Appendices.[1]

§A1. Kuranishi structure.

In this book we use the moduli spaces of pseudo-holomorphic discs of various kinds and their fundamental chains to define various operators, homomorphisms and etc.. We need to appropriately perturb our moduli spaces so that they have "correct dimension" (that is the index of the linearized operator of pseudo-holomorphic curve equation) to define their fundamental chains.

A priori, our moduli spaces are not necessarily transversal. We use the general framework developed in [FuOn99II] (See [LiTi98], [Rua99], [Sie96] for related works.), where we introduced the notion of *Kuranishi structure*. Typical examples of the space with Kuranishi structure are the moduli spaces of various kinds, in which case Kuranishi neighborhoods are constructed using the finite dimensional reduction of the differential equation defining the moduli space. However the definition of Kuranishi structure applies to more general circumstances than that of studying the moduli space.

In the framework of Kuranishi structures, we use a multisection of the obstruction bundle, as a multi-valued perturbation of the originally given equation, for the construction of a perturbed moduli space of correct dimension and of the fundamental chain of the space. We call this fundamental chain a *virtual fundamental chain*. This machinery works in many different circumstances arising in geometry. In this section we give a rather complete and self-contained account on the story of Kuranishi structures needed for the applications in this book. We provide most of the proofs of the basic properties of the Kuranishi structure except the proof of some technical lemma for which we refer to the original paper [FuOn99II]. The definition of Kuranishi structure (and that of multisection) is a rather elementary notion whose understanding requires nothing more than basic knowledge of topology and geometry.

Most of the contents of this appendix consist of a review of materials from [FuOn99II]. So the reader who is familiar with Kuranishi structure does not need to read §A1.1. However we add a few new points in subsections §A1.2, §A1.3 and §A1.6 to those already in [FuOn99II]. Namely, we give the definition of the fiber product of spaces with Kuranishi structure in §A1.2 and that of finite group action on Kuranishi structure in §A1.3. Then in §A1.6, we discuss some special case of stacks arising in relation to the finite group action.

The boundaries of various moduli spaces that we inductively construct arise as the fiber product of other moduli spaces that are constructed in the earlier induction steps. (See §29.) So the fiber product enters in an essential way in

[1] version November 10, 2007

the precise description of the boundaries of the moduli spaces. We note that the fiber product is also used in the compactification of the moduli space in a similar inductive fashion in [FuOn99II] for the construction of Gromov-Witten invariant on general symplectic manifolds. However for the case of Gromov-Witten invariants one can work in the level of homology which enables us to choose perturbations (i.e., multisections) *individually* at each stage and hence the usage of fiber products in the compactification was not so essential. On the other hand, in the circumstance of this book where we need to work in the level of chains, it is *the essential point* to make the system of multisections constructed in different inductive steps *compatible to one another*. Therefore it is crucial to give the precise meaning of the compatibility where a careful treatment of the fiber product of moduli spaces is needed in relation to the transversality and the gluing.

A brief outline of the contents of this appendix is in order. We give a quick review of the materials on the Kuranishi structure from [FuOn99II] in §A1.1 and give the definition of fiber product in §A1.2 and that of a finite group action on the Kuranishi structure in §A1.3. The action of a finite group and its quotient space with Kuranishi structure is used in §41. Actually such a notion was already used in [FuOn99II, §23] but we provide a more precise description which is needed in §41.

We add some discussions about the smoothness of the coordinate transformation of the Kuranishi structure in §A1.4. In §A1.5 we examine several (counter) examples which illustrate some delicate issues arising in the definitions of Kuranishi structures and of orbifolds.

In §A1.6, we discuss a 'purely ineffective' version of orbifold. Namely a 'space' which is obtained by 'gluing' 'quotient spaces' M/G where the action of G on M is *trivial*. Especially we study 'vector bundles' on such a 'space'. Such a space naturally appears when we study the subset $X^{\cong}(\Gamma)$ of an orbifold X consisting of the points whose isotropy groups are isomorphic to Γ. In §35 we need to study the 'normal bundle' of $X^{\cong}(\Gamma)$ in X. We introduce such a notion in §A1.6. The story we describe there can be regarded as a special case of the theory of stack, which is due to Grothendick and is classical (goes back to 50's or 60's) at least in its scheme version. So this is really not new. However we provide full details in an elementary manner without using category theory, which is usually the common language used in the definition of stacks. We believe this direct elementary approach is useful for some of the readers who are not familiar to abstract algebraic geometry.

There were some technical errors in [FOOO00] and an error in [FuOn99II] related to the materials of this section. All of these errors are corrected in this version and none of them are essential. For the convenience of readers who already studied [FOOO00], [FuOn99II], we identify these errors and their remedies in §A1.7. For other readers, this section is not necessary to read.

A1.1. Review of the definition of the Kuranishi structure and multisec-

tion.

First of all, we define the Kuranishi structure (with corners). Here we rephrase the definitions in [FuOn99II] without using the phrase "smooth embedding of orbifolds". Let X be a compact metrizable space and $p \in X$.

Definition A1.1. A *Kuranishi neighborhood* of p in X is a quintet $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ such that

(A1.2.1) V_p is a smooth manifold of finite dimension, which may or may not have boundary or corner.
(A1.2.2) E_p is a real vector space of finite dimension.
(A1.2.3) Γ_p is a finite group acting smoothly and effectively on V_p and has a linear representation on E_p.
(A1.2.4) s_p is a Γ_p equivariant smooth map $V_p \to E_p$.
(A1.2.5) ψ_p is a homeomorphism from $s_p^{-1}(0)/\Gamma_p$ to a neighborhood of p in X.

We put $U_p = V_p/\Gamma_p$ and says that U_p is a Kuranishi neighborhood. We sometimes say that V_p is a Kuranishi neighborhood by an abuse of notation.

We call $E_p \times V_p \to V_p$ the *obstruction bundle* and s_p the *Kuranishi map.* For $x \in V_p$, denote by $(\Gamma_p)_x$ the isotropy subgroup at x, i.e.,

$$(\Gamma_p)_x = \{\gamma \in \Gamma_p | \gamma x = x\}.$$

In case V_p has boundary or corners, we say that $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ is a *Kuranishi structure with boundary or corner.*

Let us take a point $o_p \in V_p$ with $s_p(o_p) = 0$ and $\psi([o_p]) = p$. We may and will assume that o_p is fixed by all elements of Γ_p.

Definition A1.3. Let $(V_p, E_p, \Gamma_p, \psi_p, s_p)$, $(V_q, E_q, \Gamma_q, \psi_q, s_q)$ be Kuranishi neighborhoods of $p \in X$ and $q \in \psi_p(s_p^{-1}(0)/\Gamma_p)$, respectively. We say a triple $(\hat\phi_{pq}, \phi_{pq}, h_{pq})$ a *coordinate change* if

(A1.4.1) h_{pq} is an injective homomorphism $\Gamma_q \to \Gamma_p$.
(A1.4.2) $\phi_{pq} : V_{pq} \to V_p$ is an h_{pq} equivariant smooth embedding from a Γ_q invariant open neighborhood V_{pq} of o_q to V_p, such that the induced map $\underline{\phi}_{pq} : V_{pq}/\Gamma_q \to V_p/\Gamma_p$ is injective.
(A1.4.3) $(\hat\phi_{pq}, \phi_{pq})$ is an h_{pq} equivariant embedding of vector bundles $E_q \times V_{pq} \to E_p \times V_p$.
(A1.4.4) $\hat\phi_{pq} \circ s_q = s_p \circ \phi_{pq}$. Here and hereafter we sometimes regard s_p as a section $s_p : V_p \to E_p \times V_p$ of trivial bundle $E_p \times V_p \to V_p$.
(A1.4.5) $\psi_q = \psi_p \circ \underline{\phi}_{pq}$ on $(s_q^{-1}(0) \cap V_{pq})/\Gamma_q$. Here $\underline{\phi}_{pq}$ is as in (A1.4.2).
(A1.4.6) h_{pq} restricts to an isomorphism $(\Gamma_q)_x \to (\Gamma_p)_{\phi_{pq}(x)}$ for any $x \in V_{pq}$.

Combining Condition (A1.4.6) and the injectivity of $\underline{\phi}_{pq} : V_{pq}/\Gamma_q \to V_p/\Gamma_p$, we find

(A1.4.6') If $(\gamma\phi_{pq}(V_{pq})) \cap (\phi_{pq}(V_{pq})) \neq \emptyset$ and $\gamma \in \Gamma_p$, then $\gamma \in h_{pq}(\Gamma_q)$.

If we choose V_q sufficiently small, (A1.4.6) is equivalent to the condition that h_{pq} restricts to an isomorphism $\Gamma_q = (\Gamma_q)_{o_q} \to (\Gamma_p)_{\phi_{pq}(o_q)}$. In our construction of Kuranishi structure on the moduli spaces, Γ_p corresponds to the automorphism group of the stable map possibly with marked points. Thus (A1.4.6) is satisfied in our case.

Definition A1.5. A *Kuranishi structure* on X assigns a Kuranishi neighborhood $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ for each $p \in X$ and a coordinate change $(\hat\phi_{pq}, \phi_{pq}, h_{pq})$ for each $q \in \psi_p(s_p^{-1}(0)/\Gamma_p)$ such that the following holds.

(A1.6.1) $\dim V_p - \operatorname{rank} E_p$ is independent of p.
(A1.6.2) If $r \in \psi_q(s_q^{-1}(0)/\Gamma_q)$, $q \in \psi_p(s_p^{-1}(0)/\Gamma_p)$ then there exists $\gamma_{pqr} \in \Gamma_p$ such that

$$h_{pq} \circ h_{qr} = \gamma_{pqr} \cdot h_{pr} \cdot \gamma_{pqr}^{-1}, \quad \phi_{pq} \circ \phi_{qr} = \gamma_{pqr} \cdot \phi_{pr}, \quad \hat\phi_{pq} \circ \hat\phi_{qr} = \gamma_{pqr} \cdot \hat\phi_{pr}.$$

Here the second equality holds on $\phi_{qr}^{-1}(V_{pq}) \cap V_{qr} \cap V_{pr}$ and the third equality holds on $E_r \times (\phi_{qr}^{-1}(V_{pq}) \cap V_{qr} \cap V_{pr})$.

We remark that (A1.6.2) is equivalent to the condition that

$$\underline{\phi}_{pq} \circ \underline{\phi}_{qr} = \underline{\phi}_{pr}.$$

(We can prove this equivalence by using the effectivity of the Γ_p action.)

We can define the notion of germ of Kuranishi structure. But we do not use it in this book. We regard two Kuranishi structures

$$(\{(V_p, E_p, \Gamma_p, \psi_p, s_p)_p \mid p \in X\},\ \{(\hat\phi_{pq}, \phi_{pq}, h_{pq}) \mid p, q \in \psi_p(s_p^{-1}(0)/\Gamma_p)\})$$

and

$$(\{(V'_p, E'_p, \Gamma'_p, \psi'_p, s'_p)_p \mid p \in X\},\ \{(\hat\phi'_{pq}, \phi'_{pq}, h'_{pq}) \mid p, q \in \psi'_p(s'^{-1}_p(0)/\Gamma_p)\})$$

are the same if and only if all the data are the same, namely $V_p = V'_p, E_p = E'_p, \cdots, h_{pq} = h'_{pq}$.

We call $\dim V_p - \operatorname{rank} E_p$ the *virtual dimension* (or dimension) of the Kuranishi structure.

In case $K \subset X$ we say U is a *Kuranishi neighborhood* of K if $U = \{V_{p_i}/\Gamma_{p_i}\}$, $K \subset \bigcup_i \psi_{p_i}(s_{p_i}^{-1}(0)/\Gamma_{p_i})$.

An *orbifold* structure on X is, by definition, a Kuranishi structure on X such that $E_p = 0$ for all p.

Example A1.7. Let W_1 be a Banach manifold, W_2 be a Banach space and let $F : W_1 \to W_2$ be a smooth (non-linear) Fredholm map. We assume that there is an action of a discrete group G on W_1 and also G has a linear representation on W_2, such that F is equivariant.

We consider the situation when X is a moduli space of solutions of an equation

$$F(w) = 0. \tag{A1.8}$$

We assume that the action of G on the set of solutions of (A1.8) is properly discontinuous. (We remark that the set of solutions of (A1.8) is locally compact since F is Fredholm.)

We consider the space : $X = \{w \in W_1 \mid F(w) = 0\}/G$. Typically (A1.8) is a nonlinear elliptic partial differential equation.

In this situation, we can define a Kuranishi structure of X as follows.

Let w be a solution of (A1.8). We consider its linearized equation as follows. Let $u \in T_w W_1$. We put

$$D_w F(u) = \frac{d}{dt} F(w + tu)|_{t=0}.$$

Here $D_w F : T_w W_1 \to W_2$. (We identify $T_0 W_2 \cong W_2$.)

In case $D_w F : T_w W_1 \to W_2$ is surjective, X is a smooth manifold in a neighborhood of w by the implicit function theorem. Hence V_w (a Kuranishi neighborhood of $[w]$) can be chosen as a neighborhood of w in X and $E_w = 0$.

We consider the case when $D_w F : T_w W_1 \to W_2$ is not surjective but is a Fredholm operator. We choose a finite dimensional linear subspace $E_w \subset W_2$ such that

$$\mathrm{Im}(D_w F) + E_w = W_2. \tag{A1.9}$$

We now define V_w to be the set of all u in a neighborhood $U(w)$ of w such that

$$F(u) \equiv 0 \mod E_w. \tag{A1.10}$$

Γ_w is the set of automorphisms of w. Namely $\Gamma_w = \{g \in G \mid gw = w\}$. Since we assumed that the action of G on the set of solutions of (A1.8) is properly discontinuous, it follows that Γ_w is a finite group. If u satisfies (A1.10), we put

$$s_w(u) = F(u) \in E_w.$$

Then s_w defines a map $V_w \to E_w$. If $u \in V_w$ and $s_w(u) = 0$, then u is a solution of (A1.8) and hence its equivalence class $\psi_w(u)$ is an element of X. We can choose V_w small enough so that (A1.9) and the implicit function theorem imply that V_w is a smooth manifold. Thus we obtained a Kuranishi neighborhood $(V_w, E_w, \Gamma_w, \psi_w, s_w)$.

If w' is in the image of ψ_w, then we may choose $E_{w'}$ so that $E_{w'} \subset E_w$. (See the discussion at the end of §29.4.) In this case $F(u) \equiv 0 \mod E_{w'}$ implies (A1.10).

Hence $V_{w'} \cap U(w)$ is a submanifold of V_w. Therefore by putting $V_{ww'} = V_{w'} \cap U(w)$, we have $\phi_{ww'} : V_{ww'} \to V_w$. Moreover since $E_{w'} \subset E_w$, we have a bundle embedding $\hat\phi_{ww'}$. Furthermore, we may choose V_w enough small such that the group of automorphism $\Gamma_{w'}$ of elements of V_w is contained in the group Γ_w, the group of automorphism of w. We thus obtain a coordinate change $(\hat\phi_{ww'}, \phi_{ww'}, h_{ww'})$.

We remark that in general $\dim V_p \neq \dim V_q$. So it may happen that the coordinate change exists only in one direction. Our idea to obtain fundamental chain for space with Kuranishi structure is to perturb the Kuranishi map s_p so that it is transversal to zero. The perturbed Kuranishi map should be compatible with coordinate change. As in the usual proof of transversality theorem, we will construct perturbation of Kuranishi map in each of the Kuranishi neighborhood inductively. Since coordinate change may exist only in one direction, we need to find a clever choice of the order of the Kuranishi neighborhoods to work out the induction. The choice of good coordinate system below provides such a clever choice.

Lemma A1.11. ([FuOn99II, Lemma 6.3]) *Let X be a space with Kuranishi structure. Then there exists a finite set $P \subset X$, an order $<$ on P, and a Kuranishi neighborhood $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ of p for each $p \in P$, with the following properties.*

(A1.12.1) *If $q < p$, $\psi_p(s_p^{-1}(0)/\Gamma_p) \cap \psi_q(s_q^{-1}(0)/\Gamma_q) \neq \emptyset$, then there exists*

$$(V_{pq}, \hat\phi_{pq}, \phi_{pq}, h_{pq})$$

where :
(A1.12.1.1) *V_{pq} is a Γ_q invariant open subset of V_q such that V_{pq}/Γ_q contains $\psi_q^{-1}(\psi_p(s_p^{-1}(0)/\Gamma_p) \cap \psi_q(s_q^{-1}(0)/\Gamma_q))$,*
(A1.12.1.2) *h_{pq} is an injective homomorphism $\Gamma_q \to \Gamma_p$ with its image $(\Gamma_p)_{\phi_{pq}(q)}$,*
(A1.12.1.3) *$\phi_{pq} : V_{pq} \to V_p$ is an h_{pq} equivariant smooth embedding such that the induced map $V_{pq}/\Gamma_q \to V_p/\Gamma_p$ is injective,*
(A1.12.1.4) *$(\hat\phi_{pq}, \phi_{pq})$ is an h_{pq} equivariant embedding of vector bundles $E_q \times V_{pq} \to E_p \times V_p$,*
(A1.12.1.5) $\hat\phi_{pq} \circ s_q = s_p \circ \phi_{pq}, \qquad \psi_q = \psi_p \circ \underline{\phi}_{pq}.$

(A1.12.2) *If $r < q < p$, $\psi_p(s_p^{-1}(0)/\Gamma_p) \cap \psi_q(s_q^{-1}(0)/\Gamma_q) \cap \psi_r(s_r^{-1}(0)/\Gamma_r) \neq \emptyset$, then there exists $\gamma_{pqr} \in \Gamma_p$ such that*

$$h_{pq} \circ h_{qr} = \gamma_{pqr} \cdot h_{pr} \cdot \gamma_{pqr}^{-1}, \quad \phi_{pq} \circ \phi_{qr} = \gamma_{pqr} \cdot \phi_{pr}, \quad \hat\phi_{pq} \circ \hat\phi_{qr} = \gamma_{pqr} \cdot \hat\phi_{pr}.$$

Here the second equality holds on $\phi_{qr}^{-1}(V_{pq}) \cap V_{qr} \cap V_{pr}$, and the third equality holds on $E_r \times (\phi_{qr}^{-1}(V_{pq}) \cap V_{qr} \cap V_{pr})$.

(A1.12.3)

$$\bigcup_{p \in P} \psi_p(s_p^{-1}(0)/\Gamma_p) = X.$$

We call $\{(V_p, E_p, \Gamma_p, \psi_p, s_p)\}_{p\in P}$ a *good coordinate system.*

The proof of Lemma A1.11 is rather technical and is omitted here. See [FuOn99II] pp 957-958.

In this book we use virtual fundamental chains of the space with Kuranishi structure. A virtual fundamental chain may be regarded as a chain of some infinite dimensional space into which all the Kuranishi neighborhoods are embedded. However this picture is not so useful for our purpose of this book. We use the Kuranishi structure over a space X from which there is a map to another space, say Y, and use the virtual fundamental chain as a chain in the space Y. In our circumstances the map will be an evaluation map from the moduli spaces.

Namely we work in the following situation.

Definition A1.13. Consider the situation of Lemma A1.11. Let Y be a topological space. A family $\{f_p\}$ of Γ_p-equivariant continuous maps $f_p : V_p \to Y$ is said to be a *strongly continuous map* if

$$f_p \circ \phi_{pq} = f_q$$

on V_{pq}. A strongly continuous map induces a continuous map $f : X \to Y$. We will ambiguously denote $f = \{f_p\}$ when the meaning is clear.

When Y is a smooth manifold, a strongly continuous map $f : X \to Y$ is defined to be smooth if all $f_p : V_p \to Y$ are smooth. We say that it is *weakly submersive* if each of f_p is a submersion.

In the construction of this book, the evaluation maps from various moduli spaces are weakly submersive, i.e., the restriction of the evaluation map to each boundary, corner of the moduli space is also weakly submersive.

