Chapter 10. Lagrangian Surgery and holomorphic discs.[1]

The purpose of this chapter is to prove Theorem Z and to study several examples arising from the study of affine Lagrangian tori and their Lagrangian surgery in the flat symplectic tori. This chapter provides explicit examples of various constructions that have been carried out in the previous chapters. Theorem Z describes how the moduli space of pseudo-holomorphic $(k+1)$-gons with Lagrangian boundary condition is related to that of the pseudo-holomorphic k-gons under the Lagrangian surgery. In the case of affine Lagrangian tori in the flat symplectic tori, the moduli space of pseudo-holomorphic polygons are discussed in [Fuk02III]. However our study of the moduli space of pseudo-holomorphic polygons for the purpose of this chapter does not depend on [Fuk02III].

A brief outline of the contents of each of the sections of this chapter is in order.

In §54, we recall the construction of Lagrangian surgery with a precise description of the way we do the surgery. This will be used for the later study of the metamorphosis of the moduli spaces of holomorphic polygons under the surgery. In this section, we also clarify the multiplicity one condition that appears in Theorem Z. In §55 we restate Theorem Z in a more precise and detailed way and state some of its generalizations. In §56 - 57 we use these to discuss various examples arising from Lagrangian surgery of affine Lagrangian tori. In §58 we review some basic properties of the moduli space of pseudo-holomorphic polygons. The proof of Theorem Z is then carried out in §59 - 62 using a gluing argument. As in [FuOh97], we first need to construct a local model of pseudo-holomorphic discs to be implanted into a small neighborhood of the point at which we perform the surgery. We describe the moduli space of such pseudo-holomorphic discs in §59 - 60. In §61, we use the local model constructed in §59-60 to smooth-off a corner of the pseudo-holomorphic triangle. In §62, we show that the pseudo-holomorphic discs constructed in §61 exhaust all such pseudo-holomorphic discs near the given pseudo-holomorphic triangle and complete the proof of Theorem Z.

The analytic details of §61-62 are closely related to those discussed in the context of 'symplectic field theory' in the literature. (See [BEHWZ03] for example.) Partly because the rigorous foundation of 'symplectic field theory' is not yet established at the time of writing this book and also because we are unable to find the literature containing a rigorous proof of what we need, we give detailed and self-contained proofs *without* relying on the literature : Especially the details of the surjectivity result like the one proven in §62 are rarely given in the literature, while this is the most delicate and difficult part of the corresponding matters. In this regard, we like to mention that the idea of separately estimating the 'horizontal' and 'vertical' energies of the holomorphic maps in the setting of the symplectization is essential for this purpose. Such an idea is originally due to Hofer [Hof93].

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In the earlier part of this book, we have been trying to prove the results as general as possible. On the other hand in this chapter, we sometimes put some inessential restrictions on the almost complex structure J in order to simplify the argument. Such restrictions, for example Assumption 54.20, could be certainly removed from Theorem Z but with paying the price of making the volume of the current book even bigger. Because the main purpose of this chapter is to illustrate the constructions of this book, we do not attempt to deal with such analytic details but to restrict ourselves to the cases that we need for the purpose of providing rigorous explanation of our examples given in §56 - 57.

§54. Lagrangian surgery and local structure of pseudo-holomorphic polygons.

The main purpose of this section is to review Lagrangian surgery and fix notations. We also give the precise statement on the multiplicity one condition in Theorem Z and review the structure of tangent cones at the vertices of pseudo-holomorphic polygons. The materials in this section are largely a review of known results in the literature. We organize them in the way suitable for our study of the metamorphosis of the moduli space of pseudo-holomorphic $(k+1)$-gons to k-gons under the Lagrangian surgery.

54.1. Lagrangian surgery in symplectic geometry.

In §54.1 and §54.2 we review Lagrangian surgery. In §54.1 we discuss the standard Lagrangian surgery studied in the symplectic geometry. In §54.2 we include the effect of the presence of almost complex structure. We refer to [LaSi91], [Pol91I] for some applications of Lagrangian surgery to the study of topology of Lagrangian submanifolds. We will be interested in the analytical aspects related to the Lagrangian surgery and pseudo-holomorphic discs. Because of this, we need to describe the Lagrangian surgery in relation to the presence of almost complex structures compatible with the symplectic form.

