Loop space and holomorphic disc

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Many parts are joint work with
Oh-Ohta-Ono

L : closed oriented manifold.

We assume that it is spin.

$\mathcal{L}(L)$: Loop space of L

S^1 acts on $\mathcal{L}(L)$: $(t \cdot \ell)(s) = \ell(t+s)$

Theorem 1 (Chas-Sullivan, ..., F)

$H_1(\mathcal{L}(L))$ has a structure of L infinity algebra.

L infinity algebra = A homotopy version of Lie algebra

M : closed symplectic manifold

L : Lagrangian submanifold of M

$$\Lambda = \left\{ \sum a_i q^{\lambda_i} \middle| \lambda_i \in \boldsymbol{R}, \quad \lambda_i \to +\infty \right\}$$

$$\Lambda_+ = \left\{ \sum a_i q^{\lambda_i} \middle| \lambda_i \in \boldsymbol{R}_+, \quad \lambda_i \to +\infty \right\}$$

Theorem 2

There exists b in $H(\mathfrak{L}(L); \Lambda_+)$ such that

$$\sum_{k=1}^{\infty} \mathbf{l}_k (b \cdots b) = 0$$

An analogue of Maurer-Cartan equation $\quad db + \frac{1}{2}\{b,b\} = 0$

$$\Lambda[q^{-1}] = \left\{ \sum a_i q^{\lambda_i} \,\middle|\, \lambda_i \in \boldsymbol{R}, \quad \lambda_i \to +\infty \right\}$$

Theorem 3

If $F : M \longrightarrow M$ be a Hamiltonian diffeomo. with

$$F(L) \cap L = \emptyset$$

then, there exists B in $H(\mathcal{L}(L); \Lambda[q^{-1}])$ *with*

$$\sum_{k=0}^{\infty} \mathbf{l}_{k+1}(B, b\cdots, b) \equiv [L] \mod \Lambda_+$$

[L] = the homology class of all constant loops

L infinity algebra

C : graded vector space

$C[1]$: degree shift $(C[1]^d = C^{d+1})$.

$$E_k C[1] = \frac{C[1] \otimes \cdots \otimes C[1]}{S_k}$$

S_k : symmetric group of order $k!$ acts by

$$\sigma(x_1 \otimes \cdots \otimes x_k) = \pm x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}$$

$$EC[1] = \bigoplus_k E_k C[1]$$ is a coalgebra (cocommutative and coassociative)

Definition

L infinity structure on C is a coderivation

$$\mathbf{d} : \mathrm{E}C[1] \longrightarrow \mathrm{E}C[1]$$

such that

$$\mathbf{d} \circ \mathbf{d} = 0.$$

It is equivalent to give series of operations

$$\mathbf{l}_k : E_k C[1] \to C[1]$$

with

$$\sum_{\sigma \in S_n} \sum_{k+\ell=n+1} \pm \frac{n!}{k!\ell!} \mathbf{l}_\ell \left(\mathbf{l}_k (x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}) \otimes x_{\sigma(k+1)} \otimes \cdots \otimes x_{\sigma(n)} \right) = 0$$

L infinity relation

Example of L infinity relation

- $\mathbf{l}_1 \circ \mathbf{l}_1 = 0$ We have a homology group. $H(C;\mathbf{l}_1)$