We need some more conditions for a space with Kuranishi structure X to have a fundamental chain. Let

$$x \in \psi_p(s_p^{-1}(0)/\Gamma_p) \cap \psi_q(s_q^{-1}(0)/\Gamma_q) \subset X,$$

where $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ and $(V_q, E_q, \Gamma_q, \psi_q, s_q)$ are Kuranishi neighborhoods. Let $x = \psi_p(x_p) = \psi_q(x_q)$. We consider $d_{x_p}s_p : T_{x_p}V_p \to (E_p)_{x_p}$ and $d_{x_q}s_q : T_{x_q}V_q \to (E_q)_{x_q}$. They may be regarded as tangential complexes of our 'moduli space'. In fact if s_p is transversal to 0 at x_p, then X is a manifold in a neighborhood of x_p and $T_{x_p}X = \mathrm{Ker}(d_{x_p}s_p)$. The condition below implies that the tangential complex $d_{x_p}s_p : T_{x_p}V_p \to (E_p)_{x_p}$ can be glued to give a virtual bundle on X.

Consider the situation of Lemma A1.10. We identify a neighborhood of $\phi_{pq}(V_{pq})$ in V_p with a neighborhood of the zero section of the normal bundle $N_{V_{pq}}V_p \to V_{pq}$, using an exponential map of an appropriate Riemannian metric. We take the fiber derivative of the Kuranishi map s_p along the fiber direction and obtain a homomorphism

$$d_{\mathrm{fiber}}s_p : N_{V_{pq}}V_p \to E_p \times V_{pq}$$

which is an h_{pq}-equivariant bundle homomorphism.

Definition A1.14. We say that the space with Kuranishi structure X *has a tangent bundle* if $d_{\rm fiber} s_p$ induces a bundle isomorphism

$$N_{V_{pq}} V_p \cong \frac{E_p \times V_{pq}}{\hat\phi_{pq}(E_q \times V_{pq})} \tag{A1.15}$$

as Γ_q-equivariant bundles on V_{pq}.

Note that Condition (A1.4.6) guarantees that the normal bundle of the smooth orbifold embedding $V_{pq}/\Gamma_q \to V_p/\Gamma_p$ is defined as an orbi-bundle. (See Definition A1.33 for the definition of orbi-bundle.)

We remark that the homomorphism (A1.15) does not depend on the choice of Riemannian metric which we use to identify a given tubular neighborhood with an open set of the normal bundle.

By definition, the following diagram commutes for each $x \in \phi_{qr}^{-1}(V_{pq}) \cap V_{qr} \cap V_{pr}$.

$$\begin{CD}
(N_{V_{qr}}V_q)_x @>{\gamma_{pqr}\cdot d\phi_{pq}}>> (N_{V_{pr}}V_p)_x @>>> (N_{V_{pq}}V_p)_{\phi_{qr}(x)} \\
@VVV @VVV @VVV \\
\dfrac{E_q}{\hat\phi_{qr,x}(E_r)} @>{\gamma_{pqr}\cdot\hat\phi_{pq}}>> \dfrac{E_p}{\hat\phi_{pr,x}(E_r)} @>>> \dfrac{E_p}{\hat\phi_{pq,\phi_{qr}(x)}(E_q)}
\end{CD} \tag{A1.16}$$

Here and hereafter $\hat\phi_{qp,x} : E_q \to E_p$ is the restriction of the bundle map $\hat\phi_{pq}$ to the fiber of x.

Definition A1.17. Let X be a space with Kuranishi structure which has a tangent bundle. We say that the Kuranishi structure on X is *oriented* if we have a trivialization of

$$\Lambda^{\rm top} E_p^* \otimes \Lambda^{\rm top} TV_p$$

which is compatible with isomorphism (A1.15).

Example A1.18. Let us consider the situation of Example A1.7. We show that this Kuranishi structure has a tangent bundle : Let w' is in the image of ψ_w. We have $E_{w'} \subset E_w$. Hence if $u \in V_{ww'} \subset V_w$ then

$$T_u V_w = (D_u P)^{-1}(E_w), \quad T_u V_{w'} = (D_u P)^{-1}(E_{w'}).$$

Therefore, since $\mathrm{Im}(D_v P) + E_{w'} = W_2$, an isomorphism

$$\frac{T_u V_w}{T_u V_{w'}} \cong \frac{E_w}{E_{w'}}$$

is induced by dF. The isomorphism (A1.15) is induced from this isomorphism. Since the Kuranishi map is a restriction of F in this example, the above isomorphism is induced by the Kuranishi map.

Now we describe the construction of the virtual fundamental chain in [FuOn99II]. For this purpose we need the notion of multisection. We assume that Γ acts on a manifold V and a vector space E. A symmetric group $\mathfrak{S}_n$ of order $n!$ acts on the product E^n by

$$\sigma(x_1, \cdots, x_n) = (x_{\sigma(1)}, \cdots, x_{\sigma(n)}).$$

Let $S^n(E)$ be the quotient space $E^n/\mathfrak{S}_n$. Then Γ action on E induces one on $S^n(E)$. The map $E^n \to E^{nm}$ defined by

$$(x_1, \cdots, x_n) \mapsto (\underbrace{x_1, \cdots, x_1}_{m \text{ times}}, \underbrace{x_2, \cdots, x_2}_{m \text{ times}}, \cdots, \underbrace{x_n, \cdots, x_n}_{m \text{ times}})$$

induces a Γ equivariant map $S^n(E) \to S^{nm}(E)$.

Definition A1.19. An n-multisection s of $\pi : E \times V \to V$ is a Γ-equivariant map $V \to S^n(E)$. We say that it is *liftable* if there exists $\widetilde{s} = (\widetilde{s}_1, \cdots, \widetilde{s}_n) : V \to E^n$ such that its composition with $\pi : E^n \to S^n(E)$ is s. (We do not assume $\widetilde{s}$ to be Γ equivariant.) Each of $\widetilde{s}_1, \cdots, \widetilde{s}_n$ is said to be a *branch* of s.

If $s : V \to S^n(E)$ is an n multisection, then it induces an nm multisection for each m by composing it with $S^n(E) \to S^{nm}(E)$.

An n multisection s is said to be equivalent to an m multisection s' if the induced nm multisections coincide to each other. An equivalence class by this equivalence relation is said to be a *multisection.*

A liftable multisection is said to be *transversal* to zero if each of its branch is transversal to zero.

A family of multisections s_ϵ is said to *converge* to s as $\epsilon \to 0$ if there exists n such that s_ϵ is represented by an n-multisection s_ϵ^n and s_ϵ^n converges to a representative of s.

From now on we assume all the multisections are liftable unless otherwise stated.

Lemma A1.20. *For any multisection s of $\pi : E \times V \to V$, which is transversal to zero in a Kuranishi neighborhood of $K \subset V/\Gamma$, there exists a family of multisections s_ϵ converging to s in the C^0-sense such that s_ϵ is transversal to zero and $s_\epsilon = s$ in a neighborhood of K.*

Proof. Let m be the order of Γ. Let s be represented by $\pi \circ \widetilde{s}^n$ where $\widetilde{s}^n : V \to E^n$. Then, by the usual transversality theorem, there exists $\widetilde{s}_\epsilon = (\widetilde{s}_{\epsilon,i})_{i=1,\cdots,n}$ which converges to $\widetilde{s}^n$, is transversal to zero, and coincides with $\widetilde{s}^n$ on $\widetilde{K}$. (Here $\widetilde{K} \subset V$ such that $\widetilde{K}/\Gamma = K$.) We define $\widetilde{s}_\epsilon : V \to E^{nm}$ by $(\sigma \circ \widetilde{s}_{\epsilon,i} \circ \sigma^{-1})_{i=1,\cdots,n,\,\sigma\in\Gamma}$. (Here $m = \#\Gamma$.) The equivalence class of $\widetilde{s}_\epsilon$ is the required multisection. □

Definition A1.21. Let us consider the situation of Lemma A1.11. We assume that our Kuranishi structure has a tangent bundle. Suppose $p, q \in P$, $q < p$ and suppose we have multisections s'_p, s'_q of $E_p \times V_p$, $E_q \times V_q$ respectively. We now define the compatibility between them.

Now consider the embedding $\phi_{pq} : V_{pq} \to V_p$ in (A1.12.1.3). We identify its normal bundle $N_{V_{pq}}V_p$ with a tubular neighborhood of $\phi_{pq}(V_{pq})$. For each $x \in V_{pq}$ we fix a splitting

$$E_p \cong \hat{\phi}_{pq,x}(E_q) \oplus \frac{E_p}{\hat{\phi}_{pq,x}(E_q)}. \tag{A1.22}$$

For each $y \in N_{V_{pq}}V_p$, we obtain an element $1(y)$ of $\frac{E_p}{\hat{\phi}_{pq,\pi(x)}(E_q)}$ by using the isomorphism (A1.15).

Then, on $N_{V_{pq}}V_p$, we define a multisection $s'_q \oplus 1$ as follows. Let $(\widetilde{s}'_{q,1}, \cdots, \widetilde{s}'_{q,n})$ be a representative of s'_q. We put (using (A1.22))

$$\begin{aligned}(\widetilde{s}'_q \oplus 1)(y) &= (\widetilde{s}'_{q,1}(\pi(y)) \oplus 1(y), \cdots, \widetilde{s}'_{q,n}(\pi(y)) \oplus 1(y)) \\ &\in \left(\hat{\phi}_{pq,x}(E_q) \oplus \frac{E_p}{\hat{\phi}_{pq,\pi(y)}(E_q)}\right)^n \cong E_p^n.\end{aligned}$$

We denote by $s'_q \oplus 1$ the equivalence class of $\widetilde{s}'_q \oplus 1$.

Now we say that s'_p is *compatible* with s'_q if the restriction of s'_p to $N_{V_{pq}}V_p$ coincides with $s'_q \oplus 1$.

We remark that the (single valued) section s_p (that is a Kuranishi map) is compatible with s_q in the sense defined above. This follows from the fact that (A1.15) is induced by the fiber derivative of the Kuranishi map.

A *multisection* $\mathfrak{s} = \{s'_p\}_{p\in P}$ is a compatible system of multisections ($s'_p : V_p \to E_p$). If X is a space with Kuranishi structure, then a multisection of X is, by definition, a multisection of its obstruction bundle.

Theorem A1.23. *Consider the circumstance of Lemma* A1.11*. Suppose the Kuranishi structure over X has a tangent bundle. Then there exists a family of multisections $\mathfrak{s}_\epsilon = \{s'_{p,\epsilon}\}$ such that it converges to $\{s_p\}_{p\in P}$ (the Kuranishi map) and such that $s'_{p,\epsilon}$ are transversal to* 0 *for all $\epsilon > 0$.*

Moreover if $K \subset X$ is a compact set and if $\{s''_{p,\epsilon}\}$ is a family of multisections in a Kuranishi neighborhood of K that converges to $\{s_p\}_{p\in P}$ and is transversal to 0 *on K, then we can take the family $\{s'_{p,\epsilon}\}_{p,\epsilon}$ so that it coincides with $\{s''_{p,\epsilon}\}$ on some Kuranishi neighborhood of K.*

Proof. We can construct $\{s'_{p,\epsilon}\}_{p,\epsilon}$ on V_p inductively on the order $<$ on P. Each inductive step is done by Lemma A1.20 as follows. Let us assume that $\{s_{q,\epsilon}\}_{q\in P}$ is defined for $q < p$. Then, on the union of $N_{V_{pq}}V_p$ ($q < p$), the multisection s_p is already determined as $\widetilde{s}'_{q,\epsilon} \oplus 1$ which was described in Definition A1.21. (If $r < q < p$, then we can prove $\widetilde{s}'_{q,\epsilon} \oplus 1 = \widetilde{s}'_{r,\epsilon} \oplus 1$ on $N_{V_{pq}}V_p \cap N_{V_{pr}}V_p$ by using commutativity of (A1.16).) We can now construct $\widetilde{s}'_{p,\epsilon}$ by Lemma A1.20. $\square$

Let Y be a topological space and $\{f_p\}$ ($f_p : V_p \to Y$) be a strongly continuous map. We will construct a virtual fundamental chain, which is a chain in Y using the multisections $\mathfrak{s}_\epsilon$ produced in Theorem A1.23. This construction is now in order.

We put

$$\begin{cases} \widetilde{(s'_{p,\epsilon})^{-1}}(0) = \{y \in V_p \mid \text{there exists a branch } \widetilde{s}'_{p,\epsilon,i} \text{ of } s'_{p,\epsilon,i} \text{ such that } \widetilde{s}'_{p,\epsilon,i}(y) = 0\}, \\ (s'_{p,\epsilon})^{-1}(0) = \widetilde{(s'_{p,\epsilon})^{-1}}(0)/\Gamma_p. \end{cases}$$

The compatibility implies that

$$\underline{\phi}_{pq}\left(\frac{\widetilde{(s'_{q,\epsilon})^{-1}}(0) \cap V_{pq}}{\Gamma_q}\right) = \frac{\widetilde{(s'_{p,\epsilon})^{-1}}(0) \cap \Gamma_p \cdot (N_{V_{pq}} V_q)}{\Gamma_p}, \tag{A1.24}$$

where $\underline{\phi}_{pq} : V_{pq}/\Gamma_q \to V_p/\Gamma_p$. We put

$$((s'_\epsilon)^{-1}(0))_{\mathrm{set}} = \bigcup_{p \in P} (s'_{p,\epsilon})^{-1}(0)/\sim,$$

where $\sim$ implies that we identify $y \in (s'_{q,\epsilon})^{-1}(0)$ with $\underline{\phi}_{pq}(y) \in (s'_{p,\epsilon})^{-1}(0)$.

For each p we choose n_p and $\widetilde{s}'_{p,\epsilon} = (\widetilde{s}'_{p,\epsilon,1}, \cdots, \widetilde{s}'_{p,\epsilon,n_p}) : V_p \to E_p^{n_p}$ such that $\pi \circ \widetilde{s}'_{p,\epsilon}$ (where $\pi : E_p^{n_p} \to S^{n_p}(E_p)$ is the projection) is a representative of $s'_{p,\epsilon}$. For $y \in \widetilde{(s'_{p,\epsilon})^{-1}}(0)$, we put

$$\mathrm{val}_p(\pi(y)) = \#\{i \mid \widetilde{s}'_{p,\epsilon,i}(y) = 0\}, \tag{A1.25}$$

where $\pi : \widetilde{(s'_{p,\epsilon})^{-1}}(0) \to (s'_{p,\epsilon})^{-1}(0)$.

Lemma A1.26. ([FuOn99II] Lemma 6.9) *For a generic choice of s'_ϵ, the space $((s'_\epsilon)^{-1}(0))_{\mathrm{set}}$ has a smooth triangulation such that for each simplex Δ_a^m in the triangulation, there exists p_a such that $\Delta_a^m \subset (s'_{p_a,\epsilon})^{-1}(0)$ and val_{p_a} is constant on $\mathrm{Int}\Delta_a^m$.*

The proof is rather technical and is omitted, cf. [FuOn99II] p. 946.

Now assume that X is oriented. Let d be the virtual dimension of X. For each d dimensional simplex Δ_a^d of $((s'_\epsilon)^{-1}(0))_{\mathrm{set}}$ we define its multiplicity as follows. (We remark that the dimension of $((s'_\epsilon)^{-1}(0))_{\mathrm{set}}$ is d by transversality.)

Definition A1.27. Let us consider the situation of Lemma A1.26. Let $\pi(y) \in \mathrm{Int}\,\Delta_a^d$, where $y \in V_{p_a}$ and $\pi : V_{p_a} \to V_{p_a}/\Gamma_{p_a}$. Let $\widetilde{s}'_{p_1,\epsilon} = (\widetilde{s}'_{p_a,\epsilon,1}, \cdots, \widetilde{s}'_{p_a,\epsilon,n_{p_a}}) : V_{p_a} \to E_{p_a}^{n_{p_a}}$ be as in the definition of val_{p_a}. For each i with $s'_{p_a,\epsilon,i}(y) = 0$ we define $\epsilon_i \in \{\pm 1\}$ as follows. Since $\mathfrak{s}_\epsilon$ is transversal to 0 it follows that

$$d_y s'_{p_a,\epsilon,i} : T_y V_{p_a} \to E_{p_a}$$

is a surjection. Using the trivialization of

$$\Lambda^{\mathrm{top}} E_{p_a}^* \otimes \Lambda^{\mathrm{top}} T_y V_{p_a}$$

induced by the orientation, we can assign an orientation on $(d_y s'_{p_a,\epsilon,i})^{-1}(0)$ as in §45.1, Case (4) in Chapter 9. We put $\epsilon_i = 1$ if this orientation on $(d_y s'_{p_a,\epsilon,i})^{-1}(0)$ coincides with the one on Δ_a^d and $\epsilon_i = -1$ if not. Now we define

$$mul_{\Delta_a^d} = \frac{\sum \epsilon_i}{n_{p_a} \cdot \#\Gamma_{p_a}} \in \mathbb{Q},$$

here the sum in the numerator is taken over all i with $s'_{p_a,\epsilon,i}(y) = 0$. Using the fact that val_{p_a} is constant, we can prove that $mul_{\Delta_a^d}$ is independent of $y \in \Delta_a^d$. Using the compatibility, we can prove that $mul_{\Delta_a^d}$ is independent of p_a such that $\Delta_a^m \subset (s'_{p_a,\epsilon})^{-1}(0)$.

Now we give the definition of the *virtual fundamental chain* $f_*[X]$ in Y.

Definition A1.28.

$$f_*[X] = \sum_a mul_{\Delta_a^d} f_{p_a *}[\Delta_a^d]. \tag{A1.29}$$

Here $f_*[X]$ is a $\mathbb{Q}$-singular chain of Y.

Next we consider the case where X has boundary or corner.

Definition A1.30. $x \in X$ is said to be in the codimension k corner of X if $x = \psi_p(\pi\widetilde{x})$ where $\widetilde{x} \in V_p$ is in the codimension k corner of V_p and $\pi : V_p \to V_p/\Gamma_p$ is a projection. (We remark that V_p is a manifold with corner.) We denote by ∂X the set of all x in the codimension 1 corner (that is the boundary). We also write $S_k X$ be the set of all elements of X in the codimension k corner.

We define a Kuranishi structure on $S_k X$ as follows : If $p \in S_k X$ and the Γ_p-action induces an effective action on $S_k V_p$ we take $(S_k V_p, E_p, \Gamma_p, s_p|_{S_k V_p})$ as its Kuranishi neighborhood where $S_k V_p$ is the codimension k corner of V_k.

If the induced Γ_p-action on $S_k V_p$ is not effective, we increase both the Kuranishi neighborhood and obstruction bundle by adding the same representation of Γ_p so that the action to the Kuranishi neighborhood is effective. (This process requires some care. See Remark A1.102.)

Lemma A1.31. *Denote by $\partial f = f|_{\partial X}$. Then*

$$\partial f_*[X] = f_*[\partial X].$$

The proof is easy and is left to the reader. We can use Lemma A1.31 to show the following.

Lemma A1.32. *If $\partial X = \emptyset$, then $f_*[X]$ is a cycle. Its homology class is independent of the choice of perturbation $\mathfrak{s}_\epsilon$.*

Proof. The first half is immediate from Lemma A1.31. To prove the second half we remark that $X \times [0,1]$ has Kuranishi structure with boundary such that $\partial(X \times [0,1]) = X \times \{0,1\}$. Let us assume that we have two choices of $\mathfrak{s}_\epsilon$, say $\mathfrak{s}_\epsilon^0, \mathfrak{s}_\epsilon^1$. We can find $\mathfrak{s}_\epsilon^+$ on $X \times [0,1]$ whose restriction to $X \times \{t_0\}$ ($t_0 = 0,1$) coincides with $s_\epsilon^{t_0}$. (Theorem A1.23). Second half of Lemma A1.32 then is immediate from Lemma A1.31. □

We next review the definition of bundle system in [FuOn99II]. (We used this notion in §41, §42.)

