_2(L^{(1)}, L^{(0)}; p, q) \times \pi_2(L^{(1)}, L^{(0)}; q, *) \to \pi_2(L^{(1)}, L^{(0)}; p, *), \\ &\pi_2(X; L^{(1)}) \times \pi_2(L^{(1)}, L^{(0)}; p, *) \to \pi_2(L^{(1)}, L^{(0)}; p, *), \\ &\pi_2(L^{(1)}, L^{(0)}; p, *) \times \pi_2(X; L^{(0)}) \to \pi_2(L^{(1)}, L^{(0)}; p, *). \end{aligned} \tag{8.23}$$

We denote them by $\#$.

Definition 8.28. We consider the moduli space of maps satisfying (1) - (3) above and of homotopy class $C_- \in \pi_2(L^{(1)}, L^{(0)}; p, *)$ and satisfying the following equation :

$$\frac{\partial \varphi}{\partial \tau} + J_{-\tau,t}\left(\frac{\partial \varphi}{\partial t}\right) = 0. \tag{8.24}$$

We denote it by

$$\mathcal{M}^{\text{reg}}(L^{(1)}, L^{(0)}; p, *; C_-).$$

We include interior and boundary marked points and compactify it. We then get the moduli space $\mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; p, *; C_-)$.

We define evaluation maps

$$ev = (ev^+, ev^{(1)}, ev^{(0)}) : \mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; p, *; C_-) \to X^\ell \times (L^{(1)})^{k_1} \times (L^{(0)})^{k_0},$$

and

$$ev_{+\infty} : \mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; p, *; C_-) \to L(u).$$

Here

$$ev_{+\infty}(\varphi) = \lim_{\tau \to +\infty} \varphi(\tau, t).$$

Using ev^+ we take fiber product with $D(\mathbf{p})$ and obtain $\mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; p, *; C_-; \mathbf{p})$.

Lemma 8.29. $\mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; p, *; C_-; \mathbf{p})$ *has an oriented Kuranishi structure with boundary. Its boundary is a union of the following four types of fiber product as the space with Kuranishi structure.*

(1)
$$\mathcal{M}_{k_1',k_0';\ell'}(L^{(1)}, L^{(0)}; p, q; B'; \mathbf{p}_1) \times \mathcal{M}_{k_1'',k_0'';\ell''}(L^{(1)}, L^{(0)}; q, *; C_-''; \mathbf{p}_2).$$
Here the notations are the same as Lemma 8.9 (1) *and* (8.11).

(2)
$$\mathcal{M}_{k_1'+1;\ell'}(L(u); \beta'; \mathbf{p}_1) \ {}_{ev_0}\times_{ev_i^{(1)}} \ \mathcal{M}_{k_1'',k_0;\ell''}(L^{(1)}, L^{(0)}; p, *; C_-''; \mathbf{p}_2).$$
Here the notations are the same as Lemma 8.9 (2) *and* (8.11).

(3)
$$\mathcal{M}_{k_1,k_0';\ell'}(L^{(1)}, L^{(0)}; p, *; C_-'; \mathbf{p}_1) \ {}_{ev_i^{(0)}}\times_{ev_0} \ \mathcal{M}_{k_0''+1;\ell''}(L(u); \beta''; \mathbf{p}_2).$$
Here the notations are the same as Lemma 8.9 (3) *and* (8.11).

(4)

$$\mathcal{M}_{k_1',k_0';\ell'}(L^{(1)},L^{(0)};p,*;C'_-;\mathbf{p}_1)_{ev_{-\infty}}\times_{ev_0}\mathcal{M}_{k_1''+k_0''+1;\ell''}(L(u);\beta'';\mathbf{p}_2),$$

where $k_j'+k_j''=k_j$, $\ell'+\ell''=\ell$, $\beta'\#C''_-=C_-$ *and* (8.11).

The proof is the same as one in [FOOO2] section 29.4.

We define

$$\mathrm{Comp}:\pi_2(L^{(1)},L^{(0)};p,*)\times Hom(\mathcal{L}_p^{(0)},\mathcal{L}_p^{(1)})\to\mathbb{C}$$

as follows. Let $\sigma\in Hom(\mathcal{L}_p^{(0)},\mathcal{L}_p^{(1)})$ and $C_-\in\pi_2(L^{(1)},L^{(0)};p,*)$. Then

$$\mathrm{Comp}(C_-,\sigma)v=\mathrm{Pal}_{\partial_1C_-}\circ\sigma\circ\mathrm{Pal}^{-1}_{\partial_0C_-}(v),\tag{8.25}$$

where $v\in\mathcal{L}_{\lim_{\tau\to+\infty}\varphi(\tau,t)}$ and we use the notation of (8.12).

Let $\sigma\in Hom(\mathcal{L}_p^{(0)},\mathcal{L}_p^{(1)})$, $C_-\in\pi_2(L^{(1)},L^{(0)};q,*)$, $B'\in\pi_2(L^{(1)},L^{(0)};p,q)$ and $\beta_j\in\pi_2(X,L^{(j)})$. Then we have

$$\begin{aligned}\mathrm{Comp}(B',\mathrm{Comp}(C_-,\sigma))&=\mathrm{Comp}(B'\#C_-,\sigma),\\ \mathrm{Comp}(\beta_0\#C_-,\sigma)&=\rho(\beta_0)\mathrm{Comp}(C_-,\sigma),\\ \mathrm{Comp}(C_-\#\beta_1,\sigma)&=\rho(\beta_1)\mathrm{Comp}(C_-,\sigma).\end{aligned}\tag{8.26}$$

Now let $C_-\in\pi_2(L^{(1)},L^{(0)};p,*)$, $\mathbf{p}\in Map(\ell,\underline{B})$ and $h_i^{(j)}$ $(i=1,\cdots,k_j)$ be differential forms on $L^{(j)}$ and $\sigma\in Hom(\mathcal{L}_p^{(0)},\mathcal{L}_p^{(1)})$. We will define an element

$$\mathfrak{g}_{\ell;k_1,k_0;C_-}(D(\mathbf{p});h_1^{(1)},\cdots,h_{k_1}^{(1)};\sigma;h_1^{(0)},\cdots,h_{k_0}^{(0)})\in\Omega(L(u))\otimes\Lambda.\tag{8.27}$$

We will define it as

$$\begin{aligned}&\mathfrak{g}_{\ell;k_1,k_0;C_-}(D(\mathbf{p});h_1^{(1)},\cdots,h_{k_1}^{(1)};\sigma;h_1^{(0)},\cdots,h_{k_0}^{(0)})\\ &=\frac{1}{\ell!}\mathrm{Comp}(C_-,\sigma)((ev_{+\infty})_!)(ev^{(1)*}h^{(1)}\wedge ev^{(0)*}h^{(0)}).\end{aligned}\tag{8.28}$$

Here $(ev_{+\infty})_!$ is the integration along the fiber of the map

$$ev_{+\infty}:\mathcal{M}_{k_1,k_0;\ell}(L^{(1)},L^{(0)};p,*;C_-;\mathbf{p})^{\mathfrak{s}}\to L(u)\tag{8.29}$$

of the appropriately perturbed moduli space. More precise definition is in order.

We can inductively define a multisection on $\mathcal{M}_{k_1,k_0;\ell}(L^{(1)},L^{(0)};p,*;C_-;\mathbf{p})$ so that this is transversal to 0 and is compatible with other multisections we have constructed in the earlier stage of induction. We can prove it in the same way as Lemma 8.25.

However it is impossible to make the evaluation map (8.29) a submersion in general by the obvious dimensional reason if we just use multisections over the moduli space $\mathcal{M}_{k_1,k_0;\ell}(L^{(1)},L^{(0)};p,*;C_-;\mathbf{p})$: We need to enlarge the base by considering a *continuous family* of multisections. This method was introduced in section 33 [FOOO2] for example and the form we need here is detailed in section 12 [Fu3]. We recall the detail of this construction in Appendix of the present paper for readers' convenience. More precisely we take $M_s=L(u)^{k_0+k_1}$, $\mathcal{M}=\mathcal{M}_{k_1,k_0;\ell}(L^{(1)},L^{(0)};p,*;C_-;\mathbf{p})$, $M_t=L(u)$, $ev_s=(ev^{(1)},ev^{(0)})$, $ev_t=ev_{+\infty}$ and apply Definition 12.16. Then, the next lemma follows from Lemma 12.19 in Appendix.

Lemma 8.30. *There exists a continuous family $\{\mathfrak{s}_\alpha\}$ of multisections on our moduli space $\mathcal{M}_{k_1,k_0;\ell}(L^{(1)},L^{(0)};p,*;\beta;\mathbf{p})$ so that it is compatible in the sense of Definition 12.12 and is also compatible with the multisections constructed before in the inductive process at the boundaries described in Lemma 8.29. Moreover (8.29) is a submersion.*

By Definition 12.16, the integration along the fiber (8.28) (or smooth correspondence map) is defined. Now we have finished the description of the element (8.27). This assignment induces a homomorphism

$$\mathfrak{g}_\beta : B((\Omega(L^{(1)})\,\widehat{\otimes}\,\Lambda_0)[1])\otimes\left(\bigoplus_{p\in L^{(1)}\cap L^{(0)}} Hom(\mathcal{L}_p^{(0)},\mathcal{L}_p^{(1)})\otimes_{\mathbb{C}}\Lambda\right)\otimes B((\Omega(L^{(0)})\,\widehat{\otimes}\,\Lambda_0)[1]) \to (\Omega(L(u))\,\widehat{\otimes}\,\Lambda)[1].$$

Now we define

$$\mathfrak{g} : CF((L^{(1)},\rho),(L^{(0)},\rho);\Lambda)\to\Omega(L(u))\widehat{\otimes}\Lambda$$

by

$$\mathfrak{g}(\sigma)=\sum_\beta T^{\omega\cap C_-/2\pi}\mathfrak{g}_\beta(e^{\mathfrak{b}}\otimes e^{\mathfrak{x}_+}\otimes\sigma\otimes e^{\mathfrak{x}_+}). \tag{8.30}$$

With these preparation, we can prove the following lemma in the same way as Lemma 8.27 using Lemmata 12.18 and 12.20. So its proof is omitted.

Lemma 8.31. $\mathfrak{g}$ *is a chain map.*

Proposition 8.32. $\mathfrak{f}\circ\mathfrak{g}$ *and* $\mathfrak{g}\circ\mathfrak{f}$ *are chain homotopic to the identity.*

Proof. We will prove that $\mathfrak{g}\circ\mathfrak{f}$ is chain homotopic to the identity. Let S_0 be a sufficiently large positive number. (Say $S_0=10$.) For $S>S_0$ we put

$$\chi_S(\tau)=\begin{cases}\chi(-\tau-S) & \tau\le 0,\\ \chi(\tau-S) & \tau\ge 0.\end{cases}$$

We will extend it to $0\le S\le S_0$ so that $\chi_0(\tau)=0$.

We consider maps $\varphi:\mathbb{R}\times[0,1]\to X$ such that the following holds :

(1) $\lim_{\tau\to-\infty}\varphi(\tau,t)$ converges to a point in $L(u)$ and is independent of t.
(2) $\lim_{\tau\to+\infty}\varphi(\tau,t)$ converges to a point in $L(u)$ and is independent of t.
(3) $\varphi(\tau,0)\in L^{(0)}$, $\varphi(\tau,1)\in L^{(\chi_S(\tau))}$.

We denote by $\pi_2(L^{(1)},L^{(0)};*,*;S)$ the set of homotopy classes of such maps. There exists a natural isomorphism $\pi_2(L^{(1)},L^{(0)};*,*;S)\cong\pi_2(X,L(u))$,

$$[\varphi]\mapsto[\varphi'],\qquad\text{where }\varphi'(\tau,t)=\psi^{-1}_{\chi_S(\tau)}(\varphi(\tau,t)).$$

Here we recall $L^{(1)}=\psi_1(L(u))$, $L^{(0)}=L(u)$. Therefore we will denote an element of $\pi_2(L^{(1)},L^{(0)};*,*;S)$ again by β as for the case of $\pi_2(X,L(u))$.

We have the obvious gluing maps

$$\begin{aligned}&\pi_2(L^{(1)},L^{(0)};*,p)\times\pi_2(L^{(1)},L^{(0)};p,*;S)\to\pi_2(L^{(1)},L^{(0)};*,*;S),\\&\pi_2(X;L^{(1)})\times\pi_2(L^{(1)},L^{(0)};*,*;S)\to\pi_2(L^{(1)},L^{(0)};*,*;S),\\&\pi_2(L^{(1)},L^{(0)};*,*;S)\times\pi_2(X;L^{(0)})\to\pi_2(L^{(1)},L^{(0)};*,*;S)\end{aligned}\tag{8.31}$$

which we denote all by $\#$.

