Informal note

on

Topology, Geometry
and
Topological Field Theory

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§ 0 Introduction

This note is an informal discussion on Topology - Geometry and Topological field theory. The author would like emphasise that this is an informal note. It means that several parts of the contents are yet far away from being a rigorous mathematics. (Of course results quoted in this note as theorems are all rigorous.) Also the author does not yet have a systematic study on some of the topics mentioned here

The author would like to explain what is the target of his recent (and coming) researchs or what he is dreamed of recently.

Our discussion is related to the following topics : “quatization of the concept of spaces” : “duality between Analysis and Geometry” : “Homological algebra and Homotopical algebra”.

Some of them do not yet have a well defined mathematical sence. We only have some phenomena or evidense which suggests their possibility.

§ 1 Duality between Geometry and Analysis

Let us start from the following "definition".

"Definition" 1.1[1] Let M be a space.
A *geometry* of M is studying a map from another space Y to M.
An *analysis* on M is studying a map from M to another space Y.

Example 1.2 Let $M = \boldsymbol{R}^2$ be the Euclidean plane. Then usual Euclidean geometry is studying lines, triangles, circles etc. They are maps from S^1 or $\boldsymbol{R}$ to $M = \boldsymbol{R}^2$.

The analysis on $M = \boldsymbol{R}^2$ is the study of functions of two variables $f(x,y)$, and so is the study of maps from $M = \boldsymbol{R}^2$ to $\boldsymbol{R}$.

From this "definition" it is quite clear what the author mean by "Gemetry is dual to Analysis". Of course everything is well known in Example 1.2. Let us mention less obvious exapmles.

The first example is well known De-Rham's theorem. Here :

[Geometry] The singular or simplicial homology theory $H_k(M)$ is defined by considering cycles, that is studying maps $\Delta \to M$ from a simplex Δ to M.

[Analysis] De-Rham cohomology is described by $\Lambda^k(M)$, the set of all k forms, and the homomorphism $d^k : \Lambda^k \to \Lambda^{k+1}$. Namely $H^k_{Dr}(M) = \dfrac{\ker d^k}{\operatorname{Im} d^{k-1}}$.

De-Rham's theorem implies that $Hom(H_k(M), \boldsymbol{R}) \cong H^k_{Dr}(M)$, the duality between geometry and analysis. We remark that this duality is basic in electro magnetism, that is the abelian gauge theory. We go back to this example in § 3.

Let me mention examples discovered more recently . That is related to the scaler curvature. There is basically two methods are known to study manifold of positive scaler curvature. One is minimal surface theory and the other is Dirac operator. I only mention small parts of them.

[Geometry] Minimal surface theory. Let M be a 3 manifold of positive scaler curvature. A theorem by D.Fisher-Coblbrie, R.Schoen and S.Yau says that if $\Sigma \to M$ is a (stable) minimal surface, then Σ is diffeomorphc to S^2. Using also existence result of minimal surface in a given homotopy class, we obtain a topological constraint of M.

Analysis: Let M be an even dimensional spin manifold. We then have a bundles of positive and negative spinors S^+, S^-. Using Riemannian metric we have an elliptic operator $D : S^+ \to S^-$, the Dirac operator. A theorem of Lichnerowitz implies that, if the Scaler

[1] M. Gromov mentioned this kind "Definition" in his talk at meeting of Japan Mathematical Society. He is not responsible to any nonsense in this note.

curvature of M is positive then kernel and cokernel of Dirac operator vanishes. On the other hand, the Atiyah-Singer index theorem implies that $\dim \ker D - \dim \operatorname{co}\ker K$ is a topological invariant of M, described explicitely by characteristic classes of M. Joining them we obtain a topological constraint of manifold of positive scaler curvature.

Thus Minimal surface and Dirac operator are the two basic methods to study manifolds of positive curvature. And, in fact, they imply the same result sometime, but by a different proof. It seems that there is no good explanation of the relation between these two methods. They are dual in our sence.

Problem 1.3 Find a more mathematical way to describe the duality between minimal surface and Dirac operator.

There is a similar phenomena related to positive mass conjecture.

The next example is a mirror symmetry. I can not discribe it precisely, partly because I do not have a enough knowledge on what is going on the study of Mirror symmetry, and partly because the problem is yet widely open.

Geometry: Let M be a 3 dimensional Kähler manifold. Assume the Ricci curvature of M vanishes. We consider the space

$$\mathcal{M}_k(M) = \left\{ \varphi : S^2 \to M \,\middle|\, J\varphi = \varphi J, \varphi^*[\omega] \cap [S^2] = k \right\}$$

Here ω is a Kähler form on M and J are complex structures of M and S^2. By the assumption on Ricci curvature, the virtual dimension of this space is independent of k and is equal to 0. By counting its order we obtain some numbers. This construction can be generalized to give a quantum deformation of cup product.

Analysis: Let M be as above. We consider the sheaf cohomology $H^1(M;\mathcal{O}(T))$. Here $\mathcal{O}(T)$ is the sheaf of holomorphic sections of tangent bundles. This cohomology is given by harmonic theory. There is a cubic map called Yukawa coupling $H^1(M;\mathcal{O}(T)) \otimes H^1(M;\mathcal{O}(T)) \otimes H^1(M;\mathcal{O}(T)) \to \mathbf{C}$.

