Symplectic Geometry of Langlangian submanifold

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Symplectic Geometry ?

Origin $\longrightarrow$ Hamiltonian Dynamics

$q_1, \cdots, q_n$ position

$p_1, \cdots, p_n$ momentum

$H(q_1, \cdots, q_n; p_1, \cdots, p_n; t)$ Hamiltonian

$$\begin{cases} \dfrac{dq_i}{dt} = \dfrac{\partial H}{\partial p_i} \\ \dfrac{dp_1}{dt} = -\dfrac{\partial H}{\partial q_i} \end{cases}$$

Hamiltonian's equation

Hamilton’s equation is invariant of the coordinate change

$$\begin{cases} Q_i = Q_i(q_1, \cdots, q_n, p_1, \cdots, p_n) \\ P_i = P_i(q_1, \cdots, q_n, p_1, \cdots, p_n) \end{cases}$$

$$\Updownarrow$$

$$\sum dP_i \wedge dQ_i = \sum dp_i \wedge dq_i$$

canonical transformaiton = symplectic diffeomorphism

Symplectic manifold $X = \bigcup U_i$

U_i has local coordinate $q_1, \cdots, q_n, p_1, \cdots, p_n$

coordinate change is symplectic diffeomorphism

$\Updownarrow$

$\omega = \sum dp_i \wedge dq_i$ is globally defined. symplectic form

$d\omega = 0$ $\omega \wedge \cdots \wedge \omega =$ volume form.

Two important sources of symplectic Geometry

(1) Hamiltonian dynamics

(2) Algebraic or Kahler geometry

(1) Hamiltonian dynamics

$X = T^* M$ cotangent bundle.