We consider a three-parameter family of compatible almost complex structures $J_{S,\tau,t}$ given by
$$J_{S,\tau,t} = \psi^*_{t\chi_S(\tau)} J.$$
Then it satisfies :
$$J_{S,\tau,t} = \begin{cases} J_{-\tau-S,t} & \tau \text{ is sufficiently small and } S \geq S_0, \\ J_{\tau-S,t} & \tau \text{ is sufficiently large and } S \geq S_0, \\ J & t = 0, \\ \psi^*_{\chi_S(\tau)} J & t = 1, \\ J & S = 0. \end{cases} \tag{8.32}$$

Definition 8.33. Consider the moduli space of maps satisfying (1) - (3) above and of homotopy class $\beta \in \pi_2(L^{(0)}, L^{(1)}; *, *)$ and satisfying the following equation
$$\frac{\partial \varphi}{\partial \tau} + J_{S,\tau,t}\left(\frac{\partial \varphi}{\partial t}\right) = 0. \tag{8.33}$$
For each $0 \leq S < \infty$, we denote the moduli space by
$$\mathcal{M}^{\rm reg}_S(L^{(1)}, L^{(0)}; *, *; \beta)$$
We also put
$$\begin{aligned} &\mathcal{M}^{\rm reg}_{+\infty}(L^{(1)}, L^{(0)}; *, *; \beta) \\ &= \bigcup_{p \in L^{(1)} \cap L^{(0)}} \bigcup_{C'_+ \# C''_- = \beta} (\mathcal{M}^{\rm reg}(L^{(1)}, L^{(0)}; *, p; C'_+) \times \mathcal{M}^{\rm reg}(L^{(1)}, L^{(0)}; p, *; C''_-)) \end{aligned}$$
and define
$$\mathcal{M}^{\rm reg}(L^{(1)}, L^{(0)}; *, *; \beta; para) = \bigcup_{S \in [0, +\infty]} (\{S\} \times \mathcal{M}^{\rm reg}_S(L^{(1)}, L^{(0)}; *, *; \beta)).$$
We can also include interior and boundary marked points and compactify the corresponding moduli space which then gives rise to the moduli space
$$\mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; *, *; \beta; para).$$

We define evaluation maps
$$ev = (ev^{\rm int}, ev^{(1)}, ev^{(0)}) : \mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; *, *; \beta) \to X^\ell \times (L^{(1)})^{k_1} \times (L^{(0)})^{k_0},$$
and
$$ev_{\pm\infty} : \mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; *, *; \beta) \to L(u).$$
Here
$$ev_{\pm\infty}(\varphi) = \lim_{\tau \to \pm\infty} \varphi(\tau, t).$$
Using $ev^{\rm int}$ we take fiber product with $D(\mathbf{p})$ and obtain $\mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; *, *; \beta; para; \mathbf{p})$.

Lemma 8.34. *$\mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; *, *; \beta; para; \mathbf{p})$ has an oriented Kuranishi structure with corners. Its boundary is a union of the following six types of fiber products as the space with Kuranishi structure :*

(1)
$$\mathcal{M}_{k'_1+1;\ell'}(L(u); \beta'; \mathbf{p}_1) \ {}_{ev_0}\times_{ev^{(1)}_i} \ \mathcal{M}_{k''_1,k_0;\ell''}(L^{(1)}, L^{(0)}; *, *; para; \beta''; \mathbf{p}_2).$$
Here the notations are the same as Lemma 8.9 (2) *and* (8.11).

(2)

$$\mathcal{M}_{k_1,k_0';\ell'}(L^{(1)},L^{(0)};*,*;para;\beta';\mathbf{p}_1)\ {}_{ev_i^{(0)}}\times_{ev_0}\ \mathcal{M}_{k_0''+1;\ell''}(L(u);\beta'';\mathbf{p}_2).$$

Here the notations are the same as Lemma 8.9 (3) *and* (8.11).

(3)

$$\mathcal{M}_{k_1',k_0';\ell'}(L^{(1)},L^{(0)};*,*;\beta';para;\mathbf{p}_1)_{ev_{+\infty}}\times_{ev_0}\mathcal{M}_{k_1''+k_0''+1;\ell''}(L(u);\beta'';\mathbf{p}_2),$$

where $k_j'+k_j''=k_j$, $\ell'+\ell''=\ell$, $\beta'\#\beta''=\beta$ *and* (8.11).

(4)

$$\mathcal{M}_{k_1'+k_0'+1;\ell'}(L(u);\beta';\mathbf{p}_1)_{ev_0}\times_{ev_{-\infty}}\mathcal{M}_{k_1'',k_0'';\ell''}(L^{(1)},L^{(0)};*,*;\beta'';para;\mathbf{p}_2),$$

where $k_j'+k_j''=k_j$, $\ell'+\ell''=\ell$, $\beta'\#\beta''=\beta$ *and* (8.11).

(5)

$$\mathcal{M}_{k_1',k_0';\ell'}(L^{(1)},L^{(0)};*,p;\mathbf{p}_1;C_+')\times\mathcal{M}_{k_1'',k_0'';\ell''}(L^{(1)},L^{(0)};p,*;\mathbf{p}_2;C_-'')$$

where $k_j'+k_j''=k_j$, $\ell'+\ell''=\ell$, $C_+'\#C_-''=\beta$ *and* (8.11).

(6) *A space* $\widetilde{\mathcal{M}}_{k_1+k_0+2;\ell}(L(u);\beta;\mathbf{p})$. *There exists an* $\mathbb{R}$ *action on it such that the quotient space is* $\mathcal{M}_{k_1+k_0+2;\ell}(L(u);\beta;\mathbf{p})$.

Proof. The proof is similar to the proofs of Lemma 8.29 etc.

We remark that the case $S=\infty$ corresponds to (5).

The case when $S=0$ corresponds to (6). In fact $\chi_0(\tau,t)=0$. So the boundary condition reduces $\varphi(\partial(\mathbb{R}\times[0,1]))\subset L(u)$ and the equation (8.33) is J holomorphicity. The τ-translations define an $\mathbb{R}$-action on the moduli space at the part $S=0$. The quotient space is the moduli space of holomorphic discs with boundary and interior marked points.

To construct a Kuranishi chart in a neighborhood of $S=\infty$, we need to choose a smooth structure of $[0,\infty]$ at ∞. We can do this so that the coordinate change of the Kuranishi structure is smooth using the standard exponential decay estimate : Namely, for a sufficiently large S, every element of $\mathcal{M}_S^{\rm reg}(L^{(1)},L^{(0)};*,*;\beta)$, together with its S-derivatives, is close to an element of $\mathcal{M}_\infty^{\rm reg}(L^{(1)},L^{(0)};*,*;\beta)$ in the order of Ce^{-cS}. We can prove this estimate in a way similar to the proof of Lemma A1.58 section A.1 [FOOO2]. □

Lemma 8.35. *There exists a continuous family* $\mathfrak{s}$ *of multisections on our moduli space* $\mathcal{M}_{k_1,k_0;\ell}(L^{(1)},L^{(0)};*,*;\beta;para;\mathbf{p})$ *such that it is compatible in the sense of Definition* 12.12 *and also compatible with the one constructed before in the induction process at the boundaries described in Lemma* 8.34. *Moreover* $ev_{\pm\infty}$ *are submersions on the moduli space perturbed by this family.*

Proof. The proof is the same as the proof of Lemma 8.30. □

We use $\pi_2(L^{(1)},L^{(0)};*,*;S)\cong\pi_2(X,L(u))$ to define $\rho:\pi_2(L^{(1)},L^{(0)};*,*;S)\to\mathbb{C}\setminus\{0\}$, as the composition

$$\pi_2(L^{(1)},L^{(0)};*,*;S)\to\pi_2(X,L(u))\to\pi_1(L(u))\xrightarrow{\rho}\mathbb{C}\setminus\{0\}.$$

There is an obvious compatibility relation of this ρ and other ρ's and Comp's we defined before through $\#$.

Now let $\beta \in \pi_2(L^{(1)}, L^{(0)}; *, *; S)$, $\mathbf{p} \in Map(\ell, \underline{B})$ and $h_i^{(j)}$ $(i = 1, \cdots, k_j)$ be differential forms on $L^{(j)}$ and h is another differential form on $L(u)$. We will define an element

$$\mathfrak{h}_{\beta;\ell;k_1,k_0}(D(\mathbf{p}); h_1^{(1)}, \cdots, h_{k_1}^{(1)}; h; h_1^{(0)}, \cdots, h_{k_0}^{(0)}) \in \Omega(L(u))\widehat{\otimes}\Lambda. \tag{8.34}$$

by

$$\begin{aligned}&\mathfrak{h}_{\beta;\ell;k_1,k_0}(D(\mathbf{p}); h_1^{(1)}, \cdots, h_{k_1}^{(1)}; h; h_1^{(0)}, \cdots, h_{k_0}^{(0)})\\ &= \frac{1}{\ell!}\rho(\beta)((ev_{+\infty})_!)(ev^{(1)*}h^{(1)} \wedge ev_{-\infty}^{*}h \wedge ev^{(0)*}h^{(0)}).\end{aligned} \tag{8.35}$$

Here $(ev_{+\infty})_!$ is the integration along fiber of the map

$$ev_{+\infty} : \mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; *, *; \beta; para; \mathbf{p})^{\mathfrak{s}} \to L(u) \tag{8.36}$$

of our moduli space which is perturbed by the continuous family $\mathfrak{s}$ of perturbations given in Lemma 8.35. More precisely we apply Definition 12.16 to $M_s = L(u)^{k_0+1+k_1}$, $M_t = L(u)$, $\mathcal{M} = \mathcal{M}_{k_1,k_0;\ell}(L^{(1)}, L^{(0)}; *, *; \beta; para; \mathbf{p})$, and $ev_s = (ev^{(1)}, ev_{-\infty}, ev^{(0)})$, $ev_t = ev_{+\infty}$. We then obtain (8.36).

The family of the maps $\mathfrak{h}_{\beta;\ell;k_1,k_0}$ induce a homomorphism

$$\begin{aligned}\mathfrak{h}_\beta : B((\Omega(L^{(1)}) \widehat{\otimes} \Lambda_0)[1]) \otimes (\Omega(L(u)) \widehat{\otimes} \Lambda))[1] \otimes B((\Omega(L^{(0)}) \widehat{\otimes} \Lambda_0))[1])&\\ \longrightarrow \Omega(L(u))[1]\widehat{\otimes}\Lambda.&\end{aligned}$$

Now we define

$$\mathfrak{h} : \Omega(L(u)) \widehat{\otimes} \Lambda \to \Omega(L(u)) \widehat{\otimes} \Lambda$$

by

$$\mathfrak{h}(h) = \sum_\beta T^{\omega\cap\beta/2\pi}\mathfrak{h}_\beta(e^{\mathfrak{b}} \otimes e^{\mathfrak{x}_+} \otimes h \otimes e^{\mathfrak{x}_+}). \tag{8.37}$$

Lemma 8.36. *$\mathfrak{h}$ is a chain homotopy from the identity to $\mathfrak{g} \circ \mathfrak{f}$.*

Proof. Lemma 12.18 implies that $d \circ \mathfrak{h} + \mathfrak{h} \circ d$ is a sum of terms which are obtained from each of (1) - (6) of Lemma 8.34 in the same way as (8.35), (8.37).

Using the fact $\mathfrak{q}_\rho(e^{\mathfrak{b}}, e^{\mathfrak{x}_+})$ is a harmonic zero form in the same way as the proof of Lemma 8.19, we can show that the contributions of (1) and (2) vanish. The contributions of (3) and (4) are

$$(\mathfrak{m}_1^{\mathfrak{b},\mathfrak{x}} - d) \circ \mathfrak{h}$$

and

$$\mathfrak{h} \circ (\mathfrak{m}_1^{\mathfrak{b},\mathfrak{x}} - d)$$

respectively. The contribution of (5) is $\mathfrak{g} \circ \mathfrak{f}$. The contribution of (6) vanishes in the case when $\beta \neq 0$, because of extra $\mathbb{R}$ symmetry. The case $\beta = 0$ gives rise to the identity.