Definition A1.33. Let X be a space with Kuranishi structure. A *bundle system* on X is the following objects : Let $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ be a Kuranishi neighborhood of $p \in X$.

(A1.34.1) For each p, we have a pair $F_{1,p}, F_{2,p}$ of vector bundles over V_p such that Γ_p acts on $F_{1,p}, F_{2,p}$.
(A1.34.2) If $q \in \psi(V_p/\Gamma_p)$, then, for $i = 0,1$, we have an h_{pq} equivariant embeddings of vector bundles $\hat\phi_{i,pq} : F_{i,q} \to F_{i,p}$ over $\phi_{pq} : V_q \to V_p$.

Moreover we have an isomorphism

$$\Phi_{pq} \;:\; \frac{F_{1,p}|_{\phi_{pq}(V_q)}}{\hat\phi_{1,pq}(F_{1,q})} \cong \frac{F_{2,p}|_{\phi_{pq}(V_q)}}{\hat\phi_{2,pq}(F_{2,q})}.$$

(A1.34.3) If $r \in \psi_q(V_q/\Gamma_q) \subseteq \psi_p(V_p/\Gamma_p)$, then $\hat\phi_{i,pq} \circ \hat\phi_{i,qr} = \gamma_{pqr} \cdot \hat\phi_{i,pr}$.
(A1.34.4) If $r \in \psi_q(V_q/\Gamma_q) \subseteq \psi_q(V_p/\Gamma_p)$, then the following diagram commutes.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \frac{F_{1,q}|_{\phi_{qr}(V_r)}}{\hat\phi_{1,pr}(F_{1,r})} & \xrightarrow{\hat\phi_{1,pq}} & \frac{F_{1,p}|_{\gamma_{pqr}\cdot\phi_{pr}(V_r)}}{\gamma_{pqr}\cdot\hat\phi_{1,pr}(F_{1,r})} & \xrightarrow{\pi\circ\gamma_{pqr}^{-1}} & \frac{F_{1,p}|_{\phi_{pq}(V_q)}}{\hat\phi_{1,pq}(F_{1,q})} & \longrightarrow & 0 \\
 & & \Phi_{qr}\Big\downarrow & & \gamma_{pqr}\circ\Phi_{pr}\circ\gamma_{pqr}^{-1}\Big\downarrow & & \Phi_{pq}\Big\downarrow & & \\
0 & \longrightarrow & \frac{F_{2,q}|_{\phi_{qr}(V_r)}}{\hat\phi_{2,pr}(F_{2,r})} & \xrightarrow{\hat\phi_{2,pq}} & \frac{F_{2,p}|_{\gamma_{pqr}\cdot\phi_{pr}(V_r)}}{\gamma_{pqr}\cdot\hat\phi_{2,pr}(F_{2,r})} & \xrightarrow{\pi\circ\gamma_{pqr}^{-1}} & \frac{F_{2,p}|_{\phi_{pq}(V_q)}}{\hat\phi_{2,pq}(F_{2,q})} & \longrightarrow & 0
\end{array}$$

In case X is an orbifold, we call a pair $(F_{1,p}, \hat\phi_{1,pq})$ an *orbi-bundle* on X if it satisfies (A1.34) with $F_{2,p} = 0$. In other words, an orbi-bundle is a pair $(\{F_{1,p}\}, \{\hat\phi_{1,pq}\})$ where $F_{1,p}$ is a Γ_p equivariant vector bundle on V_p and $\hat\phi_{1,pq}$ is an h_{pq}-equivariant isomorphism of vector bundles such that $\hat\phi_{1,pq} \circ \hat\phi_{1,qr} = \gamma_{pqr} \cdot \hat\phi_{1,pr}$.

Example A1.35. If the Kuranishi structure on X has a tangent bundle, then $F_{1,p} = TV_p$, $F_{2,p} = E_p \times V_p$ define a bundle system. In fact the isomorphism Φ_{pq} is isomorphism (A1.15). The commutativity in (A1.34.4) is commutativity of (A1.16).

A1.2. Fiber product.

We next define a fiber product of Kuranishi structure. We assume that Y_i are smooth manifolds, X_i are the spaces with Kuranishi structures, and the notations $(V_p^i, E_p^i, \Gamma_p^i, \psi_p^i, s_p^i; V_{pq}^i, \hat\phi_{pq}^i, \phi_{pq}^i, h_{pq}^i)$ are as in Lemma A1.11. ($i = 1, \cdots, I$.) Let $\{f_{i,p_i}\} : X_i \to Y_i$ be smooth strongly continuous maps. *We assume that they are weakly submersive.* We put $Y = \prod Y_i$. Let W be a manifold with corner and $f : W \to Y$ be a smooth map. Let $f_i : X_i \to Y_i$ be the continuous map induced from $\{f_{i,p_i}\}$. We put

$$
\begin{aligned}
Z &= \prod_i X_i \times_Y W : \\
&= \left\{ ((p_i), p) \in \prod_i X_i \times W \,\middle|\, (f_1(p_1), \cdots, f_n(p_n)) = f(p) \right\}.
\end{aligned} \tag{A1.36}
$$

We will define a Kuranishi structure on Z.

Definition A1.37. Let $\vec p = ((p_i), p) \in Z$ and $V_{p_i}^i$ be Kuranishi neighborhoods of p_i. We put

$$
V_{\vec p} := \left\{ ((x_i), x) \in \prod_i V_{p_i}^i \times W \,\middle|\, (f_{1,p_1}(x_1), \cdots, f_{n,p_n}(x_n)) = f(x) \right\}. \tag{A1.38}
$$

Since f_{i,p_i} are submersions, it follows that $V_{\vec p}$ is a smooth manifold for each $\vec p$. We put $E_{\vec p} = \prod E_{p_i}^i$ and $\Gamma_{\vec p} = \prod \Gamma_{p_i}^i$. It is easy to see that $\Gamma_{\vec p}$ acts on $E_{\vec p} \times V_{\vec p} \to V_{\vec p}$. We can define a map $s_{\vec p} : V_{\vec p} \to E_{\vec p}$ using $s_{p_i}^i$ in an obvious way. A map $\psi_{\vec p} : (s_{\vec p})^{-1}(0)/\Gamma_{\vec p} \to Z$ is induced from $\psi_{p_i}^i$.

It is straightforward to check that $(V_{\vec p}, E_{\vec p}, \Gamma_{\vec p}, \psi_{\vec p}, s_{\vec p})$ is a Kuranishi neighborhood of $\vec p$ and that they are glued to define a Kuranishi structure on Z.

Lemma A1.39. *If the Kuranishi structures on X_i have tangent bundles, so does the Kuranishi structure on Z. If the Kuranishi structures on X_i, and the manifolds Y_i, W are oriented, so is the Kuranishi structure on Z.*

Proof. Let $q_i < p_i$ and $x_i \in \psi_{p_i}(s_{p_i}^{-1}(0)/\Gamma_{p_i}) \cap \psi_{q_i}(s_{q_i}^{-1}(0)/\Gamma_{q_i}) \subset X_i$. We put $x_i = \psi_{p_i}(\pi(x_{i,p_i})) = \psi_{q_i}(\pi(x_{i,q_i}))$. We assume $\vec q = ((q_i), q)$, $\vec x = ((x_i), x)$, $\vec p = ((p_i), p)$ are elements of Z. By (A1.15), Kuranishi maps induce isomorphisms

$$
\frac{T_{x_{i,p_i}} V_{p_i}^i}{(d_{x_{i,q_i}} \phi_{p_i q_i})(T_{x_{i,q_i}} V_{q_i}^i)} \cong \frac{E_{p_i}^i}{\hat\phi_{p_i q_i, x_i}(E_{q_i}|_{V_{p_i q_i}^i})}. \tag{A1.40}
$$

We put $\vec x_{\vec p} = ((x_{i,p_i}), x) \in V_{\vec p}$, $\vec x_{\vec q} = ((x_{i,q_i}), x) \in V_{\vec q}$. Since there exist exact sequences

$$
0 \to T_{\vec x}(V_{\vec p}) \to \bigoplus_i T_{x_{i,p_i}} V_{p_i}^i \oplus T_x W \to \bigoplus_i T_{x_i} Y_i \to 0
$$

$$
0 \to T_{\vec x}(V_{\vec q}) \to \bigoplus_i T_{x_{i,q_i}} V_{q_i}^i \oplus T_x W \to \bigoplus_i T_{x_i} Y_i \to 0,
$$

it follows that we have an isomorphism

$$\text{(A1.41)} \qquad \frac{T_{\vec{x}_{\vec{p}}}V_{\vec{p}}}{(d_{\vec{x}_{\vec{q}}}\phi_{\vec{p}\vec{q}})(T_{\vec{x}_{\vec{q}}}V_{\vec{q}})} \cong \bigoplus_i \frac{T_{x_{i,p_i}}V^i_{p_i}}{(d_{x_{i,q_i}}\phi_{p_iq_i})(T_{x_{i,q_i}}V^i_{q_i})}.$$

Therefore (A1.40) implies

$$\text{(A1.42)} \qquad \frac{T_{\vec{x}_{\vec{p}}}V_{\vec{p}}}{(d_{\vec{x}_{\vec{p}}}\phi_{\vec{p}\vec{q}})(T_{\vec{x}_{\vec{q}}}V_{\vec{q}})} \cong \bigoplus_i \frac{E^i_{p_i}}{\hat{\phi}_{p_iq_i,x_i}(E_{q_i}|_{V^i_{p_iq_i}})} = \frac{E_{\vec{p}}}{\hat{\phi}_{\vec{p}\vec{q},\vec{x}}(E_{\vec{q}}|_{V_{\vec{p}\vec{q}}})},$$

which is induced by the differential of the Kuranishi map. We have thus proved existence of the tangent bundle. The proof of orientability is similar. □

We remark that the map $Z \to W$ is induced by a strongly continuous map, which we write $\hat{f} = (f_i) \times_Y W$.

Lemma A1.43. *We assume that W and Y are compact without boundary and orientation and $\partial X_i = \emptyset$. Then we have*

$$f^*(PD(f_{1*}([X_1])) \cup \cdots \cup PD(f_{I*}([X_I]))) = \pm PD(\hat{f}_*([Z])).$$

Here PD is the Poincaré duality.

The proof is easy and is left to the reader. (We do not use Lemma A1.43 in this book.)

Let $f_i : X_i \to Y$ $(i = 1, 2)$ be strongly continuous maps and $Z = X_1 \times_Y X_2 = \{(x_1, x_2) \in X_1 \times X_2 \mid f_1(x_1) = f_2(x_2)\}$. If X_1,X_2 have Kuranishi structure and f_1, f_2 are weakly submersive, then we may regard $Z = (X_1 \times X_2) \times_{Y^2} Y$ where $Y \to Y^2$, $y \mapsto (y, y)$. Then Z has a Kuranishi structure. Actually it is enough to assume that one of f_1,f_2 is weakly submersive.

Remark A1.44. (1) We define fiber product only in case when f_i are weakly submersive. We can reduce the general case to this case by modifying the Kuranishi structure. Namely, we can increase $V^i_{p_j}$ and $E^i_{p_j}$ at the same time so that f_{i,p_j} are submersions as follows. Let $(V^i_{p_j}, E^i_{p_j}, \Gamma^i_{p_j}, \psi^i_{p_j}, s^i_{p_j})$ be a Kuranishi neighborhood of $p_j \in X_i$. Consider the graph $\Gamma_{f_{i,p_j}} \subset V^i_{p_j} \times Y$ and take its tubular neighborhood $U(\Gamma_{f_{i,p_j}})$. The normal bundle $N(U(\Gamma_{f_{i,p_j}}))$, whose the unit disc bundle is identified with $U(\Gamma_{f_{i,p_j}})$, is naturally isomorphic to the restriction of pr_2^*TY, where pr_ℓ is the projection from $V^i_{p_j} \times Y$ to the ℓ-th factor. Then $(U(\Gamma_{f_{i,p_j}}), pr_2^*TY|_{U(\Gamma_{f_{i,p_j}})}, \Gamma^i_{p_j}, \psi^i_{p_j} \circ pr_1, s^i_{p_j} \oplus s_{can})$ is a Kuranishi neighborhood of $p_j \in X_i$. Here, $\Gamma^i_{p_j}$ acts trivially on the second factor Y, and s_{can} is the tautological section of $N(U(\Gamma_{f_{i,p_j}}))$ over $U(\Gamma_{f_{i,p_j}})$. Since f_{i,p_j} coincides with $pr_2 \circ (id, f_{i,p_j})$ and the restriction of pr_2 to $U(\Gamma_{f_{i,p_j}})$ is clearly a submersion, we can make f_i weakly submersive. Alternatively, we can use the above observation that the fiber product $X_1 \times_Y X_2$ is identified with $Z = (X_1 \times X_2)_{f_1 \times f_2} \times_{Y \times Y} {}_{\Delta} Y$, where $\Delta : Y \to Y \times Y$ is the diagonal embedding. We also remarked that this fiber product carries a Kuranishi structure, if either $f_1 \times f_2$ or Δ is weakly submersive. Denote by $U(Y)$ a tubular neighborhood of $\Delta(Y)$ in $Y \times Y$ and by $N_Y(Y \times Y)$ its normal bundle. Then $(U(Y), N_Y(Y \times Y), s_{can})$ gives a Kuranishi structure of Y, where s_{can} is the tautological section. Note that $U(Y) \to Y$ is an open embedding, especially a submersion. From this observation Δ is weakly submersive. Thus Z carries a Kuranishi structure.

(2) It seems possible to generalize the construction to the case when Y is not necessarily a manifold but has a Kuranishi structure, under an appropriate assumption. We would then have a category whose objects are the spaces with Kuranishi structure and whose morphisms are a kind of strongly continuous smooth maps. Such a category would be a smooth analog of the category of schemes or stacks. (We remark that the fiber product of schemes is always well-defined and plays an important role in the theory of schemes.) We are not trying to rigorously define it in this book since we do not use it.

A1.3. Finite group action and the quotient space.

We next define the action of a finite group on the space with Kuranishi structure and its quotient space. We used it in §§41,42.

Definition A1.45. Let $\varphi : X \to X$ be a homeomorphism of a space X with Kuranishi structure. We say that it induces an *automorphism of Kuranishi structure* if the following holds : Let $p \in X$ and $p' = \varphi(p)$. Then, for the Kuranishi neighborhoods $(V_p, E_p, \Gamma_p, \psi_p, s_p)$, $(V_{p'}, E_{p'}, \Gamma_{p'}, \psi_{p'}, s_{p'})$ of p and p' respectively, there exist $\rho_p : \Gamma_p \to \Gamma_{p'}$, $\varphi_p : V_p \to V_{p'}$, and $\hat\varphi_p : E_p \to E_{p'}$ such that

(A1.46.1) ρ_p is an isomorphism of groups.
(A1.46.2) φ_p is a ρ_p equivariant diffeomorphism.
(A1.46.3) $\hat\varphi_p$ is a ρ_p equivariant bundle isomorphism which covers φ_p.
(A1.46.4) $s_{p'} \circ \varphi_p = \hat\varphi_p \circ s_p$.
(A1.46.5) The restriction of φ_p to $s_p^{-1}(0)$ induces a homeomorphism $s_p^{-1}(0)/\Gamma_p \to s_{p'}^{-1}(0)/\Gamma_{p'}$, which we write $\underline{\varphi}_p$. Then we have

$$\psi_{p'} \circ \underline{\varphi}_p = \varphi \circ \psi_p.$$

We assume ρ_p, φ_p, $\hat\varphi_p$ are compatible with the coordinate changes of Kuranishi structure in the following sense : Let $q \in \psi_p(s_p^{-1}(0)/\Gamma_p)$ and $q' \in \psi_{p'}(s_{p'}^{-1}(0)/\Gamma_{p'})$ such that $\varphi(q) = q'$. Let $(\hat\phi_{pq}, \phi_{pq}, h_{pq})$, $(\hat\phi_{p'q'}, \phi_{p'q'}, h_{p'q'})$ be the coordinate changes. Then there exists $\gamma_{pqp'q'} \in \Gamma_{p'}$ with the following properties :

(A1.47.1) $\rho_p \circ h_{pq} = \gamma_{pqp'q'} \cdot (h_{p'q'} \circ \rho_q) \cdot \gamma_{pqp'q'}^{-1}$.
(A1.47.2) $\varphi_p \circ \phi_{pq} = \gamma_{pqp'q'} \cdot (\phi_{p'q'} \circ \varphi_q)$.
(A1.47.3) $\hat\varphi_p \circ \hat\phi_{pq} = \gamma_{pqp'q'} \cdot (\hat\phi_{p'q'} \circ \hat\varphi_q)$.

We call $((\rho_p, \varphi_p, \hat\varphi_p)_p; \varphi)$ an *automorphism of our Kuranishi structure.*

We say an automorphism $((\rho_p, \varphi_p, \hat\varphi_p)_p; \varphi)$ is *conjugate* to $((\rho'_p, \varphi'_p, \hat\varphi'_p)_p; \varphi')$ if $\varphi = \varphi'$ and if there exists $\gamma_p \in \Gamma_{\varphi(p)}$ for each p such that

(A1.48.1) $\rho'_p = \gamma_p \cdot \rho_p \cdot \gamma_p^{-1}$.
(A1.48.2) $\varphi'_p = \gamma_p \cdot \varphi_p$.
(A1.48.3) $\hat\varphi'_p = \gamma_p \cdot \hat\varphi_p$.

The *composition* of the two automorphisms is defined by

$$((\rho^1_p, \varphi^1_p, \hat\varphi^1_p)_p; \varphi^1) \circ ((\rho^2_p, \varphi^2_p, \hat\varphi^2_p)_p; \varphi^2) = ((\rho^1_{\varphi^2(p)} \circ \rho^2_p, \varphi^1_{\varphi^2(p)} \circ \varphi^2_p, \hat\varphi^1_{\varphi^2(p)} \circ \hat\varphi^2_p)_p; \varphi^1 \circ \varphi^2).$$

We can easily check that the right hand side is compatible with coordinate change in the sense of (A1.47) if so is left hand side.

We can also easily check that the composition is compatible with the conjugation.

We say that the *lift is orientation preserving* if it is compatible with the trivialization of $\Lambda^{\mathrm{top}} E_p^* \otimes \Lambda^{\mathrm{top}} TV_p$.

Let $\mathrm{Aut}(X)$ be the set of all conjugacy classes of the automorphisms of X and $\mathrm{Aut}_0(X)$ be the set of all conjugacy classes of the orientation preserving automorphisms of X. They become groups by composition.

Let G be a finite group which acts on a compact space X. We assume that X has a Kuranishi structure. We say that G *acts* on X (as a space of Kuranishi structure) if, for each element of $g \in G$, the homeomorphism $x \mapsto gx$, $X \to X$ is lifted to an automorphism g_* of the Kuranishi structure and the composition of g_* and h_* is conjugate to $(gh)_*$.

In other words an action of G to X is a homomorphism $G \to \mathrm{Aut}(X)$.

An *involution* of a space with Kuranishi structure is a $\mathbb{Z}_2$ action.

Lemma A1.49. *If a finite group G acts on a space X with Kuranishi structure then the quotient space X/G has Kuranishi structure.*

If X has a tangent bundle and the action preserves it, then the quotient space has a tangent bundle. If X is oriented and the action preserves the orientation, then the quotient space has an orientation.

Proof. We first assume that the action of G is effective.

Let $g \in G$ and $p \in X$ and $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ be its Kuranishi neighborhood. We put

$$G_p = \{g \in G \mid gp = p\}.$$

By definition, the element $g \in G_p$ induces

$$\varphi_{g,p} : V_p \to V_p, \quad \hat\varphi_{g,p} : E_p \times V_p \to E_p \times V_p, \quad \rho_{g,p} : \Gamma_p \to \Gamma_p.$$

We identify $G_p = \{\varphi_{g,p} \mid g \in G_p\}$.