$q_1, \cdots, q_n$ local coordinate of M

$p_1, \cdots, p_n$ coordinate of the cotangent vector

$$\omega = \sum dp_i \wedge dq_i$$

symplectic form

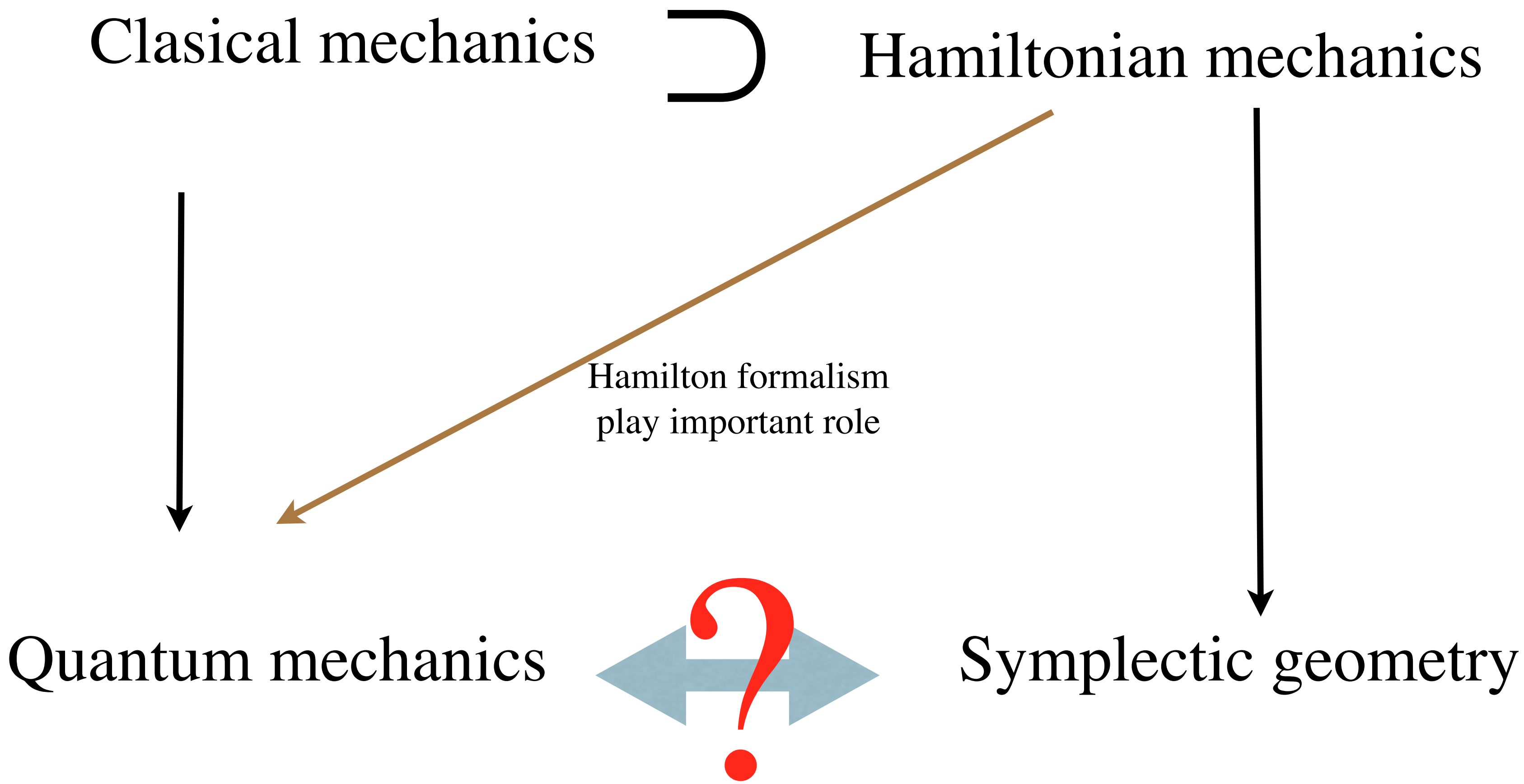
(2) Algebraic or Kahler geometry

Solution set of polynomial equation
has a symplectic structure
(Fubini-Study form)

Example

$$X = \left\{ (x, y, z, w) \,\middle|\, x^5 + y^5 + z^5 + w^5 = 1 \right\}$$

(Take closure in projective space.)


Clasical mechanics
⊃
Hamiltonian mechanics
Hamilton formalism
play important role
Quantum mechanics
?
Symplectic geometry

Global Symplectic Geometry ?

It is not clear whether
Global Symplectic Geometry
is related to the origin of symplectic geometry,
that is Physics.

On the other hand,
from Mathematical point of view
Local symplectic geometry is trivial.

Riemannian geometry

R_{ijkl} curvature (how locally spacetime curves.)

The most important quantity of Riemannian geometry.

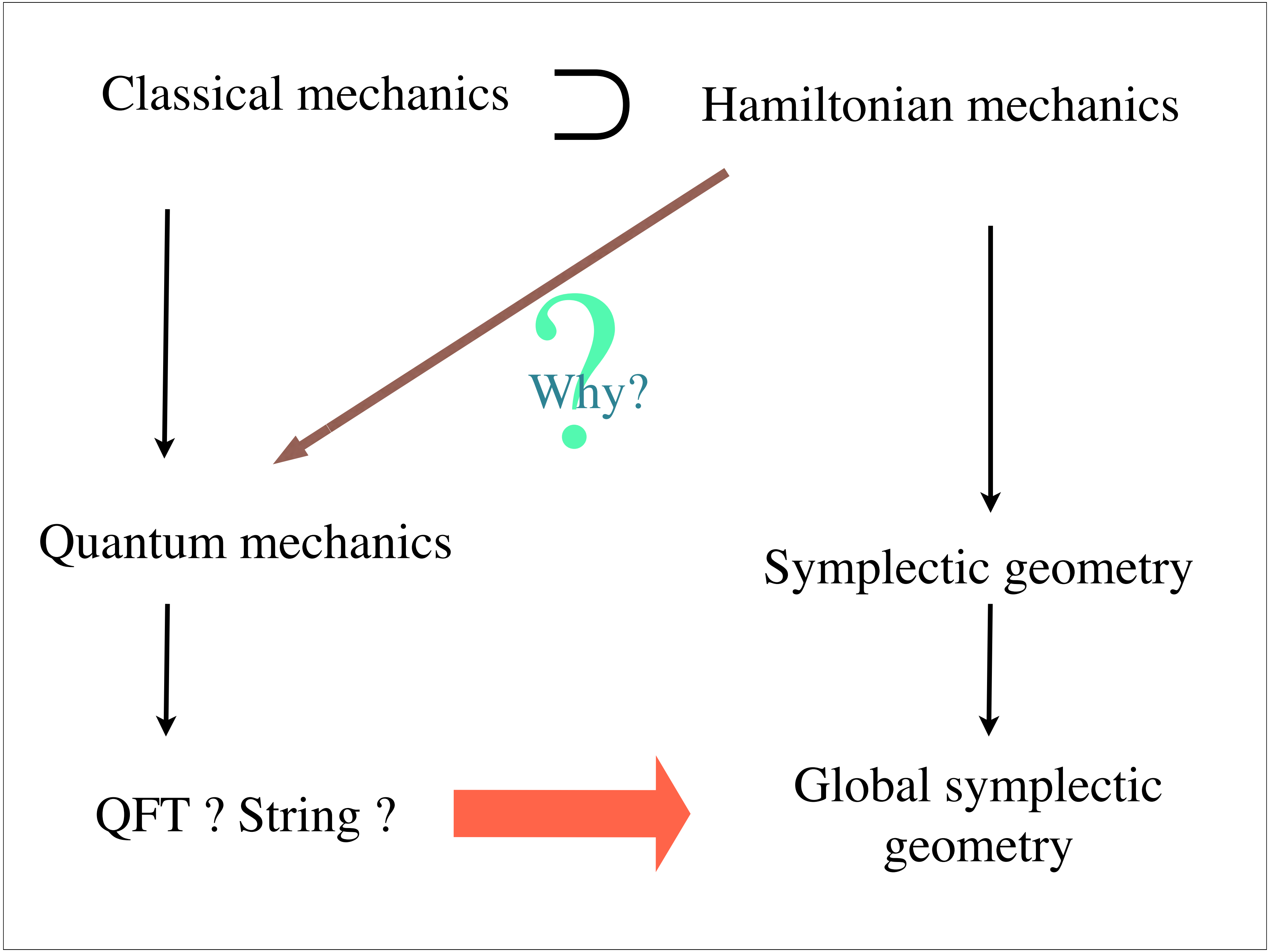
There is no curvature in symplectic geometry.