Let L_1 and L_2 be a pair of oriented Lagrangian submanifolds in (M, ω) that intersect transversely at p_{12}. We fix an ordering of the pair as (L_1, L_2). We can always choose a Darboux chart in a neighborhood U of p_{12}, $I : U \to V \subset \mathbb{C}^n$ so that $I(p_{12}) = 0$,

$$I(L_1 \cap U) = \mathbb{R}^n \cap V, \quad I(L_2 \cap U) = \sqrt{-1}\,\mathbb{R}^n \cap V.$$

The proof follows from a version of Darboux theorem (see [Theorem 7.1, Wei71]) but strongly relies on the following well-known fact in symplectic linear algebra whose proof we omit.

Lemma 54.1. *The linear symplectic group $Sp(2n)$ acts transitively on the set of transversal pairs of Lagrangian subspaces.*

We would like to point out that $U(n) \subset Sp(2n)$ does not act transitively on the set of such pairs. (See Lemma 54.10.)

Let ϵ be a real number sufficiently close to 0. We choose the function $f_\epsilon : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ defined by

$$f_\epsilon(x) = \epsilon \log |x|, \tag{54.2}$$

and denote by $H_\epsilon \subset \mathbb{C}^n$ the graph of $df_\epsilon(x)$.

This is a Lagrangian submanifold in $T^*(\mathbb{R}^n \backslash \{0\}) \cong \mathbb{C}^n \backslash \sqrt{-1}\,\mathbb{R}^n$ which is asymptotic to $\sqrt{-1}\,\mathbb{R}^n$ as $|x| \to 0$, and to $\mathbb{R}^n$ as $|x| \to \infty$. Noting that we have

$$df_\epsilon(x) = \epsilon \frac{x \cdot dx}{|x|^2} = \frac{\epsilon}{|x|^2} \sum_{j=1}^n x_j dx_j \tag{54.3}$$

we can write

$$H_\epsilon = \left\{ (z_1, \cdots, z_n) \;\middle|\; y_j = \frac{\epsilon x_j}{|x|^2},\, j = 1, \cdots, n \right\} \tag{54.4}$$

in coordinates. Here we denote the complex coordinates of $\mathbb{C}^n$ as $z_j = x_j + \sqrt{-1}\, y_j$ for $j = 1, \cdots, n$.

Let $\tau : \mathbb{C}^n \to \mathbb{C}^n$ be the reflection along the diagonal

$$\Delta = \{(z_1, z_2, \cdots, z_n) \,|\, x_i = y_i\},$$

i.e., be the map

$$(x_1 + \sqrt{-1}\, y_1, \cdots, x_n + \sqrt{-1}\, y_n) \mapsto (y_1 + \sqrt{-1}\, x_1, \cdots, y_n + \sqrt{-1}\, x_n).$$

We remark that (54.4) implies $|x|^2|y|^2 = |\epsilon|^2$, hence we also have

$$H_\epsilon = \left\{ (z_1, \cdots, z_n) \;\middle|\; x_j = \frac{\epsilon y_j}{|y|^2},\, j = 1, \cdots, n \right\}$$

In other words $\tau(H_\epsilon) = H_\epsilon$. Note $\inf\{|\vec{z}| \mid \vec{z} \in H_\epsilon\} = \sqrt{2|\epsilon|}$.