Let $\Gamma_{[p]}$ be the group generated by G_p and Γ_p in the group of diffeomorphisms of V_p.

We claim that an element of the group $\Gamma_{[p]}$ is written uniquely as a composition $g \circ \gamma$ where $g \in G_p$ and $\gamma \in \Gamma_p$. To prove the claim we proceed as follows.

For $g_1, g_2 \in G_p$, let $\gamma_{g_1,g_2} \in \Gamma_p$ be an elements such that

$$g_1(g_2(x)) = \gamma_{g_1,g_2}((g_1g_2)(x)).$$

Such γ_{g_1,g_2} exists since the composition of g_1 and g_2 is conjugate to g_1g_2. It follows that, the product $\circ$ on $\Gamma_{[p]}$ is given by the following formula

$$g_1 \circ g_2 = \gamma_{g_1,g_2} \circ (g_1g_2), \quad g \circ \gamma = \rho_{g,p}(\gamma) \circ g$$

for $g, g_1, g_2 \in G_p$ and $\gamma \in \Gamma_p$. This implies our claim.

It follows that $\Gamma_{[p]}$ is a finite group. Moreover, there exists an exact sequence

$$1 \to \Gamma_p \to \Gamma_{[p]} \to G_p \to 1. \tag{A1.50}$$

The action of each element of G_p on Γ_p induced by the exact sequence of (A1.50) is given by $g \mapsto \rho_{g,p}$. (Note $g \mapsto \rho_{g,p}$ is not a homomorphism.). The exact sequence (A1.50) may not split. Actually γ_{g_1,g_2} corresponds to the 2 cocycle which determines the extension (A1.50).

We can lift the action of $\Gamma_{[p]}$ on V_p to its action on $E_p \times V_p$ since the relations among the maps $\hat\varphi_{g,p}$ $(g \in G_p)$ and $\gamma \in \Gamma_p$ are described by the same $\rho_{p,g}$ and γ_{g_1,g_2}. The Kuranishi map $s_p : V_p \to E_p$ is $\Gamma_{[p]}$ equivariant. Since $V_p/\Gamma_{[p]} \cong (V_p/\Gamma_p)/G_p$, it follows that $\psi_p : s_p^{-1}(0)/\Gamma_p \to X$ induces $\psi_{[p]} : {s_p}^{-1}(0)/\Gamma_{[p]} \to X/G$ which is a homeomorphism onto a neighborhood of $[p]$. Thus we constructed a Kuranishi neighborhood $(V_p, E_p, \Gamma_{[p]}, \psi_{[p]}, s_p)$ of $[p] \in X/G$.

It is straightforward to construct a coordinate change of this Kuranishi neighborhood and to check the required properties in Definition A1.3 and Definition A1.5. The proof about the tangent bundle and the orientation is obvious from their definitions.

When G is not effective, we fix an effective finite dimensional representation $G \to GL(E)$. We replace the Kuranishi neighborhood $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ by $(V_p \times E, E_p \times E, \Gamma_p, \psi_p, s_p \oplus id)$. Then the G-action on this Kuranishi neighborhood is effective and hence the above construction applies. The proof of Lemma A1.49 is now complete. □

Remark A1.51. (1) In the circumstance in Chapter 8 §38-43 where we use Proposition A1.49, we can construct Kuranishi structure so that the $\mathbb{Z}_{2^m}$ action on the moduli space is induced by an action on its Kuranishi structure.

(2) In the circumstance of Example A1.7, we assume that there exists a group G' containing G as a normal subgroup of finite index and that G actions on W_1, W_2

extend to G' actions and F is G' equivariant. Then we can choose the Kuranishi structure on $X = F^{-1}(0)/G$ so that it becomes H equivariant where $H = G'/G$. Moreover the quotient Kuranishi structure on X/H of Lemma A1.49 is isomorphic to the Kuranishi structure on $F^{-1}(0)/G'$.

(3) The definition of the group action can be generalized to the case of continuous group in an obvious way, cf. §22 in [FuOn99II]. Then Lemma A1.49 is generalized under the assumption that the action is proper and the isotropy group is finite. We do not need this generalization in this book so we do not try to prove it.

A1.4. A remark on smoothness of coordinate transform.

In the definition of Kuranishi structure we require the coordinate transform $\phi_{pq} : V_q \to V_p$ to be smooth. (See (A1.4.2).) In the actual construction of Kuranishi structure on the moduli space we use, this smoothness is apparent as far as the points p, q lie in the interior of the moduli space. This corresponds to the case where the domain of the corresponding stable map is nonsingular. The smoothness is less apparent when they correspond to the stable map whose domain is singular. In this subsection we clarify this point mainly for the completeness' sake : this point of the smoothness of the Kuranishi map s was already mentioned but not spelled out in [FuOn99II, Remark 13.16]. (We remark that smoothness of coordinate change was not used in [FuOn99II], since only 0 and 1 dimensional moduli space was used there. In other words, in Situation 30.2 mentioned in §30, we do not need it.) The authors would like to thank Melissa Liu who called our attention of this point during her visit of Kyoto in 2003.

To highlight the main point, we consider the case of the moduli space $\mathcal{M}_{k+1}(\beta)$ and restrict ourselves to the case discussed in §29.3. The discussion of the other cases are similar.

Let $\mathbf{p}_j = (\Sigma^{(j)}, \vec{z}^{(j)}, w^{(j)})$, $j = 1, 2$ be two elements of $\mathcal{M}_{k+1}(\beta)$. We put the following additional assumptions for simplicity of exposition :

(1) $\Sigma^{(j)}$ has exactly one boundary singular point and splits into

$$\Sigma^{(j)} = \Sigma_1^{(j)} \cup \Sigma_0^{(j)},$$

where each of $\Sigma_i^{(j)}$ is the disc.

(2) They are of the same combinatorial type, namely there exists a homeomorphism $(\Sigma^{(1)}, \vec{z}^{(1)}) \to (\Sigma^{(2)}, \vec{z}^{(2)})$.

(3) The restriction of $w^{(1)}$ to $\Sigma_0^{(1)}$ (resp. $\Sigma_1^{(1)}$) is homotopic and close to the restriction of $w^{(2)}$ to $\Sigma_0^{(2)}$ (resp. $\Sigma_1^{(2)}$) so that the inclusion (A1.52) below holds.

Without loss of generalities, we may assume that the zero-th marked points of them are on $\Sigma_0^{(1)}$ or $\Sigma_0^{(2)}$ respectively, and let the double point be the ℓ-th marked point of $\Sigma_0^{(1)}$ or $\Sigma_0^{(2)}$. We put

$$(\Sigma^{(j)}, \vec{z}^{(j)}, w^{(j)}) = (\Sigma_1^{(j)}, \vec{z}_1^{(j)}, w_1^{(j)}) \# (\Sigma_0^{(j)}, \vec{z}_0^{(j)}, w_0^{(j)})$$

and

$$(\Sigma_1^{(j)}, \vec{z}_1^{(j)}, w_1^{(j)}) \in \mathcal{M}_{k_1+1}(\beta_1), \qquad (\Sigma_0^{(j)}, \vec{z}_0^{(j)}, w_0^{(j)}) \in \mathcal{M}_{k_0+1}(\beta_0).$$

Following the notations of §29.3 we denote

$$(\Sigma_i^{(j)}, \vec{z}_i^{(j)}) = (X(\mathfrak{t}_i^{(j)}, \ell_i^{(j)}), \vec{p}^{(j),i}), \quad (i = 0, 1,\ j = 1, 2).$$

In §29.4, we took the finite dimensional spaces

$$E_{(\mathfrak{t}_i^{(j)}, \ell_i^{(j)}, w_i^{(j)})} \subset \mathcal{E}^{0,p}_{(\mathfrak{t}_i^{(j)}, \ell_i^{(j)}, w_i^{(j)})}$$

so that the map (29.26) becomes surjective. They determine the obstruction bundles

$$E_{\mathbf{p}_j} = E_{(\Sigma^{(j)}, \vec{z}^{(j)}, w^{(j)})} \cong E_{(\mathfrak{t}_1^{(j)}, \ell_1^{(j)}, w_1^{(j)})} \oplus E_{(\mathfrak{t}_0^{(j)}, \ell_0^{(j)}, w_0^{(j)})}$$

and the Kuranishi neighborhood $U_{\mathbf{p}_j}$ is the set of solutions of the equation (29.22). As in §29.4, we adopt the normalization put in Appendix of [FuOn99II], when at least one of the marked Riemann surfaces $\Sigma_0^{(j)}$ and $\Sigma_1^{(j)}$ is unstable. Namely we add an interior marked point $z_i^{(j),\mathrm{int}} \in \mathrm{Int}(\Sigma_i^{(j)})$ where the maps $w_i^{(j)}$ is an immersion. For each i, j we also take an $(2n-2)$ dimensional submanifold $Y_i^{(j)}$ which intersects with $w_i^{(j)}(\Sigma_i^{(j)})$ transversely at $w_i^{(j)}(z_i^{(j),\mathrm{int}})$.

Now we assume that $\mathbf{p}_2 \in U_{\mathbf{p}_1}$. Then by the choice of $E_{\mathbf{p}_j}$ explained in §29.4 we may assume

$$E_{\mathbf{p}_2} \subset E_{\mathbf{p}_1}. \tag{A1.52}$$

(We identify the subspace $E_{\mathbf{p}_2}$ of $\bigoplus_i \mathcal{E}^{0,p}_{(\mathfrak{t}_i^{(2)}, \ell_i^{(2)}, w_i^{(2)})}$ with a subspace of $\bigoplus_i \mathcal{E}^{0,p}_{(\mathfrak{t}_i^{(1)}, \ell_i^{(1)}, w_i^{(1)})}$ via the parallel transport along the minimal geodesics as we did in §29.)

By (A1.52) we have

$$\phi_{\mathbf{p}_1,\mathbf{p}_2} \;:\; U_{\mathbf{p}_2} \subset U_{\mathbf{p}_1}. \tag{A1.53}$$

(A1.53) provides our coordinate transformation. To show that it is smooth, we need to put a coordinate chart on $U_{\mathbf{p}_j}$. We review how the coordinate chart is defined in §29.3. Let us put

$$\mathbf{p}_j^{(i)} = (\Sigma_i^{(j)}, \vec{z}_i^{(j)}, w_i^{(j)}) \in \mathcal{M}_{k_i+1}(\beta_i).$$

We use
$$E_{\mathbf{p}_j^{(i)}} = E_{(\mathfrak{t}_i^{(j)}, \ell_i^{(j)}, w_i^{(j)})}$$
as an obstruction bundle and obtain a Kuranishi neighborhood $(U_{\mathbf{p}_j^{(i)}}, E_{\mathbf{p}_j^{(i)}}, s_{\mathbf{p}_j^{(i)}})$ of $\mathbf{p}_j^{(i)}$ in $\mathcal{M}_{k_i+1}(\beta_i)$. (In the current case of our interest, the automorphism group is trivial.) Then by construction we may take

$$U_{\mathbf{p}_j} \cong U_{\mathbf{p}_j^{(1)}} \times_L U_{\mathbf{p}_j^{(0)}} \times (C, \infty] \tag{A1.54}$$

for some large C. Here the last factor $(C,\infty]$ is the smoothing parameter of the singularity of $\Sigma^{(j)}$.

Let us discuss this parametrization more in detail in case the domain $\Sigma_i^{(j)}$ is unstable. We assume $k_1 = 0$, $k_0 = 1$. In other words, we glue the 0-th (the only) marked point of $\Sigma_1^{(j)}$ to the 1-st marked point of $\Sigma_0^{(j)}$. (The 0-th marked point of $\Sigma_0^{(j)}$ will become the 0-th marked point of the glued disc.) This is the most delicate case where both of the components are unstable. As we mentioned above we take interior marked points $z_i^{(j),\mathrm{int}} \in \mathrm{Int}(\Sigma_i^{(j)})$. We identify the domain $\Sigma_1^{(j)}$ with

$$Z_1 := \{z \in \mathbb{C} \mid |z| \le 1,\ \mathrm{Re} z \le 0\} \cup \{z \in \mathbb{C} \mid |\mathrm{Im} z| \le 1, \mathrm{Re} z \ge 0\}, \tag{A1.55.1}$$

and $\Sigma_0^{(j)}$ with

$$Z_0 := \{z \in \mathbb{C} \mid |z| \le 1,\ \mathrm{Re} z \ge 0\} \cup \{z \in \mathbb{C} \mid |\mathrm{Im} z| \le 1, \mathrm{Re} z \le 0\}. \tag{A1.55.2}$$

Under this identification, $z_i^{(j),\mathrm{int}}$ corresponds to $0 \in \mathbb{C}$, the 0-th (the only) marked point of $\Sigma_1^{(j)}$ to $+\infty$, and the first marked point of $\Sigma_0^{(j)}$ to $-\infty$.

Since we consider a compact family, the 0-th marked point of $\Sigma_0^{(j)}$ moves in a compact subset of $\partial\Sigma_0^{(j)}$. We choose a constant C that is sufficiently large compared to the distance of the 0-th marked point of $\Sigma_0^{(j)}$ from the origin in $Z_- \cong \Sigma_0^{(j)}$.

For each $T \ge C$ we identify $5T + \sqrt{-1}s \in \Sigma_1^{(j)}$ with $-5T + \sqrt{-1}s \in \Sigma_0^{(j)}$ to obtain $\Sigma_T^{(j)}$. We then perform the construction of §29 to obtain a map from $\Sigma_T^{(j)}$ to M when the corresponding evaluation maps coincide. This defines a map

$$\phi_{\mathbf{p}_1,\mathbf{p}_2} \ :\ U_{\mathbf{p}_2^{(1)}} \times_L U_{\mathbf{p}_2^{(0)}} \times (C,\infty] \to U_{\mathbf{p}_1^{(1)}} \times_L U_{\mathbf{p}_1^{(0)}} \times (C',\infty], \tag{A1.56}$$

if C is sufficiently large compared to C'. Under the identification (A1.54) this map is nothing but (A1.53).

On the other hand, by the definition of Kuranishi neighborhoods, there exists a coordinate change
$$\overline{\phi}_{\mathbf{p}_1^{(i)},\mathbf{p}_2^{(i)}} \ :\ U_{\mathbf{p}_2^{(i)}} \to U_{\mathbf{p}_1^{(i)}}.$$
We define
$$\phi'_{\mathbf{p}_1,\mathbf{p}_2} \ :\ U_{\mathbf{p}_2^{(1)}} \times_L U_{\mathbf{p}_2^{(0)}} \times (C,\infty] \to U_{\mathbf{p}_1^{(1)}} \times_L U_{\mathbf{p}_1^{(0)}} \times (C',\infty]$$
by
$$\phi'_{\mathbf{p}_1,\mathbf{p}_2}(\mathbf{x}_1, \mathbf{x}_0, T) = (\overline{\phi}_{\mathbf{p}_1^{(1)},\mathbf{p}_2^{(1)}}(\mathbf{x}_1), \overline{\phi}_{\mathbf{p}_1^{(0)},\mathbf{p}_2^{(0)}}(\mathbf{x}_0), T).$$

Proposition A1.57.

(A1.58.1) *The restriction of* $\phi_{\mathbf{p}_1,\mathbf{p}_2}$ *in* (*A*1.56) *induces a smooth map*

$$U_{\mathbf{p}_2^{(1)}} \times_L U_{\mathbf{p}_2^{(0)}} \times (C,\infty) \to U_{\mathbf{p}_1^{(1)}} \times_L U_{\mathbf{p}_1^{(0)}} \times (C',\infty).$$

(A1.58.2) *At* $T=\infty$ *the map* $\phi_{\mathbf{p}_1,\mathbf{p}_2}$ *coincides with* $\phi'_{\mathbf{p}_1,\mathbf{p}_2}$.

(A1.58.3) *There exist* $c_k > 0$ *and* $C_k > 0$ *depending only on* k *such that*

$$\left|\phi'_{\mathbf{p}_1,\mathbf{p}_2} - \phi_{\mathbf{p}_1,\mathbf{p}_2}\right|_{C^k} \le C_k \exp(-c_k T).$$

The first two statement is obvious from construction given in §29.3. We will prove (A1.58.3) later in this subsection.

Now we put

$$s = 1/T.$$

It easily follows that $\phi_{\mathbf{p}_1,\mathbf{p}_2}$ is smooth with respect to this coordinate up to the boundary of $U_{\mathbf{p}_2^{(1)}} \times_L U_{\mathbf{p}_2^{(0)}} \times (C,\infty]$.

Note the C^k norm in (A1.58.3) involves both the derivatives along $U_{\mathbf{p}_j}$ direction and T direction.

Proof of Proposition A1.57. Recall we are considering only the case $k_1 = 0$ and $k_0 = 1$. We use the notation introduced above and in §29.3. We first take a relatively compact subset $W_i^{(j)}$ such that each element of $E_{\mathbf{p}_j^{(i)}}$ is supported in it. (See (29.14.2).) We start with a family of pseudo-holomorphic maps $w^{(j)}_{i,\mathbf{x}_i}$ parameterized by $\mathbf{x}_i \in U_{\mathbf{p}_2^{(i)}}$. Then for sufficiently large T we obtain an approximate solution

$$w^{(j)\prime}_{T;\mathbf{x}_1,\mathbf{x}_0} = w^{(j)}_{1,\mathbf{x}_1} \sharp_T w^{(j)}_{0,\mathbf{x}_0} \;:\; \Sigma_T^{(j)} \longrightarrow M$$

by gluing two maps using a partition of unity. (This is the map w'_T defined just below Figure 29.4.) Then by the argument of §29.3, we obtain a genuine solution $w^{(j)}_{T;\mathbf{x}_1,\mathbf{x}_0}$ of the equation (29.22) parameterized by $\mathbf{x}_1, \mathbf{x}_0$ and T.

Now the main technical lemma we need to prove Proposition A1.57 is Lemma A1.59 below which provides an estimate of difference between $w'_{T;\mathbf{x}_1,\mathbf{x}_0}$ and $w_{T;\mathbf{x}_1,\mathbf{x}_0}$ together with all of its derivatives. In order to define appropriate norm we use for the estimates, we proceed as follows :

We take compact subsets $W_i \subset Z_i \cong \Sigma_i^{(j)}$ and take a finitely many but a big number $\mathcal{N}$ of points in it. We then consider the evaluation map $ev_{W_i} : Map(\Sigma_T^{(j)}, M) \to M^{\mathcal{N}}$. We remark that our choice of W_i is independent of $j = 1,2$. We may choose it so that the restriction map

$$(\mathbf{x}_1,\mathbf{x}_0) \mapsto ev_{W_i}(w^{(j)}_{1,\mathbf{x}_1} \sharp_T w^{(j)}_{0,\mathbf{x}_0}) \,; \quad U_{\mathbf{p}_2^{(1)}} \times_L U_{\mathbf{p}_2^{(0)}} \to M^{\mathcal{N}}$$

defines an embedding, for each T. (Note that if we choose W_i so that they are disjoint from the neck region, this map does not depend on T.)

Lemma A1.59.

$$\left|\frac{\partial^{k+\ell}}{\partial T^k \partial \mathbf{x}^\ell}(ev_{W_i}(w^{(j)}_{T;\mathbf{x}_1,\mathbf{x}_0}) - ev_{W_i}(w^{(j)\prime}_{T;\mathbf{x}_1,\mathbf{x}_0}))\right| < C_{k,\ell}\exp\left(-c_{k,\ell}T\right),$$

where $c_{k,\ell} > 0$, $C_{k,\ell} > 0$. *(Here and hereafter* $\frac{\partial^\ell}{\partial \mathbf{x}^\ell}$ *denotes the derivatives with respect to* $\mathbf{x}_0, \mathbf{x}_1$.*)*

Since the target of ev_{W_i} is a manifold, it is an abuse of notation to write $-$ in the above formula. However since it will be proved that the two elements $ev_{W_i}(w_{T;\mathbf{x}_1,\mathbf{x}_0})$ and $ev_{W_i}(w'_{T;\mathbf{x}_1,\mathbf{x}_0})$ are sufficiently close to each other, we can obviously make sense out of the expression via the exponential map.