(Dauboux's theorem 19th century.)

There is nontrivial
Global Symplectic Geometry.

This is highly nontrivial fact and was established by using

"string theory"


Classical mechanics
⊃
Hamiltonian mechanics
?
Why?
Quantum mechanics
Symplectic geometry
QFT ? String ?
Global symplectic geometry

I will discuss geometry of

Lagrangian submanifold

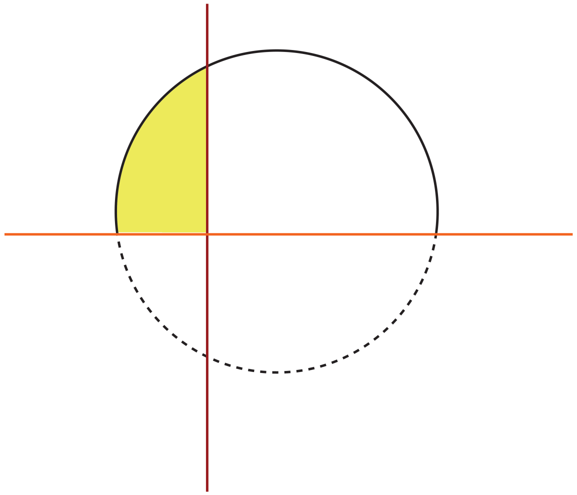
as an example of nontrivial
Global Symplectic geometry.

Lagrangian submanifold

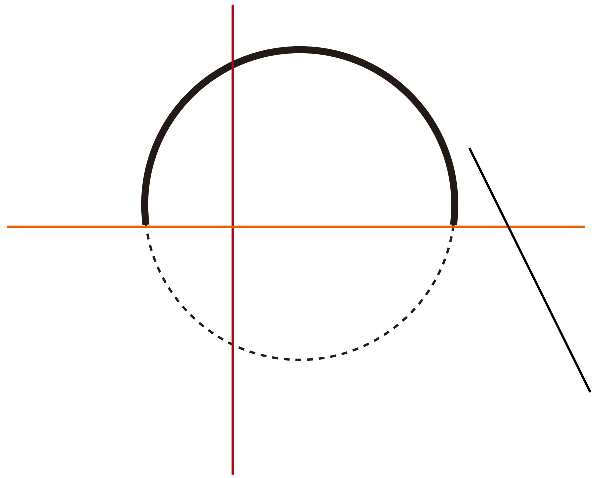
Example $\qquad f(q_1,\cdots,q_n)$ a function of $q_1,\cdots,q_n$.

$$p_i = \frac{\partial f}{\partial q_i}, \qquad i = 1,\cdots,n$$

defines an n dimensional submanifold,
Lagrangian submanifold L.