Next we consider a function $\rho : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\rho = \begin{cases} \log r - |\epsilon| & \text{if } r \le \sqrt{|\epsilon|} S_0 \\ \log \sqrt{|\epsilon|} S_0 & \text{if } r \ge 2\sqrt{|\epsilon|} S_0 \end{cases}$$
$$\rho'(r) \ge 0, \quad \rho''(r) \le 0,$$

here S_0 is a sufficiently large number, which will be fixed at the begining of §61.5. ϵ is chosen so that $\sqrt{|\epsilon|}S_0$ is sufficiently small. We then define the function $\widetilde{f}_\epsilon : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by

$$\widetilde{f}_\epsilon(x) = \epsilon\rho(|x|). \tag{54.5}$$

Consider the graph $\mathrm{Graph}\, d\widetilde{f}_\epsilon$ and define a Lagrangian submanifold H'_ϵ so that the following holds :

(54.6.1) $\tau(H'_\epsilon) = H'_\epsilon$.
(54.6.2)

$$\begin{aligned}\{(x_1 + \sqrt{-1}\, y_1, \cdots, x_n + \sqrt{-1}\, y_n) \mid \forall i\ x_i \geq y_i\} \cap H'_\epsilon \\ = \{(x_1 + \sqrt{-1}\, y_1, \cdots, x_n + \sqrt{-1}\, y_n) \mid \forall i\ x_i \geq y_i\} \cap \mathrm{Graph}\, d\widetilde{f}_\epsilon.\end{aligned}$$

Figure 54.1

By construction, H'_ϵ is invariant under τ and $H'_\epsilon = \mathbb{R}^n \cup \sqrt{-1}\,\mathbb{R}^n$ outside the ball $B^{2n}(2\sqrt{|\epsilon|}S_0)$ around 0 in $I(U) \subset \mathbb{R}^{2n}$. Therefore for a given ordered pair (L_1, L_2), we can construct a Lagrangian submanifold $L_\epsilon \subset M$ such that

$$L_\epsilon - U = L_1 \cup L_2 - U, \quad I(L_\epsilon \cap U) = H'_\epsilon \cap V.$$

Definition 54.7. For a given ordered pair (L_1, L_2), we call L_ϵ the *Lagrangian submanifold obtained from L_1 and L_2 by Lagrangian surgery* at $p_{12} \in L_1 \cap L_2$ and write $L_1 \#_\epsilon L_2 = L_\epsilon$.

Note that, if we change the ordering of the pair L_1, L_2 at $p \in L_1 \cap L_2$ and change the sign of ϵ at the same time, then the resulting Lagrangian submanifolds are isomorphic. In fact, we have

$$\mathbb{R}^n \#_\epsilon \sqrt{-1}\,\mathbb{R}^n = \sqrt{-1}\mathbb{R}^n \#_{-\epsilon} \mathbb{R}^n.$$

We call the pre-image

$$(L_1\#_\epsilon L_2)\cap U = I^{-1}(H'_\epsilon \cap V)$$

a *Lagrangian handle* and its meridian sphere S^{n-1} a *vanishing cycle* of the Lagrangian surgery L_ϵ.

We say that the pair L_1, L_2 or its associated Lagrangian surgery $L_1\#_\epsilon L_2$ with $\epsilon > 0$ is *positive* at p_{12} if

$$T_{p_{12}}L_1 \oplus T_{p_{12}}L_2 = (-1)^{n(n-1)/2+1}T_{p_{12}}M$$

as an oriented vector space and *negative* otherwise. (Here we equip $T_{p_{12}}M$ with the symplectic orientation.) For example, for $L_1 = \mathbb{R}$, $L_2 = \sqrt{-1}\mathbb{R} \subset \mathbb{C}$ with the standard orientation on $\mathbb{R}, \sqrt{-1}\mathbb{R}$, the Lagrangian surgery $L_1\#_\epsilon L_2$, $\epsilon > 0$, is negative. (This example is directly extended to the case of $L_1 = \mathbb{R}^n, L_2 = \sqrt{-1}\mathbb{R}^n \subset \mathbb{C}^n$.) It is easy to check that only the positive surgery allows to glue the orientations on L_1 and L_2 to have the surgery $L_1\#_\epsilon L_2$ carry a compatible orientation. (In the case of $L_1 = \mathbb{R}$, $L_2 = \sqrt{-1}\mathbb{R} \subset \mathbb{C}$, it is easy to see that it is impossible to give an orientation of $L_1\#_\epsilon L_2$, which is compatible with both standard orienatations of L_i. Similar remark also holds for $L_1 = \mathbb{R}^n, L_2 = \sqrt{-1}\mathbb{R}^n \subset \mathbb{C}^n$)

We remark that $L_1\#_\epsilon L_2$ is not even isotopic to $L_2\#_\epsilon L_1$ (or $L_1\#_{-\epsilon} L_2$) in general even when both are orientable. On the other hand, it is easy to check that $L_1\#_\epsilon L_2$ are Lagrangian isotopic to one another for different ϵ's with the same signs. However they are not Hamiltonian isotopic to one another in general.