The mirror conjecture asserts that the Geometry of M in the above sence is equivalent to Analysis of $\hat{M}$. Here $\hat{M}$ is another Ricci flat Kähler 3 fold which is called the Mirror of M.

Our fourth example is a theorem proved recently by C. Taubes [T] . Let (M,ω) be a symplectic 4 manifold.

Geometry: (Gromov-Witten invariant). We fix an almost complex structure J on M compatible with the symplectic structure of M. Namely we assume that $g(X,Y) = \omega(JX,Y)$ is a Riemannian metric. Let $x \in H^2(M;\mathbf{Z})$. Let Σ^g be the closed surface of genus g. We put

$$\mathcal{M}_g(M,x) = \frac{\{\varphi:(\Sigma^g, J_\Sigma) \to M \mid J\varphi = \varphi J_\Sigma,\ \varphi_*[\omega] = PD(x)\}}{\sim}$$

Here PD is the Poincare duality and $(\varphi, J_\Sigma) \sim (\varphi', J'_\Sigma)$ if and only if there exists a diffeomorphism $\psi : \Sigma^g \to \Sigma^g$ such that $(\varphi, J_\Sigma) = (\varphi'\psi, \psi^* J'_\Sigma)$. By Riemann-Roch's theorem, one can find the virtual dimention of $\mathcal{M}_g(M,x)$. We consider the order of this set $\mathcal{M}_g(M,x)$ counted with sign when the dimension is 0. This is by definition the Gromov-Witten invariant $Z_{GW}(M,x)$. (g is automatically determined.)

Analysis: (Seiberg-Witten invariant) Let $x \in H^2(M;\mathbf{Z})$ and L be a complex line bundle such that $c^1(L) = x$. We consider $Spin_{\mathbf{C}}$ structure P of M such that $L = P \times_{Spin_{\mathbf{C}}(4)} U(1)$. The group $Spin_{\mathbf{C}}(4)$ has (complex 2 dimensional) representations $\Delta^\pm$. We put $S^\pm = P \times_{Spin_{\mathbf{C}}(4)} \Delta^\pm$. Given a connection A on L there is a Dirac operator $D_A : \Gamma(S^+) \to \Gamma(S^-)$. Also there is a quadratic map $\Phi : \Gamma(S^+) \to \Gamma(\Lambda_2^+(M))$, where $\Lambda_2^+(M)$ is the bundle of self-adjoint 2 forms. The Monopole moduli space is defined by

$$\frac{\{(A,u) \mid D_A u = 0, \Phi(u) = dA + *dA\}}{\mathcal{G}}.$$

Here $\mathcal{G}$ is the Gauge group $Aut(L) = Map(M,U(1))$. Counting the order of this moduli space (with sign), when its virtual dimension is 0, we get the Seiberg -Witten invariant. $Z_{SW}(x)$.

Taubes' theorem assert $Z_{GW}(M,x) = Z_{SW}(x)$.

Remark 1.4 The above therem by Taubes seems to be closely related to Problem 1.3. Namely the Seiberg-Witten invariant is defined by using monopole equation which is a nonlinear version of Dirac operator. Also Seiberg- Witten invariant is closely related to scaler curvature. On the other hand, Gromov-Witten invariant is defined by using pseudo holomorphic curve equation. Holomorphic curve is a minimal surface. And Witten [W2] means that Gromov-Witten invariant is the twisted and topological version of the minimal surface theory (or harmonic map).

Now let M be a manifold. We consider $C(M)$, the space of all functions on M. The basic idea of noncommutative geometry by A.Connes and others is to consider noncomutative ring instead of the commutative ring $C(M)$. In our terminology, this can be regarded as an "analytic approach" to generalize the concept of spaces.

Problem 1.5 Are there any "geometric approach" which is dual to noncommutative geometry ?

Probably there is already an answer, that is the **String theory**. Let me remark that generalization of the concepts of spaces is the basic problem to quatize the gravity, and string theory is regarded as a good candidate of quatum gravity.

§ 2 Topological field theory and homologycal-homotopical algebra

It seems to me that the idea of String theory to generalize the concept of space is as follows. Let M be a space. We consider the space of maps $Map(\Sigma, M)$, from surface Σ to M. We consider an "integral"

$$\int_f e^{L(f)} \mathcal{D}f \tag{2.1}$$

Here $\mathcal{D}f$ is the "Feynmann measure", and $L(f)$ is a Lagrangian, a function on $Map(\Sigma, M)$, which we do not specify. The integral is taken over the space $Map(\Sigma, M)$ or its submanifold.

$L(f)$ may depend also on additional structures on M and Σ, for example vector bundle on M, submanifold of M or Σ etc. By chaning such additional structures on M or Σ and also the topological type of Σ (the genus of Σ), we obtain a system of numbers. (We fix topology and geometry of M.) We regard them as "correlation function of 2 dimensional field theory". This field theory is two dimensional, since (in our terminology) we are discussing analysis on Σ.