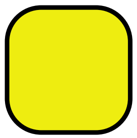


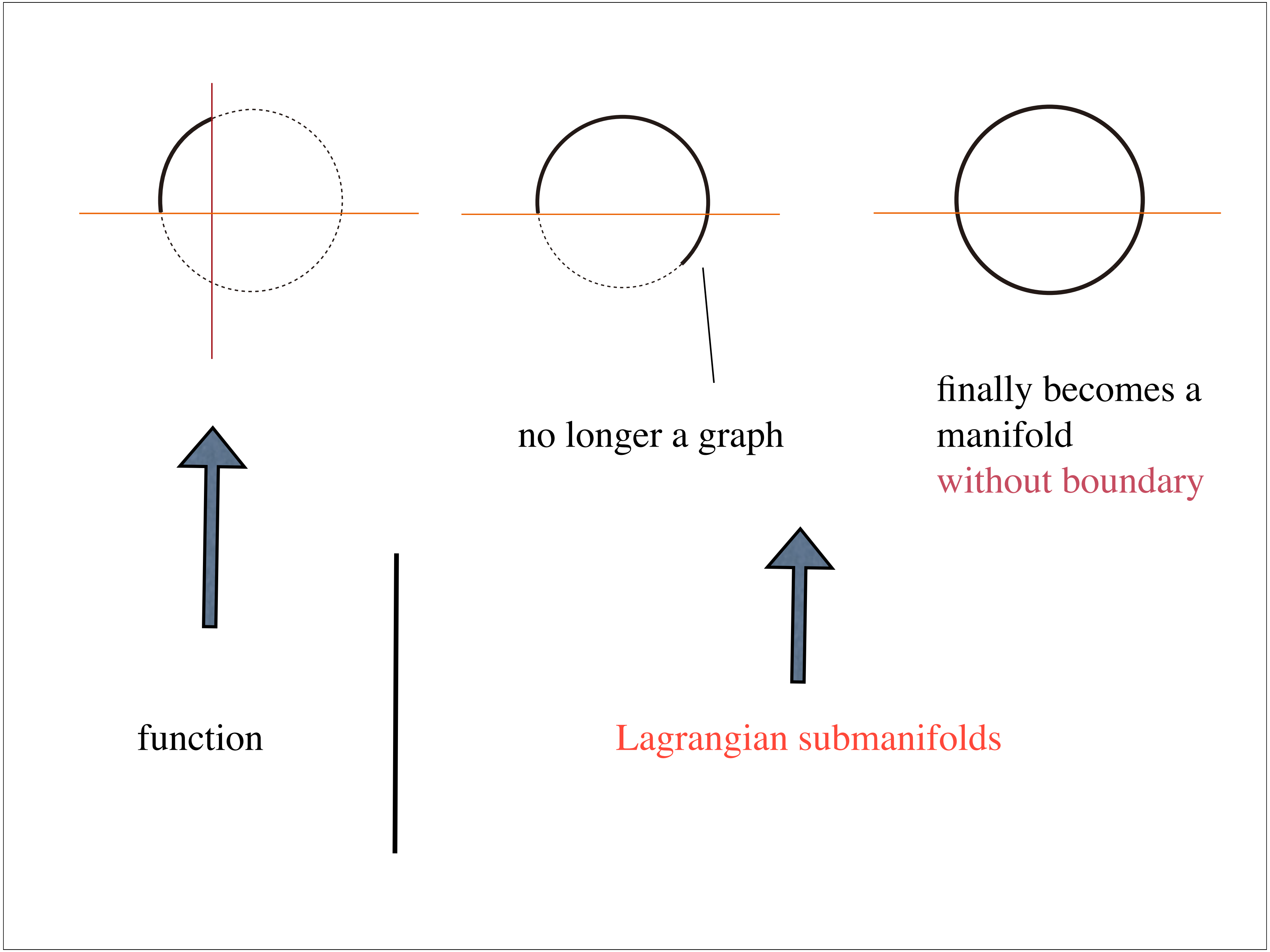
q



graph of $p = \dfrac{\partial f}{\partial q}$.

$f(q) = \text{area}$





no longer a graph
finally becomes a
manifold
without boundary
function
Lagrangian submanifolds

Definition:

$L \subset X$ a submanifold of X.

L is a Lagrangian submanifold.

$\Updownarrow$

$\omega = 0$ on L.

$\dim L = \frac{1}{2} \dim X$

Role of Lagrangian submanifold

Lagrangian submanifold of T^*M is a generalization of a function on M

Symplectic diffeomorphism: $X \to X$ is a Lagrangian submanifold of $X \times X$

$\boldsymbol{R}^n \subset \boldsymbol{C}^n$ is a Lagrangian submanifold

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Lagrangian submanifold is the correct boundary condition for open string.