Remark 54.8. If L_1, L_2 are spin manifolds, the surgered manifold $L_1\#_\epsilon L_2$ is also a spin manifold. However, there is a slightly delicate issue about the choice of spin structures. We explain the way to obtain the spin structure. (The argument can be extended to relative spin pair or tuples of Lagrangian submanifolds.)

Let X be a spin manifold, which is not necessarily connected. Let $p_i \in X$ and D_i small discs around p_i, $i = 1, 2$. Consider the operation of attaching a 1-handle

$$Z = X \times [0,1] \cup_h D^n \times [1,2],$$

where $h : D^n \times \{i\} \to D_i \times \{1\}$ is the attaching map. After smoothing the corner of Z, we obtain a cobordism between X and a new manifold X'. For a spin structure on X, the spin structure on X' is described as follows.

Give the orientation on $D^n \times \{i\}$ as open subsets of the boundary of $D^n \times [1,2]$, $i = 1, 2$. Denote by $P_{spin}(X), P_{spin}(D^n \times [1,2])$ the principal spin bundles, i.e., the spin structures of X and $D^n \times [1,2]$, respectively. Pick a lift $\iota_{p_i} : P_{spin}(X)|_{p_i} \to P_{spin}(D^n \times [1,2])|_{(0,i)}$ of $h_*^{-1} : T_{p_i}X \to T_{(0,i)}D^n \subset T_{(0,i)}D^n \times [1,2]$. Clearly $(\iota_{p_1}, \iota_{p_2})$ and $(-\iota_{p_1}, -\iota_{p_2})$ derive the same spin structure. Here -1 denotes the non-trivial element in $(\ker : Spin(n) \to SO(n))$. Then h and $(\iota_{p_1}, \iota_{p_2})/\{\pm 1\}$ determines a spin structure on Z, hence a spin structure on X' in its boundary.

If p_1, p_2 belong to different connected components X_1 and X_2, the spin structure is independent of the choice of $(\iota_{p_1}, \iota_{p_2})$. Since there is an automorphism of the principal spin bundle, which is identity except on the component X_1 and is given by the right multiplication by $-1 \in (\ker : Spin(n) \to SO(n))$ on X_1. This action changes ι_{p_1} to $-\iota_{p_1}$ with keeping ι_{p_2} invariant. Hence the spin structure is uniquely determined by the spin structure on X and the attaching map h.

However, when p_1 and p_2 belong to the same connected component, $(\iota_{p_1}, \iota_{p_2})$ and $(\iota_{p_1}, -\iota_{p_2})$ derive different spin structures. This point is important, for example, when we consider singular Lagrangian fibrations with nodal singular fibers. Suppose that there is an irreducible nodal singular fiber, which is spin. Regular fibers around it are obtained as Lagrangian surgery. In the 2-dimensional case, the monodromy is given by the Dehn twist along the vanishing cycle. Thus we find that the spin structure obtained in the above construction is not perserved under the monodromy.

Remark 54.9. The discussion on the surgery of this chapter is related to homological mirror symmetry in the following way. Let L_1, L_2 be a pair of Lagrangian submanifolds in a symplectic manifold M. Consider, for example, that M is Calabi-Yau 3 fold for which we have a mirror complex manifold $M^\dagger$. We then consider Lagrangian submanifolds L_i whose Maslov classes vanish. Suppose that L_i are unobstructed and have mirror objects $\mathcal{E}(L_i)$ on $M^\dagger$ which are objects of the derived category of coherent sheaves. Let L be another Lagrangian submanifold of M with vanishing Maslov class whose mirror is $\mathcal{E}(L)$. We furthermore assume that L_1 intersects with L_2 at one point p_{12} transversely.