- If $\mathbf{l}_k = 0$ for $k \neq 2$
  L infinity relation becomes

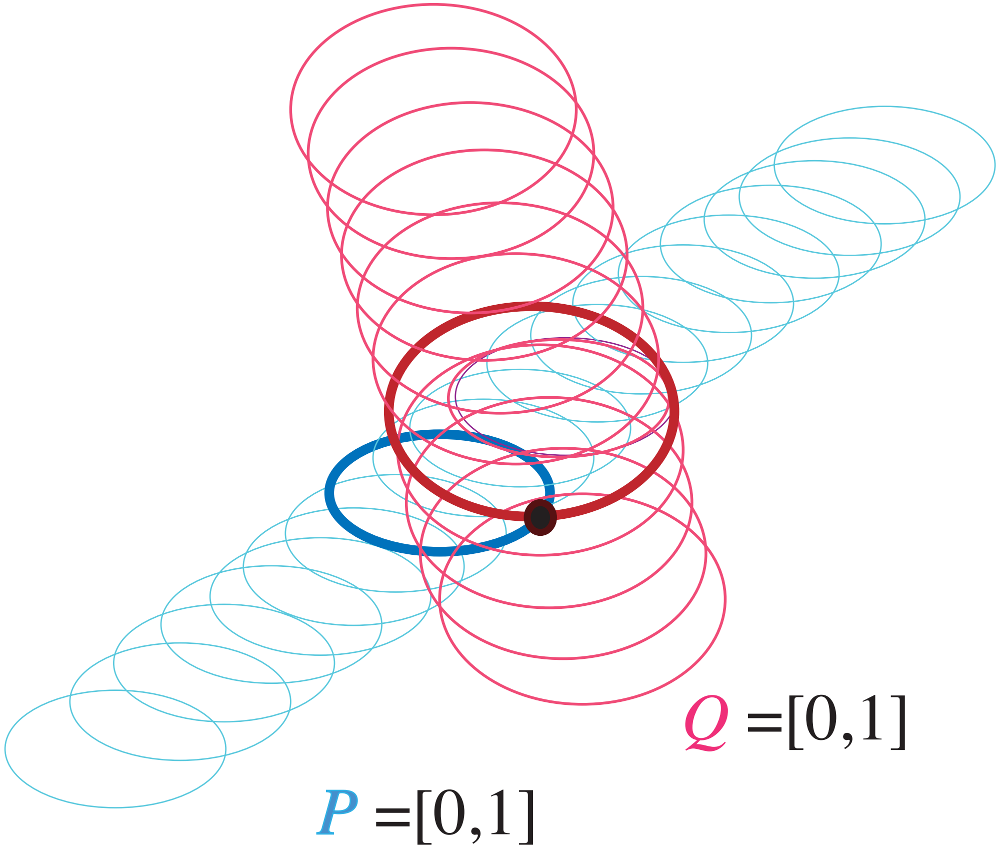
$$\pm\mathbf{l}_2\left(\mathbf{l}_2(x \otimes y) \otimes z\right) \pm \mathbf{l}_2\left(\mathbf{l}_2(y \otimes z) \otimes x\right) \pm \mathbf{l}_2\left(\mathbf{l}_2(z \otimes x) \otimes y\right) = 0$$

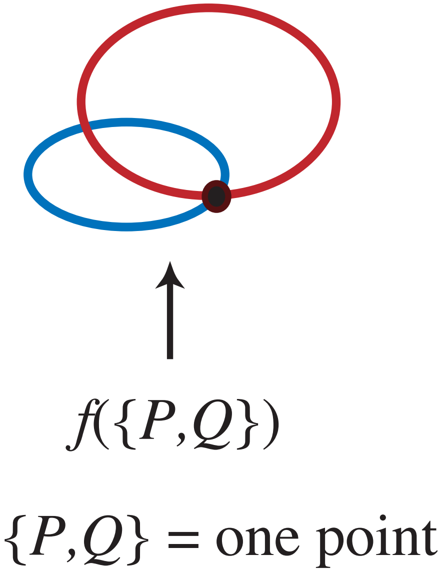
graded Jacobi

L infinity structure on $H(\mathfrak{L}(L))$

Intersection theory of chains on loop space

$$F : P \longrightarrow \mathfrak{L}(L) \qquad G : Q \longrightarrow \mathfrak{L}(L)$$





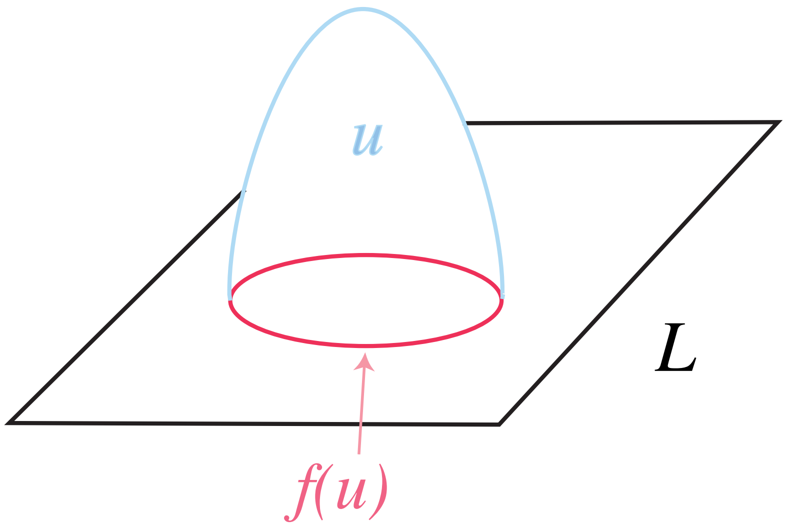
Definition of $b \in H(\mathfrak{L}(L))$