Thus if we could justify (2.1) for appropriate $L(f)$, we would get a "functor"

$$\{\text{Spaces}\} \Rightarrow \{\text{2 dimensional field theories}\}. \tag{2.2}$$

Here the author does not know the definitino of "the category of 2 dimensional field theory", because that is exactly the target of the research, which seems very hard and take a lot of time.

Conjecture 2.1 The functor is locally injective. Namely there is no deformation of spaces which gives the same field theory.

If this conjecture is true , then one may say that "2 dimensional field theory" is a generalization of the concept of spaces .

However it seems still too much difficult at this point to attack this kind of problems and to make these constructions mathematically rigorous. So instead of studying Geometry we start with Topology, which is somewhat easier. The schema is

Geometry	$\Leftrightarrow$	(Actual) Field theory
Topology	$\Leftrightarrow$	Topological Field theory

More precisely, the functor $\{\text{metric spaces}\} \Rightarrow \{\text{2 dimensional field theories}\}$, is an (actual) Field theory. And $\{\text{topological spaces}\} \Rightarrow \{\text{2 dimensional field theories}\}$ is a topological field theory.

Now we consider the problem, what is the definition of "2 dimensional field theory" we need for this purpose. The author believe that answering this question is exactly the algebraic topology. To explain why, let us recall what was the basic idea of algebraic topology.

We, for a moment, give up to replace all informations of topological spaces by algebras and try to replace spaces as much as possible by algebras. Namely our purpose here is to approximate spaces by algebras. The first step is :

Example 2.2 The homology theory is to approximate the space by chain complex $C_*(X)$.

Thus the strategy to generalize the concepts of the space is as follows. First replace the space (manifold) by algebraic objects as much as possible. Then generalize these algebraic objects.

Remark 2.3 This approach is, in fact, not so much satisfactory, since what we are looking for is a geometric (not algebraic) notion generalizing the concept of spaces. But this is the only approach I can try for a moment.

Remark 2.4 We again recall that there is one established way to replace space by algebra. That is to replace the space X by $C(X)$ the ring of functions. But as we mentioned, this is analytic approach. What we are looking for is an approach dual to this one.

The next step to approximat space by its (co) homology is to involve cup product and those kinds of product structure. In the analytic way, this was achieved by De-Rham homotopy theory or rational homotopy theory by D. Quillen - D. Sullivan. We recall

Definition 2.5 Two simply connected spaces M and M' are said to be $\boldsymbol{Q}$ homotopy equivalent if there exists N and maps $f : N \to M$, $f' : N \to M'$ such that f and f' induce isomorphism on homotopy groups $\otimes \boldsymbol{Q}$.

Roghly (and inprecisely) speaking, two spaces are $\boldsymbol{Q}$ homotopy equivalent, if and only if "they are homotopy equivalent up to finite ambiguity" .

The idea of rational homotopy theory is to approximate space by differential graded algebra (DGA) , the cochain complex with product structure. This is performed by using differential forms. We recall

Definition 2.6 $(\Lambda^k, d, \cdot)$ is said to be a differential graded algebra (DGA) if. $(\Lambda^k, \cdot)$ is a graded ring satisfying graded commutativity. Namely $\Lambda^k \cdot \Lambda^{k'} \subseteq \Lambda^{k+k'}$, $a \cdot b = -(-1)^{\deg a \deg b} b \cdot a$. Here $\deg a = k$ if $a \in \Lambda^k$. (Λ^k, d) is a chain complex (that is

$d:\Lambda^k \to \Lambda^{k+1}$, $dd=0$.) Finally d is a differential that is $d(a\cdot b)=da\cdot b+(-1)^{\deg a}a\cdot db$.

Let M be a manifold. Then the ring of differential forms $(\Lambda^k(M),d,\wedge)$ is a DGA. Thus we have a functor

(2.7) $$\{\text{manifolds}\}\to\{\text{DGA}\}$$

Remark 2.8 In fact, the DGA $(\Lambda^k(M),d,\wedge)$ itself does not work well for this construction, since it is not finitely generated and too big. Sullivan's idea in [Su], is to triangulate the manifold and use piecewise linear form with rational coefficient. We do not discuss this point here.

Then in inprecise way, the result by D.Quillen and D.Sullivan is described as

"Theorem" 2.9 The functor induces an isomorphism

$$\frac{\{\text{spaces}\}}{\text{homotopy equivalent}}\cong\frac{\{\text{DGA}\}}{\text{homotopy equivalent}}$$

Remark 2.10 I put **""** in Theorem 2.9 only because the way written here is not precise. The theorem is rigorously established in their papers.

Again this is analytic approach, since it uses differential form. What is dual and geometric approach to it. ? It is to consider tree and maps from it. Here we recall

Definition 2.11 A simplicial complex T if it is one dimensional and simply connected.

(See Table 1 for example.) Then dual in "Geometry" to the construction of DGA is as follows

: $\dim\Sigma=1$ Consider the space of maps $Map(T,M)$ from a tree to M.
: $\dim\Sigma=2$ Consider the space of maps $Map(S^2,M)$ from a Riemann sphere with marked points to M.

These ideas are discussed in more detail in [Fu1], [Fu2], so we do not repeart it here. Here are tables describing main ideas of this note.