We assume that $L_1 \cap L_2 \cap L = \emptyset$ and L is transversal to L_1 and L_2. We assume also that there exists no pseudo-holomorphic triangle as in Figure 54.2 below. We have

$$(L \cap L_1) \cup (L \cap L_2) = L \cap (L_1 \#_\epsilon L_2)$$

if $\epsilon > 0$ is sufficiently small. Since there is no pseudoholomorphic discs as in Figure 54.2, we can show (by an easier analogy of Theorem Z) that $CF(L; L_1)$ is a subcomplex of $CF(L; L_1 \#_\epsilon L_2)$. Moreover we have the following long exact sequence

$$(*) \qquad \to HF(L; L_1) \to HF(L; L_1 \#_\epsilon L_2) \to HF(L; L_2) \to$$

where the connecting homomorphism

$$HF(L; L_2) \to HF(L; L_1)$$

is induced by

$$[x] \mapsto \mathfrak{m}_2([p_{12}], [x]).$$

Here $\mathfrak{m}_2$ is the composition of the A_∞-category, which is defined by counting holomorphic triangles [Fuk02III].

Figure 54.2

The exact sequence (*) can be interpreted in the mirror side by the distinguished triangle

$$\mathcal{E}(L_1) \to \mathcal{E}(L_1 \#_\epsilon L_2) \to \mathcal{E}(L_2) \to \mathcal{E}(L_1)[1].$$

This observation was made in [FOOO00] and [Fuk02III]. A similar observation was made independently by R. Thomas in [Tho01]. See also §38.4 of [HoVa03].

We remark that in case $L_1 = S^n$ and L_1 intersects with L_2 at one point transversely, $L_1 \#_\epsilon L_2$ is the image of the L_2 by the Dehn twist centered at L_1. In this case the above exact sequence coincides with one by Seidel [Sei03I].

54.2. Lagrangian surgery in almost Kähler geometry.

When we study pseudo-holomorphic maps together with the Lagrangian surgery, we need to describe the Lagrangian surgery in the almost Kähler setting (M, ω, J). In this section, we relate the model handle $H_\epsilon \subset \mathbb{C}^n$ implanted in the surgery to a particular Lagrangian submanifold used in [HaLa82], [Law89], [ThYa02]. This particular model is useful for our later analysis of metamorphosis of the moduli space of pseudo-holomorphic polygons under the Lagrangian surgery.

Let $\mathbb{C}^n$ be the standard complex vector space with standard complex structure J_0 and standard symplectic structure ω_0. Namely

$$J_0\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \qquad \omega_0 = \sum dx_i \wedge dy_i,$$

where $z_i = x_i + \sqrt{-1} y_i$ $(i = 1, \cdots, n)$ is the standard coordinate of $\mathbb{C}^n$.

Let $V_1, V_2 \subset \mathbb{C}^n$ be a transversal ordered pair of oriented Lagrangian linear subspaces. Recall Corollary 2.6 which reads that there exists a unique symmetric unitary matrix $A_i \in U(n)$ such that

$$A_i \cdot \mathbb{R}^n = V_i \quad \text{for } i = 1,\, 2.$$

The following lemma is an easy consequence of this whose proof is omitted.

Lemma 54.10. *There exists a unique collection of angles*

$$0 < \alpha_1 \leq \cdots \leq \alpha_n < \pi$$

and a matrix $A \in U(n)$ such that $A(V_1) = \mathbb{R}^n$ and

$$A(V_2) = \left\{ \left(e^{\alpha_1\sqrt{-1}}v_1, \cdots, e^{\alpha_n\sqrt{-1}}v_n\right) \,\middle|\, v_1, \cdots, v_n \in \mathbb{R}\right\},$$

as an oriented vector space. Here we define an orientation of the right hand side by the isomorphism

$$(v_1, \cdots, v_n) \mapsto \left(e^{\alpha_1\sqrt{-1}}v_1, \cdots, e^{\alpha_n\sqrt{-1}}v_n\right).$$

We call $\alpha_1, \cdots, \alpha_n$ the *Kähler angles* between V_1 and V_2.