$\beta \in \pi_2(M, L)$

$$P(\beta) = \left\{ u : (D^2, \partial D^2) \to (M, L) \,\middle|\, u \text{ is holomorphic, } [u] = \beta \right\}$$

$$f : P(\beta) \to \mathfrak{L}(L)$$

$$f(u) = u|_{\partial D^2} : S^1 \to L$$

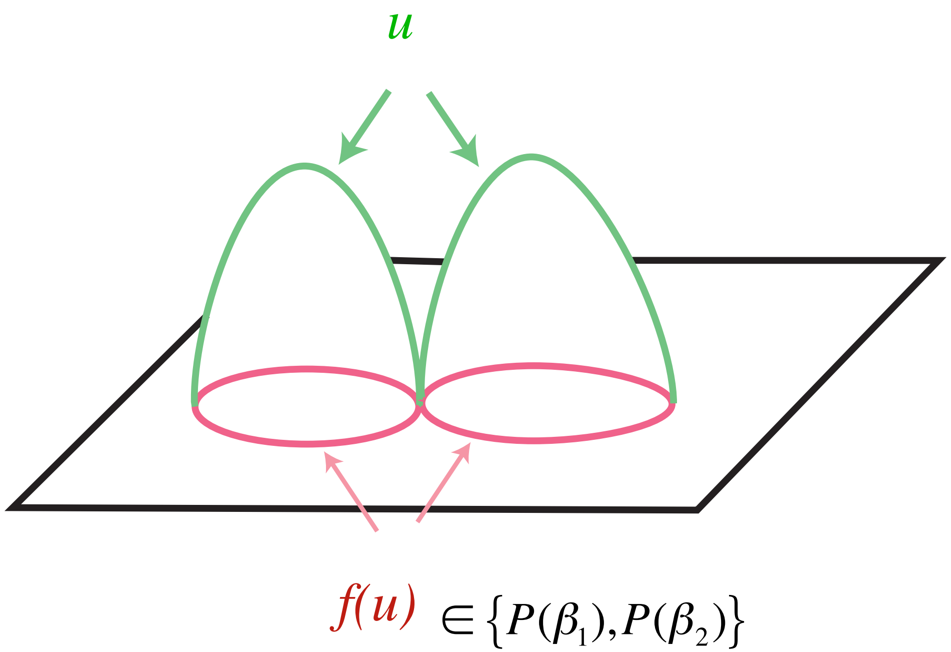


$$\boxed{b = \sum_{\beta} (P(\beta), f)\, q^{\beta \cap \omega}}$$

ω : symplectic form,

b satisfies Maurer-Cartan

$$\partial b + \mathbf{l}_2(b,b) = 0$$



$$u_i \in P(\beta)$$

$$\lim_{i\to\infty} u_i = u \in \partial(P(\beta))$$

Maurer-Cartan equation

$$\updownarrow$$

$$x \mapsto d_b(x) = dx + \{b, x\} \quad \text{satisfies} \quad d_b \circ d_b = 0$$

Open string $\longleftrightarrow$ Gauge theory

Loop space

Witten (1992), Cattaneo Frohlich, Jurg, and F.

Applications :

Theorem 4

Let L be a Lagrangian submanifold of C^n such that

$$\pi_k(L) = 1 \quad k \neq 1.$$

Then there exists $\Gamma \subseteq \pi_1(L)$ *a finite index subgroup such that*

$$\Gamma \cong \mathbf{Z} \times G$$

Corollary 5

Let L be a Lagrangian submanifold of C^3
assume L is irreducible,
then

$$L \cong \Sigma \times S^1$$

(diffeomorphic).

Corollary 6 (Audin's conjecture)

Let T^n be a Lagrangian submanifold of C^n.
Then the Maslov index homomorphism

$$\pi_1(L) \to 2\mathbf{Z}$$

is surjective.