§ 3 Degree, Linking number and Vassiliev invariant

To mention something more down to earch, we recall here the work by Gauss, Vassiliev, Bar-Nathan, Kontsevitch, Bott-Taubes, D.Thurston etc. In this section, we follow Bott-Taubes' paper [BT] and D.Thurston's lecture [Th] in many points.

Let us first consider the mapping degree of a map $f : M^n \to N^n$ between two closed oriented manifolds of the same dimension. There are two simple ways to define it :

(3.1.1) Let $p \in N$ be the regular value of f. Then $f^{-1}(p)$ consists of finitely many points. For each $q \in f^{-1}(p)$, we consider the differential $Df_q : T_q M \to T_p N$ of f. We put $\varepsilon_q = 1$ if Df_q preseaves orientation and $\varepsilon_q = -1$ if not. The mapping degree of f is then $\deg f = \sum_{q \in f^{-1}(p)} \varepsilon_q$.

(3.1.2) We choose any n-form ω on N such that $\int_N \omega \neq 0$. Then mapping degree of f is equal to $\deg f = \frac{\int_M f^* \omega}{\int_N \omega}$.

In our terminology (3.1.1) is a geometric way and (3.1.2) is an analytic way to define mapping degree. Their coincidence is a consequence of De-Rham's theorem.

To unify two definition one cay use (as De-Rham did) the notion of current. We recall that a k current T on an n dimensional manifold M is a map $T : \Lambda^{n-k}(M) \to \boldsymbol{R}$, which is $\boldsymbol{R}$-linear and is continuous with respect to the C^∞-topology. k form ω is regarded as a k current T_ω by $T_\omega(v) = \int_M \omega \wedge v$. An oriented $n-k$ dimensional submanifold N is regarded as a k current T_N by $T_N(v) = \int_N v$.

Using current we can rewrite (3.1.1) as $\deg f = \frac{\int_M f^* T_p}{\int_N T_p}$. In this way the description is completely parallel to (3.1.2). We remark that the pull back of the current $f^* T$ is not always well defined. In fact in the case $T = T_N$ for a submanifold N, the pull back $f^* T$ is well defined if and only if f is transversal to N. This is the condition we assumed in (3.1.1).

Now we use this exercise to study Gauss' formula for linking number. Let ω_{S^2} be the volume element of S^2 such that $\int_{S^2} \omega_{S^2} = 1$. We conisider a map $G : \left(\boldsymbol{R}^3 \times \boldsymbol{R}^3\right)(\Delta \to S^2$ by

$$G(x, y) = \frac{x - y}{\| x - y \|}.$$

Here Δ stands for the diagonal.

Lemma 3.2 $d\left(G^* \omega_{S^2}\right) = T_\Delta$.

Proof: Of course this is well known and very old lemma (probably goes back to Gauss.) But the author wants to prove it since the proof suggests the relation of the lemma to the compactification of configuration space due to Fulton-Macphearson [FM] and Kontsevitch [Ko1].

First we "complactify" $\left(\boldsymbol{R}^3 \times \boldsymbol{R}^3\right)(\Delta$ as follows. We take direct product $\left(\left(\boldsymbol{R}^3 \times \boldsymbol{R}^3\right)(\Delta\right) \times S^2$ and a map $\left(\boldsymbol{R}^3 \times \boldsymbol{R}^3\right)(\Delta \to \left(\left(\boldsymbol{R}^3 \times \boldsymbol{R}^3\right)(\Delta\right) \times S^2$, $(x,y) \to ((x,y), G(x,y))$. The closure of the image of this map is a manifold with boundary. We write it $\overline{\left(\boldsymbol{R}^3 \times \boldsymbol{R}^3\right)(\Delta}$. Obviously $G : \left(\boldsymbol{R}^3 \times \boldsymbol{R}^3\right)(\Delta \to S^2$ is extended to $\overline{\left(\boldsymbol{R}^3 \times \boldsymbol{R}^3\right)(\Delta}$. The boundary $\partial \overline{\left(\boldsymbol{R}^3 \times \boldsymbol{R}^3\right)(\Delta}$ of $\overline{\left(\boldsymbol{R}^3 \times \boldsymbol{R}^3\right)(\Delta}$ is identified to the unit sphere bundle $S(\boldsymbol{R}^3)$ of the tangent bundle of $\boldsymbol{R}^3$.

Now let $v \in \Lambda^3(\boldsymbol{R}^3 \times \boldsymbol{R}^3)$ be the arbitrary test function. Using, Stokes' theorem, we calculate

$$
\begin{aligned}
&\left(dG^* T_{\omega_{S^2}}\right)(v) \\
&\quad = \int_{\left(\boldsymbol{R}^3 \times \boldsymbol{R}^3\right)(\Delta} dv \wedge G^* \omega_{S^2} \\
&\quad = \int_{\overline{\left(\boldsymbol{R}^3 \times \boldsymbol{R}^3\right)(\Delta}} dv \wedge G^* \omega_{S^2} \\
&\quad = \int_{\overline{\left(\boldsymbol{R}^3 \times \boldsymbol{R}^3\right)(\Delta}} d\left(v \wedge G^* \omega_{S^2}\right) - \int_{\overline{\left(\boldsymbol{R}^3 \times \boldsymbol{R}^3\right)(\Delta}} v \wedge dG^* \omega_{S^2} \\
&\quad = \int_{\partial \overline{\left(\boldsymbol{R}^3 \times \boldsymbol{R}^3\right)(\Delta}} v \wedge G^* \omega_{S^2} \\
&\quad = \int_{\Delta} v
\end{aligned}
$$

The lemma follows.