Definition 54.11. For a transversal pair L_1, L_2 of oriented Lagrangian submanifolds, we define their *Kähler angles at* $p_{12} \in L_1 \cap L_2$ to be the Kähler angles between the tangent spaces $T_{p_{12}}L_1$, $T_{p_{12}}L_2$.

Referring to [ThYa02] for the description of the Lagrangian surgery for general Kähler angles, we restrict ourselves to the case where all Kähler angles of Λ are the same, i.e.,

$$\alpha_1 = \cdots = \alpha_n =: \alpha, \quad 0 < \alpha < \pi.$$

We closely follow the presentation of Thomas and Yau [ThYa02] below with some notational changes.

To any given embedded curve $\gamma : I \to \mathbb{C}$ and $I \subset \mathbb{R}$ a connected interval, we associate a Lagrangian submanifold

$$L_\gamma = \{(\gamma(t)a_1, \cdots, \gamma(t)a_n) \mid t \in I, \quad a = (a_j)_{j=1}^n \in S^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}^n\}.$$

Under this notation, $\mathbb{R}^n$ is represented by the curve $\gamma_1(r) = (r, 0)$, $r \in [0, \infty)$ and Λ by

$$\gamma_2(r) = e^{\sqrt{-1}\alpha}r,\ r \in [0, \infty).$$

The Lagrangian surgery $\mathbb{R}^n \#_\epsilon \Lambda$ for $\epsilon > 0$ is represented by any smoothing off the cone at the origin of $\gamma_1 \cup \gamma_2$ which stays inside the cone

$$\{re^{\sqrt{-1}\theta} \mid r > 0,\, \theta \in [0, \alpha]\}$$

and coincides with $\gamma_1 \cup \gamma_2$ outside a compact set.

For the case $\epsilon < 0$, the Lagrangian surgery $\mathbb{R}^n \#_\epsilon \Lambda$ is represented by a similar smoothing of $\gamma_1 \cup \gamma_2$ which stays inside the cone

$$\{re^{\sqrt{-1}\theta} \mid r > 0,\, \theta \in [\alpha, \pi]\}$$

instead.

Figure 54.3

For the later purpose, we will use the model for the neck in the transition region by the following curve

(54.12.1) $$\gamma_\epsilon^\alpha = \left\{ re^{\sqrt{-1}\theta} \in \mathbb{C} \,\middle|\, |2\epsilon|^{\frac{\pi}{2\alpha}} = r^{\frac{\pi}{\alpha}} \sin\left(\frac{\pi\theta}{\alpha}\right),\, \theta \in (0, \alpha) \right\} \quad \epsilon > 0,$$

(54.12.2)
$$\gamma_\epsilon^\alpha = \left\{ re^{\sqrt{-1}\theta} \in \mathbb{C} \,\middle|\, |2\epsilon|^{\frac{\pi}{2\alpha}} = r^{\frac{\pi}{\alpha}} \sin\left(\frac{\pi(\theta - \alpha)}{\pi - \alpha}\right),\, \theta \in (\alpha, \pi) \right\} \quad \epsilon < 0.$$

These give rise to the Lagrangian submanifolds H_ϵ^α by

(54.12.3) $$H_\epsilon^\alpha = \gamma_\epsilon^\alpha \cdot S_{\mathbb{R}^n}^{n-1} \subset \mathbb{C}^n.$$

H_ϵ^α becomes *special Lagrangian* submanifolds when $\alpha = \frac{\pi}{2}$ which are precisely the local model constructed by Harvey-Lawson [HaLa82] and also used by Lawler [Law89]. When $\alpha = \frac{\pi}{2}$, this coincides with the local model given in (54.4), i.e., $H_\epsilon = H_\epsilon^{\frac{\pi}{2}}$.