In sum, we use Stokes' formula to conclude

$$\mathfrak{h} \circ \mathfrak{m}_1^{\mathfrak{b},\mathfrak{x}} + \mathfrak{m}_1^{\mathfrak{b},\mathfrak{x}} \circ \mathfrak{h} = \mathfrak{g} \circ \mathfrak{f} - id.$$

We use the composition formula in Appendix (Lemma 12.20) to prove the above formulae. The proof of Lemma 8.36 is now complete. □

We have thus proved that $\mathfrak{g} \circ \mathfrak{f}$ is chain homotopic to the identity. We can prove $\mathfrak{f} \circ \mathfrak{g}$ is chain homotopic to identity in the same way. The proof of Proposition 8.32 is now complete. □

The proof of Proposition 8.21 is complete. Hence we have also completed the proof of Proposition 3.15 also. □

Remark 8.37. We gave the proof of the above proposition using de Rham cohomology. In [FOOO2] we gave a proof based on singular cohomology. Strictly speaking we only discussed in the case when $b \in H^1(L(u);\Lambda_+)$ in [FOOO2]. But using Cho's idea of shifting the constant term by non-unitary flat connection, the proof of [FOOO2] can be easily generalized to the present situation of $H^1(L(u);\Lambda_0)$. In fact Theorem 2.5 was proved in section 13 or 22 [FOOO2] by proving a statement similar to Proposition 8.21 from which we can derive Proposition 3.15.

Since we use de Rham cohomology to calculate the potential function in this paper, we need to rely on Lemma 6.12 to get results on the displacement out of the proof in [FOOO2], which uses the singular cohomology version.

The approach using de Rham cohomology is shorter but we cannot treat the results with $\mathbb{Q}$-coefficients, at least at the time of writing this article. Theretofore we need to use singular homology version for that purpose. It might be possible to develop the $\mathbb{Q}$ de Rham theory for the purpose. We remark that by Lemma 6.8, $\mathfrak{PO}^u(\mathfrak{b}; y_1, \cdots, y_n)$ is defined over the $\mathbb{Q}$ coefficients. To study quantum cohomology $QH(X;\Lambda_0(\mathbb{Q}))$ this de Rham version will be enough.

Proof of Theorem 3.19. This is a straightforward combination of the proofs of Proposition 3.15 and Theorem J [FOOO2]. □

9. Domain of definition of potential function with bulk

The purpose of this section is to prove Theorem 3.11. Theorem 3.11 is not used in the other part of this paper except in section 11 but will be used in Part III of this series of papers.

Proof of Theorem 3.11. We recall that $D_1, \cdots, D_m$ are of complex codimension one in X and $D_{m+1}, \cdots, D_B$ are of higher complex codimension. Let $\mathbf{p} \in Map(\ell, \underline{B})$. We put

$$|\mathbf{p}|_{\rm high} = \#\{j \mid \mathbf{p}(j) > m\}.$$

Lemma 9.1. *For any E we have*

$$\sup\{|\mathbf{p}|_{\rm high} \mid c(\beta, \mathbf{p}) \neq 0,\ \beta \cap \omega < E\} < C(E),$$

where $C(E)$ depends only on E and X.

Proof. If $|\mathbf{p}|_{\rm hight} = \mathcal{N}$ and $c(\beta,\mathbf{p}) \neq 0$ then $2\mathcal{N} \le \mu(\beta)$ by the dimension counting. The lemma then follows from Proposition 6.1 (5) and Gromov's compactness. □

We denote by $Map(\ell_+, \underline{B} \setminus \underline{m})$ be the set of the maps $\{1, \cdots, \ell_+\} \to \underline{B} \setminus \underline{m} = \{m+1, \cdots, B\}$. We put

$$M_+ = \bigcup_{\ell_+} Map(\ell_+, \underline{B} \setminus \underline{m}).$$

For $\mathbf{p}_+ \in Map(\ell_+, \underline{B} \setminus \underline{m})$ and $\ell_1, \cdots, \ell_m$ we define $\mathbf{p} = (\ell_1, \cdots, \ell_m; \mathbf{p}_+)$ by

$$\mathbf{p}(i) = \begin{cases} j & \text{if } \ell_1 + \cdots + \ell_{j-1} < i \le \ell_1 + \cdots + \ell_j, \\ \mathbf{p}_+(i - \sum_{j=1}^m \ell_j) & \text{if } i > \sum_{j=1}^m \ell_j. \end{cases} \tag{9.1}$$

Lemma 9.2. *If* $\mathbf{p} = (\ell_1, \cdots, \ell_m; \mathbf{p}_+)$ *then*

$$c(\mathbf{p}; \beta) = c(\mathbf{p}_+; \beta) \prod_{i=1}^{m} (\beta \cap D_i)^{\ell_i}.$$

Proof. By the dimensional reason

$$\dim \mathcal{M}_{1,|\mathbf{p}_+|}(L(u); \beta; \mathbf{p}_+) = n$$

and $c(\mathbf{p}_+; \beta)$ is the degree of the map

$$ev_0 : \mathcal{M}_{1,|\mathbf{p}_+|}(L(u); \beta; \mathbf{p}_+) \to L(u). \tag{9.2}$$

Note $\mathcal{M}_{1,|\mathbf{p}_+|}(L(u); \beta; \mathbf{p}_+)$ after perturbation is a space with triangulation and the weight in $\mathbb{Q}$, which is defined by the multiplicity and the order of the isotropy group. So it has a fundamental cycle over $\mathbb{Q}$.

We fix a regular value $p_0 \in L(u)$ of (9.2). Let

$$ev_0^{-1}(p_0) = \{\varphi_j \mid j = 1, \cdots, K\}$$

be its preimage. Each of its elements contributes to $c(\mathbf{p}_+; \beta)$ by $\epsilon_j \in \mathbb{Q}$ so that $\sum \epsilon_j = c(\mathbf{p}_+; \beta)$.

We remark that our counting problem to calculate $c(\mathbf{p}; \beta)$ is well-defined in the sense of Lemma 6.8. Therefore we can perform the calculation in the homology level to find that each of φ_j contributes $\epsilon_j \prod_{i=1}^{m} (\beta \cap D_i)^{\ell_i}$ to $c(\mathbf{p}; \beta)$. The lemma follows. $\square$

Now we are ready to complete the proof of Theorem 3.11. By Lemmata 7.1, (7.5), 9.2 and 6.5 (6), we find

$$\begin{aligned}
&\mathfrak{PO}^u(w_1, \cdots, w_B; \mathfrak{x}) \\
&= \sum_{\beta} \sum_{\mathbf{p}_+ \in M_+} \sum_{\ell_1, \cdots, \ell_m} \left(\frac{(\ell_1 + \cdots + \ell_m + |\mathbf{p}_+|)!}{\ell_1! \cdots \ell_m! |\mathbf{p}_+|!} \right) \frac{w_{\mathbf{p}_+(1)} \cdots w_{\mathbf{p}_+(|\mathbf{p}_+|)}}{(\ell_1 + \cdots + \ell_m + |\mathbf{p}_+|)!} \\
&\qquad T^{\beta \cap \omega / 2\pi} c(\mathbf{p}_+; \beta) \left(\prod_{i=1}^{m} (\beta \cap D_i)^{\ell_i} \right) w_1^{\ell_1} \cdots w_1^{\ell_m} \exp(\partial \beta \cap \mathfrak{x}) \\
&= \sum_{\beta} \sum_{\mathbf{p}_+ \in M_+} \frac{c(\mathbf{p}_+; \beta)}{|\mathbf{p}_+|!} w_{\mathbf{p}_+(1)} \cdots w_{\mathbf{p}_+(|\mathbf{p}_+|)} T^{\beta \cap \omega / 2\pi} \\
&\qquad \mathfrak{w}_1^{\beta \cap D_1} \cdots \mathfrak{w}_m^{\beta \cap D_m} y_1(u)^{\partial \beta \cap \mathbf{e}_1} \cdots y_n(u)^{\partial \beta \cap \mathbf{e}_n}.
\end{aligned} \tag{9.3}$$

By Lemma 9.1 this series converges on $\mathfrak{v}_T^u$-adic topology for any u.

Remark 9.3. In the second equality in (9.3) we use

$$\sum_{k=0}^{\infty} \frac{w_i^k}{k!} = \mathfrak{w}_i \tag{9.4}$$

(9.4) is actually the definition of the formal variable $\mathfrak{w}_i$. If we replace the formal variable w_i by a number $c_i \in \mathbb{C}$ and $\mathfrak{w}_i$ by $\mathfrak{c}_i = e^{c_i} \in \mathbb{C}$, then the second equality (9.3) still holds. However the convergence in the left hand side of

$$\sum_{k=0}^{\infty} \frac{c_i^k}{k!} = \mathfrak{c}_i$$

is with respect to the usual Archimedean topology of $\mathbb{C}$ and is *not* with respect to the non-Archimedean topology we are using here.

Now we examine the dependence of this sum on u's. Firstly, through the isomorphism $\psi_u : H^*(T^n;\mathbb{Z}) \to H^*(L(u);\mathbb{Z})$, we may regard β or β_j are independent of u and so are the coefficients a_j's. Secondly by the structure theorem, Proposition 6.1, the moduli spaces associated to a given β are all isomorphic and so can be canonically identified when $u \in \mathrm{Int}P$ varies. Thirdly the two factors $T^{\beta\cap\omega/2\pi}$ and $y_1(u)^{\partial\beta\cap\mathbf{e}_1}\cdots y_n(u)^{\partial\beta\cap\mathbf{e}_n}$ depending on u can be combined into

$$T^{\beta\cap\omega/2\pi}y_1(u)^{\partial\beta\cap\mathbf{e}_1}\cdots y_n(u)^{\partial\beta\cap\mathbf{e}_n} = \prod_{j=1}^{m}(z_j(u))^{a_j}$$

where $\beta = \sum_{j=1} a_j\beta_j$ in $H_2(X, L(u))$ and then Lemma 3.7 showed that $z_j(u)\circ\psi_u$ are independent of $u \in \mathrm{Int}P$. Therefore the composition $\mathfrak{PO}^u \circ \psi_u$ are a function defined on $\mathcal{A}(\Lambda_+)\times H^1(T^n;\Lambda_0)$ independent of u's.

The proof of Theorem 3.11 is complete. □

We recall that X is nef if and only if every holomorphic sphere $w : S^2 \to X$ satisfies $w_*[S^2]\cap c_1(X) \geq 0$. In the nef case we can prove the following statement which is somewhat similar to Proposition 4.11.

Proposition 9.4. *If X is nef and $\mathfrak{b}$ is as in* (4.10)*, then we have*

$$\mathfrak{PO}^u(\mathfrak{b};y) = \sum_{l=1}^{K}\sum_{j=1}^{a(l)} T^{S_l}(\exp(\mathfrak{b}_{l,j})+c_{l,j}(\mathfrak{b}))y^{\vec{v}_{i(l,j)}} + \sum_{i=\mathcal{K}+1}^{m} T^{\ell_i(u)}(1+c_i(\mathfrak{b}))y^{\vec{v}_i}, \tag{9.5}$$

where $c_i(\mathfrak{b}), c_{l,j}(\mathfrak{b}) \in \Lambda_+$.

Proof. Let $\beta \in H_2(X, L(u);\mathbb{Z})$ with $\mu(\beta) = 2$. We assume $\mathcal{M}^{\mathrm{main}}_{1;\ell}(L(u),\beta)$ is nonempty. Let

$$\beta = \sum_{i=1}^{m} k_i\beta_i + \sum_j \alpha_j$$

be as in Proposition 6.1 (5). Since $\alpha_j \cap c_1(X) \geq 0$ by assumption, it follows from the condition $\mu(\beta) = 2$ that there exists unique i such that $k_i = 1$ and other k_i is zero. Moreover $\alpha_j \cap c_1(X) = 0$. Hence if β is not β_i then

$$\beta = \beta_i + \sum_j \alpha_j.$$

This β contributes

$$cT^{\sum_j \alpha_j\cap[\omega]/2\pi}T^{\ell_i(u)}y^{\vec{v}_i},$$

to $\mathfrak{PO}^u(\mathfrak{b};y)$. The rest of the proof is the same as the proof of Proposition 4.11. □

10. Euler vector field

The formula (9.3) derived in the previous section yields an interesting consequence that is related to the Euler vector field on a Frobenius manifold and to our potential function. In Part III of this series of papers, we will further discuss the Frobenius manifold structure on the quantum cohomology and on the Jacobian ring of our potential function and their relationship. (See Remark 10.3.)

For $i = 1, \cdots, B$ let d_i be the degree of $D_i \in \mathcal{A}$. (That is twice of the real codimension of the corresponding faces of P.) In case $d_i = 2$ (that is $i \le m$) we put

$$2r_i = [D_i] \cap \mu_{L(u)} \in \mathbb{Z}.$$

Here $\mu_{L(u)} \in Hom(\pi_2(X, L(u)); \mathbb{Z}) \cong H^2(X, L(u); \mathbb{Z})$ is the Maslov index.