Before proving Lemma A1.59, we now complete the proof of Proposition A1.57 using the lemma.

We note that since we chose $Y_i^{(j)}$'s so that they intersect at $w_i^{(j)}(z_i^{(j),int})$ and set $z_i^{(j),int} = 0$ in the identification of $\Sigma_i^{(j)} \cong Z_i$ given in (A1.55), the two T-coordinates for $j = 1$, 2 do not coincide. We now estimate their difference as $T \to \infty$.

Consider the T-projection of the coordinate change $\phi_{\mathbf{p}_1,\mathbf{p}_2}$,

$$\mathfrak{T} := \mathrm{pr}_T \circ \phi_{\mathbf{p}_1,\mathbf{p}_2} \ : \ (\mathbf{x}_1, \mathbf{x}_0, T) \mapsto \mathfrak{T}((\mathbf{x}_1, \mathbf{x}_0, T)).$$

In general $\mathfrak{T}((\mathbf{x}_1, \mathbf{x}_0, T)) \neq T$. However we now prove the estimate

$$|\mathfrak{T}((\mathbf{x}_1, \mathbf{x}_0, T)) - (T + h(\mathbf{x}_1, \mathbf{x}_0))|_{C^k} < C_k \exp\left(-c_k T\right), \tag{A1.60.1}$$

for some smooth function $h(\mathbf{x}_1, \mathbf{x}_0)$ on $U_{\mathbf{p}_2^{(1)}} \times U_{\mathbf{p}_2^{(0)}}$. Here the C^k norm in the left hand side includes both T and $\mathbf{x}_0, \mathbf{x}_1$ derivatives.

We first take the intersection of $w^{(1)\prime}_{T;\mathbf{x}_1,\mathbf{x}_0}$ (here $j = 1$) with $Y_i^{(2)}$ (here $j = 2$). We thus fix a complex structure of the domain $\Sigma_T^{(j)}$ equipped with additional marked points. We use it to obtain $\mathfrak{T}'((\mathbf{x}_1, \mathbf{x}_0, T))$. Using Lemma A1.59 we find

$$|\mathfrak{T}((\mathbf{x}_1, \mathbf{x}_0, T)) - \mathfrak{T}'((\mathbf{x}_1, \mathbf{x}_0, T))| < C_k \exp\left(-c_k T\right).$$

On the other hand, by construction, it is easy to see that

$$|\mathfrak{T}'((\mathbf{x}_1, \mathbf{x}_0, T)) - (T + h(\mathbf{x}_1, \mathbf{x}_0))|_{C^k} < C_k \exp\left(-c_k T\right),$$

where $h(\mathbf{x}_1, \mathbf{x}_0)$ is a smooth function on $U_{\mathbf{p}_2^{(1)}} \times U_{\mathbf{p}_2^{(0)}}$. Indeed we can prove this inequality by using exponential decay estimate near the end of the biholomorphic map $\Sigma_i^{(1)} \to \Sigma_i^{(2)}$ which sends $z_i^{(1),\mathrm{int}}$ to $z_i^{(2),\mathrm{int}}$. We thus obtain (A1.60.1).

We next compare $ev_{W_i}(w^{(1)\prime}_{T;\mathbf{x}_1,\mathbf{x}_0})$ with $ev_{W_i}(w^{(2)\prime}_{T;\mathbf{x}'_1,\mathbf{x}'_0})$ where

$$(\mathbf{x}'_1, \mathbf{x}'_0) = \overline{\phi}_{\mathbf{p}_1,\mathbf{p}_2}(\mathbf{x}_1, \mathbf{x}_0).$$

By the construction of coordinate change $\overline{\phi}_{\mathbf{p}_1,\mathbf{p}_2}$ and by (A1.60.1), we easily obtain

$$\left|\frac{\partial^{k+\ell}}{\partial T^k \partial \mathbf{x}^\ell}(ev_{W_i}(w^{(1)\prime}_{T;\mathbf{x}_1,\mathbf{x}_0}) - ev_{W_i}(w^{(2)\prime}_{T;\mathbf{x}'_1,\mathbf{x}'_0}))\right| < C_{k,\ell}\exp(-c_{k,\ell}T). \tag{A1.60.2}$$

Therefore (A1.60.1), (A1.60.2) and Lemma A1.59 immediately imply Proposition A1.57. □

We can prove Lemma A1.59 in a straightforward way by examining the details of the proof of §29.3. Since estimating the T-derivatives may not be so standard, we give a detailed proof of the exponential estimates in Lemma A1.59 below for completeness.

Before going to the proof, we recall that the method of the proof in §29.3 imitates the one in [Fuk96II]. The proofs of surjectivity of the linearized operator in Proposition 29.27 of this book and the one stated in [Fu96II, Lemma 8.5], follow Donaldson's ' 'alternating method' in [Don86]. The method of §29.3 is slightly different which uses the implicit function theorem. (However at the most important ingredient, Lemma 29.32, of the proof, we use the same idea as the alternating method.) In [Fu96II] the alternating method was used to study the linear part of the equation, while Donaldson used the alternating method directly to solve non-linear partial differential equation in his original argument [Don86] which is the anti-self-dual equation for his case.

To see the T dependence of the construction in a transparent way, it seems simpler to use the alternating method directly rather than to put a part of the proof in a black box of functional analysis (that is implicit function theorem). All the techniques involved in the proof have become by now standard since it first appeared more than 20 years ago. For example, Donaldson's paper [Don86] appeared in 1986 which has become a classic for those working on geometric PDE seriously. Nevertheless, we describe the main scheme of alternating method here in our circumstance. After that, we examine details of the constructions used in the method and derive the statement of Lemma A1.59. Especially we will explain how the T-derivatives can be estimated in this way. It seems that the estimate of T-derivative is easier if we use cylindrical coordinate than the standard coordinate near the nodal points on the domain.

Now we are ready to prove Lemma A1.59.

Proof of Lemma A1.59. First of all, we may assume that the $(2n-2)$-dimensional submanifolds Y_0, Y_1 are totally geodesic with respect to the Riemannian metric. In the following argument, we consider the space of mappings with the constraints that $w(z_i^{\rm int}) \in Y_i$, $i = 0, 1$. Then the tangent vector at w is a section V of $(w^*TM, w|_{\partial\Sigma}^* TL)$ such that $V(z_i^{\rm int}) \in T_{w(z_i^{\rm int})}Y_i$. (Here Σ is either Σ_i, $i = 0, 1$ or Σ_T.)

We start with the approximate solution $w^{(j)\prime}_{T;\mathbf{x}_1,\mathbf{x}_0}$ constructed in the proof of Proposition A1.57. From now on we omit j and write $w'_{T;\mathbf{x}_1,\mathbf{x}_0}$ since we can work for each of j separately for the proof of Lemma A1.59.

(Cycle 0) We put

$$w'_{T;\mathbf{x}_1,\mathbf{x}_0} = w'_{T;\mathbf{x}_1,\mathbf{x}_0;0}.$$

(There are no (Step 0-1) and (Step 0-2) in Cycle 0, because we already start with the approximate solution. See the corresponding steps in the later cycle described below.)

(Step 0-3) : (Apply $\overline{\partial}$ operator to the approximate solution.) We consider the $\overline{\partial}$ derivative

$$\overline{\partial}(w'_{T;\mathbf{x}_1,\mathbf{x}_0;0}) \in \Gamma(\Sigma_T; \Lambda^{0,1} \otimes (w'_{T;\mathbf{x}_1,\mathbf{x}_0;0})^*TM).$$

This is exponentially small in the sense of Lemma 29.29. We denote it by Err_0 :

$$\mathrm{Err}_0 = \overline{\partial}(w'_{T;\mathbf{x}_1,\mathbf{x}_0;0}).$$

(Step 0-4) : (Decompose the error term into 2-pieces.) We use the operator $J^S_{*,i}$ $(i = 0, 1)$ which was defined just below Figure 29.4 to obtain elements

$$\mathrm{Err}_{0,i} = J^0_{*,i}(\mathrm{Err}_0) \in \Gamma(\Sigma_i, \Lambda^{0,1} \otimes (w'_{T;\mathbf{x}_1,\mathbf{x}_0;0;i})^*TM), \quad i = 0, 1.$$

Here we put $S = 0$ in $J^S_{*,i}$. Let us explain the notation $w'_{T;\mathbf{x}_1,\mathbf{x}_0;0;i}$. We remark that $J^0_{*,i}$ for $i = 0$, 1 are operators which decomposes the section into two pieces by a partition of unity cuting along $[-1, 1] \times [0, 2]$.

Therefore after we cut the section into two, the support of the section $J^0_{*,1}(\mathrm{Err}_0)$ lies outside $[5T + 1, \infty) \times [0, 1] \subset \Sigma_1$. (Note that Σ_1 is denoted by $X(\mathfrak{t}_1, \ell_1)$ in §29.3 and contains $[0, \infty) \times [0, 1]$.) We define a map $w'_{T;\mathbf{x}_1,\mathbf{x}_0;0;1} : \Sigma_1 \to M$ so that it coincides with $w'_{T;\mathbf{x}_1,\mathbf{x}_0;0}$ outside $[10T - 1, \infty) \times [0, 1]$ where $w'_{T;\mathbf{x}_1,\mathbf{x}_0;0}$ is defined, and extend it over $[10T - 1, \infty) \times [0, 1]$ by an appropriate partition of unity via the exponential map at the corresponding nodal point. We also define $w'_{T;\mathbf{x}_1,\mathbf{x}_0;0;0} : \Sigma_0 \to M$ in a similar way. Note that the part $[10T - 1, \infty) \times [0, 1]$ is not used in (Step n-2) and (Step n-4) so it does not matter how we extend as far as we choose it so that its T-derivatives etc are exponentially small. We can check this point inductively on n, the number of cycles.

By a reason we will explain later (after (A1.62)) we complete the 0-th cycle here and enter to the next cycle, (Cycle 1).

(Cycle 1)

(Step 1-1) : (Solve the equations on the 2-parts.) We apply a version of Lemma 29.20 here. Namely we use the surjectivity of

$$\pi \circ D_{w'_{T;\mathbf{x}_1,\mathbf{x}_0;0;1}, w'_{T;\mathbf{x}_1,\mathbf{x}_0;0;0}} \overline{\partial} \; : \; \mathcal{X} \to \mathcal{E}^{0,p}_{\Sigma_1}/E_1 \oplus \mathcal{E}^{0,p}_{\Sigma_0}/E_0. \tag{A1.61}$$

We remark (A1.61) is slightly different from the operator appearing in Lemma 29.20. That is, we use $w'_{T;\mathbf{x}_1,\mathbf{x}_0;0;i}$ in place of w_i which was used in Lemma 29.20. The reason of this difference is that we are doing a nonlinear version of the alternating argument of Donaldson while Lemma 29.20 was used for its linear version. However since the difference between them is exponentially small, (A1.61) is still surjective.

We apply a right inverse of (A1.61) to $(\mathrm{Err}_{0,1}, \mathrm{Err}_{0,0})$. We then obtain

$$V_{1,i} \in \Gamma((\Sigma_i, z_i^{\rm int}), ((w'_{T;\mathbf{x}_1,\mathbf{x}_0;0;i})^*TM, TY_i)), \qquad i = 0,1.$$

(Step 1-2) : (Glue the solutions.) We use $(V_{1,1}, V_{1,0})$ in the same way as we constructed w'_T just below Figure 29.4. More precisely, we glue them as in (29.23). (We take $S = 2T$ in (29.3).) We then obtain an element of

$$\Gamma((\Sigma_T, z_1^{\rm int}, z_0^{\rm int}), ((w'_{T;\mathbf{x}_1,\mathbf{x}_0;0})^*TM, TY_1, TY_0).$$

Applying the exponential map $\exp_{w'_{T;\mathbf{x}_1,\mathbf{x}_0;0}}$ thereto, we obtain a new map

$$w'_{T;\mathbf{x}_1,\mathbf{x}_0;1} \ : \ \Sigma_T \to M,$$

such that $w'_{T;\mathbf{x}_1,\mathbf{x}_0;1}(z_i^{\rm int}) \in Y_i$, $i = 0,1$. We remark that we use a partition of unity cutting along $[2T-1, 2T+1] \times [0,1] \subset \Sigma_T$ which corresponds to $[7T-1, 7T+1] \times [0,1] \subset \Sigma_0$. This region is by $2T$ farther to the direction towards the end than the region cut along which we used a partition of unity in (Step n-4). This difference of $2T$ is the crucial ingredient which makes the error term drop by the order exponentially small with respect to T. This point appears both in Donaldson's argument [Don86] and in the proof of Lemma 29.32 by the similar fashion.

(Step 1-3) : (Apply $\overline{\partial}$ operator to the approximate solution.) We consider the $\overline{\partial}$ derivative

$$\mathrm{Err}_1 = \overline{\partial}(w'_{T;\mathbf{x}_1,\mathbf{x}_0;1}) \in \Gamma(\Sigma_T; \Lambda^{0,1} \otimes (w'_{T;\mathbf{x}_1,\mathbf{x}_0;1})^*TM)$$

in the same way as (Step 0-3).

(Step 1-4) : (Decompose the error term into 2-pieces.) We use the operator $J^S_{*,i}$ $(i = 0,1)$ in the same way as (Step 0-4) to obtain element

$$\mathrm{Err}_{1,i} = J^0_{*,i}(\mathrm{Err}_1) \in \Gamma(\Sigma_i, \Lambda^{0,1} \otimes (w'_{T;\mathbf{x}_1,\mathbf{x}_0;1;i})^*TM), \qquad i = 0,1.$$

The maps $w'_{T;\mathbf{x}_1,\mathbf{x}_0;1;i}$ are defined in the same way as Step 0-4.

Now we come back to the same situation as Step 1-1. So we can start the next cycle :

(Cycle 2) (Step 2-1) => (Step 2-2) => (Step 2-3) => (Step 2-4)

and proceed to (Cycle n) in the same way.

In the same way as [Don86] (with its minor adaptation to the Bott-Morse case in [Fuk96]) we can show that the limit

$$\lim_{n\to\infty}(w'_{T;\mathbf{x}_1,\mathbf{x}_0;n})$$

is a pseudo-holomorphic map which is our desired $w_{T;\mathbf{x}_1,\mathbf{x}_0}$.

We remark that at the beginning of each (Cycle n) we have elements of

$$\Gamma(\Sigma_i, \Lambda^{0,1}\otimes (w'_{T;\mathbf{x}_1,\mathbf{x}_0;n-1;i})^*TM), \qquad i=0,1. \tag{A1.62}$$

Here the domain Σ_i is independent of T. The bundle $(w'_{T;\mathbf{x}_1,\mathbf{x}_0;n-1;i})^*TM$ may depend on T. But the difference is small in the exponential order of T (as we can prove inductively at the same time). So they can be identified to each other by taking a trivialization of the tangent bundle of the target space M locally. Namely for each cycle we start with elements of (A1.62) and at the end of the cycle we obtain an element of the same function space. So it makes sense to take T derivation at each cycle. (This is the reason why we start each of the cycle as we described above.)

Now we are in the position to estimate the T derivatives of the sections appearing at each step. (The estimates of the terms which do not involve T derivatives are quite obvious from the description above.)

We first consider (Step n-2), (Step n-3) and (Step n-4). We remark that these steps are local processes, which are gluing sections by partitions of unity, applying exponential maps, and applying the $\overline{\partial}$ operator. None of these processes themselves involve the T-parameter. but some functions etc appearing in the processes contain the T parameter. Therefore they become the sources where non zero T-derivatives can appear. By examining the construction, we can check that there are two sources from which such non zero T-derivatives can appear :

(A) We use partitions of unity that depend on T. Taking the T-derivatives of the terms involving the partitions of unity generate terms of the form $\partial^k\chi_T/dT^k$. Such terms are bounded in C^∞ norm. Therefore multiplying them is a bounded operator. We then note that we multiply the derivatives to a section whose exponential decay is already established by an earlier stage of induction. Therefore these terms can be estimated in the exponential order.

(B) The maps $w'_{T;\mathbf{x}_1,\mathbf{x}_0;n}$ or $w'_{T;\mathbf{x}_1,\mathbf{x}_0;n;i}$ ($i=0,1$) appear in the course of the construction. They are T-dependent because T appears at the earlier stage of induction. However we can inductively prove that they are exponentially small in local weighted $W^{1,p}$ norm used here. (See §29.3 for the definition of the norm.) Therefore by taking the T-derivative we obtain a multiplication operator by a section whose local $W^{1,p}$ norm is of exponentially small order. Using the boundedness of the product map $W^{1,p}\times W^{1,p}\to L^p$ we conclude that the resulting terms in this way is also exponentially small.

We next consider (Step n-1). This step is slightly more non-trivial since it is non-local and uses the right inverse of (A1.61). To study this step, we consider the T-derivative of (A1.61). We remark that the leading term of (A1.61) is the Cauchy-Riemann operator and does not contain T. The 0-th order part of the operator (A1.61) does contain T because the maps $w'_{T;\mathbf{x}_1,\mathbf{x}_0;n;i}$ contain T. By the same reason as (B) above, the T-derivative of (A1.61) is a multiplication operator by a section whose weighted $W^{1,p}$ norm is exponentially small. Here we note that the function spaces in (A1.61) do not depend on T at all in (Step n-1). Thus by a standard argument (for example by using the Neumann series) we can find its right inverse whose T derivatives are exponentially small with respect to the operator norm for the operators $L^p \to W^{1,p}$).

Therefore the proof of Lemma A1.59 is now complete. □

Remark A1.63. If we identify the double of the glued discs with a closed Riemann surface, then the glued surface is regarded locally (around the singular point) as a solution of $z_1 z_2 = r$, where $r = \exp(-cT)$. So (A1.58.3) can be written as

$$\left| \frac{\partial^k \phi_{\mathbf{p}_1,\mathbf{p}_2}}{\partial r^k} \right| < r^{c_k - k}.$$

This is the same type of estimate which is satisfied by a function such as $f(r) = r^\lambda$. In that sense our estimate is natural. On the other hand, it is *not* enough for the smoothness of $\phi_{\mathbf{p}_1,\mathbf{p}_2}$ with respect to (a power of) r. Our choice of coordinate here is

$$r = e^{-1/s}.$$

In this coordinate $f(s) = e^{-\lambda/s}$ is certainly smooth.

One might expect the smoothness of $\phi_{\mathbf{p}_1,\mathbf{p}_2}$ with respect to some power of r. (This seems to be the case when we can work in the category of real algebraic geometry, for example.) We do not try to improve our estimate here since a sharper estimate is not necessary for our purpose, though such an estimate looks interesting of its own. The authors have recently learned (at the year 2007) that several people are trying to prove this sharper estimate.

A1.5. Some counter examples.

Example A1.64. We first give an example related to the definition of the tangent bundle (Definition A1.14). Let $X = [-2, 2]$. We put $V_1 = \mathbb{C} \times [-1, 2]$, $E_1 = \mathbb{C}$, $\Gamma_1 = \{1\}$. We define $s_1(z, t) = z^2$ and $\psi_1(0, t) = t$, $\psi_1 : s_1^{-1}(0) \to X$. We next put $V_2 = [-2, 0]$ $E_2 = 0$, $\Gamma_2 = \{1\}$, $s_2 \equiv 0$. $\psi_2(t) = t$.

The coordinate change ϕ_{12} is $t \mapsto (0, t)$, $[-1, 0] = V_{12} \to \mathbb{C} \times [-1, 0] \subset V_1$.