Now let $\ell_i : S^1 \to \boldsymbol{R}$ ($i = 1, 2$) be the link, that is we assume that they are embeddings and that the images are disjoint to each other. We then obtain a map $\Phi : S^1 \times S^1 \to S^2$ by $\Phi(s,t) = G(\ell_1(s), \ell_2(t))$. The linking number of this link is by definition is the degree of this map. They are calculated by two different ways :

(3.3.1) We consider a generic point $p \in S^2$ and count the order of the set $\Phi^{-1}(p)$ with sign. (See (3.1.1) for the way to define sign.)

(3.3.2) We calculate the integral $\int_{S^2} \Phi^* \omega_{S^2}$.

They give the same number. It implies the famous formula by Gauss :

$$\text{(3.4)} \qquad Lk(\ell_1,\ell_2) = \int_0^1 \int_0^1 \frac{\left(\left(\ell_1(s)-\ell_2(s)\right)\times \frac{d\ell_1}{ds}\right)\bullet \frac{d\ell_2}{dt}}{\|\ell_1(s)-\ell_2(s)\|^3} dsdt$$

It is easy to see that (3.3.2) gives the right hand side of (3.4). To see that 3.3.1 gives the left hand side, we remark the following two facts.

(a) If $(\ell_1(s),\ell_2(t))$ is unlink, we can put one of them in the region $x >> 0$, $|y|,|z| < 1$, and the other in $x << 0$, $|y|,|z| < 1$. It is then easy to see that 3.3.1 is zero in that case.

(b) We study how many the number in 3.3.1 changes when ℓ_1 cross ℓ_2 and go other side. It is easy to see by geometric observation that it changes by ± 1.

Combining these two observation we are done.

Remark 3.5 There is third choice for the current representing the fundamental class of S^2. Namely we take $S^1 \subseteq S^2$, and consider the current $d\alpha \wedge T_{S^1}$. Here $\alpha : S^1 \to R/Z$ is the diffeomorphism. Using this current we obtain a formula similar to Kontsevitch integral (See [Ba2]) to give linking number. (This is pointed out by D.Thurston and D.Bar-Nathan in [Th].)

So far everything yet is classical and well known. But this formalism can be generalized directly to study Vassiliev invariant. Let $\ell : S^1 \to \mathbf{R}^3$ be a knot. Following Bott-Taubes we put

$$C_4 = \left\{ (t_1,t_2,t_3,t_4) \in \left(S^1\right)^4 \,\middle|\, t_1,t_2,t_3,t_4 \text{ respects the cyclic order} \right\}.$$

We have a map $\Phi_4 : C_4 \to \left(S^2\right)^2$, $(t_1,t_2,t_3,t_4) \mapsto \left(G(\ell(t_1),\ell(t_3)), G(\ell(t_2),\ell(t_4))\right)$. Unfortunately the space C_4 has a boundary. Hence the mapping degree of Φ_4 is not well defined. One take the following

$$C_{3,1} = \left\{ (t_1,t_2,t_3,x) \in \left(S^1\right)^3 \times \mathbf{R}^3 \,\middle|\, \begin{array}{l} t_1,t_2,t_3 \text{ respects the cyclic order} \\ x \neq \ell(t_1),\ell(t_2),\ell(t_3) \end{array} \right\}$$

We then have a map $\Phi_{3,1} : C_{3,1} \to \left(S^2\right)^3$, $(t_1,t_2,t_3,x) \mapsto (G(\ell(t_1),x), G(\ell(t_2),x), G(\ell(t_3),x))$. Bott-Taubes proved that the combining them one get a well defined a degree. (Their approach is based on De-Rham theory. D. Thurston gave a version of counting number of the inverse of one point.)

They use a compactification of the configulation spaces C_4, $C_{3,1}$. Those compactification is a generalization of $\overline{\left(\mathbf{R}^3 \times \mathbf{R}^3\right) (\Delta}$ discussed in the proof of Lemma 3.2.

We mention the relation of this construction to graph. The configulation space C_4 is related to the graph 1 and $C_{3,1}$ is related to graph 2 below.