Definition 10.1. We define the *Euler vector field* $\mathfrak{E}$ on $\mathcal{A}$ by

$$\mathfrak{E} = \sum_{i=m+1}^{B} \left(1 - \frac{d_i}{2}\right) w_i \frac{\partial}{\partial w_i} + \sum_{i=1}^{m} r_i \frac{\partial}{\partial \mathfrak{w}_i}.$$

Theorem 10.2. *The directional derivative $\mathfrak{PO}^u$ along the vector field $\mathfrak{E}$ satisfies*

$$\mathfrak{E}(\mathfrak{PO}^u) = \mathfrak{PO}^u.$$

Proof. The proof is similar to the proof of a similar identity for the case of the Gromov-Witten potentials. (See [Dub] for example.) Let

$$\begin{aligned}
\mathfrak{E}_1 &= \sum_{i=m+1}^{B} \left(1 - \frac{d_i}{2}\right) w_i \frac{\partial}{\partial w_i}, \\
\mathfrak{E}_2 &= \sum_{i=1}^{m} r_i \frac{\partial}{\partial \mathfrak{w}_i}, \\
\mathfrak{PO}^u_{\beta,1} &= \sum_{\mathbf{p}_+ \in M_+} \frac{c(\mathbf{p}_+; \beta)}{|\mathbf{p}_+|!} w_{\mathbf{p}_+(1)} \cdots w_{\mathbf{p}_+(|\mathbf{p}_+|)}, \\
\mathfrak{PO}^u_{\beta,2} &= \mathfrak{w}_1^{\beta \cap D_1} \cdots \mathfrak{w}_m^{\beta \cap D_m}.
\end{aligned}$$

Since $\dim \mathcal{M}_{1,|\mathbf{p}_+|}(L(u), \beta; \mathbf{p}_+) = n$, it follows that

$$n - 2 + \mu_{L(u)}(\beta) + \sum_i (2 - \deg \mathbf{p}_+(i)) = n.$$

Therefore

$$\mathfrak{E}_1(\mathfrak{PO}^u_{\beta,1}) = \left(1 - \frac{\mu_{L(u)}(\beta)}{2}\right) \mathfrak{PO}^u_{\beta,1}.$$

On the other hand, we have

$$\mathfrak{E}_2(\mathfrak{PO}^u_{\beta,2}) = \frac{\mu_{L(u)}(\beta)}{2} \mathfrak{PO}^u_{\beta,2}$$

by definition. Theorem 10.2 now follows from (9.3). □

Remark 10.3. In Part III of this series of papers, we will prove the isomorphism

$$\Phi : (H(X; \Lambda_0), \cup^{\mathfrak{b}}) \cong \frac{\Lambda_0^P\{y, y^{-1}\}}{\left(y_i \frac{\partial \mathfrak{PO}_{\mathfrak{b}}}{\partial y_i} : i = 1, \cdots, n\right)}, \tag{10.1}$$

for arbitrary compact toric manifold (which is not necessarily Fano). Here the product $\cup^{\mathfrak{b}}$ in the left hand side is defined by the formula

$$\langle \mathfrak{a}_1 \cup^{\mathfrak{b}} \mathfrak{a}_2, \mathfrak{a}_3 \rangle_{\mathrm{PD}} = \sum_{\alpha \in H_2(X;\mathbb{Z})} \sum_{\ell=0}^{\infty} \frac{T^{\alpha \cap \omega / 2\pi}}{\ell!} GW_{\alpha, \ell+3}(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{b}^{\otimes \ell})$$

where

$$GW_{\alpha,m}(\mathfrak{c}_1,\cdots,\mathfrak{c}_m)=\int_{\mathcal{M}_m(\alpha)} ev^*(\mathfrak{c}_1\times\cdots\times\mathfrak{c}_m),$$

$\mathcal{M}_m(\alpha)$ is the moduli space of the stable maps of genus 0 with m marked points in homology class α, and $ev:\mathcal{M}_m(\alpha)\to X^m$ is the evaluation map. ($\langle,\rangle_{\rm PD}$ denotes the Poincaré duality.)

The isomorphism (10.1) is defined as follows. We choose a lift

$$\bigoplus_{d\neq 0} H^d(X;\Lambda_0)\cong \mathcal{H}(\lambda_0)\subset \mathcal{A}(\Lambda_0).$$

Using its basis $\mathbf{f}_a$ we write the element of $\mathcal{H}(\lambda_0)$ as $\sum_a w_a\mathbf{f}_a$. Then (10.1) sends $\mathbf{f}_a$ to

$$\left[\left(\frac{\partial}{\partial w_a}\mathfrak{PO}\right)(\mathfrak{b};y)\right].$$

We can prove that this map is a ring isomorphism by an argument which elaborates the discussion outlined in Remark 5.14 [FOOO3]. We will work it out in detail in Part III.

We will also prove in Part III that if $\mathfrak{PO}_{\mathfrak{b}}$ has only nondegenerate critical point, then (10.1) sends Poincaré duality to the residue pairing. Here residue pairing is defined as follows : By nondegeneracy assumption we have a ring isomorphism

$$\frac{\Lambda_0^P\{y,y^{-1}\}}{\left(y_i\frac{\partial\mathfrak{PO}^u_{\mathfrak{b}}}{\partial y_i}:i=1,\cdots,n\right)}\otimes_{\Lambda_0}\Lambda\cong\prod_{p\in{\rm Crit}(\mathfrak{PO}_{\mathfrak{b}})}\Lambda. \tag{10.2}$$

(See Proposition 6.9 [FOOO3]. It can be generalized to the non-Fano case.) Here ${\rm Crit}(\mathfrak{PO}_{\mathfrak{b}})$ is the set of critical points of $\mathfrak{PO}_{\mathfrak{b}}$. Let 1_p be the unit $\in\Lambda$ in the factor corresponding to p. We then put

$$\langle 1_p,1_q\rangle_{\rm res}=\begin{cases}0 & \text{if } p\neq q,\\ \frac{1}{\det {\rm Hess}_p\mathfrak{PO}^u_{\mathfrak{b}}} & \text{if } p=q.\end{cases}$$

Here

$${\rm Hess}_p\mathfrak{PO}^u_{\mathfrak{b}}=\left(\frac{\partial^2\mathfrak{PO}^u_{\mathfrak{b}}}{\partial x_i\partial x_j}\right)(\mathfrak{x})$$

is the Hessian matrix at $\mathfrak{x}=(\mathfrak{x}_1,\cdots,\mathfrak{x}_n)$, $e^{\mathfrak{x}_i}=\mathfrak{y}_i$, $e^{x_i}=y_i$, and $(T^{u_1}\mathfrak{y}_1,\cdots,T^{u_n}\mathfrak{y}_n)=p$. Then we have :

$$\langle\mathfrak{c},\mathfrak{d}\rangle_{\rm PD}=\langle\Phi(\mathfrak{c}),\Phi(\mathfrak{d})\rangle_{\rm res} \tag{10.3}$$

The proof of (10.3), which we will give in Part III, uses the moduli space of pseudo-holomorphic annuli bordered to our Lagrangian fiber $L(u)$.

In the mean time, here we illustrate the identity (10.3) for the simple case $X=\mathbb{C}P^1$, $\mathfrak{b}=\mathbf{0}$. (See [Ta].) Its moment polytope is $[0,1]$. The potential function is :

$$\mathfrak{PO}^u_{\mathbf{0}}(y)=T^uy+T^{1-u}y^{-1}.$$

The critical points are given at $u=1/2$ and $y=\pm 1$. We denote them by p_+,p_- respectively. We have

$${\rm Hess}_{p_+}\mathfrak{PO}^{1/2}_{\mathbf{0}}=2T^{1/2},\quad {\rm Hess}_{p_-}\mathfrak{PO}^{1/2}_{\mathbf{0}}=-2T^{1/2}.$$

(Note we here take $x=\log y$ as a variable.) Therefore

$$\langle 1_{p_+},1_{p_+}\rangle_{\rm res}=T^{-1/2}/2,\quad \langle 1_{p_-},1_{p_-}\rangle_{\rm res}=-T^{-1/2}/2,\quad \langle 1_{p_+},1_{p_-}\rangle_{\rm res}=0.$$

We consider $PD[pt] \in H^2(\mathbb{C}P^1)$ and identify it with $[\pi^{-1}(0)]$. Then the isomorphism (10.1) sends $PD[pt]$ to $T^u y \mod \left(y\frac{\partial \mathfrak{PO}_0^u}{\partial y}\right)$. At $u = 1/2$, the latter becomes $T^{1/2}(1_{p_+} - 1_{p_-})$ in the Jacobian ring, which can be easily seen from the identity

$$T^{1/2}y = \frac{1}{2}\left((1+y) - (1-y)\right)T^{1/2}.$$

On the other hand $PD[\mathbb{C}P^1] \in H^0(\mathbb{C}P^1)$ is the unit and so becomes $1_{p_+} + 1_{p_-}$. We have

$$\langle T^{1/2}(1_{p_+} - 1_{p_-}), 1_{p_+} + 1_{p_-}\rangle_{\rm res} = 1.$$

This is consistent with the corresponding pairing

$$\langle PD[pt], PD[\mathbb{C}P^1]\rangle_{\rm PD} = (PD[pt] \cup PD[\mathbb{C}P^1]) \cap [\mathbb{C}P^1] = 1$$

in the quantum cohomology side.

We recall that collection of a product structure on the tangent space, residue pairing, Euler vector field, and the unit consists of the data which determine Saito's flat structure (that is, the structure of Frobenius manifold) [Sa].

11. Deformation by $\mathfrak{b} \in \mathcal{A}(\Lambda_0)$

In sections 4 and 8, we used the bulk deformation of Lagrangian Floer cohomology by the divisor cycles $\mathfrak{b} \in \mathcal{A}(\Lambda_+)$. Actually using the result of section 9, most of the argument there can be generalized to the case when $\mathfrak{b} \in \mathcal{A}(\Lambda_0)$ by a minor modification. In this section we discuss this and some new phenomena appearing in the deformation by $\mathfrak{b} \in \mathcal{A}(\Lambda_0)$. In this section we consider the case $R = \mathbb{C}$. (See Remark 11.5, however.) We write Λ_0 etc. in place of $\Lambda_0(\mathbb{C})$ etc.. We first remark that the potential function $\mathfrak{PO}^u_{\mathfrak{b}}(y_1, \cdots, y_n) = \mathfrak{PO}^u(\mathfrak{b}; y_1, \cdots, y_n)$ itself is defined for $\mathfrak{b} \in \mathcal{A}(\Lambda_0)$ in section 9. But the definition of the leading term equation(4.6), Definition 4.3 need some minor modification which is in order. We put

$$\mathfrak{b} = \sum \mathfrak{b}_a D_a \in \mathcal{A}(\Lambda_0)$$

and consider its zero order term

$$\mathfrak{b}_a \equiv \overline{\mathfrak{b}}_a \mod \Lambda_+$$

where $\overline{\mathfrak{b}}_a \in \mathbb{C}$. We put

$$(\mathfrak{PO}^u_{\mathfrak{b}})_l = \sum_{r=1}^{a(l)} \exp(\overline{\mathfrak{b}}_{i(l,r)}) y^{\vec{v}_{l,r}} \in \mathbb{C}[y_{1,1}, \cdots, y^{-1}_{l,d(l)}]. \tag{11.1}$$

We define the leading term equation for

$$y_{l,s}\frac{\partial \mathfrak{PO}^u_{\mathfrak{b}}}{\partial y_{l,s}} = 0 \tag{11.2}$$

for $\mathfrak{b} \in \mathcal{A}(\Lambda_0)$ in the same way as Definition 4.3 by using (11.1) in place of (4.6).

We remark that only $\overline{\mathfrak{b}}_a$, $a = 1, \cdots, m$ appears in (11.1). In other words, coefficients of the cohomology classes D_a of degree > 2 do not affect the leading term equation.

Lemma 11.1. *Lemma 4.4 holds for $\mathfrak{b} \in \mathcal{A}(\Lambda_0)$ also.*

Proof. The formula (9.3) implies that the coefficient of $y^{\vec{v}_{l,r}}$ $(r = 1, \cdots, a(l))$ in $\mathfrak{PO}^u_{\mathfrak{b}}$ is $T^{S_l}\exp(\overline{\mathfrak{b}}_{i(l,r)})$. The rest of the proof is the same as the proof of Lemma 4.4. □

The leading term equation is of the form

$$0 = \sum_{r=1}^{a(l)} \exp(\overline{\mathfrak{b}}_{i(l,r)}) y^{\vec{v}_{l,r}} \vec{v}_{l,r}.$$

By varying $\overline{\mathfrak{b}}_i$ the coefficients $\exp(\overline{\mathfrak{b}}_i)$ can assume all elements from $\mathbb{C} \setminus \{0\}$.