Independently claimed by Cielibak and Eliashberg

Idea of the proof

$$\pi_k(L) = 1 \quad k \neq 1$$

$$\downarrow$$

Contradiction by comparing degree

$H(\mathcal{L}(L))$ is small

$$\updownarrow$$

Need nontrivial element in the left hand side.

$$\uparrow$$

The right hand side is non zero.

$$\uparrow$$

$$\sum_{k=0}^{\infty} \mathbf{l}_{k+1}(B, b \cdots, b) \equiv [L] \mod \Lambda_+ \qquad \text{(Theorem 3)}$$

Theorem 3

If $F: M \longrightarrow M$ *be a Hamiltonian diffeomo. with*

$$F(L) \cap L = \emptyset$$

then, there exists B *in* $H(\mathcal{L}(L);\Lambda[q^{-1}])$ *with*

$$\sum_{k=0}^{\infty} \mathbf{l}_{k+1}(B, b \cdots b) \equiv [L] \mod \Lambda_+$$

$[L]$ = the homology class of all constant loops

Theorem 7

Let L be a Lagrangian submanifold of M. Assume

- $\dim L$ *is even.*
- L *admits a metric of negative sectional curvature.*

Let $F: M \longrightarrow M$ Hamiltonian diffeo. with

- $F(L) \pitchfork L$.

Then

$$\#\big(F(L) \cap L\big) \geq \sum_k \operatorname{rank} H_k(L)$$

There is an earlier related work by Viterbo

Theorem 8 (Based on the work by Cho-Oh,Cho)

Let M be a toric Fano manifold with moment map $\mathrm{pr}: M \longrightarrow B$. *Put* $T(v) = \mathrm{pr}^{-1}(v)$.

Then there exists $v \in B$ *such that for any Hamiltonian diffeomorphism* $F: M \longrightarrow M$.

$$F(T(v)) \cap T(v) \neq \emptyset$$

There is also an earlier related work by Entov-Polterovich

Application to the structure of super potential

$$b = \sum_{\beta} (P(\beta), f) q^{\beta \cap \omega} = \sum_{\beta} b(\beta) q^{\beta \cap \omega}$$

$$b(\beta) \in H(\mathfrak{L}(L))$$

$b(\beta)$ can be viewed as a (super) function on cohomology group of L

Super potential of L

Iterated integral of Chen

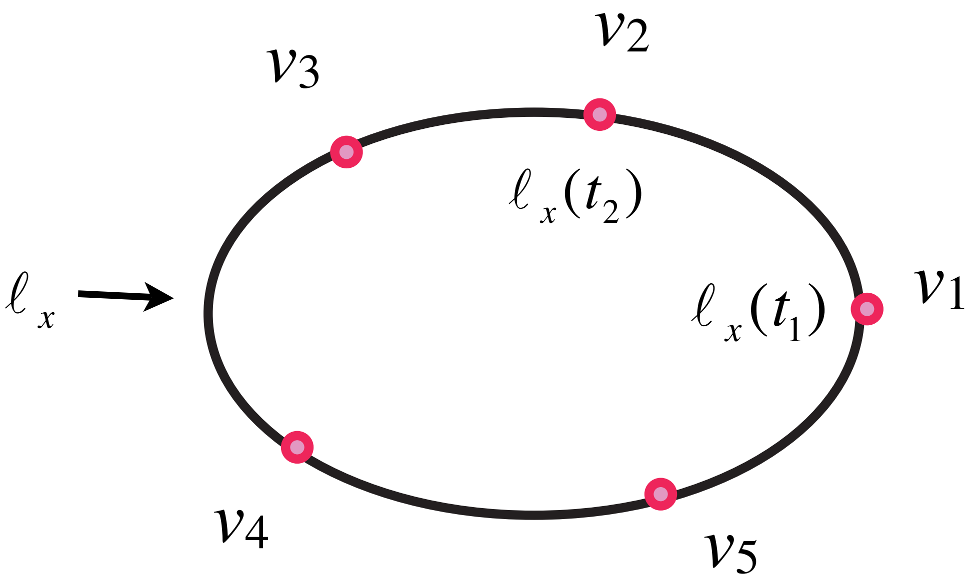
$$b(\beta) = (P(\beta), f), \quad f : P(\beta) \to \mathcal{L}(L)$$