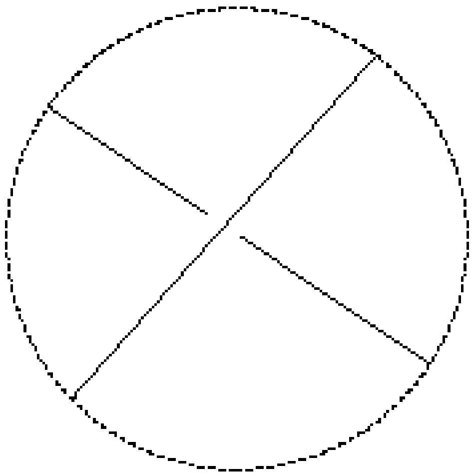

Figure 1

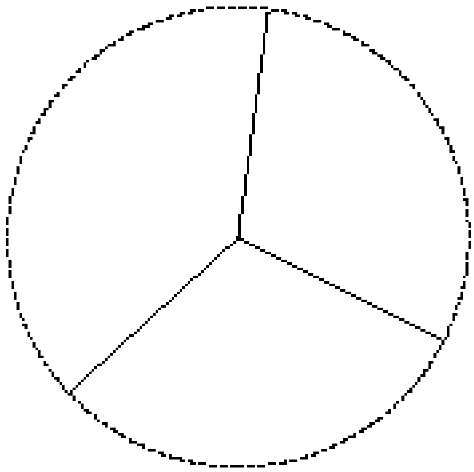

Figure 2

§4 Invariant of 3 manifolds and Morse homotopy

We want to generalization the construction of the last section to general 3 manifolds. Maybe the most direct generalization is as follows. We assume that our 3 manifold M is a homology 3 sphere. We fix a frame of M, that is a trivialization of the tangent bundle TM. We fix a point p of M and put $M^\circ = M \setminus \{p\}$.

Lemma 4.1 There exists a map $G : M^\circ \times M^\circ \setminus \Delta \to S^2$ which extends to a map from $\overline{M^\circ \times M^\circ \setminus \Delta}$ to S^2 such that its restriction to $\partial \overline{M^\circ \times M^\circ \setminus \Delta} = SM^\circ$ is obtained by the given framing of TM.

Here $\overline{M^\circ \times M^\circ \setminus \Delta}$ is defined in a similar way as $\overline{(\boldsymbol{R}^3 \times \boldsymbol{R}^3) \setminus \Delta}$. The proof of Lemma 4.1 is an easy obstruction theory. However in fact we need some more condition on the behavior of the map $G : M^\circ \times M^\circ \setminus \Delta \to S^2$ at the neighborhood of infinity of M°. (Namely in the neighborhood of the point p of M .)

We consider the 2 form $G^* \omega_{S^2}$ on $M^\circ \times M^\circ \setminus \Delta$. In a way similar to Lemma 3.2 we can prove $d\left(G^* \omega_{S^2}\right) = T_\Delta$. It seems likely that one can obtain an invariant of knot using this form in a way similar to Bott-Taubes.

Remark 4.2 M.Kontsevitch [Ko1] discussed 3 manifold invariant using 2 form ω on $M^\circ \times M^\circ \setminus \Delta$ such that $d\omega = T_\Delta$. Probably the construction of such a form he had in mind is the one discussed above. The way taken by Axerlod- Singer [AS] is to use Riemannian metric and Green's function to find such a form ω on $M^\circ \times M^\circ \setminus \Delta$ such that $d\omega = T_\Delta$. It is easy to see that the form obtained above is directly related to the framing of M.

To study 3 manifold invariant (instead of knot invariant) using this 2 form, we consider the following diagram.

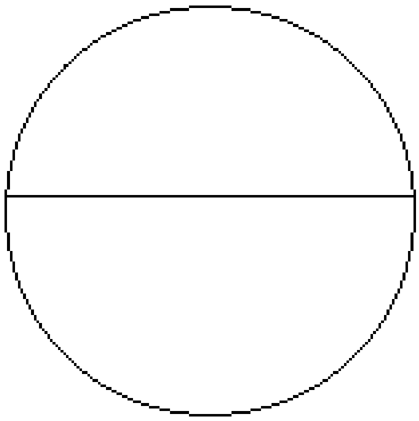

Figure 3

It means that we consider the following integral

(4.3) $$\int_{((x_1,y_1),(x_1,y_1),(x_1,y_1))\in\left(M^\circ\times M^\circ(\Delta\right)^3} \omega(x_1,y_1)\wedge\omega(x_2,y_2)\wedge\omega(x_3,y_3)$$

One need correction term to get well defined invariant. See Axerlod- Singer [AS] , for detail.

Problem 4.4 Find another version of the way to define the same invariant by counting argument using the map $G : M^\circ \times M^\circ (\Delta \to S^2$.

The author [Fu4], found counting argument to define similar (and most likely the same) invariant, in a bit different way from using the map $G : M^\circ \times M^\circ (\Delta \to S^2$. This approach is based on Morse theory and is roughly described as follows. We choose 3 generic functions f_1, f_2, f_3 on M. We fix a Riemannian metric on M. And let $\varphi_t^{f_i}$ be the one parameter group of transformations associated to the gradient vector field of f_i. We use the moduli space

(4.5) $$\mathcal{M}_{MS}(f_1, f_2, f_3) = \left\{(t_1,t_2,t_3,x,y)\in \boldsymbol{R}_+^3\times M^3 \middle| \varphi_{t_i}^{f_i}(x) = y \quad i = 1,2,3\right\}$$

Counting its order we obtain a number. We also need correction term to obtrain a well defined invariant. We omit the detail which is discussed in [Fu3], [Fu4].