Definition 11.2. A system of polynomial equations

$$0 = \sum_{r=1}^{a(l)} C_{i(l,r)} y^{\vec{v}_{l,r}} \vec{v}_{l,r}$$

with $C_{i(l,r)} \in \mathbb{C} \setminus \{0\}$, $l = 1, \cdots K$ is called a *generalized leading term equation.*

Now Theorem 4.7 is generalized as follows.

Proposition 11.3. *The following two conditions for u are equivalent to each other.*

1. *There exists a generalized leading term equation of $\mathfrak{PO}_0^u$, which has a solution $y_{l,j} \in \mathbb{C} \setminus \{0\}$.*
2. *There exists $\mathfrak{b} \in H(\Lambda_0)$ such that $\mathfrak{PO}_{\mathfrak{b}}^u$ has a critical point on $(\Lambda_0 \setminus \Lambda_+)^n$.*

Proof. (2) $\Rightarrow$ (1) follows from Lemma 11.1. Let us assume that the generalized leading term equation with $C_{i(l,r)}$ as a coefficient has a solution $\mathfrak{y}_{l,s} \in \mathbb{C} \setminus \{0\}$. We put $\overline{\mathfrak{b}}_{i(l,r)} = \log C_{i(l,r)}$. Then we can add higher order term in the same way as the proof of Theorem 4.7 to obtain $\mathfrak{b}$ such that $\mathfrak{y}_{l,s}$ is a solution of (11.2). □

Proposition 11.4. *Theorem* 3.12 *holds for* $\mathfrak{b} \in \mathcal{A}(\Lambda_0)$.

Proof. Using a similar formula as (9.3) the proof is the same as the proof of Theorem 3.12 given in section 8. We omit the detail. □

Remark 11.5. Let R be a field such that $\mathbb{Q} \subset R \subset \mathbb{C}$. Even if we assume $\mathfrak{b} = \sum \mathfrak{b}_a D_a \in \mathcal{A}(\Lambda_0(R))$, it does not imply

$$(\mathfrak{PO}_{\mathfrak{b}}^u)_l \in R[y_{1,1}, \cdots, y_{l,d(l)}^{-1}]. \tag{11.3}$$

In fact $\exp(\overline{\mathfrak{b}}_{i(r,s)})$ may not be an element of R. (This point is related to Remark 9.3.) An appropriate condition for (11.3) to hold is

$$\exp(\mathfrak{b}_i) \in \Lambda_0(R)$$

for $i = 1, \cdots, m$.

Example 11.6. We put

$$P = \{(u_1, u_2) \mid 0 \le u_1, u_2,\ u_1 + u_2 \le 1,\ u_2 \le 2/3\}.$$

P is a moment polytope of monotone one point blow up of $\mathbb{C}P^2$. We consider $\mathbf{u} = (1/3, 1/3)$. $L(\mathbf{u})$ is a monotone Lagrangian submanifold. We put

$$D_2 = \pi^{-1}(\{(u_1, u_2) \in P \mid u_2 = 0\}).$$

Let $\mathfrak{b}_c = (\log c)[D_2]$. where $c \in \mathbb{C} \setminus \{0\}$. Proposition 4.9 implies

$$\mathfrak{PO}^{\mathbf{u}}(\mathfrak{b}_c; y_1, y_2) = T^{1/3}(y_1 + c y_2 + y_2^{-1} + (y_1 y_2)^{-1}).$$

Thus the critical point is given by

$$1 - y_1^{-2} y_2^{-1} = 0 = c - y_2^{-2} - y_1^{-1} y_2^{-2}.$$

The first equation gives $y_2 = y_1^{-2}$. Hence the second equation becomes

$$c - y_1^4 - y_1^3 = 0. \tag{11.4}$$

(11.4) has a nonzero multiple root $y_1 = -3/4$ if $c = -27/256$.

Namely if $\mathfrak{b} = (\log(-27/256))[D_2]$, then $\mathfrak{PO}^{\mathbf{u}}_{\mathfrak{b}}$ has a degenerate critical point of type A_2.

Example 11.7. We again consider the example of two points blow up in section 5. Namely its moment polytope is (5.1) with $\beta = \frac{1-\alpha}{2}$. We consider the point $\mathbf{u} = (\beta, \beta)$. We put $D_2 = \pi^{-1}(\{(u_1, u_2) \in P \mid u_2 = 0\})$, and consider $\mathfrak{b}_c = (\log c)[D_2]$. We have

$$\mathfrak{PO}^{\mathbf{u}}(\mathfrak{b}_c; y_1, y_2) = T^{\beta}(cy_2 + y_2^{-1} + y_1 + y_1 y_2) + T^{1-\beta} y_1^{-1} y_2^{-1}.$$

The (generalized) leading term equation is

$$c - y_2^{-2} + y_1 = 0 = 1 + y_2.$$

It has a nonzero solution $(1-c, -1)$ if $c \neq 1$. Hence there exists b such that

$$HF((L(\mathbf{u}), (\mathfrak{b}_c, b)), (L(\mathbf{u}), (\mathfrak{b}_c, b)); \Lambda) \neq 0$$

if and only if $c \neq 1$.

If we deform only by $\mathfrak{b} \in \Lambda_+$ then $c = 1$. Namely there is no such b with nontrivial Floer cohomology. We remark that $L(\mathbf{u})$ is bulk-balanced in the sense of Definition 3.13 since it is a limit of balanced fibers.

The authors do not know an example of $L(u)$ that carries a pair $(\mathfrak{b}, b)$ with $\mathfrak{b} \in \mathcal{A}(\Lambda_0)$, $b \in H^1(L; \Lambda_0)$ for which we have

$$HF((L(u), (\mathfrak{b}, b)), (L(u), (\mathfrak{b}, b)); \Lambda) \neq 0,$$

but which is not bulk-balanced in the sense of Definition 3.13.

12. Appendix : Continuous family of multisections

In this section we review the techniques of using a continuous family of multisections and integration along the fiber on their zero sets so that smooth correspondence by spaces with Kuranishi structure induces a map between de Rham complex.

This technique is not new and is known to various people. In fact [Ru], section 16 [Fu1] use a similar technique and section 33 [FOOO2], [Fu2], [Fu3] contain almost the same argument as we describe below. We include the details here for reader's convenience which we used in section 8.

Let $\mathcal{M}$ be a space with Kuranishi structure and $ev_s : \mathcal{M} \to M_s$, $ev_t : \mathcal{M} \to M_t$ be strongly continuous smooth maps. (See Definition 6.6 [FO] and the description below.) (Here s and t stand for source and target, respectively.) We assume our smooth manifolds M_s, M_t are compact and oriented without boundary. We also assume $\mathcal{M}$ has a tangent bundle and is oriented in the sense of Kuranishi structure. (See Definition A1.14 [FOOO2] and the description below.)

Remark 12.1. We may relax the orientability assumption above by using local coefficients in the same way as section A2 [FOOO2]. We do not discuss it here since we do not need this generalization in this paper.

We include the case when $\mathcal{M}$ has a boundary or corner. We assume that ev_t is weakly submersive. (See A1.13 [FOOO2] and the description below.) In this situation we will construct the map

$$(\mathcal{M}; ev_s, ev_t)_* : \Omega^k M_s \to \Omega^{k+\dim M_t - \dim \mathcal{M}} M_t. \tag{12.1}$$

We call (12.1), the *smooth correspondence map associated to* $(\mathcal{M}; ev_s, ev_t)$.

The space $\mathcal{M}$ is covered by a finite number of Kuranishi charts $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$, $\alpha \in \mathfrak{A}$. They satisfy the following :

Condition 12.2. (1) V_α is a smooth manifold (with boundaries or corners) and Γ_α is a finite group acting effectively on V_α.
(2) $\mathrm{pr}_\alpha : E_\alpha \to V_\alpha$ is a finite dimensional vector bundle on which Γ_α acts so that pr_α is Γ_α- equivariant.
(3) s_α is a Γ_α equivariant section of E_α.
(4) $\psi_\alpha : s_\alpha^{-1}(0)/\Gamma_\alpha \to \mathcal{M}$ is a homeomorphism to its image.
(5) The union of $\psi_\alpha(s_\alpha^{-1}(0)/\Gamma_\alpha)$ for various α is $\mathcal{M}$.

We assume that $\{(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha) \mid \alpha \in \mathfrak{A}\}$ is a good coordinate system, in the sense of Definition 6.1 [FO] or Lemma A1.11 [FOOO2]. This means the following : The set $\mathfrak{A}$ has a partial order $<$, where either $\alpha_1 \le \alpha_2$ or $\alpha_2 \le \alpha_1$ holds for $\alpha_1, \alpha_2 \in \mathfrak{A}$ if

$$\psi_{\alpha_1}(s_{\alpha_1}^{-1}(0)/\Gamma_{\alpha_1}) \cap \psi_{\alpha_2}(s_{\alpha_2}^{-1}(0)/\Gamma_{\alpha_2}) \ne \emptyset.$$

Let $\alpha_1, \alpha_2 \in \mathfrak{A}$ and $\alpha_1 \le \alpha_2$. Then, there exists a Γ_{α_1}-invariant open subset $V_{\alpha_2,\alpha_1} \subset V_{\alpha_1}$, a smooth embedding

$$\varphi_{\alpha_2,\alpha_1} : V_{\alpha_2,\alpha_1} \to V_{\alpha_2}$$

and a bundle map

$$\widehat{\varphi}_{\alpha_2,\alpha_1} : E_{\alpha_1}|_{V_{\alpha_2,\alpha_1}} \to E_{\alpha_2}.$$

which covers $\varphi_{\alpha_2,\alpha_1}$. Moreover there exists an injective homomorphism

$$\widehat{\widehat{\varphi}}_{\alpha_2,\alpha_1} : \Gamma_{\alpha_1} \to \Gamma_{\alpha_2}.$$

We require that they satisfies the following

Condition 12.3. (1) The maps $\varphi_{\alpha_2,\alpha_1}$, $\widehat{\varphi}_{\alpha_2,\alpha_1}$ are $\widehat{\widehat{\varphi}}_{\alpha_2,\alpha_1}$-equivariant.
(2) $\varphi_{\alpha_2,\alpha_1}$ and $\widehat{\widehat{\varphi}}_{\alpha_2,\alpha_1}$ induce an embedding of orbifold

$$\overline{\varphi}_{\alpha_2,\alpha_1} : \frac{V_{\alpha_2,\alpha_1}}{\Gamma_{\alpha_1}} \to \frac{V_{\alpha_2}}{\Gamma_{\alpha_2}}. \tag{12.2}$$

(3) We have

$$s_{\alpha_2} \circ \varphi_{\alpha_2,\alpha_1} = \widehat{\varphi}_{\alpha_2,\alpha_1} \circ s_{\alpha_1}.$$

(4) We have

$$\psi_{\alpha_2} \circ \overline{\varphi}_{\alpha_2,\alpha_1} = \psi_{\alpha_1}$$

on

$$\frac{V_{\alpha_2,\alpha_1} \cap s_{\alpha_1}^{-1}(0)}{\Gamma_{\alpha_1}}.$$

(5) If $\alpha_1 < \alpha_2 < \alpha_3$ then
$$\varphi_{\alpha_3,\alpha_2} \circ \varphi_{\alpha_2,\alpha_1} = \varphi_{\alpha_3,\alpha_1},$$
on $\varphi^{-1}_{\alpha_2,\alpha_1}(V_{\alpha_3,\alpha_2})$.
$$\widehat{\varphi}_{\alpha_3,\alpha_2} \circ \widehat{\varphi}_{\alpha_2,\alpha_1} = \widehat{\varphi}_{\alpha_3,\alpha_1},$$
and
$$\widehat{\widehat{\varphi}}_{\alpha_3,\alpha_2} \circ \widehat{\widehat{\varphi}}_{\alpha_2,\alpha_1} = \widehat{\widehat{\varphi}}_{\alpha_3,\alpha_1},$$
hold in the similar sense.

(6) $V_{\alpha_2,\alpha_1}/\Gamma_{\alpha_1}$ contains $\psi^{-1}_{\alpha_1}(\psi_{\alpha_1}(s^{-1}_{\alpha_1}(0)/\Gamma_{\alpha_1}) \cap \psi_{\alpha_2}(s^{-1}_{\alpha_2}(0)/\Gamma_{\alpha_2}))$.

Condition 12.4. The condition that $\mathcal{M}$ has a tangent bundle means the following : the differential of s_{α_2} in the direction of the normal bundle induces a bundle isomorphism
$$ds_{\alpha_2} : \frac{\varphi^*_{\alpha_2,\alpha_1} TV_{\alpha_2}}{TV_{\alpha_2,\alpha_1}} \to \frac{\widehat{\varphi}^*_{\alpha_2,\alpha_1} E_{\alpha_2}}{E_{\alpha_1}}.$$
We say $\mathcal{M}$ is oriented if V_α, E_α is oriented, the Γ_α action is orientation preserving, and ds_α is orientation preserving.