Put : $f(x) = \ell_x : S^1 \to L \ (x \in P(\beta))$

$$ev_k : P(\beta) \times \left(S^1\right)^k \to L^k$$

$$ev_k(x; t_1, \cdots, t_k) = \left(\ell_x(t_1) \cdots, \ell_x(t_k)\right)$$

v_i : Harmonic forms on L



$x \in P(\beta)$

$t_i \in S^1$

$$\boxed{\Psi_{\beta,k}(v_1,\cdots,v_k) = \int_{P(\beta)\times(S^1)^k} ev_k^*(v_1 \wedge \cdots \wedge v_k)}$$

$$\Psi = \sum_{\beta,k} \frac{1}{k!} q^{\omega\cap\beta} \Psi_{\beta,k}$$

super potential : $H(L;\mathbf{Q}) \rightarrow \Lambda$

Let $x_1,\cdots,x_a$ be variables of $H^1(L)$.

Let $z_1,\cdots,z_c$ be variables of $\bigoplus_{k\neq 1} H^k(L)$.

Put $y_i = e^{x_i}$.

Theorem 9

There exist Laurant polynomials $\psi_i(y_1,\cdots,y_a,z_1,\cdots,z_c)$ *such that*

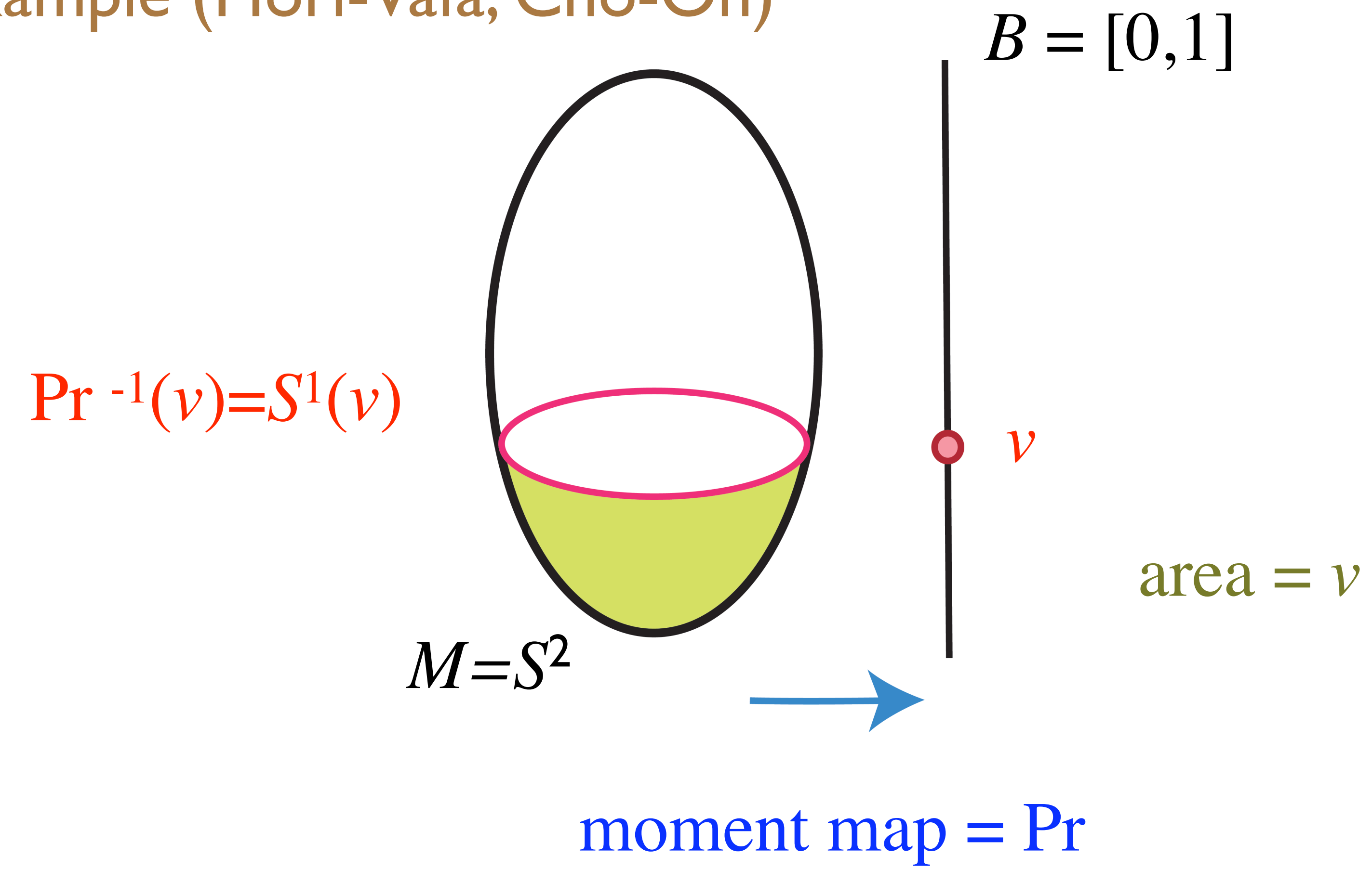
$$\Psi = \sum_i q^{\lambda_i}\psi_i$$

$b \in H(\mathfrak{L}(L))$ is well defined up to gauge equivalence.

$$\downarrow$$

Ψ is well defined up to change of variables.

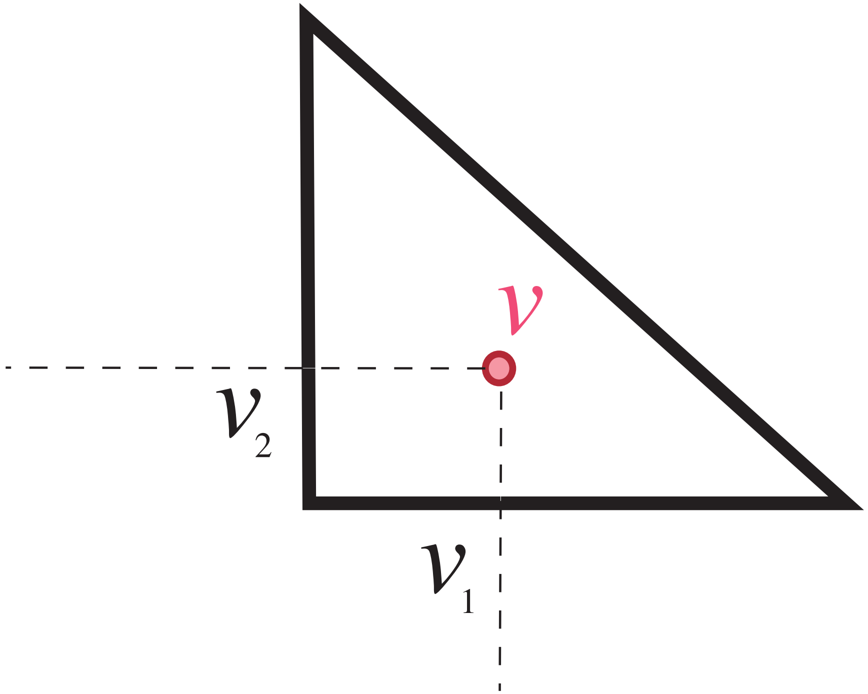
Example (Hori-Vafa, Cho-Oh)



$$\Psi_{S^1(v)}(x) = e^x q^v + e^{-x} q^{1-v} \qquad x \in H^1(S^1(v))$$

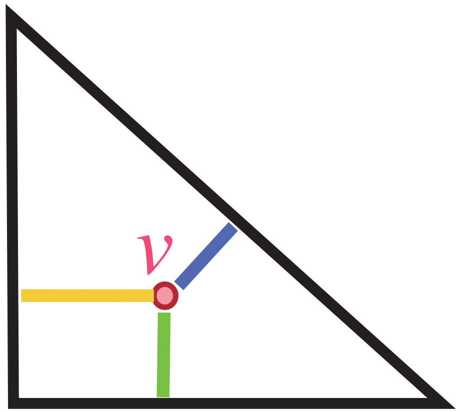
$$M = CP^2 \xrightarrow{\text{Pr = moment map}} \textit{Triangle in } R^2$$