Remark 4.6 It is sometimes simpler to work with local coefficient. Namely we consider a flat Lie algebra bundle η on M such that $H^*(M;\eta) = 0$. This implies that the current T_Δ represented by the diagonal in $M\times M$ is a boundary in this local coefficient. So we do not need to remove one point p to find a current ω such that $d\omega = T_\Delta$.

Remark 4.7 The graph in Figure 1.3 is regarded as a Feymann diagram. Then, the invariant we have been discussed is a 2 loop amplitude. There is an interaction in this case. In fact the interaction is related to the product structure. In Formula 4.3 it is related to the wedge product of the forms. And in Formula 4.5 it is related to the intersection theory.

Originally E.Guadagnini-M.Martinelli-M.Mintchev, D.Bar-Nathan, Axerlod-Singer, Kontsevitch discovered this kind of construction as a pertubation theory for Chern-Simons Gauge theory or Witten's invariant. It is Kontsevitch who obsearved its relation to homotopical algebra and to compactification of configulation space.

Finally we remark that their is probably a family version of this construction, as is pointed out by [AS][Ko]. Namely we can construct invariants of fibre bundle whose fiber is M.

§5 Relation to string theory

We next describe the relation of the construction in the last section to String theory. We have discussed this topic in [Fu5] last section and [FO]. So we do not go into detail. Let us consider the cotangent bundle T^*M of M. It has a canonical symplectic structure ω. We choose and fix an almost complex structure J of T^*M such that $\omega(JV,W) = g(V,W)$ is a Riemannian metric on T^*M. For each function f on M we associate a graph Λ_f of its differential df. Λ_f is a Lagrangian submanifold of T^*M. Namely $\omega|_{\Lambda_f} = 0$. We consider surface with boundary $\Sigma_{0,3} = D^2 - \cup D^2$ (Figure 4.)

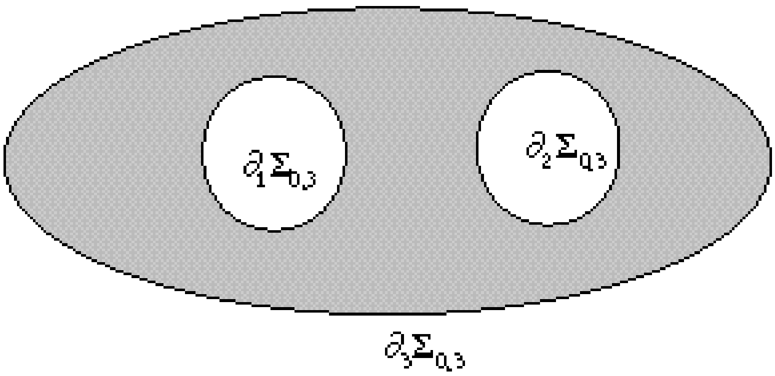

Figure 4

Then we consider the moduli space

$$\mathcal{M}_{SY}(f_1, f_2, f_3) = \frac{\left\{ (\varphi, J_{\Sigma_{0,3}}) \;\middle|\; \begin{array}{l} \varphi : \Sigma_{0,3} \to T^*M \\ J_{\Sigma_{0,3}} \text{ is a complex strucuture of } \Sigma_{0,3} \\ J\varphi = \varphi J_{\Sigma_{0,3}}, \\ \varphi(\partial_i \Sigma_{0,3}) \subseteq \Lambda_{f_i} \; i = 1,2,3 \end{array} \right\}}{\sim}$$

Here $\partial_i \Sigma_{0,3}$ $(i = 1,2,3)$ are components of the boundary of $\Sigma_{0,3}$. The equivalence relation $\sim$ is defined as follows.

$$(\varphi, J_{\Sigma_{0,3}}) \sim (\varphi', J'_{\Sigma_{0,3}}) \Leftrightarrow \exists \psi \quad \varphi' = \varphi\psi,\ J'_{\Sigma_{0,3}} = \psi^* J_{\Sigma_{0,3}}$$

We consider the order (counted with sign) of this moduli space. This is one of the correlation function of open string theory of T^*M. We add various correction term to obtain an (topological) invariant of M.

This invariant is related to one in the last section as follows :

Theorem5.1 (Fukaya-Oh) $\mathcal{M}_{SY}(\varepsilon f_1, \varepsilon f_2, \varepsilon f_3) = \mathcal{M}_{MS}(f_1, f_2, f_3)$ for sufficiently small ε.

There is a similar result which assert the ration between rational homotopy type and 0 loop amplitude of open string on the cotangent bundle ([FO]). In that case, we replace $\mathcal{M}_{SY}(\varepsilon f_1, \varepsilon f_2, \varepsilon f_3)$ by the moduli space of maps from D^2 to T^*M with appropriate bounary condition and $\mathcal{M}_{MS}(f_1, f_2, f_3)$ with the moduli space of maps from tree to M.

In the simplest case, namely in the case when the graph is a line and Riemann surface is D^2 with two marked points on the boundary, this result was proved by Floer and used in his calculation of Floer homology of Lagrangian intersection.