A strongly continuous smooth map $ev_t : \mathcal{M} \to M_t$ is a family of Γ_α invariant smooth maps
$$ev_{t;\alpha} : V_\alpha \to M_t \tag{12.3}$$
which induces
$$\overline{ev}_{t;\alpha} : V_\alpha/\Gamma_\alpha \to M_t$$
such that
$$\overline{ev}_{t;\alpha_2} \circ \overline{\varphi}_{\alpha_2,\alpha_1} = \overline{ev}_{t;\alpha_1}$$
on $V_{\alpha_2,\alpha_1}/\Gamma_\alpha$. (Note Γ_α action on M_t is trivial.) $ev_s : \mathcal{M} \to M_s$ consists of a similar family, $ev_{s;\alpha} : V_\alpha \to M_s$.

Our assumption that ev_t is weakly submersive means that each of $ev_{t;\alpha}$ in (12.3) is a submersion.

We next review on the multisections. (See section 3 [FO].) Let $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$ be a Kuranishi chart of $\mathcal{M}$. For $x \in V_\alpha$ we consider the fiber $E_{\alpha,x}$ of the bundle E_α at x. We take its l copies and consider the direct product $E^l_{\alpha,x}$. We take the quotient thereof by the action of symmetric group of order $l!$ and let $\mathcal{S}^l(E_{\alpha,x})$ be the quotient space. There exists a map
$$tm_m : \mathcal{S}^l(E_{\alpha,x}) \to \mathcal{S}^{lm}(E_{\alpha,x}),$$
which sends $[a_1, \cdots, a_l]$ to
$$[\underbrace{a_1, \cdots, a_1}_{m \text{ copies}}, \cdots, \underbrace{a_l, \cdots, a_l}_{m \text{ copies}}].$$
A smooth *multisection* s of the orbibundle
$$E_\alpha \to V_\alpha$$
consists of an open covering
$$\bigcup_i U_i = V_\alpha$$
and s_i which maps $x \in U_i$ to $s_i(x) \in \mathcal{S}^{l_i}(E_{\alpha,x})$. They are required to have the following properties.

Condition 12.5. (1) U_i is Γ_α-invariant. s_i is Γ_α-equivariant. (We remark that there exists an obvious map

$$\gamma : \mathcal{S}^{l_i}(E_{\alpha,x}) \to \mathcal{S}^{l_i}(E_{\alpha,\gamma x})$$

for each $\gamma \in \Gamma_\alpha$.)

(2) If $x \in U_i \cap U_j$ then we have

$$tm_{l_j}(s_i(x)) = tm_{l_i}(s_j(x)) \in \mathcal{S}^{l_i l_j}(E_{\alpha,\gamma x}).$$

(3) s_i is *liftable and smooth* in the following sense. For each x there exists a smooth section $\tilde{s}_i$ of $\underbrace{E_\alpha \oplus \cdots \oplus E_\alpha}_{l_i \text{ times}}$ in a neighborhood of x such that

$$\tilde{s}_i(y) = (s_{i,1}(y), \cdots, s_{i,l_i}(y)), \quad s_i(y) = [s_{i,1}(y), \cdots, s_{i,l_i}(y)]. \tag{12.4}$$

We identify two multisections $(\{U_i\}, \{s_i\}, \{l_i\})$, $(\{U'_i\}, \{s'_i\}, \{l'_i\})$ if

$$tm_{l_j}(s_i(x)) = tm_{l'_i}(s'_j(x)) \in \mathcal{S}^{l_i l'_j}(E_{\alpha,\gamma x})$$

on $U_i \cap U'_j$. We say $s_{i,j}$ to be a *branch* of s_i in the situation of (12.4).

We next discuss continuous family of multisections and their transversality. Let W_α be a finite dimensional smooth oriented manifold and consider the pull-back bundle

$$\pi_\alpha^* E_\alpha \to W_\alpha \times V_\alpha$$

under the projection $\pi_\alpha : W_\alpha \times V_\alpha \to V_\alpha$. The action of Γ_α on W_α is, by definition, trivial.

Definition 12.6. (1) A W_α-parameterized family $\mathfrak{s}_\alpha$ of multisections is by definition a multisection of $\pi_\alpha^* E_\alpha$.

(2) We fix a metric of our bundle E_α. We say $\mathfrak{s}_\alpha$ is ϵ-close to s_α in C^0 topology if the following holds. Let $(w,x) \in W_\alpha \times V_\alpha$. Then for any branch $\mathfrak{s}_{\alpha,i,j}$ of $\mathfrak{s}_\alpha$ we have

$$|\mathfrak{s}_{\alpha,i,j}(w,\cdots) - s_\alpha(\cdots)|_{C^0} < \epsilon$$

in a neighborhood of x.

(3) $\mathfrak{s}_\alpha$ is said to be transversal to 0 if any branch $\mathfrak{s}_{\alpha,i,j}$ of $\mathfrak{s}_\alpha$ is transversal to 0.

(4) Let $f_\alpha : V_\alpha \to M$ be a Γ_α-equivariant smooth map. We assume that $\mathfrak{s}_\alpha$ is transversal to 0. We then say that $f_\alpha|_{\mathfrak{s}_\alpha^{-1}(0)}$ is a submersion if the following holds : Let $(w,x) \in W_\alpha \times V_\alpha$. Then for any branch $\mathfrak{s}_{\alpha,i,j}$ of $\mathfrak{s}_\alpha$ the restriction of

$$f_\alpha \circ \pi_\alpha : W_\alpha \times V_\alpha \to M$$

to

$$\{(w,x) \mid \mathfrak{s}_{\alpha,i,j}(w,x) = 0\} \tag{12.5}$$

is a submersion. We remark that (12.5) is a smooth manifold by our assumption.

Remark 12.7. In case $\mathcal{M}$ has a boundary or a corner, so does (12.5). In this case we require that the restriction of f_α to each of the stratum of (12.5) is a submersion.

Lemma 12.8. *We assume that $f_\alpha : V_\alpha \to M$ is a submersion. Then there exists W_α such that for any ϵ there exists a W_α-parameterized family $\mathfrak{s}_\alpha$ of multisections which is ϵ close to s_α, transversal to 0 and such that $f_\alpha|_{\mathfrak{s}_\alpha^{-1}(0)}$ is a submersion.*

If $\mathfrak{s}_\alpha$ is already given and satisfies the required condition on a neighborhood of a Γ_α invariant compact set $K_\alpha \subset V_\alpha$, then we may extend it to the whole V_α without changing it on K_α.

In the course of the proof of Lemma 12.8 we need to shrink V_α slightly. We do not mention it explicitly.

Proof. We may choose W_α to be a vector space of sufficiently large dimension so that there exists a surjective bundle map

$$\mathrm{Sur} : W_\alpha \times V_\alpha \to E_\alpha. \tag{12.6}$$

We remark that (12.6) is not necessarily Γ_α-equivariant. We put

$$\mathfrak{s}_\alpha^{(1)}(w,x) = \mathrm{Sur}(w,x) + s_\alpha(x).$$

We put

$$\mathfrak{s}_\alpha^{(2)}(w,x) = [\gamma_1 \mathfrak{s}'_\alpha(w,x), \cdots, \gamma_g \mathfrak{s}'_\alpha(w,x)]$$

where $\Gamma_\alpha = \{\gamma_1, \cdots, \gamma_g\}$. $\mathfrak{s}^{(2)}$ defines a multisection on $W_\alpha \times V_\alpha$ which is transversal to 0 by construction. Moreover since $(\mathfrak{s}_\alpha^{(2)})^{-1}(0) \to V_\alpha$ is a submersion it follows from assumption that $f_\alpha|_{(\mathfrak{s}_\alpha^{(2)})^{-1}(0)}$ is a submersion. By replacing W_α to a small neighborhood of 0, we can choose $\mathfrak{s}_\alpha^{(2)}$ which is sufficiently close to s_α.

The last part of the lemma can be proved by using an appropriate partition of unity in the same way as section 3 [FO]. □

Now let θ_α be a smooth differential form of compact support on V_α. We assume that θ_α is Γ_α-invariant. Let $f_\alpha : V_\alpha \to M$ be a Γ_α equivariant submersion. (The Γ_α action on M is trivial.) Let $\mathfrak{s}_\alpha$ satisfy the conclusion of Lemma 12.8. We put a smooth measure ω_α on W_α of compact support with total mass 1. By fixing an orientation on W_α we regard ω_α as a differential form of top degree. We have

$$\int_{W_\alpha} \omega_\alpha = 1. \tag{12.7}$$

We next define *integration along the fiber*

$$((V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha), (W_\alpha, \omega_\alpha), \mathfrak{s}_\alpha, f_\alpha)_*(\theta_\alpha) \in \Omega^{\deg\theta_\alpha + \dim M - \dim \mathcal{M}}(M).$$

Let $(U_i, \mathfrak{s}_{\alpha,i})$ be a representative of $\mathfrak{s}_\alpha$. Namely $\{U_i \mid i \in I\}$ is an open covering of $W_\alpha \times V_\alpha$ and s_α is represented by $\mathfrak{s}_{\alpha,i}$ on U_i. By the definition of the multisection, U_i is Γ_α-invariant. We may shrink U_i, if necessary, so that there exists a lifting $\tilde{\mathfrak{s}}_{\alpha,i} = (\tilde{\mathfrak{s}}_{\alpha,i,1}, \cdots, \tilde{\mathfrak{s}}_{\alpha,i,l_i})$ as in (12.4).

Let $\{\chi_i \mid i \in I\}$ be a partition of unity subordinate to the covering $\{U_i \mid i \in I\}$. By replacing χ_i with its average over Γ_α we may assume χ_i is Γ_α-invariant.

We put

$$\tilde{\mathfrak{s}}_{\alpha,i,j}^{-1}(0) = \{(w,x) \in U_i \mid \tilde{\mathfrak{s}}_{\alpha,i,j}(w,x) = 0\}. \tag{12.8}$$

By assumption $\tilde{\mathfrak{s}}_{\alpha,i,j}^{-1}(0)$ is a smooth manifold and

$$f_\alpha \circ \pi_\alpha|_{\tilde{\mathfrak{s}}_{\alpha,i,j}^{-1}(0)} : \tilde{\mathfrak{s}}_{\alpha,i,j}^{-1}(0) \to M \tag{12.9}$$

is a submersion.

Definition 12.9. We define

$$\begin{aligned}&((V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha), (W_\alpha, \omega_\alpha), \mathfrak{s}_\alpha, f_\alpha)_*(\theta_\alpha)\\ &= \frac{1}{\#\Gamma_\alpha}\sum_{i=1}^{I}\sum_{j=1}^{l_i}\frac{1}{l_i}(f_\alpha\circ\pi_\alpha|_{\tilde{\mathfrak{s}}_{\alpha,i,j}^{-1}(0)})_!((\chi_i\pi_\alpha^*\theta_\alpha\wedge\omega_\alpha)|_{\tilde{\mathfrak{s}}_{\alpha,i,j}^{-1}(0)}).\end{aligned} \tag{12.10}$$

Here $(f_\alpha\circ\pi_\alpha|_{\tilde{\mathfrak{s}}_{\alpha,i,j}^{-1}(0)})_!$ is the integration along fiber of the smooth submersion (12.9).

Lemma 12.10. *The right hand side of* (12.10) *depends only on* $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$, $(W_\alpha, \omega_\alpha)$, $\mathfrak{s}_\alpha$, f_α, *and* θ_α *but independent of the following choices :*

1. *The choice of representatives* $(\{U_i\}, \mathfrak{s}_{\alpha,i})$ *of* $\mathfrak{s}_\alpha$.
2. *The lifting* $\tilde{\mathfrak{s}}_{\alpha,i}$.
3. *The partition of unity* χ_i.

Proof. The proof is straightforward generalization of the proof of well-definedness of integration on manifold, which can be found in the text book of manifold theory, and is left to the leader. □

So far we have been working on one Kuranishi chart $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$. We next describe the compatibility conditions among the W_α-parameterized families of multisections for various α. During the construction we need to shrink V_α a bit several times. We will not mention explicitly this point henceforth.

Let $\alpha_1 < \alpha_2$. We use an appropriate Γ_{α_2} invariant Riemannian metric on V_{α_2} to define the exponential map

$$\mathrm{Exp}_{\alpha_2,\alpha_1} : \varphi_{\alpha_2,\alpha_1}^* B_\epsilon V_{\alpha_2} \to V_2. \tag{12.11}$$

(Here $B_\epsilon V_{\alpha_2}$ is the ϵ neighborhood of the zero section of TV_{α_2}.)