This defines a Kuranishi structure on X. The normal bundle $N_{V_{12}}V_1$ is obviously trivial. Hence it is isomorphic to $E_1/\hat{\phi}_{12}(E_2)$. So the isomorphism (A1.5) exists but is *not induced by s.*

The multiplicity at $\{2\}$ is 2 and at $\{-2\}$ is 1. The virtual fundamental chain is not homologous to zero. Therefore, by Lemma A1.31, we can not modify our Kuranishi structure of X outside the boundary, so that it has a tangent bundle.

Example A1.65. We next consider compatibility condition (A1.6.2). Let us consider an orbifold $\mathbb{C}/\mathbb{Z}_2$ where $\mathbb{Z}_2$ action is by $z \mapsto -z$. We find that $(\mathbb{C}/\mathbb{Z}_2) \setminus \{0\}$ is isomorphic to $\mathbb{C} \setminus \{0\}$ as an orbifold. We glue it with $(\mathbb{C} \cup \{\infty\}) \setminus \{0\}$ to obtain an orbifold X. This is an example of bad orbifold, that is, an orbifold that is not a global quotient of a manifold by a discrete group action.

A coordinate at 0 is given by $V_0/\Gamma_0 = \mathbb{C}/\mathbb{Z}_2$. Let $p_t = [\exp(2\pi t\sqrt{-1})] \in \mathbb{C}/\mathbb{Z}_2$. A small disc $\mathrm{Int}D^2$ with trivial Γ, is its coordinate chart. The coordinate change from this chart to V_0/Γ_0 is given by

$$z \mapsto \pm \exp(2\pi t\sqrt{-1}) + z. \tag{A1.66}$$

Here we may choose either $+$ or $-$ in the above formula.

Let us move t from 0 to 1 continuously. If we do not put γ_{pqr} in compatibility condition (A1.6.2) then we can not change $\pm$ in (A1.66). Namely if we fix it for some t then we are not allowed to change it. However $t = 0$ and $t = 1/2$ corresponds to the same point. This is a contradiction. Namely we need to include $\gamma_{pqr} = -1$ to Definition A1.5 (A1.6.2) to include this example to an orbifold.

We remark that we need $\gamma_{pp',qq'}$ etc. in Definition A1.45 by the same reason.

Remark A1.67. An orbifold can be regarded as a *groupoid* which is a category all of whose morphisms are isomorphisms. (This fact is classical and goes back to the early days when the theory of stack was started by Grothendieck. As the first and fourth named authors mentioned in [FuOn99I], the theory of Kuranishi structure is an analog of the theory of stack.)

A map between two orbifolds corresponds to a functor between the two categories. In the category theory, it is crucial to tell the notion of two objects, morphisms etc. being *equal* from that of being *isomorphic*. The condition (A1.6.2) corresponds to the condition that two functors are equivalent (that is, *isomorphic*). The isomorphism with $\gamma_{pqr} \equiv 1$ exactly corresponds to the *equality*. In order to define the notion of stack we need to glue local objects. The compatibility condition of gluing maps is, by definition, that the composition of gluing maps is *equivalent to* the third composition map. This coincides with the condition (A1.6.2) in our definition. In fact, our exposition would be more systematic, if we used the notion of 2-category. However since we do not need this abstraction for the applications in this book, we do not use the language of 2-category but state the gluing condition explicitly as the coincidence condition of equivalence classes.

We next provide an example related to the discussion in §35. Let X be an orbifold and Γ be a finite group. We define

$$X^{\cong}(\Gamma) = \{x \in X \mid I_x \cong \Gamma\}.$$

In §35 we need to consider a normal bundle $N_{X^{\cong}(\Gamma)}X$ of $X^{\cong}(\Gamma)$ in X. At first sight, one might expect that there exists a vector bundle $N_{X^{\cong}(\Gamma)}X$ over the topological space $X^{\cong}(\Gamma)$ together with a Γ action on $N_{X^{\cong}(\Gamma)}X$ such that $N_{X^{\cong}(\Gamma)}X/\Gamma$ is diffeomorphic to a neighborhood of $X^{\cong}(\Gamma)$ in X. However such a vector bundle $N_{X^{\cong}(\Gamma)}X$ does not exist in general. In fact we have the following counter example.

Example A1.68. We consider $\mathbb{C}^n \times S^1$ with $\mathbb{Z}_p$ action defined by

$$[k] \cdot (z, [t]) = (\exp(2\pi\sqrt{-1}k/p)z, [t]). \tag{A1.69}$$

Here $[k] \in \mathbb{Z} \mod p$, $[t] \in \mathbb{R}/\mathbb{Z} = S^1$.

We define an isomorphism

$$F : (\mathbb{C}^n \times S^1)/\mathbb{Z}_p \to (\mathbb{C}^n \times S^1)/\mathbb{Z}_p$$

by

$$F([z, [t]]) = [\exp(2\pi\sqrt{-1}t/p)z, [t]].$$

We take two copies $(\mathbb{C}^n \times D^2_{\pm})/\mathbb{Z}_p$ of $(\mathbb{C}^n \times D^2)/\mathbb{Z}_p$ where $\mathbb{Z}_p$ action is similar to (A1.69). We identify

$$(\mathbb{C}^n \times S^1)/\mathbb{Z}_p = (\mathbb{C}^n \times \partial D^2_{+})/\mathbb{Z}_p$$

with

$$(\mathbb{C}^n \times S^1)/\mathbb{Z}_p = (\mathbb{C}^n \times \partial D^2_{-})/\mathbb{Z}_p$$

by F and obtain an orbifold X.

In this example $X^{\cong}(\mathbb{Z}_p) = S^2$. The normal bundle $N_{X^{\cong}(\mathbb{Z}_p)}X$ does not exist since F does not lift to a bundle isomorphism : $\mathbb{C}^n \times S^1 \to \mathbb{C}^n \times S^1$.

A1.6. Singular locus as a stack and its normal bundle.

As Example A1.68 shows, the normal bundle of the singular locus $X^{\cong}(\Gamma)$ does not exist as a vector bundle with a Γ action, in general. On the other hand, in §35 we need to use normal bundle $X^{\cong}(\Gamma)$ to define normally conical perturbations. For this purpose we define a notion of normal bundle in the sense of stack. We restrict our discussion of the stack to the case we use for this purpose. Related material is discussed in various references such as [Bry93], [Gir70]. The discussion here is related to the phenomenon that occurs when we remove the effectivity of the Γ_p action from (A1.2.3).

A1.6.1. Sheaves of category over a group.

Let G be a group. We consider the category $\underline{G}$ which has only one object $*$ and morphism $\underline{G}(*, *) = G$. Let M be a topological space and $\mathcal{U} = \{U_i \mid i \in I\}$ be an open covering of M. We assume that $U_{i_1} \cap \cdots \cap U_{i_k}$ $(i_1, \cdots, i_k \in I)$ are either empty or contractible. Namely we take a good covering.

Definition A1.70. (1) A *sheaf of category of* $\underline{G}$ *on* $(M,\mathcal{U})$ consists of pair

$$(\{h_{ij}\}, \{\gamma_{ijk}\})$$

of an isomorphism

$$h_{ij} : G \to G \qquad \text{for each } U_i \cap U_j \neq \emptyset$$

and an element

$$\gamma_{ijk} \in G \qquad \text{for each } U_i \cap U_j \cap U_k \neq \emptyset$$

such that the following compatibility conditions (A1.71.1) and (A1.71.2) hold for every $U_i \cap U_j \cap U_k \cap U_l \neq \emptyset$:

$$h_{ij} \circ h_{jk} = \gamma_{ijk} \cdot h_{ik} \cdot \gamma_{ijk}^{-1}. \tag{A1.71.1}$$

$$\gamma_{ijk} \cdot \gamma_{ikl} = h_{ij}(\gamma_{jkl}) \cdot \gamma_{ijl}. \tag{A1.71.2}$$

See Figure A1.1. (We will explain how (A1.71) follows from the definition of a stack in the categorical context in Remark A1.81 (5).)

Figure A1.1

(2) $(\{h_{ij}\},\{\gamma_{ijk}\})$ is said to be *isomorphic* to $(\{h'_{ij}\},\{\gamma'_{ijk}\})$ if there exist $\psi_i \in \mathrm{Aut}(G)$ for each i, $\mu_{ij} \in G$ for each $U_i \cap U_j \neq \emptyset$, $h''_{ij} \in \mathrm{Aut}(G)$ for each $U_i \cap U_j \neq \emptyset$, and $\gamma''_{ijk} \in G$ for each $U_i \cap U_j \cap U_k \neq \emptyset$ such that

$$h''_{ij} = \psi_i \circ h_{ij} \circ \psi_j^{-1}, \tag{A1.72.1}$$

$$\gamma''_{ijk} = \psi_i(\gamma_{ijk}), \tag{A1.72.2}$$

$$\mu_{ij} \cdot h''_{ij} \cdot \mu_{ij}^{-1} = h'_{ij}, \tag{A1.72.3}$$

$$\mu_{ij} \cdot h''_{ij}(\mu_{jk}) \cdot \gamma''_{ijk} = \gamma'_{ijk} \cdot \mu_{ik}. \tag{A1.72.4}$$

See Figure A1.2. We call a pair $(\{\mu_{ij}\},\{\psi_i\})$ an *isomorphism* : $(\{h_{ij}\},\{\gamma_{ijk}\}) \to (\{h'_{ij}\},\{\gamma'_{ijk}\})$. We will prove in Lemma A1.75 that we can compose isomorphism and 'isomorphic' defines an equivalence relation.

Figure A1.2

(3) We denote by $Sh((M,\mathcal{U});\underline{G})$ the set of all isomorphism classes of sheaf of categories of $\underline{G}$ on $(M,\mathcal{U})$.

To illustrate the meaning of (A1.71) we show the following :

Lemma A1.73. *Let $\{Y_i\}_i$ be a collection of sets on each of which G acts effectively and there is a point with trivial isotropy group. Let $h_{ij} : G \to G$ be group isomorphisms, γ_{ijk} elements of G, and $\phi_{ij} : Y_j \to Y_i$ be the maps that are injective and h_{ij}-equivariant. Suppose*

$$\phi_{ij} \circ \phi_{jk} = \gamma_{ijk} \cdot \phi_{ik}. \tag{A1.74}$$

Then γ_{ijk} satisfies (A1.71).

Proof. Let $g \in G$, and $y \in Y_k$ with trivial isotropy group. Then $\phi_{ik}(y)$ also has a trivial isotropy group because ϕ_{ik} are assumed to be injective and h_{ik}-equivariant.

Again by the h_{ij}-equivariance of the map ϕ_{ik}, we obtain

$$\begin{aligned}\gamma_{ijk} \cdot h_{ik}(g) \cdot \phi_{ik}(y) &= \gamma_{ijk} \cdot \phi_{ik}(g \cdot y) = \phi_{ij}(\phi_{jk}(g \cdot y)) \\ &= h_{ij}(h_{jk}(g)) \cdot \phi_{ij}(\phi_{jk}(y)) = h_{ij}(h_{jk}(g)) \cdot \gamma_{ijk} \cdot \phi_{ik}(y).\end{aligned}$$

Because $\phi_{ik}(y)$ has a trivial isotropy group, (A1.71.1) follows.

Similarly for $y \in Y_l$ we calculate

$$\begin{aligned}h_{ij}(\gamma_{jkl}) \cdot \gamma_{ijl} \cdot \phi_{il}(y) &= h_{ij}(\gamma_{jkl}) \cdot \phi_{ij}(\phi_{jl}(y)) = \phi_{ij}(\gamma_{jkl} \cdot \phi_{jl}(y)) \\ &= \phi_{ij}(\phi_{jk}(\phi_{kl}(y))) = \gamma_{ijk} \cdot \phi_{ik}(\phi_{kl}(y)) = \gamma_{ijk} \cdot \gamma_{ikl} \cdot \phi_{il}(y).\end{aligned}$$

This implies (A1.71.2). $\square$

In the definition of Kuranishi structure the groups Γ_p are assumed to be a finite group and the space V_p is a smooth manifold. One can show that effectivity of the action of Γ_p automatically implies existence of a point with trivial isotropy group. And we also assume that the map ϕ_{pq} is an h_{pq}-equivariant embedding and in particular injective. Therefore the same argument used in the proof of Lemma A1.73, implies that γ_{pqr} in Definition A1.5 satisfies (A1.71.2).

Lemma A1.75. *The relation 'isomorphism' in Definition* A1.70 *is an equivalence relation.*

Proof. We use notation of (A1.72) and put

$$(h''_{ij}, \gamma''_{ijk}) = (1, \psi_i)_*(h_{ij}, \gamma_{ijk}), \quad (h'_{ij}, \gamma'_{ijk}) = (\mu_{ij}, 1)_*(h''_{ij}, \gamma''_{ijk}).$$

We also put $(\mu_{ij}, 1)_* \circ (1, \psi_i)_* = (\mu_{ij}, \psi_i)_*$.

We remark that

$$(1, \psi_i)_* \circ (1, \psi'_i)_* = (1, \psi_i \circ \psi'_i)_*. \tag{A1.76}$$

We next claim

$$(\mu_{ij}, 1)_* \circ (\mu'_{ij}, 1)_* = (\mu_{ij} \cdot \mu'_{ij}, 1)_*. \tag{A1.77}$$

Let us prove (A1.77). We put

$$(\mu'_{ij}, 1)_*(h^1_{ij}, \gamma^1_{ijk}) = (h^2_{ij}, \gamma^2_{ijk}), \quad (\mu_{ij}, 1)_*(h^2_{ij}, \gamma^2_{ijk}) = (h^3_{ij}, \gamma^3_{ijk}).$$

Then

$$\begin{aligned}h^2_{ij} &= \mu'_{ij} \cdot h^1_{ij} \cdot (\mu'_{ij})^{-1}, \\ \gamma^2_{ijk} &= \mu'_{ij} \cdot h^1_{ij}(\mu'_{jk}) \cdot \gamma^1_{ijk} \cdot (\mu'_{ik})^{-1}.\end{aligned}$$

Therefore we have

$$h^3_{ij} = \mu_{ij} \cdot \mu'_{ij} \cdot h^1_{ij} \cdot (\mu'_{ij})^{-1} \cdot \mu_{ij}^{-1}$$

and

$$\begin{aligned}
\gamma^3_{ijk} &= \mu_{ij} \cdot h^2_{ij}(\mu_{jk}) \cdot \gamma^2_{ijk} \cdot \mu_{ik}^{-1} \\
&= \mu_{ij} \cdot \mu'_{ij} \cdot h^1_{ij}(\mu_{jk}) \cdot (\mu'_{ij})^{-1} \cdot \mu'_{ij} \cdot h^1_{ij}(\mu'_{jk}) \cdot \gamma^1_{ijk} \cdot (\mu'_{ik})^{-1} \cdot \mu_{ik}^{-1} \\
&= \mu_{ij} \cdot \mu'_{ij} \cdot h^1_{ij}(\mu_{jk} \cdot \mu'_{jk}) \cdot \gamma^1_{ijk} \cdot (\mu_{ik} \cdot \mu'_{ik})^{-1}.
\end{aligned}$$

(A1.77) is proved.

We next claim

$$(1, \psi_i)_* \circ (\mu_{ij}, 1)_* = (\psi_i(\mu_{ij}), \psi_i)_*. \tag{A1.78}$$

Let us prove (A1.78). We put

$$(\mu_{ij}, 1)_*(h^1_{ij}, \gamma^1_{ijk}) = (h^2_{ij}, \gamma^2_{ijk}), \quad (1, \psi_i)_*(h^2_{ij}, \gamma^2_{ijk}) = (h^3_{ij}, \gamma^3_{ijk}).$$

Then

$$\begin{aligned}
h^2_{ij} &= \mu_{ij} \cdot h^1_{ij} \cdot \mu_{ij}^{-1}, \\
\gamma^2_{ijk} &= \mu_{ij} \cdot h^1_{ij}(\mu_{jk}) \cdot \gamma^1_{ijk} \cdot (\mu_{ik})^{-1}.
\end{aligned}$$

Therefore we have

$$h^3_{ij} = \psi_i \circ (\mu_{ij} \cdot h^1_{ij} \cdot \mu_{ij}^{-1}) \circ \psi_j^{-1} = \psi_i(\mu_{ij}) \cdot (\psi_i \circ h^1_{ij} \circ \psi_j^{-1}) \cdot \psi_i(\mu_{ij})^{-1}$$

and

$$\gamma^3_{ijk} = \psi_i(\mu_{ij}) \cdot \psi_i(h^1_{ij}(\mu_{jk})) \cdot \psi_i(\gamma^1_{ijk}) \cdot \psi_i(\mu_{ik})^{-1}.$$

We next put

$$(1, \psi_i)_*(h^1_{ij}, \gamma^1_{ijk}) = (h^4_{ij}, \gamma^4_{ijk}), \quad (\psi_i(\mu_{ij}), 1)_*(h^4_{ij}, \gamma^4_{ijk}) = (h^5_{ij}, \gamma^5_{ijk}).$$

Then

$$h^4_{ij} = \psi_i \circ h^1_{ij} \circ \psi_j^{-1}, \quad \gamma^4_{ijk} = \psi_i(\gamma^1_{ijk}).$$

Therefore $h^5_{ij} = h^3_{ij}$ and

$$\begin{aligned}
\gamma^5_{ijk} &= \psi_i(\mu_{ij}) \cdot h^4_{ij}(\psi_j(\mu_{jk})) \cdot \gamma^4_{ijk} \cdot \psi_i(\mu_{ik})^{-1} \\
&= \psi_i(\mu_{ij}) \cdot \psi_i(h^1_{ij}(\mu_{jk})) \cdot \psi_i(\gamma^1_{ijk}) \cdot \psi_i(\mu_{ik})^{-1} = \gamma^3_{ijk}.
\end{aligned}$$

(A1.78) is proved.

(A1.76), (A1.77) and (A1.78) imply that we can compose isomorphisms. Hence the relation 'isomorphic' is transitive.

On the other hand, (A1.76) and (A1.77) imply that each isomorphism has an inverse. Hence the relation 'isomorphic' is reflexive. $\square$

Remark 1.79. We assume (h_{ij}, γ_{ijk}) satisfies (A1.71). We define (h'_{ij}, γ'_{ijk}) by (A1.72). We can check that (h'_{ij}, γ'_{ijk}) satisfies (A1.71), in a similar way as the above calculation. For example we consider the case $\psi_i = 1$ and check (A1.71.2) as follows. We have

$$\begin{aligned}
&\gamma'_{ijk} \cdot \gamma'_{ikl} \\
&= \mu_{ij} \cdot h_{ij}(\mu_{jk}) \cdot \gamma_{ijk} \cdot h_{ik}(\mu_{kl}) \cdot \gamma_{ikl} \cdot \mu_{il}^{-1} \\
&= \mu_{ij} \cdot h_{ij}(\mu_{jk}) \cdot \gamma_{ijk} \cdot h_{ik}(\mu_{kl}) \cdot \gamma_{ijk}^{-1} \cdot h_{ij}(\gamma_{jkl}) \cdot \gamma_{ijl} \cdot \mu_{il}^{-1}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&h'_{ij}(\gamma'_{jkl}) \cdot \gamma'_{ijl} \\
&= \mu_{ij} \cdot h_{ij}(\mu_{jk} \cdot h_{jk}(\mu_{kl}) \cdot \gamma_{jkl} \cdot \mu_{jl}^{-1}) \cdot \mu_{ij}^{-1} \cdot \mu_{ij} \cdot h_{ij}(\mu_{jl}) \cdot \gamma_{ijl} \cdot \mu_{il}^{-1} \\
&= \mu_{ij} \cdot h_{ij}(\mu_{jk}) \cdot \gamma_{ijk} \cdot h_{ik}(\mu_{kl}) \cdot \gamma_{ijk}^{-1} \cdot h_{ij}(\gamma_{jkl}) \cdot h_{ij}(\mu_{jl}^{-1}) \cdot \mu_{ij}^{-1} \\
&\quad \cdot \mu_{ij} \cdot h_{ij}(\mu_{jl}) \cdot \gamma_{ijl} \cdot \mu_{il}^{-1}.
\end{aligned}$$

Hence follows (A1.71.2).