D brane

Symplectic Geometry

- analogy from algebraic geometry

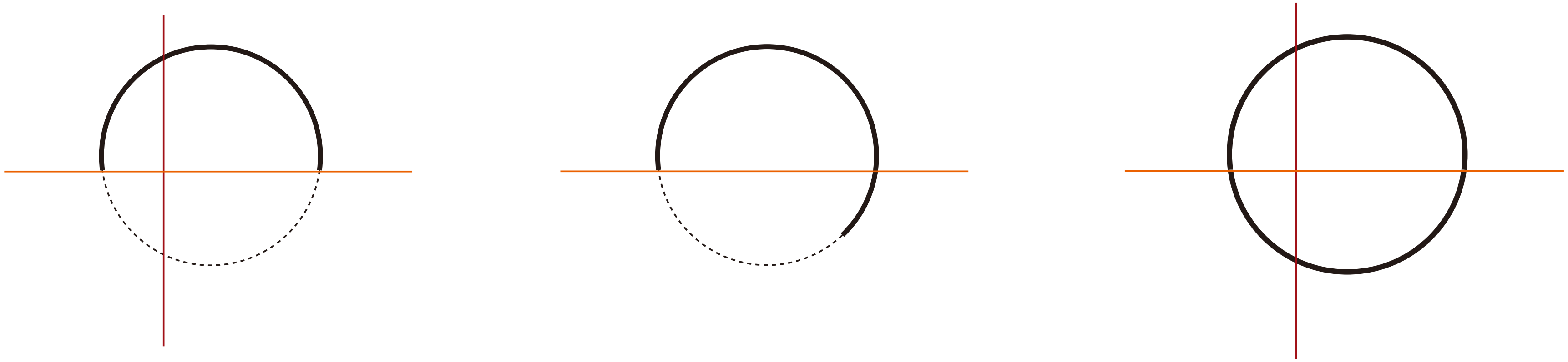
= Hamiltonian dynamics

+

Lagrangain submanifold

+

epsilon



Lagrangian submanifold of 2 dimensional Euclidean space

= Circle

Classify Lagrangian submanifolds of $\boldsymbol{C}^n$?

The first interesting case $n = 3$.

The case $n = 2$.
Answer: 2 dimensional torus. (Easy (except the case of Klein bottle))

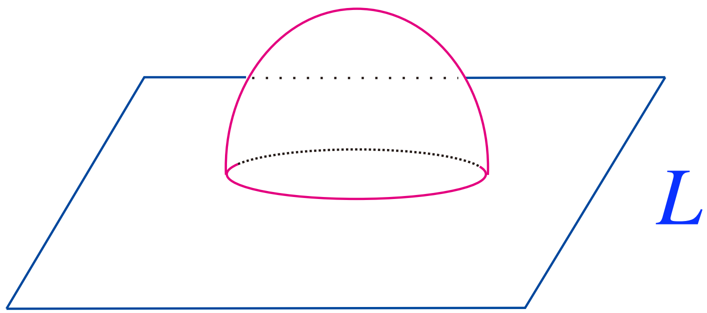
Theorem (Gromov, 1980')

3 sphere S^3 is NOT a Lagrangian submanifold of $\boldsymbol{C}^3$.

We need "open string theory" to prove this.

(I)

If L is a Lagrangian submanifolds in $R^{2n} = C^n$
then there exists a disc which bounds it.