We identify a neighborhood of the image of (12.11) with $\varphi_{\alpha_2,\alpha_1}^* B_\epsilon V_{\alpha_2}/\Gamma_{\alpha_1}$. and denote it by $U_\epsilon(V_{\alpha_2,\alpha_1}/\Gamma_{\alpha_1})$.

Using the projection

$$\mathrm{Pr}_{V_{\alpha_2,\alpha_1}} : U_\epsilon(V_{\alpha_2,\alpha_1}/\Gamma_{\alpha_1}) \to V_{\alpha_2,\alpha_1}/\Gamma_{\alpha_1}$$

we extend the orbibundle E_{α_1} to $U_\epsilon(V_{\alpha_2,\alpha_1}/\Gamma_{\alpha_1})$. Also we extend the embedding $E_{\alpha_1} \to \widehat{\varphi}_{\alpha_2,\alpha_1}^* E_{\alpha_2}$, (which is induced by $\widehat{\varphi}_{\alpha_2,\alpha_1}$) to $U_\epsilon(V_{\alpha_2,\alpha_1}/\Gamma_{\alpha_1})$.

We fix a Γ_α-invariant inner product of the bundles E_α. We then have a bundle isomorphism

$$E_{\alpha_2} \cong E_{\alpha_1} \oplus \frac{\widehat{\varphi}_{\alpha_2,\alpha_1}^* E_{\alpha_2}}{E_{\alpha_1}} \tag{12.12}$$

on $U_\varepsilon(V_{\alpha_2,\alpha_1}/\Gamma_{\alpha_1})$. We can use Condition 12.4 to modify $\mathrm{Exp}_{\alpha_2,\alpha_1}$ in (12.11) so that the following is satisfied.

Condition 12.11. If $y = \mathrm{Exp}_{\alpha_2,\alpha_1}(\tilde y) \in U_\epsilon(V_{\alpha_2,\alpha_1}/\Gamma_{\alpha_1})$ then

$$ds_{\alpha_2}(\tilde y \mod TV_{\alpha_1}) \equiv s_{\alpha_2}(y) \mod E_{\alpha_1}. \tag{12.13}$$

Let us explain the notation of (12.13). We remark that $\tilde y \in T_{\varphi_{\alpha_2,\alpha_1}(x)}V_{\alpha_2}$ for $x = \mathrm{Pr}(\tilde y) \in V_{\alpha_2,\alpha_1}$. Hence

$$\tilde y \mod TV_{\alpha_1} \in \frac{T_{\varphi_{\alpha_2,\alpha_1}(x)}V_{\alpha_2}}{T_x V_{\alpha_1}}.$$

Therefore

$$ds_{\alpha_2}(\tilde y \mod TV_{\alpha_1}) \in \frac{(E_{\alpha_2})_{\varphi_{\alpha_2,\alpha_1}(x)}}{(E_{\alpha_1})_x}.$$

(12.13) claims that it coincides with s_{α_2} modulo $(E_{\alpha_1})_x$.

We remark that Condition 12.4 implies that

$$\frac{d}{dt}(\mathrm{Exp}_{\alpha_2,\alpha_1}(t\tilde{y}))|_{t=0} \equiv \frac{d}{dt}s_{\alpha_2}(\mathrm{Exp}_{\alpha_2,\alpha_1}(t\tilde{y}))|_{t=0} \mod E_{\alpha_1}.$$

Therefore we can use implicit function theorem to modify $\mathrm{Exp}_{\alpha_2,\alpha_1}$ so that Condition 12.11 holds.

Let W_{α_1} be a finite dimensional manifold and $\mathfrak{s}_{\alpha_1}$ be a multisection of $\pi^*_{\alpha_1}E_{\alpha_1}$ on $W_{\alpha_1}\times V_{\alpha_1}$. We put $W_{\alpha_2} = W_{\alpha_1}\times W'$, where W' is to be defined later.

Definition 12.12. A multisection $\mathfrak{s}_{\alpha_2}$ of $W_{\alpha_2}\times V_{\alpha_2}$ is said to be *compatible* with $\mathfrak{s}_{\alpha_1}$ if the following holds for each $y = \mathrm{Exp}_{\alpha_2,\alpha_1}(\tilde{y}) \in U_\epsilon(V_{\alpha_2,\alpha_1}/\Gamma_{\alpha_1})$.

$$\mathfrak{s}_{\alpha_2}((w,w'),y) = \mathfrak{s}_{\alpha_1}(w,\mathrm{Pr}(\tilde{y})) \oplus ds_{\alpha_2}(\tilde{y} \mod TV_{\alpha_1}). \tag{12.14}$$

We remark that $\mathfrak{s}_{\alpha_1}(w,\mathrm{Pr}(\tilde{y}))$ is a multisection of $\pi^*_{\alpha_1}E_{\alpha_1}$ and $ds_{\alpha_2}(\tilde{y} \mod TV_{\alpha_1})$ is a (single valued) section. Therefore using (12.12) the right hand side of (12.14) is an element of $\mathcal{S}^{l_i}(E_{\alpha_2})_x$ $(x = \mathrm{Pr}(\tilde{y}))$, and hence is regarded as a multisection of $\pi^*_{\alpha_2}E_{\alpha_2}$. In other words, we omit $\widehat{\varphi}_{\alpha_2,\alpha_1}$ in (12.14).

Condition 12.11 implies that the original Kuranishi map s_α satisfies the compatibility condition (12.14). We use this and (the proof of) Lemma 12.8 and prove the following. Let $ev_t : \mathcal{M} \to M_t$ be a weakly submersive strongly smooth map. We choose a good coordinate system $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$ and let $ev_{t,\alpha} : V_\alpha \to M_t$ be a local representative of ev_t.

Lemma 12.13. *We have W_α such that for each ϵ there exists $\mathfrak{s}_\alpha$, a W_α-parameterized family of multisections with the following properties.*

1. *$\mathfrak{s}_\alpha$ is transversal to* 0.
2. *$ev_{t,\alpha}|_{\mathfrak{s}_\alpha^{-1}(0)}$ is a submersion.*
3. *$\mathfrak{s}_\alpha$ is ϵ close to s_α.*
4. *$\mathfrak{s}_{\alpha_2}$ is compatible with $\mathfrak{s}_{\alpha_1}$ for each $\alpha_1 < \alpha_2$.*

If $\{\mathfrak{s}_\alpha\}$ is already defined and satisfies (1) - (4) *on a neighborhood of a compact set $K \subset \mathcal{M}$, then we may choose $\mathfrak{s}_\alpha$ without changing it on K.*

Proof. The proof is by induction on α. (We remark that $\mathfrak{A}$ (the totality of α's) is partially ordered.) For minimal α we use Lemma 12.8 to prove existence of $\mathfrak{s}_\alpha$. If we have constructed $\mathfrak{s}_{\alpha'}$ for every α' smaller than α, then we use (12.14) to define $\mathfrak{s}_\alpha$ on a neighborhood of the images of $V_{\alpha,\alpha'}$ for various $\alpha' < \alpha$. They coincide on the overlapped part by the induction hypothesis and Condition 12.3. Condition 12.11 then implies that this is still ϵ close to s_α. Therefore we can use Lemma 12.8 (the relative version) to extend it and obtain $\mathfrak{s}_\alpha$. (We choose W' at this step.)

The proof of the last statement is similar. $\square$

We choose measures ω_α on W_α such that the measure ω_{α_2} is a direct product measure $\omega_{\alpha_1}\times\omega'$ on $W_\alpha \times W'$ if $\alpha_1 < \alpha_2$.

We next choose a partition of unity χ_α subordinate to our Kuranishi charts. To define the notion of partition of unity, we need some notation. For $\alpha_1 < \alpha_2$, we take the normal bundle $N_{V_{\alpha_1\alpha_2}}V_{\alpha_2}$ of $\varphi_{\alpha_1\alpha_2}(V_{\alpha_1\alpha_2})$ in V_{α_2}. Let $\mathrm{Pr}_{\alpha_1\alpha_2} : N_{V_{\alpha_1\alpha_2}}V_{\alpha_2} \to V_{\alpha_1\alpha_2}$ be the projection. We fix a Γ_{α_1}-invariant positive definite metric of $N_{V_{\alpha_1\alpha_2}}V_{\alpha_2}$ and let $r_{\alpha_1\alpha_2} : N_{V_{\alpha_1\alpha_2}}V_{\alpha_2} \to [0,\infty)$ be the norm with respect to this metric. We fix a sufficiently small δ and let $\chi^\delta : \mathbb{R} \to [0,1]$ be a smooth function such that

$$\chi^\delta(t) = \begin{cases} 0 & t \ge \delta \\ 1 & t \le \delta/2. \end{cases}$$

Let $U_\delta(V_{\alpha_1\alpha_2}/\Gamma_{\alpha_1})$ be the image of the exponential map. Namely

$$U_\delta(V_{\alpha_1\alpha_2}/\Gamma_{\alpha_1}) = \{\mathrm{Exp}(v) \mid v \in N_{V_{\alpha_1\alpha_2}}V_{\alpha_2}/\Gamma_{\alpha_1} \mid r_{\alpha_1\alpha_2}(v) < \delta\}.$$

We push out our function $r_{\alpha_1\alpha_2}$ to $U_\delta(V_{\alpha_1\alpha_2}/\Gamma_{\alpha_1})$ and denote it by the same symbol. It is called a *tubular distance function*. We assume appropriate compatibility condition for various tubular neighborhoods and tubular distance functions. See [Ma] and section 35.2 [FOOO2].

Let $x \in V_\alpha$. We put

$$\begin{aligned} \mathfrak{A}_{x,+} &= \{\alpha_+ \mid x \in V_{\alpha_+,\alpha},\, \alpha_+ > \alpha\} \\ \mathfrak{A}_{x,-} &= \{\alpha_- \mid [x \mod \Gamma_\alpha] \in U_\delta(V_{\alpha,\alpha_-}/\Gamma_{\alpha_-}),\, \alpha_- < \alpha\}. \end{aligned}$$

For $\alpha_- \in \mathfrak{A}_{x,-}$ we take x_{α_-} such that $\mathrm{Exp}(x_{\alpha_-}) = x$.

Definition 12.14. A system $\{\chi_\alpha \mid \alpha \in \mathfrak{A}\}$ of Γ_α-equivariant smooth functions $\chi_\alpha : V_\alpha \to [0,1]$ of compact support is said to be a partition of unity subordinate to our Kuranishi chart if :

$$\chi_\alpha(x) + \sum_{\alpha_- \in \mathfrak{A}_{x,-}} \chi^\delta(r_{\alpha\alpha_-}(x_{\alpha_-}))\chi_{\alpha_-}(\mathrm{Pr}_{\alpha_1\alpha_2}(x_{\alpha_-})) + \sum_{\alpha_+ \in \mathfrak{A}_{x,+}} \chi_{\alpha_+}(\varphi_{\alpha_+,\alpha}(x)) = 1.$$

Lemma 12.15. *There exists a partition of unity subordinate to our Kuranishi chart.*

Proof. We may assume that $\mathfrak{A}$ is a finite set since $\mathcal{M}$ is compact. By shrinking V_α if necessary we may assume that there exists V_α^- such that V_α^- is a relatively compact subset of V_α and that E_α, $\varphi_{\alpha_2,\alpha_1}$, s_α, etc restricted to V_α^- still defines a good coordinate system. We take a Γ_α invariant smooth function χ'_α on V_α which has compact support and satisfies $\chi'_\alpha = 1$ on V_α^-. We define

$$h_\alpha(x) = \chi'_\alpha(x) + \sum_{\alpha_- \in \mathfrak{A}_{x,-}} \chi^\delta(r_{\alpha\alpha_-}(x_{\alpha_-}))\chi'_{\alpha_-}(\mathrm{Pr}_{\alpha_1\alpha_2}(x_{\alpha_-})) + \sum_{\alpha_+ \in \mathfrak{A}_{x,+}} \chi'_{\alpha_+}(\varphi_{\alpha_+,\alpha}(x)).$$

Using compatibility of tubular neighborhoods and tubular distance functions, we can show that h_α is Γ_α invariant and

$$h_{\alpha_2}(\varphi_{\alpha_2,\alpha_1}(x)) = h_{\alpha_1}(x)$$

if $x \in V_{\alpha_2,\alpha_1}$. Therefore

$$\chi_\alpha(x) = \chi'_\alpha(x)/h_\alpha(x)$$

has the required properties. □

Now we consider the situation we start with. Namely we have two strongly continuous smooth maps

$$ev_s : \mathcal{M} \to M_s, \qquad ev_t : \mathcal{M} \to M_t$$

and ev_t is weakly submersive. Let h be a differential form on M_s. We choose $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$, $(W_\alpha, \omega_\alpha)$, $\mathfrak{s}_\alpha$ which satisfies (1) - (4) of Lemma 12.13. We also choose a partition of unity χ_α subordinate to our Kuranishi chart. We put

$$\theta_\alpha = \chi_\alpha(ev_s \circ \pi_\alpha)^* h \tag{12.15}$$

which is a differential form on $W_\alpha \times V_\alpha$.