Definition A1.80. Let $\mathcal{U}' = \{U'_j \mid j \in J\}$ be another covering of M and let $i(\cdot) : j \mapsto i(j)$ be a map $J \to I$ such that $U'_j \subseteq U_{i(j)}$. We define a map :

$$i(\cdot)^* : Sh((M,\mathcal{U});\underline{G}) \to Sh((M,\mathcal{U}');\underline{G})$$

by

$$i(\cdot)^*([\{h_{i_1i_2}\},\{\gamma_{i_1i_2i_3}\}]) = [\{h'_{j_1j_2}\},\{\gamma'_{j_1j_2j_3}\}]$$

where

$$h'_{j_1j_2} = h_{i(j_1)i(j_2)}, \quad \gamma'_{j_1j_2j_3} = \gamma_{i(j_1)i(j_2)i(j_3)}.$$

We thus obtain an inductive system $\mathcal{U} \mapsto Sh((M,\mathcal{U});\underline{G})$. We use inductive limit with respect to this inductive system and define :

$$Sh(M,\underline{G}) = \varinjlim Sh((M,\mathcal{U});\underline{G}).$$

An element of $Sh(M,\underline{G})$ is said to be a *sheaf of category $\underline{G}$* on M.

Remark A1.81. (1) There is more general notion that is a stack. It is defined by Grothendieck ([Grot62], [Grot71]). See also [Bry93], [Gir70]. We only consider the case when the stalk is independent of the point and is the category $\underline{G}$.

(2) In case when G is commutative, (A1.71.1) implies

$$h_{ij} \circ h_{jk} = h_{ik}.$$

Therefore it defines a G local system $\mathfrak{G}$. Then (A1.71.2) becomes

$$\gamma_{ijk} + \gamma_{ikl} = h_{ij}(\gamma_{jkl}) + \gamma_{ijl}.$$

Namely $\{\gamma_{ijk}\}$ defines a Čech cocycle in $\check{C}^2(\mathcal{U}, \mathfrak{G})$.

Next we assume that $(\{h_{ij}\}, \{\gamma_{ijk}\})$ is isomorphic to $(\{h'_{ij}\}, \{\gamma'_{ijk}\})$. Then (A1.72.1) and (A1.72.2) imply that the induced local system is isomorphic and $\{\gamma''_{ijk}\}$ is the same Čech cocycle as $\{\gamma_{ijk}\}$ under this isomorphism. (A1.72.3) and (A1.72.4) imply that

$$\gamma'_{ijk} - \gamma''_{ijk} = \mu_{ij} + h''_{ij}(\mu_{jk}) - \mu_{ik}.$$

Namely $\{\gamma'_{ijk}\}$ is cohomologous to $\{\gamma''_{ijk}\}$.

Thus

$$Sh(M, \underline{G}) \cong \bigcup_{\mathfrak{G}: G \text{ local systems}} \check{H}^2(M; \mathfrak{G})$$

in the abelian case.

(3) Usually (but not always) the effectivity of the (finite) group Γ_p action on V_p is assumed when one defines the notion of a chart (V_p, Γ_p, ψ_p) of an orbifold. On the other hand, there is no such assumptions for stacks.

Note (A1.71.1) is the same formula as the first formula of (A1.6.2) in the definition of Kuranishi structure. In (A1.6.2) we assumed only the *existence* of γ_{pqr}. Namely it is not a part of the structure. Also the formula corresponding to (A1.71.2) is not in Definition A1.5. On the other hand, in Definition A1.70 we include γ_{ijk} as a part of the structure.

Actually, in the situation of Definition A1.5 where the Γ_p action is assumed to be effective, the element γ_{pqr} satisfying (A1.6.2) is unique if it exists. Moreover a formula corresponding to (A1.71.2) can be proved. (Lemma A1.73.)

In our situation where the G action on M is trivial, γ_{ijk} is not determined from the other data and so we include it as a part of the structure. Also (A1.71.2) is put as a part of conditions.

When the notion of orbifold was discovered by Satake [Sat56], he assumed the effectivity of the action of Γ_p. When Thurston renamed Satake's V-manifold as orbifold, he did not change its mathematical contents and still assumed the effectivity of Γ_p. The authors here include the effectivity of Γ_p as a part of the definition of orbifold, since it is confusing to change the definition, 50 years after its discovery, and after it had been used in various branches of mathematics.

(4) Consider the situation of Definition A1.45. Then in a similar way as Lemma A1.73, we can prove

$$\gamma_{pqp'q'} \cdot h_{p'q'}(\gamma_{qrq'r'}) \cdot \gamma_{p'q'r'} = \rho_p(\gamma_{pqr}) \cdot \gamma_{prp'r'} \tag{A1.82}$$

where $\varphi(p) = p'$, $\varphi(q) = q'$, $\varphi(r) = r'$, $q \in \psi_p(s_p^{-1}(0)/\Gamma_p)$, $r \in \psi_q(s_q^{-1}(0)/\Gamma_q)$. Since (A1.82) is automatic, we did not put it as a part of assumptions in Definition A1.45.

In the situation where effectivity of the Γ_p action is not assumed, (A1.82) will not be automatic.

(5) Using the language of category theory, we can rewrite the definition of $Sh(M, \underline{G})$, as follows. (Our discussion below is informal since we do not use it in this book.)

We first define a category $\mathcal{O}(M)$. Its object is an open set of M. There is no morphisms from U to V if U is not a subset of V. If $U \subset V$, there exists exactly one morphism from U to V.

We next consider the 2-category $\underline{\underline{G}}$ as follows. There is only one object in it. The category of morphism from this object to itself is $\underline{G}$.

Then an element of $Sh(M, \underline{G})$ is regarded as a pseudo-functor from $\mathcal{O}(M)$ to $\underline{\underline{G}}$, in the sense of [Grot71] Exposé VI 8.

Let us explain how a pseudo-functor $\mathcal{O}(M) \to \underline{\underline{G}}$ is related to an element of $Sh(X, \underline{G})$. A pseudo-functor $\mathcal{O}(M) \to \underline{\underline{G}}$ first assigns a functor $F_{UV} : \underline{G} \to \underline{G}$ for each $U \subset V$. Such a functor is nothing but a homomorphism $\phi_{UV} : G \to G$.

If $U_3 \subset U_2 \subset U_1$, then the pseudo-functor associate a natural transformation

$$T_{U_3U_2U_1} : F_{U_3U_1} \to F_{U_3U_2} \circ F_{U_2U_1}$$

which is (in our situation) automatically an equivalence. By definition of the category $\underline{G}$, such a natural transformation is given by an element $\gamma_{U_3U_2U_1} \in G$ such that

$$\gamma_{U_3U_2U_1} \cdot \phi_{U_3U_1} = \phi_{U_3U_2} \circ \phi_{U_2U_1}.$$

This formula corresponds to (A1.71.1).

For the pair $(F_{UV}, \gamma_{U_3U_2U_1})$ to be a pseudo-functor we need to assume a compatibility condition between them, that is the commutativity of the following diagram for each $U_4 \subset U_3 \subset U_2 \subset U_1$.

$$\begin{tikzcd}
F_{U_4U_3} \circ F_{U_3U_1} \arrow[d, "(F_{U_4U_3})_*(T_{U_3U_2U_1})"] & F_{U_4U_1} \arrow[l, "T_{U_4U_3U_1}"'] \arrow[r, "T_{U_4U_2U_1}"] & F_{U_4U_2} \circ F_{U_2U_1} \arrow[d, "(F_{U_2U_1})^*(T_{U_4U_3U_2})"'] \\
F_{U_4U_3} \circ F_{U_3U_2} \circ F_{U_2U_1} \arrow[rr, equal] & & F_{U_4U_3} \circ F_{U_3U_2} \circ F_{U_2U_1}.
\end{tikzcd}$$

(See [FGIKNV05] Definition 3.10 (iv)(b) or [Grot71] Exposé VI Proposition 7.4.) The commutativity of this diagram is equivalent to

$$\gamma_{U_4U_3U_2} \cdot \gamma_{U_4U_2U_1} = \phi_{U_4U_3}(\gamma_{U_3U_2U_1}) \cdot \gamma_{U_4U_3U_1}.$$

This formula is the same as (A1.71.2). (A1.71.1) is a consequence of the fact that $T_{U_3U_2U_1}$ is a natural transformation.

We remark that, in the definition of pseudo-functor in [FGIKNV05] Definition 3.10, there are other conditions (ii), (iv)(a). In our situation, it will become

$$h_{ii}(g) = \gamma_{iii} \cdot g \cdot \gamma_{iii}^{-1}, \quad \gamma_{iij} = \gamma_{iii}, \quad \gamma_{ijj} = h_{ij}(\gamma_{jjj}).$$

They follow from (A1.71). In fact the first formula follows from (A1.71.1) by putting $i = j = k$. The second formula follows from (A1.71.2) by putting $i = j = k$. The third formula follows from (A1.71.2) by putting $j = k = l$.

We can continue and rewrite (A1.72) using category theory. We do not try to do it here. In fact the theory of stack which is well established is based on category theory, and in this subsection we try to give a self-contained account of the part of it which we need, *without* using category theory.

(6) We did *not* assume

$$h_{ii} = id, \qquad h_{ij} \circ h_{ji} = id,$$

in Definition A1.70.

There seems to be a version which assumes the above identities together with

$$\gamma_{ijk} = \gamma_{ikj}^{-1}, \qquad \gamma_{ijk} = h_{ij}(\gamma_{jki}).$$

In the abelian case, these conditions will become the condition that $h_{ij\cdots k}$ is anti-symmetric with respect to the change of indices. It is well known that we have the same Čech cohomology, whether or not we assume the anti-symmetricity.

A1.6.2. The normal bundle of the singular locus $X^{\cong}(\Gamma)$.

Now we apply the above discussion of stacks to the circumstance given in §35 in relation to the study of normal bundles of the singular locus $X^{\cong}(\Gamma)$.

Example-Definition A1.83. Let G be a finite group acting effectively and smoothly on a smooth manifold $\widetilde{X}$. Consider the orbifold $X = \widetilde{X}/G$. Let Γ be an abstract group. We put

$$X^{\cong}(\Gamma) = \{x \in \widetilde{X} \mid I_x \cong \Gamma\}/G,$$

where

$$I_x = \{g \in G \mid gx = x\}.$$

It follows from (A1.84) below that $X^{\cong}(\Gamma)$ is a smooth manifold.

We now give a construction of an element of $Sh(X^{\cong}(\Gamma), \underline{G})$. Decompose $X^{\cong}(\Gamma)$ according to the conjugacy classes of I_x and study each of them separately. For each subgroup $\Gamma_0 \subset G$ with $\Gamma_0 \cong \Gamma$, we consider

$$\widetilde{X}^{=}(\Gamma_0) = \{x \in \widetilde{X} \mid I_x = \Gamma_0\}.$$

Denote the normalizer of Γ_0 by

$$N(\Gamma_0) = \{g \in G \mid g\Gamma_0 g^{-1} = \Gamma_0\}.$$

Then $H(\Gamma_0) = N(\Gamma_0)/\Gamma_0$ acts freely on $\widetilde{X}^{=}(\Gamma_0)$ and by definition we have

$$X^{\cong}(\Gamma) = \bigcup_{\Gamma_0} \widetilde{X}^{=}(\Gamma_0)/H(\Gamma_0), \tag{A1.84}$$

where the union is taken over all conjugacy classes of subgroups of G isomorphic to Γ. We have an exact sequence

$$1 \longrightarrow \Gamma_0 \longrightarrow N(\Gamma_0) \longrightarrow H(\Gamma_0) \longrightarrow 1. \tag{A1.85}$$

We choose a sufficiently fine good covering $\mathcal{U} = \{U_i \mid i \in I\}$ of $X^{\cong}(\Gamma)$ and a lift $\widetilde{U}_i \subset \widetilde{X}^{=}(\Gamma_0)$ of U_i so that the projection $\widetilde{X} \to X$ restricts to a homeomorphism from $\widetilde{U}_i$ to U_i.

For each i, j with $U_i \cap U_j \neq \emptyset$ there exists a unique $\underline{h}_{ij} \in H(\Gamma_0)$ such that $\widetilde{U}_i \cap (\underline{h}_{ij}\widetilde{U}_j) \neq \emptyset$. We remark that $\underline{h}_{ij} \cdot \underline{h}_{jk} = \underline{h}_{ik}$ in $H(\Gamma_0)$, if $U_i \cap U_j \cap U_k \neq \emptyset$.

We choose lifts $\widetilde{\underline{h}}_{ij} \in N(\Gamma_0)$ of $\underline{h}_{ij}$.

We define $h_{ij} : \Gamma_0 \to \Gamma_0$ by

$$h_{ij}(g) = \widetilde{\underline{h}}_{ij} \cdot g \cdot \widetilde{\underline{h}}_{ij}^{-1}.$$

We define $\gamma_{ijk} \in \Gamma_0$ by

$$\gamma_{ijk} \cdot \widetilde{\underline{h}}_{ik} = \widetilde{\underline{h}}_{ij} \cdot \widetilde{\underline{h}}_{jk}.$$

Then it is easy to check (A1.71).

We can generalize the construction above and include the case of orbifold which is not necessarily a quotient of manifold globally. We do not discuss it here since we do not use it in the main application (the proof of Proposition 35.52).

We remark that we can choose $\gamma_{ijk} = 1$ if the exact sequence (A1.85) splits. But this is not always the case.

Example A1.86. Let us consider the orbifold X in Example A1.68. Then $X^{\cong}(\Gamma) = S^2$. The $\mathbb{Z}_p$ local system is necessarily trivial on S^2. So

$$Sh(S^2; \mathbb{Z}_p) \cong \check{H}^2(S^2; \mathbb{Z}_p) \cong \mathbb{Z}_p.$$

The element thereof defined in Example-Definition A1.83 is the generator of $\mathbb{Z}_p$ and hence is nonzero.

Lemma A1.87. *The element of $Sh(X^{\cong}(\Gamma), \underline{G})$ represented by (h_{ij}, γ_{ijk}) is independent of various choices involved in the construction.*

Proof. We first fix $\mathcal{U} = \{U_i \mid i \in I\}$. We change $\widetilde{U}_i$ to $\alpha_i\widetilde{U}_i$ where $\alpha_i \in N(\Gamma_0)$. We also change $\widetilde{\underline{h}}_{ij}$ to

$$\widetilde{\underline{h}}''_{ij} = \alpha_i \cdot \widetilde{\underline{h}}_{ij} \cdot \alpha_j^{-1}.$$

Then $h_{ij} \in \mathrm{Aut}(G)$ is transformed to

$$h''_{ij} = \mathrm{ad}(\alpha_i) \circ h_{ij} \circ \mathrm{ad}(\alpha_j)^{-1},$$

where ad : $N(\Gamma_0) \to \mathrm{Aut}(\Gamma_0)$ is defined by $\mathrm{ad}(g)(g') = g \cdot g' \cdot g^{-1}$. We put $\psi_i = \mathrm{ad}(\alpha_i)$ and $\mu_{ij} = 1$. Then (A1.72.1), (A1.72.3) are satisfied and hence defines an isomorphism.

We also have

$$\gamma''_{ijk} = \alpha_i \cdot \gamma_{ijk} \cdot \alpha_i^{-1} = \psi_i(\gamma_{ijk}).$$

Therefore the isomorphism class is independent of the choice of $\widetilde{U}_i$.

We next fix $\widetilde{U}_i$ and change the lift $\widetilde{\underline{h}}_{ij} \in N(\Gamma_0)$ of $\underline{h}_{ij}$. We put

$$\widetilde{\underline{h}}'_{ij} = \mu_{ij} \cdot \widetilde{\underline{h}}_{ij}.$$

Then we have

$$\gamma'_{ijk} = \widetilde{\underline{h}}'_{ij} \cdot \widetilde{\underline{h}}'_{jk} \cdot (\widetilde{\underline{h}}'_{ik})^{-1} = \mu_{ij} \cdot h_{ij}(\mu_{jk}) \cdot \gamma_{ijk} \cdot \mu_{ik}^{-1}$$

as required. The invariance under the refinement of the covering is easy to prove. $\square$

Definition A1.88. We call the structure defined by $(\{h_{ij}\}, \{\gamma_{ijk}\})$ in Example A1.83, the *standard stack structure on* $X^{\cong}(\Gamma)$.

Going back to the general case of topological space M, we next define a vector bundle on the stack defined by an element of $Sh(M, \underline{G})$. Let $(\{h_{ij}\}, \{\gamma_{ijk}\}) \in Sh((M,\mathcal{U}), \underline{G})$.

Definition A1.89. A *vector bundle* on $(\{h_{ij}\}, \{\gamma_{ijk}\}) \in Sh((M,\mathcal{U}), \underline{G})$ is a pair $(\{F_i\}, \{g_{ij}\})$ such that F_i is a vector bundle on U_i with G action and g_{ij} is an h_{ij}-equivariant bundle isomorphism

$$g_{ij} : F_j|_{U_i \cap U_j} \to F_i|_{U_i \cap U_j}$$

such that :

$$g_{ij} \circ g_{jk} = \gamma_{ijk} \cdot g_{ik}. \tag{A1.90}$$

We assume that $(\{\mu_{ij}\}, \{\psi_i\})$ is an isomorphism $(\{h_{ij}\}, \{\gamma_{ijk}\}) \to (\{h'_{ij}\}, \{\gamma'_{ijk}\})$. An *isomorphism* from a vector bundle $\mathcal{F} = (\{F_i\}, \{g_{ij}\})$ on $(\{h_{ij}\}, \{\gamma_{ijk}\})$ to a vector bundle $\mathcal{F}' = (\{F'_i\}, \{g'_{ij}\})$ on $(\{h'_{ij}\}, \{\gamma'_{ijk}\})$ is a family $\{\phi_i\}_{i \in I}$ of ψ_i-equivariant isomorphisms of vector bundles

$$\phi_i : F_i \to F'_i$$

such that

$$g'_{ij} \circ \phi_j = \mu_{ij} \cdot (\phi_i \circ g_{ij}). \tag{A1.91}$$

We say $\{\phi_i\}_{i \in I}$ is an *isomorphism* : $\mathcal{F} \to \mathcal{F}'$ *over* $(\{\mu_{ij}\}, \{\psi_i\})$.

Lemma A1.92. *The relation 'isomorphic' in Definition* A1.89 *is an equivalence relation.*

Proof. Let $(\mu_{ij},\psi_i)_*(h^1_{ij},\gamma^1_{ijk}) = (h^2_{ij},\gamma^2_{ijk})$ and $(\mu'_{ij},\psi'_i)_*(h^2_{ij},\gamma^2_{ijk}) = (h^3_{ij},\gamma^3_{ijk})$. Let $\mathcal{F}^c = (\{F^c_i\},\{g^c_{ij}\})$ be a vector bundle on $(\{h^c_{ij}\},\{\gamma^c_{ijk}\})$ and $\{\phi_i\}_{i\in I} : \mathcal{F}^1 \to \mathcal{F}^2$ and $\{\phi'_i\}_{i\in I} : \mathcal{F}^2 \to \mathcal{F}^3$ be isomorphisms over $(\{\mu_{ij}\},\{\psi_i\})$, $(\{\mu'_{ij}\},\{\psi'_i\})$, respectively. Then we can check easily by calculation that $\{\phi'_i \circ \phi_i\}_{i\in I}$ is an isomorphism $: \mathcal{F}^1 \to \mathcal{F}^3$ over $(\{\mu'_{ij}\},\{\psi'_i\}) \circ (\{\mu_{ij}\},\{\psi_i\}) = (\{\mu'_{ij}\cdot\psi'_i(\mu_{ij})\},\{\psi'_i\circ\psi_i\})$. $\square$

Lemma A1.93. *Let $\mathcal{F} = (\{F_i\},\{g_{ij}\})$ be a vector bundle on $(\{h_{ij}\},\{\gamma_{ijk}\})$ and let $(\{\mu_{ij}\},\{\psi_i\})$ be an isomorphism $(\{h_{ij}\},\{\gamma_{ijk}\}) \to (\{h'_{ij}\},\{\gamma'_{ijk}\})$. Let F'_i be a G-equivariant vector bundle on U_i and let $\phi_i : F_i \to F'_i$ be a ψ_i-equivariant bundle isomorphism. We* **define** *g'_{ij} by* (A1.91).