The author would like to mention only one part of the proof. Theorem 5.1 gives a relation between two moduli space. The space $\mathcal{M}_{MS}(f_1, f_2, f_3)$ is the space of maps from the graph (of Figure 3) to M. The space $\mathcal{M}_{SY}(\varepsilon f_1, \varepsilon f_2, \varepsilon f_3)$ is the moduli space of maps from open Riemann surface (of Figure 4) to T^*M.

The graph and the Riemann surface are given some additional structures. In the case of $\mathcal{M}_{MS}(f_1, f_2, f_3)$, we put positive numbers t_1, t_2, t_3 in definitino (4.5). They are regarded as a length of edges of the graph. In the case of $\mathcal{M}_{SY}(\varepsilon f_1, \varepsilon f_2, \varepsilon f_3)$, we take a complex structure of $\Sigma_{0,3}$.

So the first step is to find an isomorphism between moduli of metrics on the graph and moduli of complex structures of open Riemann surfaces. The similar relation was found between moduli of metric (ribbon) graphs and moduli of (closed) Riemann surface.

Our discussion thus implies that Chern-Simons Gauge theory on 3-manifold is equal to open string theory on its cotangent bundle. This is the observation of Witten [W5].

This kind of equivalence is an example of the phenomena that String theory sometimes is equivalent to the field theory of the target space. (The later is called, in that case, the string field theory.) The duality between geometry and analysis we mentioned in this kind cases is String theory = String Field theory, where the former is geometry and the later is analysis.

Probably there is another important case for this equivalence due to [BCOV]. In their case they study the closed string theory on Calabi-Yau 3 hold. (Namely complex 3 dimensional manifold with Ricci flat curvature.) Closed string theory, in this context, means counting holomorphc maps from closed Riemann surface. It seems to the author that their conclusion is that the “dual” to this construction is something related to the $\bar{\partial}$ opeartor. In the case discussed in this section, the dual to the open string is something related to the d operator. It is interesting to find some algebraic machinery which are common in 4 theories :$\bar{\partial}$,d ,closed string, open string.

We remark that the relation of homotopical algebra to closed string (the operad structure in conformal fiedl theory) is discussed also by many authors [St2] ,[HL], [Ge] etc.

§6 Illusion

The author want to state the "topological version" of Conjecture 2.1. As we sketched very briefly, we can construct system of numbers by studying maps from graph to a given manifold M. Namely we choose Morse function and count the number of maps from a graph to M such that each edge is a gradient line of one of the functions. There is a many variant of this kind of numbers. Namely we can involve homology class of the moduli space of the metric structure of graphs, also we can use various flat bundle, on target space M, and also cohomology class of M.

Moreover, there is a family version. Namely if there is a fiber bundle with M as a fiber, then we have again many numbers. In that case we can also use cohomology class of the base space.

Thus, there are quite a lot of numbers obtained in that way. The author does not know what kind of algebraic structure are enjoyed by this system of numbers. He would like to call Morse homotopy theory (of higher genus) which describes these numbers. They are topological field theory. And in the sence of section 2, they consists of a category of field theory on one dimensional space (graphs).

Conjecture 6.1 The functor

$$\{\text{smooth manifold}\} \Rightarrow \{\text{one dimensional field theory}\}$$

is injective. In other words, Morse homotopy gives complete invariant for differential topology. The family version is also true.

Maybe it is too much ambitious to write 6.1 as a conjecture. In fact there is not so many reason one can believe it. Let me give some evidence.

1: Let $\dim M \geq 5$ and M is simply connected. Then Sullivan's theorem says that M is determined up to finite ambiguity by its rational homotopy type and Pontrjagin class. We can find rational homotopy type from Morse homotopy. M.Betz and R.Cohen [BC] asserts that one can also find Pontrjagin class from Morse homotopy.

2: Let $\dim M = 3$. First there is some reason to believe that Vassiliev's invariant is a complete invariant of knot. When we take its analogy we might believe that the analogue of

Vassiliev's invariant (invariant of Chern-Simons perturbation theory) is a complete invariant of 3 manifold. It is almost sure that all Chern-Simons perturbation theory invariant come from Morse homotopy.

3. By simple dimension counting, it seems that the number of invariant we obtain from Morse homotopy rapidly decrease when the dimension becomes higher. We know from differential topology that in higher dimension we do not have so many invariant compared to the case of dimension 3 or 4.

4. Let $\dim M = 2$. Then of course there is not so many invariant for 2 manifold. So, to get something nontrivial, we need to consider the invariant of families. Namely the invariant of fibre bundle whose fiber is 2 manifold. Such an invariant is of course the cohomology of mapping class group or cohomology of moduli space of Riemann surface. On the other hand, our Morse homotopy invariant are supposed to be parametrized by the cohomology class of the moduli space of graphs. But one knows that moduli space of metric ribbon graph is the same as the moduli space of Riemann surface. So if we can find correct algabraic machinery, Conjecture 6.1 in this case, might be a trivial isomorphism. Namely Morse homotopy only gives identity map.

In fact the author does not have even very weak evience in the case when dimension is 4. In that case Taubes' result mentioned in § 1 shows Gromov-Witten invariant is a topological in invariant many cases. (Namely independent of symplectic structure.) It is not likely that this invariant comes from Morse homotopy. So in dimension 4 one may need also 2 dimensional field theory to describe smooth topology.

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