$T^2(v) = \mathrm{Pr}^{-1}(v)$

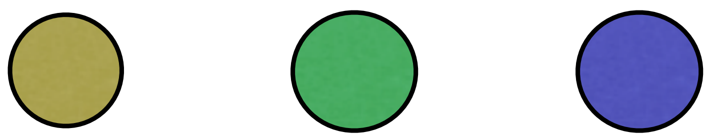


$$\Psi_{T^2(v)}(x_1, x_2) = e^{x_1} q^{v_1} + e^{x_2} q^{v_2} + e^{-x_1 - x_2} q^{1 - v_1 - v_2}$$

3 holomorphic discs (of Maslov index 2)



$$\Psi_{T^2(v)}(x_1,x_2) = e^{x_1}q^{v_1} + e^{x_2}q^{v_2} + e^{-x_1-x_2}q^{1-v_1-v_2}$$



Theorem 8

Let M *be a toric Fano manifold with moment map* $\mathrm{pr} : M \longrightarrow B$. *Put* $T(v) = \mathrm{pr}^{-1}(v)$.

Then there exists $v \in B$ *such that for any Hamiltonian diffeomorphism* $F : M \longrightarrow M$.

$$F(T(v)) \cap T(v) \neq \emptyset$$

Idea of the proof of Theorem 8

Floer homology $HF((T(v),x)),(T(v),x))) \neq 0$

(Floer, Oh, Oh-Ohta-Ono-F)

Super potential $\Psi_{T(v)}(x)$ *has a critical point at* x.

Here $\Psi_{T(v)}(x)$ is regarded as $x \in H^1(T(v)) \otimes \Lambda$
a function of $= \mathfrak{M}(L)$

Lagrangian Floer theory (FOOO)

- (M,L) $\longrightarrow$ Set $\mathfrak{M}(L)$
- $x_i \in \mathfrak{M}(L_i)$ $\longrightarrow$ $HF((L_1, x_1),(L_2, x_2))$
- $F : M \longrightarrow M$ Hamiltonian $\longrightarrow$
  - $F_* : \mathfrak{M}(L) \to \mathfrak{M}(F(L))$
  - $HF((L, x), (L, x)) = HF((L, x),(F(L), F_*(x)))$
- $L_1 \pitchfork L_2$ $\longrightarrow$ $\#(L_1 \cap L_2) \geq \operatorname{rank} HF((L_1, x_1),(L_2, x_2))$

Example (Cho-Oh)

$$M = CP^2$$

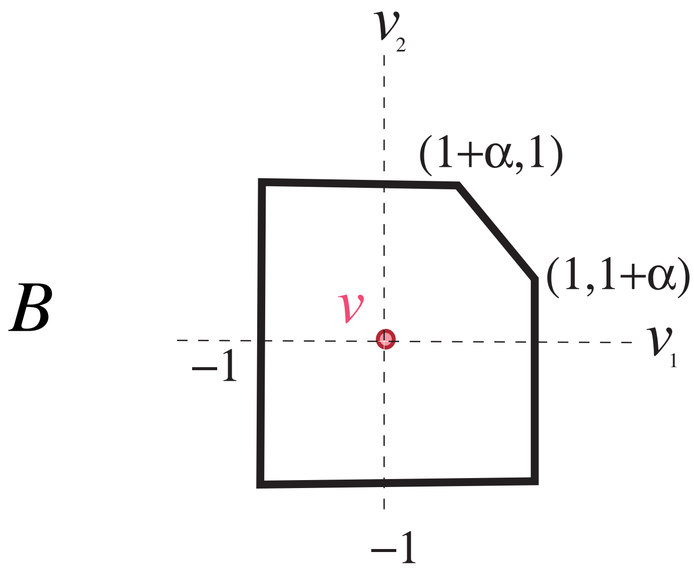
$$\Psi_{T^2(v)}(x_1, x_2) = e^{x_1}q^{v_1} + e^{x_2}q^{v_2} + e^{-x_1-x_2}q^{1-v_1-v_2}$$