Then $(\{F'_i\},\{g'_{ij}\})$ is a vector bundle on $(\{h'_{ij}\},\{\gamma'_{ijk}\})$.

Proof. It is easy to see that g'_{ij} is h'_{ij} equivariant. So it suffices to check (A1.90) for g'_{ij} and γ'_{ijk}.

We may divide the cases into $(\mu_{ij},\psi_i) = (1,\psi_i)$ and $(\mu_{ij},\psi_i) = (\mu_{ij},1)$.

In case $(\mu_{ij},\psi_i) = (1,\psi_i)$ we have

$$\begin{aligned} g'_{ij}\circ g'_{jk}\circ\phi_k &= \phi_i\circ g_{ij}\circ g_{jk} = \phi_i(\gamma_{ijk}\cdot g_{ik}) \\ &= \psi_i(\gamma_{ijk})\cdot\phi_i\circ g_{ik} = \gamma'_{ijk}\cdot g'_{ik}\circ\phi_k \end{aligned}$$

as required.

In case $(\mu_{ij},\psi_i) = (\mu_{ij},1)$ we have

$$\begin{aligned} g'_{ij}\circ g'_{jk}\circ\phi_k &= g'_{ij}\circ(\mu_{jk}\cdot(\phi_j\circ g_{jk})) \\ &= h'_{ij}(\mu_{jk})\cdot\mu_{ij}\cdot\phi_i\circ(g_{ij}\circ g_{jk}) \\ &= \mu_{ij}\cdot h_{ij}(\mu_{jk})\cdot\gamma_{ijk}\cdot\mu_{ik}^{-1}\cdot(g'_{ik}\circ\phi_k) \\ &= \gamma'_{ijk}\cdot(g'_{ik}\circ\phi_k), \end{aligned}$$

as required. The proof of the lemma is now complete. $\square$

We next discuss how a vector bundle behaves under the refinement of the covering. Let

$$i(\cdot)^*([\{h_{i_1i_2}\},\{\gamma_{i_1i_2i_3}\}]) = [\{h'_{j_1j_2}\},\{\gamma'_{j_1j_2j_3}\}].$$

See Definition A1.80. Then a vector bundle $(\{F_i\},\{g_{i_1i_2}\})$ on $(\{h_{i_1i_2}\},\{\gamma_{i_1i_2i_3}\})$ induces $(\{F'_j\},\{g'_{j_1j_2}\})$ on $(\{h'_{j_1j_2}\},\{\gamma'_{j_1j_2j_3}\})$ by

$$g'_{j_1j_2} = g_{i(j_1)i(j_2)}|_{U'_{j_1j_2}}.$$

Therefore we can define a notion of a vector bundle on a pair $(M,[\{h_{ij}\},\{\gamma_{ijk}\}])$ where $[\{h_{ij}\},\{\gamma_{ijk}\}] \in Sh(M,\underline{G})$.

Definition A1.94. We consider the situation of Example-Definition A1.83 and use the notations there. We will define the normal bundle $N_{X^{\cong}(\Gamma)}X$ over $X^{\cong}(\Gamma)$ with the standard stack structure.

We identify $U_i \subseteq X^{\cong}(\Gamma)$ with $\widetilde{U}_i \subset \widetilde{X}^{=}(\Gamma_0)$ by the projection and put $F_i = N_{\widetilde{U}_i}\widetilde{X}$, the normal bundle of $\widetilde{U}_i$ in $\widetilde{X}$. Γ_0 ($\subset G$) action on $\widetilde{X}$ induces one on F_i.

We next define g_{ij}. We have

$$(\widetilde{\underline{h}}_{ij}^{-1}\widetilde{U}_i) \cap \widetilde{U}_j \subset \widetilde{U}_j \cong U_j.$$

We identify $(\widetilde{\underline{h}}_{ij}^{-1})\widetilde{U}_i \cap \widetilde{U}_j$ with $U_i \cap U_j$. Then an open embedding

$$\widetilde{\underline{h}}_{ij}\cdot : U_i \cap U_j \to \widetilde{U}_i \cong U_i$$

is induced. It extends to a map $\widetilde{X} \to \widetilde{X}$. We then have

$$g_{ij} := (\widetilde{\underline{h}}_{ij}\cdot)_* : F_j|_{U_i \cap U_j} = N_{\widetilde{\underline{h}}_{ij}^{-1}\widetilde{U}_i}\widetilde{X} \to F_i = N_{\widetilde{U}_i}\widetilde{X}.$$

Since

$$\widetilde{\underline{h}}_{ij} \cdot \widetilde{\underline{h}}_{jk} = \gamma_{ijk} \cdot \widetilde{\underline{h}}_{ik},$$

we have (A1.90).

We put $N_{X^{\cong}(\Gamma)}X = (\{F_i\}, \{g_{ij}\})$ which we call the *normal bundle* of $X^{\cong}(\Gamma)$.

We can generalize the construction above and include the case of orbifold which is not necessarily a quotient of manifold globally. We do not discuss this here since we do not use it in our main application (the proof of Proposition 35.52).

Lemma A1.95. $(\{F_i\}, \{g_{ij}\})$ *in Definition* A1.94 *is independent of the choices involved in the construction up to isomorphism.*

Proof. We use the notation of the proof of Lemma A1.87.

We first change $\widetilde{U}_i$ to $\alpha_i \cdot \widetilde{U}_i$. Then we have

$$\alpha_i \cdot : N_{\widetilde{U}_i}\widetilde{X} \to N_{\alpha_i \cdot \widetilde{U}_i}\widetilde{X}.$$

It induces

$$\phi_i = (\alpha_i \cdot)_* : F_i \to F_i'.$$

Since $\psi_i = ad(\alpha_i)$ it follows that ϕ_i is ψ_i equivariant.

$$\widetilde{\underline{h}}_{ij}' = \alpha_i \cdot \widetilde{\underline{h}}_{ij} \cdot \alpha_j^{-1}$$

implies that

$$g_{ij}' \circ \phi_j = \phi_i \circ g_{ij}.$$

Since $\mu_{ij} = 1$ in this case, we obtain the required isomorphism.

We next fix $\widetilde{U}_i$ and change the lift $\widetilde{\underline{h}}_{ij}$ to

$$\widetilde{\underline{h}}'_{ij} = \mu_{ij} \cdot \widetilde{\underline{h}}_{ij}. \tag{A1.96}$$

In this case we have $F_i = F'_i$ and $\phi_i =$ identity. (A1.96) implies

$$g'_{ij} = (\widetilde{\underline{h}}'_{ij})_* = \mu_{ij} \cdot (\widetilde{\underline{h}}_{ij})_* = \mu_{ij} \cdot g_{ij}.$$

The invariance under the refinement of the covering is easy to prove. $\square$

Example A1.97. In the situation of Example A1.68, the normal bundle does not exist as a (usual) vector bundle on S^2. But it exists over S^2 with (nontrivial) stack structure, which corresponds to the generator of $\check{H}^2(S^2; \mathbb{Z}_p)$.

Definition A1.98. We consider the situation of Example-Definition A1.83 and let $\widetilde{E}$ be a G equivariant vector bundle on $\widetilde{X}$. It induces an orbi-bundle E on $X = \widetilde{X}/G$. We will define a vector bundle $E|_{X^{\cong}(\Gamma)}$ on $X^{\cong}(\Gamma)$. We use the notation of Example-Definition A1.83 and of Definition A1.94.

We put

$$E_i = \widetilde{E}|_{\widetilde{U}_i}$$

and regard it as a vector bundle on U_i.

As in Definition A1.94, we have an open embedding

$$\widetilde{\underline{h}}_{ij} \cdot : U_i \cap U_j \to \widetilde{U}_i \cong U_i.$$

Since $\widetilde{\underline{h}}_{ij} \in G$ it induces a bundle map

$$g_{ij} : E_j|_{U_i \cap U_j} \to E_i.$$

It follows that g_{ij} satisfies the required relation in the same way as Definition A1.94. We define

$$E|_{X^{\cong}(\Gamma)} = (\{E_i\}, \{g_{ij}\})$$

and call it the *restriction* of E to $X^{\cong}(\Gamma)$.

We can generalize the construction to the case of orbifold which may not be globally a quotient.

In the same way as Lemma A1.95, we can prove that $(\{E_i\}, \{g_{ij}\})$ is independent of the choices up to isomorphism.

Let $\mathcal{F} = (\{F_i\}, \{g_{ij}\})$ be a vector bundle over $(\{h_{ij}\}, \{\gamma_{ijk}\}) \in Sh(X, \underline{G})$. Assume that the G action on the fibers of F_i is effective. We are going to define an orbifold structure on $\mathcal{F}/G$. We define an equivalence relation $\sim$ on $\bigcup_i F_i$ as follows.

(Here F_i also denotes the total space of the vector bundle F_i.) Let $x \in F_i$ and $y \in F_j$. Then $x \sim y$ if and only if one of the following holds.

(1) $i = j$, $x = \gamma y$ for some $\gamma \in G$.

(2) $\pi(x) \in U_i \cap U_j$, $y = \gamma \cdot g_{ji}(x)$ for some $\gamma \in G$. Here $\pi : F_i \to U_i$, $\pi : F_j \to U_j$ are the projections.

It is easy to see that $\sim$ is an equivalence relation. We put

$$|\mathcal{F}/G| = \bigcup_i F_i/\sim$$

and define a quotient topology on it. $\pi : E_i \to U_i \subset X$ induces a map

$$\pi : |\mathcal{F}/G| \to X.$$

Let F be the fiber of the vector bundle F_i. F is a vector space on which G acts effectively.

Lemma A1.99. *$|\mathcal{F}/G|$ has a structure of orbifold. We denote it by $\mathcal{F}/G$. If $\mathcal{F}$ is isomorphic to $\mathcal{F}'$, then $\mathcal{F}/G$ is diffeomorphic to $\mathcal{F}'/G$ as an orbifold.*

$\pi : \mathcal{F}/G \to X$ is a locally trivial fiber bundle whose fiber is F/G and the structure group is the group of orbifold diffeomorphisms of F/G.

Proof. We fix an order on the set of indices I. Let $p \in |\mathcal{F}/G|$. We put

$$I(p) = \{i \in I \mid p = [x], x \in F_i\}.$$

Let $i = i(p) = \inf I(p)$. We put

$$V_p = \bigcap_{j \in I(p)} \pi^{-1}(V_{ji}).$$

Here V_{ji} is the image of $U_{ji} \subset \widetilde{U}_i$ by $\widetilde{X} \to X$. G acts on V_p. There is a map $\psi_p : V_p/G \to |\mathcal{F}/G|$ which sends $x \mod G \in V_p/G$ to the equivalence class of x in $|\mathcal{F}/G|$. We take (V_p, G, ψ_p) as the orbifold chart of p.

Let $q \in \psi_p(V_p/G)$. Let $j = i(q)$. Since $I(q) \supseteq I(p)$, it follows that $j = i(q) \le i(p) = i$.

By definition

$$g_{ij}(V_q) \subseteq V_p.$$

Hence $(h_{pq}, \phi_{pq}) := (h_{ij}, g_{ij}) : (G, V_q) \to (G, V_p)$ gives a coordinate change of the orbifold. By definition we have

$$h_{pq} \circ h_{qr} = \gamma_{pqr} \cdot h_{pr} \cdot \gamma_{pqr}^{-1}, \quad \phi_{pq} \circ \phi_{qr} = \gamma_{pqr} \cdot \phi_{pr}$$

where

$$\gamma_{pqr} = \gamma_{i(p)i(q)i(r)}.$$

We thus defined an orbifold structure on $|\mathcal{F}/G|$.

We omit the proof of the other part of the lemma and leave it to the interested readers. $\square$

Lemma A1.100. (Tubular neighborhood theorem) *Consider the situation of Example-Definition* A1.83. *Then* $(N_{X^{\cong}(\Gamma)}X)/\Gamma$ *is diffeomorphic to a neighborhood of* $X^{\cong}(\Gamma)$ *in* X *as an orbifold.*

Proof. We use the notation of Definition A1.94. We recall that

$$N_{X^{\cong}(\Gamma)}X = (\{E_i\},\{g_{ij}\})$$

where $E_i = N_{\widetilde{U}_i}\widetilde{X}$.

Let $\chi : [0,\infty) \to [0,\epsilon)$ be a diffeomorphism which is identity in a neighborhood of 0.

We take a G invariant Riemannian metric on $\widetilde{X}$ and use the exponential map to identify $E_i = N_{\widetilde{U}_i}\widetilde{X}$ with a neighborhood $\widetilde{U}_i$ in $\widetilde{X}$ by

$$\exp_i : v \mapsto \exp\left(\frac{\chi(\|v\|)}{\|v\|}v\right).$$

We use also the notation of the proof of Lemma A1.99. Let $p = [v] \in |(N_{X^{\cong}(\Gamma)}X)/\Gamma|$ and $i = i(p)$. Now $\exp_i$ defines a smooth map

$$f_p : V_p \to \widetilde{X}.$$

Here we recall

$$V_p = \left\{ v \in N_{\widetilde{U}_i}\widetilde{X} \,\middle|\, \pi(v) \in \bigcap_{j \in I(p)} (V_{ji}) \right\}.$$

It is easy to check that f_p defines a diffeomorphism from $(N_{X^{\cong}(\Gamma)}X)/\Gamma$ to an open neighborhood $X^{\cong}(\Gamma)$ in X as an orbifold. □

We remark that various operations of vector bundle such as Whitney sum, tensor product, Hom bundle, symmetric tensor product, etc. can be generalized to the case of vector bundle on $[\{h_{ij}\},\{\gamma_{ijk}\}] \in Sh(M;\underline{G})$. For example, if $\mathcal{F} = (\{F_i\},\{g^F_{ij}\})$ and $\mathcal{E} = (\{E_i\},\{g^E_{ij}\})$ are vector bundles on $(\{h_{ij}\},\{\gamma_{ijk}\})$ then $Hom(\mathcal{F},\mathcal{E}) = (\{Hom(F_i,E_i)\},\{g_{ij}\})$ where

$$g_{ij}(u_j) = g^E_{ij} \circ u_j \circ (g^F_{ij})^{-1}.$$

We use the next lemma in §35.

Lemma A1.101. *Let* $\mathcal{E} = (\{E_i\},\{g_{ij}\})$ *be a vector bundle on* $(\{h_{ij}\},\{\gamma_{ijk}\}) \in Sh(M,\underline{G})$. *We put*

$$E_i^G = \{v \in E_i \mid \forall \gamma \in G\ \gamma v = v\}.$$

Then they are glued by g_{ij} *to define a vector bundle on the topological space* M (*in the usual sense*).

Proof. Let $g^G_{ij} : E^G_i|_{U_{ij}} \to E^G_j$ be the restriction of g_{ij}. Then, since G action on E^G_i is trivial, it follows from (A1.90) that

$$g^G_{ij} \circ g^G_{jk} = g^G_{ik}.$$

The lemma follows. □

Remark A1.102. In some situation we need to consider subspaces $Y_p \subset V_p$ of a space X with Kuranishi structure $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ and put a Kuranishi structure on the subspace. When the action of Γ_p is not effective, we need to add some Γ_p vector space F_p to both Y_p and obstruction bundle E_p of each Kuranishi neighborhood, in order to define a Kuranishi structure on the subspace $\cup_p Y_p$. We need some care to carry this out. Namely we should glue those vector spaces by a family of linear isomorphisms $g_{pq} : U_{pq} \to Hom(F_q, F_p)$ so that

$$g_{pq} \circ g_{qr} = \gamma_{pqr} \cdot g_{pr}$$

where γ_{pqr} is the one appearing in (A1.6.2). For example, in the case of Kuranishi structure with corners, we define Kuranishi structure on codimension k corner $S_k X$ as follows. We take the normal bundle to $S_k X$ and add the normal bundle to both $S_k V_p$ and the obstruction bundle in order to make the Γ_p action effective.

A1.7. Some errors in the earlier versions and corrections thereof.

A1.7.i) In Definition A1.1, we consider only a trivial bundle $E_p \times V_p \to V_p$ as an obstruction bundle. It is not necessarily trivial as an equivariant bundle. In [FuOn99II] we allowed more general bundle. The two definitions however are equivalent, since locally any bundle is trivial and Kuranishi neighborhood of a point can be chosen arbitrary small.

A1.7.ii) In the preprint version [FOOO00] of this book, Definition A1.3 were loosely stated, when we rephrase Definition 5.3 in [FuOn99II] without using the word "smooth embedding of orbifolds". In Condition (A1.4.2), we should require that the induced map $V_{pq}/\Gamma_q \to V_p/\Gamma_p$ is an injective map. We also add Condition (A1.4.6). Note that Condition (A1.4.6) holds, if the h_{pq}-equivariant smooth embedding $\phi_{pq} : V_{pq} \to V_p$ induces a smooth embedding of orbifolds in the sense of p. 941 in [FuOn99II]. Thus Definition A1.3 above (with (A1.4.6) included) is equivalent to Definition 5.3 of [FuOn99II].

A1.7.iii) In Definition A1.5 (1.6.2) we include γ_{pqr} in the formula. In (A2.1.11) of [FOOO00], the term γ_{pqr} is missing. This is an error as Example A1.65 illustrates. On the other hand, the corresponding definition, Definition 5.3 in [FuOn99II], is correct as it is. See Remark A1.67 which is related to this point.

A1.7.iv) We point out that Definition A1.14 is slightly different from the corresponding one in [FuOn99II]. Namely in [FuOn99II] the isomorphism (A1.15) is not required to be induced by the differential of the Kuranishi map but only existence of such an isomorphism for which the diagram (A1.16) commutes. In fact, our proof of Theorem 6.4 in [FuOnII99] implicitly uses the condition that (A1.15) is induced by $d_{\text{fiber}} s_p$.

Therefore this is an error and Definition A1.14 is the correct definition of the tangent bundle. (See §A1.5.) Example A1.64 illustrates a space X with Kuranishi structure that has not the tangent bundle in the sense of this book but does in the sense of [FuOn99II]. We need to exclude X in Example A1.64 so that cobordism invariance of virtual fundamental cycle to hold.

In all the applications of the Kuranishi structure with tangent bundle to the moduli problems in [FuOn99II] and in this book, the isomorphism (A1.15) is induced by the differential of the Kuranishi map. So this error does not cause any problem.

In the proof of the existence of compatible multisection (Theorem A1.23) we extend multisection in a compatible way. Namely we require the compatibility condition in Definition A1.21. The original Kuranishi map is compatible in the sense there only if (A1.5) is induced by the Kuranishi map. When the original Kuranishi map is not compatible then we can not perturb it so that it is compatible. So (A1.5) should be induced by the Kuranishi map to prove Theorem A1.23. In (6.4.4) of [FuOn99II] the same compatibility condition was required.

If we consider the Kuranishi structure constructed in §29, both the normal bundle and the quotient of obstruction bundle are isomorphic to the difference of the two choices of E appearing (29.16). Therefore, the isomorphism (A1.5) is obtained by the differential of s.

A1.7.v) In [FOOO00 §A3] we used normal bundle to $N_{X^{\cong}(\Gamma)}X$ in a similar way as §35 of this book. At the time of writing [FOOO00], the authors overlooked the fact that the normal bundle to $N_{X^{\cong}(\Gamma)}X$, in the sense of the usual vector bundle divided by Γ, may not exist. We have given rather detailed discussion on the vector bundle over stack in §A1.6 and rectify this error in §35 of this book .

Added reference

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