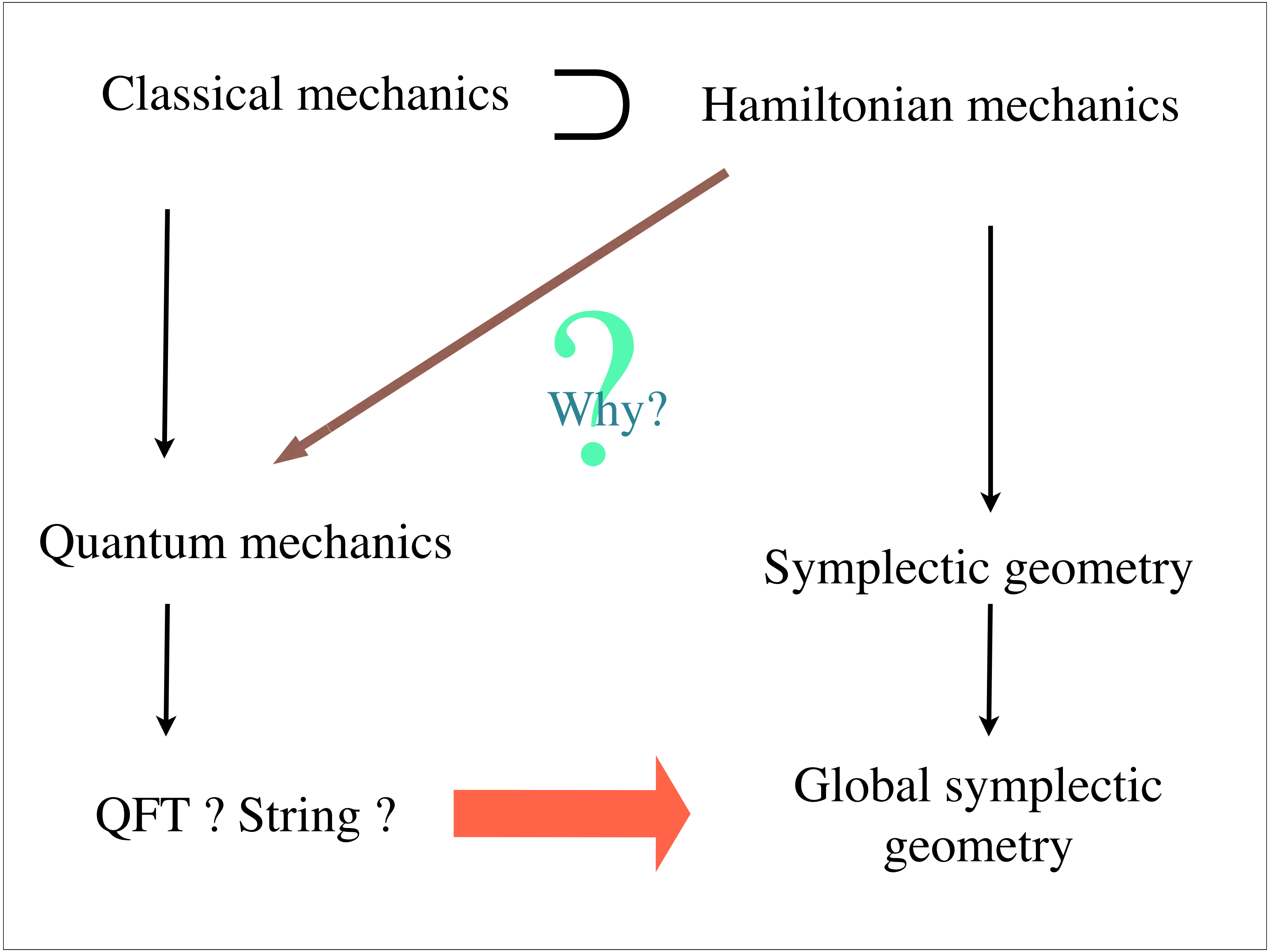
$$\varphi : \quad D^2 \to \boldsymbol{C}^n \qquad \text{holomorphic.}$$
$$\partial D^2 \to L$$

(II)

Such a disc can not exists if L is sphere S^3 because

L is S^3 $\Longrightarrow$ $\int_{D^2} \varphi^* \omega = 0$

φ holomorphic. $\Longrightarrow$ $\int_{D^2} \varphi^* \omega > 0$


Classical mechanics ⊃ Hamiltonian mechanics
Why?
Quantum mechanics
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Global symplectic geometry

To go further we need to be more systematic.

$$\boxed{\text{Approximate} \quad \text{Geometry} \quad \text{by} \quad \text{Algebra}.}$$

Poincare (begining of 20th century)

X : space

$$\Big\downarrow$$

$H(X)$: Homology group

Poincare (begining of 20th century)

X : space

$\Big\downarrow$ $=$ Algebraic topology

$H(X)$: Homology group

Poincare (begining of 20th century)

X : space

Linear story

$\Big\downarrow$

$H(X)$: Homology group

Begining of 21th century

we are now working on non Linear story

Classify the Lagrangian submanifolds of $\boldsymbol{C}^3$?

Thurston-Perelman

3 manifolds are one of the 8 types of spaces

Which among those 8 types is a Lagrangian submanifold of $\boldsymbol{C}^3$?

Answer

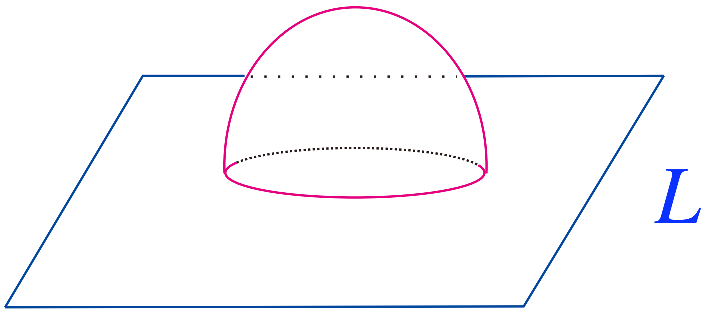
3 manifold	Lagrangian submanifold?
S^3	No(Gromov)
R^3	Yes
H^3	No(Viterbo)
$R \times S^2$	Yes
$R \times H^2$	Yes
$SL(2,R)$	No (F)
Sol	No (F)
Nil	No (F)