Definition 12.16. We define

$$(\mathcal{M}; ev_s, ev_t)_*(h) = \sum_\alpha ((V_\alpha, \Gamma_\alpha, E_\alpha, \psi_\alpha, s_\alpha), (W_\alpha, \omega_\alpha), \mathfrak{s}_\alpha, ev_{t,\alpha})_*(\theta_\alpha). \tag{12.16}$$

This is a smooth differential form on M_t.

Remark 12.17.
(1) Actually the right hand side of (12.16) *depends* on the choice of $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$, $(W_\alpha, \omega_\alpha), \mathfrak{s}_\alpha$. We write $\mathfrak{s}$ to demonstrate this choice and write $(\mathcal{M}; ev_s, ev_t, \mathfrak{s})_*(h)$.
(2) The right hand side of (12.16) is independent of the choice of partition of unity. The proof is similar to the well-definiedness of integration on manifolds.

In case $\mathcal{M}$ has a boundary $\partial\mathcal{M}$, the choices $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$, $(W_\alpha, \omega_\alpha), \mathfrak{s}_\alpha$ on $\mathcal{M}$ induces one for $\partial\mathcal{M}$. We then have the following :

Lemma 12.18 (Stokes' theorem)**.** *We have*

$$d((\mathcal{M}; ev_s, ev_t, \mathfrak{s})_*(h)) = (\mathcal{M}; ev_s, ev_t, \mathfrak{s})_*(dh) + (\partial\mathcal{M}; ev_s, ev_t, \mathfrak{s})_*(h). \tag{12.17}$$

We will discuss the sign at the end of this section.

Proof. Using the partition of unity χ_α it suffices to consider the case when $\mathcal{M}$ has only one Kuranishi chart V_α. We use the open covering U_i of V_α and the partition of unity again to see that we need only to study on one U_i. In that case (12.17) is immediate from the usual Stokes' formula. □

We consider the following situation. We assume $\mathcal{M}$ is a space with Kuranishi structure with corners. Let $\partial_c\mathcal{M}$, $c = 1, \cdots, C$ be a decomposition of the boundary $\partial\mathcal{M}$ into components. The intersection $\partial_c\mathcal{M} \cap \partial_{c'}\mathcal{M}$ is a codimension 2 stratum of $\mathcal{M}$ if it is nonempty. We denote it by $\partial_{cc'}\mathcal{M}$. (Actually there may be a case where there is a self intersection of $\partial_c\mathcal{M}$ with itself. If it occurs there is a codimension 2 stratum of $\mathcal{M}$ corresponding to the self intersection points. We write it as $\partial_{cc}\mathcal{M}$.) $\partial_c\mathcal{M}$ is regarded as a space with Kuranishi structure which we denote by the same symbol. (This is slightly imprecise in case there is a self intersection. Since the way to handle it is rather obvious we do not discuss it here.) The boundary of $\partial_c\mathcal{M}$ is the union of $\partial_{cc'}\mathcal{M}$ for various c'. (Actually we include the case $c' = c$. In that case we take two copies of $\partial_{cc}M$, which become components of the boundary of $\partial_c\mathcal{M}$.)

Now we have the following :

Lemma 12.19. *If there exists data $\mathfrak{s}_c$ as in Remark* 12.17 (1) *on each of $\partial_c\mathcal{M}$. We assume that the restriction of $\mathfrak{s}_c$ to $\partial_{cc'}\mathcal{M}$ coincides with the restriction of $\mathfrak{s}_{c'}$ to $\partial_{cc'}\mathcal{M}$. We assume a similar compatibility at the self intersection $\partial_{cc}\mathcal{M}$.*

Then there exists a datum $\mathfrak{s}$ on $\mathcal{M}$ whose restriction to $\partial_c\mathcal{M}$ is $\mathfrak{s}_c$ for each c.

Proof. Using the compatibility condition we assumed we can define $\mathfrak{s}$ in a neighborhood of the union $\partial_c\mathcal{M}$ over c. We can then extend it by using Lemma 12.13. □

We next discuss composition of smooth correspondences. We consider the following situation. Let

$$ev_{s;st} : \mathcal{M}_{st} \to M_s, \qquad ev_{t;st} : \mathcal{M}_{st} \to M_t$$

be as before such that $ev_{t;st}$ is weakly submersive. Let

$$ev_{r;rs} : \mathcal{M}_{rs} \to M_r, \qquad ev_{s;rs} : \mathcal{M}_{rs} \to M_s$$

be a similar diagram such that $ev_{s;rs}$ is weakly submersive. We use the fact that $ev_{s;rs}$ is weakly submersive to define the fiber product

$$\mathcal{M}_{rs\ ev_{s;rs}} \times_{ev_{s;st}} \mathcal{M}_{st}$$

as a space with Kuranishi structure. We write it as $\mathcal{M}_{rt}$. We have a diagram of strongly continuous smooth maps

$$ev_{r;rt} : \mathcal{M}_{rt} \to M_r, \qquad ev_{t;rt} : \mathcal{M}_{rt} \to M_t.$$

It is easy to see that $ev_{t;rt}$ is weakly submersive.

We next make choices $\mathfrak{s}^{st}$, $\mathfrak{s}^{rs}$ for $\mathcal{M}_{st}$ and $\mathcal{M}_{rs}$. It is easy to see that it determines a choice $\mathfrak{s}^{rt}$ for $\mathcal{M}_{rt}$.

Now we have :

Lemma 12.20 (Composition formula)**.** *We have the following formula for each differential form h on M_r.*

$$\begin{aligned}&(\mathcal{M}^{rt}; ev_{r;rt}, ev_{t;,rt}, \mathfrak{s}^{rt})_*(h)\\ &= ((\mathcal{M}^{st}; ev_{s;st}, ev_{t;,st}, \mathfrak{s}^{st})_* \circ (\mathcal{M}^{rs}; ev_{r;rs}, ev_{s;,rs}, \mathfrak{s}^{rs})_*)(h).\end{aligned} \tag{12.18}$$

Proof. Using a partition of unity it suffices to study locally on $\mathcal{M}^{rs}$, $\mathcal{M}^{st}$. In that case it suffices to consider the case of usual manifold, which is well-known. □

We finally discuss the signs in Lemmas 12.18 and 12.20. It is rather cumbersome to fix appropriate sign convention and show those lemmata with sign. So, instead, we use the trick of section 53.3 [FOOO2] (see also section 13 [Fu3]) to reduce the orientation problem to the case which is already discussed in Chapter 9 [FOOO2], as follows.

For generic $w \in W_\alpha$, the space $\mathfrak{s}_{\alpha,i,j}^{-1}(0) \cap (\{w\} \times U_i)$ is a smooth manifold. Hence the right hand side of (12.10) can be regarded as an average of the correspondence by $\mathfrak{s}_{\alpha,i,j}^{-1}(0) \cap (\{w\} \times U_i)$ over w. We can also represent the smooth form h by an appropriate average (with respect to certain smooth measure) of a family of currents realized by smooth singular chains. So, as far as sign concerns, it suffices to consider a current realized by a smooth singular chain. Then the right hand side of (12.10) turn out to be a current realized by a smooth singular chain which is obtained from a smooth singular chain on M_s by a transversal smooth correspondence. In fact, we may assume that all the fiber products appearing here are transversal, since it suffices to discuss the sign at the generic point where the transversality holds. Thus the problem reduces to find a sign convention (and orientation) for correspondence of the singular chains by a smooth manifold. In the situation of our application, such sign convention (singular homology version) was determined and analyzed in detail in Chapter 9 [FOOO2]. Especially the existence of an appropriate orientation that is consistent with the sign appearing in A_∞ formulae etc. was proved there. Therefore we can prove that there is a sign (orientation) convention which induces all the formulae we need with sign, in our de Rham version, as well. See section 53.3 [FOOO2] or section 13 [Fu3] for detail.

References

[Ber] B.G. Berkovich, Spectral theory and analytic geometry over non-Archimedean fields. Mathematical Surveys and Monographs, 33. American Mathematical Society, Providence, RI, 1990.

[BGR] S. Bosch, U. Güntzer and R. Remmert, Non-Archimedean Analysis. A systematic Approach to Rigid Analytic Geometry, Grundlehren der Mathematischen Wissenschaften 261, Springer-Verlag, Berline, 1984.

[Cho] C.-H. Cho, *Non-displaceable Lagrangian submanifolds and Floer cohomology with non-unitary line bundle*, preprint 2007, arXiv:0710.5454.

[CO] C.-H. Cho and Y.-G. Oh, *Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds*, Asian J. Math. 10 (2006), 773-814.

[Dub] B. Dubrovin, *Geometry of 2D topological field theories*, Lecture note in Math. 1620 (1996), 120 - 348.

[Dui] J. J. Duistermaat, *On global action-angle coordinates*, Comm. Pure Appl. Math. 33 (1980), no. 6, 687–706.

[EP] M. Entov and L. Polterovich, *Calabi quasimorphism and quantum homology,* Int. Math. Res. Not. 2003, no. 30, 1635–1676.

[Fl] A. Floer, *Morse theory for Lagrangian intersections,* J. Differ. Geom. 28 (1988), 513-547.

[Fu1] K. Fukaya, *Mirror symmetry of abelian variety and multi theta function*, J. Algebraic Geom. 11 393–512 2002

[Fu2] K. Fukaya, *Application of Floer homology of Langrangian submanifolds to symplectic topology.* in 'Morse Theoretic Methods in Nonlinear Analysis and in Symplectic Topology", 231–276, NATO Sci. Ser. II Math. Phys. Chem., 217, Springer, Dordrecht, 2006.

[Fu3] K. Fukaya, *Differentiable operad, Kuranishi correspondence,and Foundation of topological field theories based on pseudo-holomorphic curve*, preprint (2007 Feb.), to appear in Proc. of AGAQ conference (ed. O. Ceyhan, Y. Manin, M.Marcolli).

[FOOO1] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian intersection Floer theory-anomaly and obstruction,* Kyoto University preprint, 2000.

[FOOO2] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian intersection Floer theory-anomaly and obstruction,* expanded version of [FOOO1], 2006 & 2007.

[FOOO3] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagranaign Floer theory on compact toric manifolds I*, preprint, arXiv:0802.1703.

[FO] K. Fukaya and K. Ono, *Arnold conjecture and Gromov-Witten invariant*, Topology 38 (1999), no. 5, 933–1048.

[Ful] W. Fulton, Introduction to Toric Varieties, Annals of Math. Studies, 131, Princeton University Press, Princeton, 1993.

[H] H. Hofer, *On the topological properties of symplectic maps*, Proc. Royal Soc. Edinburgh 115 (1990), 25-38.

[KL] S. Katz, and C.-C. Liu, *Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc*, Adv. Theor. Math. Phys. 5 (2001), 1–49.

[KS] M. Kontsevich and Y. Soibelman, *Affine structures and non-archimedean analytic spaces*, in : The Unity of Mathematics (P. Etingof, V. Retakh, I. M. Singer, eds), pp 321-385, Progr. Math. 244, Birkhäuser 2006.

[Ma] J. Mather, *Stratifications and mappings.* Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971) 41. 195–232. Academic Press. New York. 1973.

[OsTy] Y. Ostrover and I. Tyomkin, *On the quantum homology algebra of toric Fano manifolds*, preprint, arXiv:0804.0270.

[Ru] Y. Ruan, *Virtual neighborhood and pseudoholomorphic curve.* Turkish J. Math. 1999 161–231

[Sa] K. Saito, *Period mapping associated to a primitive form*, Publ. R.I.M.S. 19 (1983), 1231 - 1261.

[Ta] A. Takahashi, *Primitive forms, Topological LG model coupled with gravity, and mirror symmetry*, preprint, arXiv:9802059.

[W] A. Weinstein, *Symplectic manifolds and their Lagrangian submanifolds*, Advances in Math. 6 (1971), 329–346 .

Department of Mathematics, Kyoto University, Kyoto, Japan
E-mail address: fukaya@math.kyoto-u.ac.jp

Department of Mathematics, University of Wisconsin, Madison, WI, USA & Korea Institute for Advanced Study, Seoul, Korea
E-mail address: oh@math.wisc.edu

Graduate School of Mathematics, Nagoya University, Nagoya, Japan
E-mail address: ohta@math.nagoya-u.ac.jp

Department of Mathematics, Hokkaido University, Sapporo, Japan
E-mail address: ono@math.sci.hokudai.ac.jp