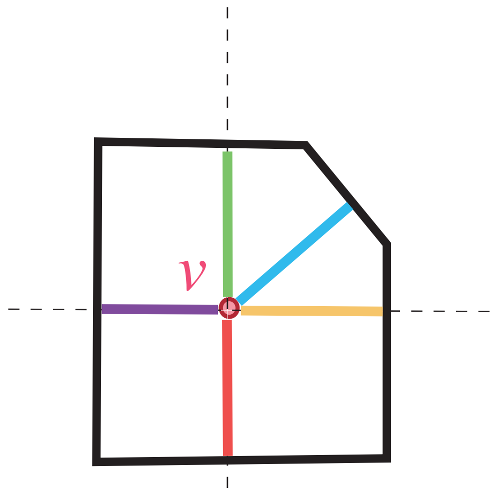
$$v_1 = v_2 = \frac{1}{3}$$

$$\Psi_{T^2(v)} = \left(e^{x_1} + e^{x_2} + e^{-x_1-x_2}\right)q^{\frac{1}{3}} = \left(y_1 + y_2 + y_1^{-1}y_2^{-1}\right)q^{\frac{1}{3}}$$

$$\boxed{\frac{\partial \Psi_{T^2(v)}}{\partial y_1} = \frac{\partial \Psi_{T^2(v)}}{\partial y_2} = 0 \longleftrightarrow y_1 = y_2, \quad y_1^3 = 1}$$

Example





$v = (0,0)$

$$\Psi_{T(v)}(x_1,x_2) = \left(e^{x_1} + e^{x_2} + e^{-x_1} + e^{-x_2}\right)q + e^{-x_1-x_2}q^{1+\alpha}$$

Critical point

$$\boxed{x_1 = x_2 = \log\left(1 + \frac{q^{\alpha}}{2} - \frac{3q^{2\alpha}}{8} + \cdots\right)}$$

Releted work and Generalization

- M Calabi-Yau and L zero Maslov (eg. special)

  The same structure theorem as Theorem 9 but the sum

  $\Psi = \sum_i q^{\lambda_i} \psi_i$ becomes infinite sum.

- (Pseudo)-holomorphic map from bordered Riemann surface with higher genus.

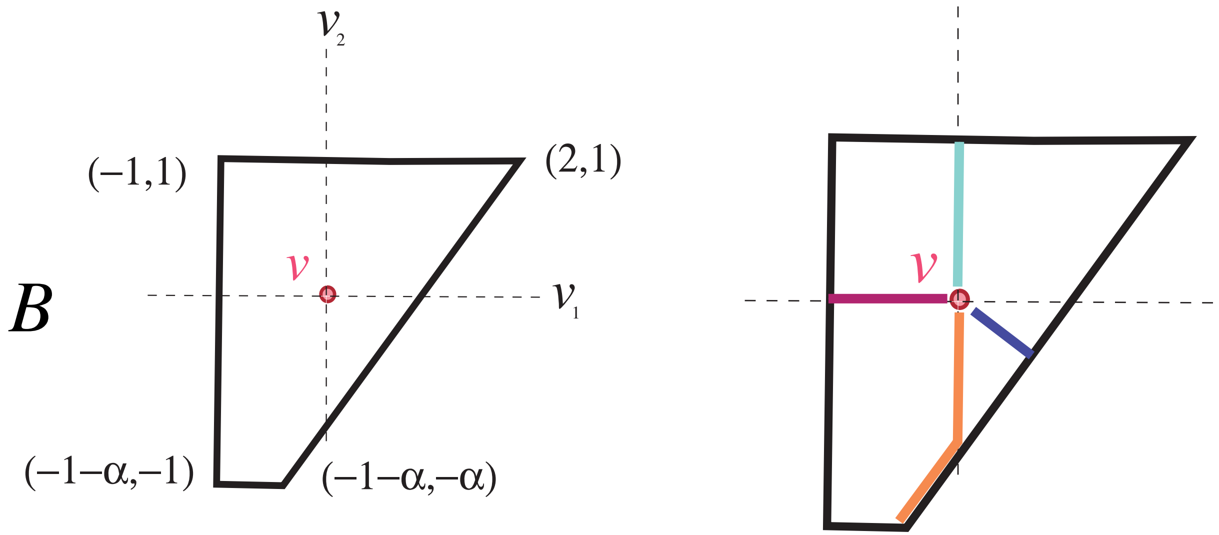
  Involutive bi-Lie-infinity structure in place of L infinity structure.

  (algebraic part : Cielibak-F- Latshov)

- Relation to Pertubative Chern-Simons.

  Some relation is visible but not yet clear.

Example



$$\Psi_{T(v)}(x_1, x_2) = \left(e^{x_1} + e^{-x_2} + e^{-x_1 - x_2}\right)q + e^{x_2}q^{1+\alpha}$$