3 manifold	Lagrangian submanifold?
S^3 : Curvature =1	No(Gromov)
R^3 : Curvature =0	Yes
H^3 : Curvature=−1	No(Viterbo)

3 manifold	Lagrangian submanifold?	
$SL(2,R)$	No (F)	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $ad - bc = 1$
Sol	No (F)	$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$
Nil	No (F)	$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$

Method : Count the discs.

= Open string



$$\varphi : \quad D^2 \longrightarrow \boldsymbol{C}^3 \qquad \text{holomorphic.}$$

$$\partial D^2 \to L$$

Count the discs.

Obtain numbers. (Many numbers)

Those sysem of numbers has a structure.

Obtain algebraic system; (something like group)

- In theoretical Physics,

  Higher (>4) dimensional spaces are (at last) begining to be studied.

- Space of dimension > 4 will never directly observable from human.

- They will be seen to us only through some system of numbers which can be checked by experiments.

- The role of higher dimensional geometry in physics here seems to be to provide a way to understand some huge list of numbers.

- This is the same as what I said about algebraic topology.

Keep going and understand
Lagrangian submanifold
by open string theory.

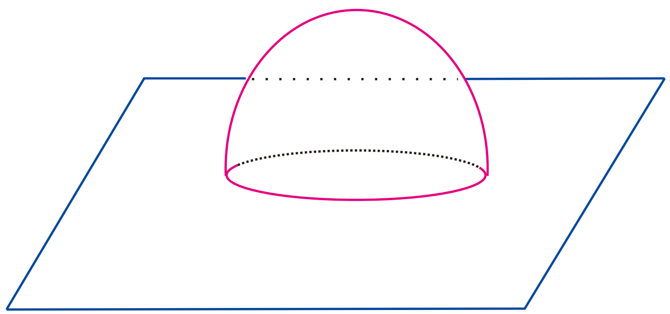
Difficulty

Counting the discs is actually a difficult problem.

Counting the discs = Counting the number of solution of Non Linear differential equations

First and essential step to show S^3 is not a Lagrangian submanifold.

If L is a Lagrangian submanifold in $R^{2n} = C^n$ then there exists a disc which bounds it.



Counting the discs is difficult.

Difficulty

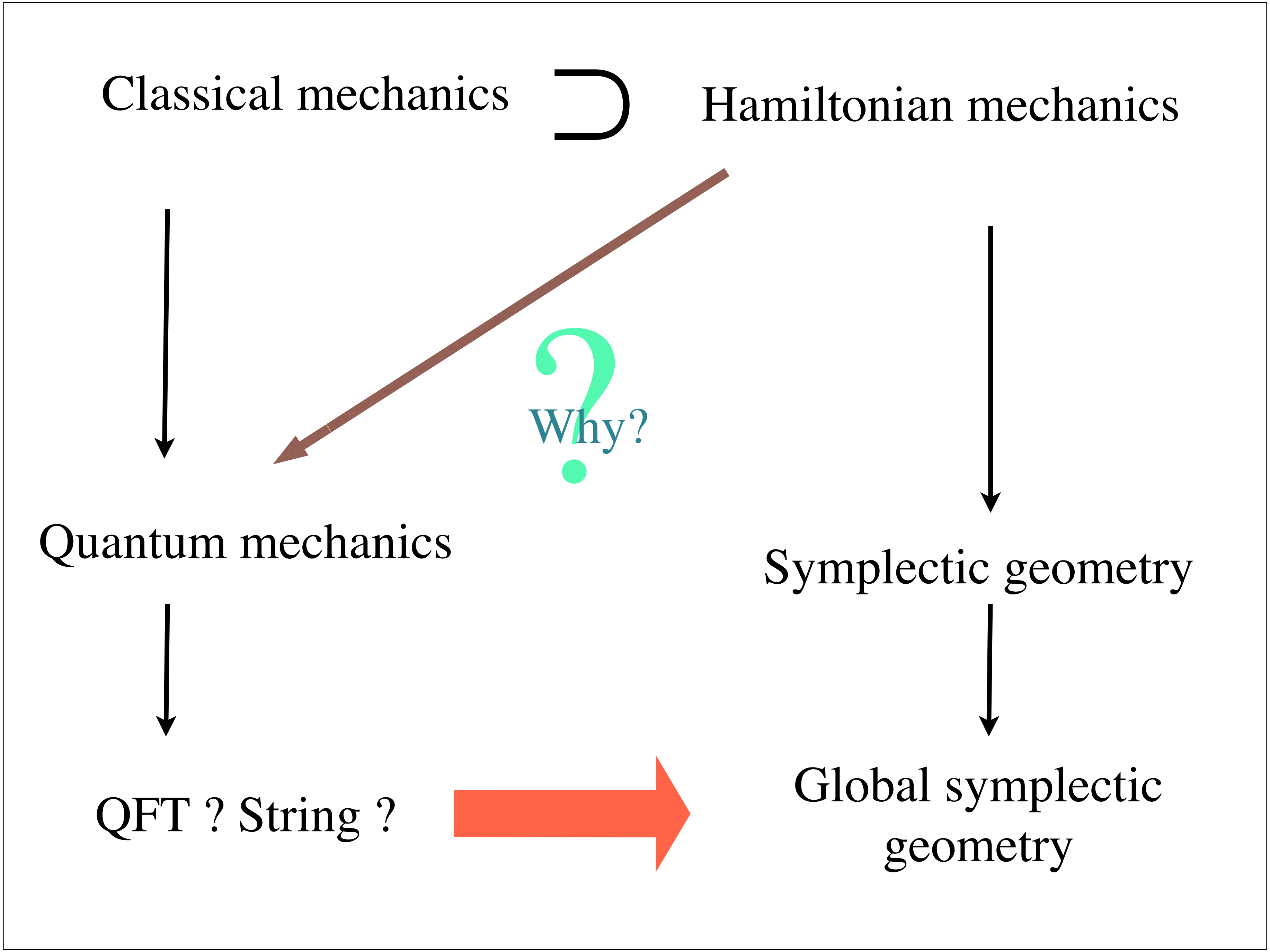
Counting the discs is actually a difficult problem.

Counting the discs = Counting the number of solutions of Non Linear differential equations

Physics helps

Mirror symmetry (discovered in 1990')

provides (potentially) a powerful tool to compute the number of discs.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(Homological) Mirror symmetry (Konsevitch 1994)

X

Symplectic manifold

L

Lagrangian submanifold
(A brane)

$\hat{X}$

Complex manifold

$E \to \hat{X}$

Holomorphic vector bundle

(Homological) Mirror symmetry

$$\Downarrow$$

Difficult problem of counting discs

becomes

Attackable problem of complex geometry

(Homological) Mirror symmetry

Difficult problem of counting discs

Global,   Non Linear

becomes

Attackable problem of complex geometry

Local,   Linear

Difficult problem of counting discs

Global, Non Linear

Non perturbative

becomes

Attackable problem of complex geometry

Local, Linear

Perturbative

Theorem (Seidel-Smith-F, Nadler)

Compact Lagrangian submanifold L of T^*M is the same as $M \subset T^*M$ as D-brane

if

L, M are simply conneted and L is spin.

A special case of a version of a conjecture by Arnold.

L is the same as M as D-brane

implies in particular

○ Homolog group of L is homology group of M.

○ $[L] = [M]$ in $H(T^*M)$.

Arnold conjectured stronger conclusion in 1960’s.

(Homological) Mirror symmetry (Kontsevitch 1992)

X

Symplectic manifold

L

Lagrangian submanifold
(A brane)

$\hat{X}$

Complex manifold

$E \rightarrow \hat{X}$

Holomorphic vector bundle

In our case $X = T^*M$ is noncompact and situaltion is slightly different.

T^*M Symplectic manifold L Lagrangian submanifold (A brane)	$\Leftrightarrow$	$\hat{X} = T^*M$ (or M) $E \to M$ Flat vector bundle

- String theory of T^*M is gauge theory on M. (Witten 1990')
- M is simply connected $\Rightarrow$ Flat bundle on M is trivial.

- By proving a (small) part of (homological) Mirror symmetry conjecture we get new insight on Lagrangian submanifolds.
- Then we enhance conjecture and make it more precise and richer.
- Solving some more parts we get another insight.
- Conjecture now is becoming richer and richer contain many interesting and attackable open problems.

I want to keep going and understand

Global symplectic geometry

by using the ideas from

String theory.

Hamiltonian Dynamics

$H : T^{*}M \to R$

Global symplectic manifold X

Quantum mechanics

$$\sqrt{-1}\,\frac{\partial \psi}{\partial t} = H\left(q, \sqrt{-1}\,\partial/\partial q\right)\psi$$

?