We can modify H_ϵ^α and construct $(H_\epsilon^\alpha)'$ in the same way as §54.1 as follows. Let us consider the case $\epsilon > 0$. (The case $\epsilon < 0$ is similar.) We consider a function

$$\theta(r) : [\sqrt{2|\epsilon|}, \infty) \to [0, \alpha/2]$$

such that

(54.13.1) $\theta(r) = 0$ for $r \geq 2S_0\sqrt{|\epsilon|}$.

(54.13.2) If $r \leq S_0\sqrt{|\epsilon|}$, then

$$|2\epsilon|^{\frac{\pi}{2\alpha}} = r^{\frac{\pi}{\alpha}} \sin\left(\frac{\pi\theta(r)}{\alpha}\right).$$

In particular $\theta(\sqrt{2|\epsilon|}) = \alpha/2$.

(54.13.3)
$$\frac{d\theta}{dr} \leq 0.$$

And we put

(54.14.1)
$$(\gamma_\epsilon^\alpha)' = \left\{re^{\sqrt{-1}\theta(r)} \,\Big|\, r \in [\sqrt{2|\epsilon|}, \infty)\right\} \cup \left\{re^{\sqrt{-1}(\alpha-\theta(r))} \,\Big|\, r \in [\sqrt{2|\epsilon|}, \infty)\right\}.$$

We then define

(54.14.2)
$$(H_\epsilon^\alpha)' = (\gamma_\epsilon^\alpha)' \cdot S_{\mathbb{R}^n}^{n-1} \subset \mathbb{C}^n.$$

Figure 54.4.

54.3. The tangent cones of a pseudo-holomorphic polygon at its corners.

In this subsection, we consider pseudo-holomorphic polygons that appear in the definition of A_∞ category of a symplectic manifold [Fuk93,Fuk02II]. Especially we state a result on the structure of the image of general pseudo-holomorphic polygons near the corner. This then will be used to study some singular perturbation problem in relation to the Floer cohomology of the Lagrangian surgery later in this chapter.

Let J be an almost complex structure compatible with ω on M. The triple (M, ω, J) then defines an almost Kähler structure. Let $\mathfrak{L} = (L_0, L_1, L_2, \cdots, L_k)$ be a $(k+1)$-tuple of compact Lagrangian submanifolds in (M, ω) that intersect pairwise transversely. Let $(\Sigma, \vec{u})$ denote an element in $\mathcal{M}^{b,\text{main}}_{k+1}$ (see Definition 2.20) ($\vec{u} = (u_{01}, \cdots, u_{(k-1)k}, u_{k0})$) and denote by $\overline{u_{(j-1)j} u_{j(j+1)}}$ the segment of $\partial\Sigma$ between $u_{(j-1)j}$ and $u_{j(j+1)}$ for $j = 0, \cdots, k$.

Let $w : \Sigma \to M$ be a map that satisfies the boundary condition

$$w(\overline{u_{(j-1)j} u_{j(j+1)}}) \subset L_j \tag{54.15.1}$$

$$w(u_{j(j+1)}) \in L_j \cap L_{j+1}. \tag{54.15.2}$$

We denote by $\mathcal{M}(\mathfrak{L}, \vec{u}, J)$ the set of J-holomorphic maps that satisfy (54.15). (See §58 for further discussion on this moduli space.)

Let $p_{12} \in L_1 \cap L_2$ and assume that the Kähler angles α_i $(i = 1, \cdots, n)$ between L_1 and L_2 at p_{12} are all the same. We denote the common angle by $\alpha = \alpha_1 = \cdots = \alpha_n$. Using Lemma 54.10, we can always choose a Darboux chart in a neighborhood U of p_{12}, $I : U \to V \subset \mathbb{C}^n$ so that $I(p_{12}) = 0$,

$$I(L_1 \cap U) = \mathbb{R}^n \cap V, \quad I(L_2 \cap U) = e^{\sqrt{-1}\alpha}\, \mathbb{R}^n \cap V = \Lambda, \tag{54.16.1}$$

and $J(p_{12}) = (I^* J_0)(p_{12})$, i.e.,

$$D_{p_{12}} I \circ J = J_0 \circ D_{p_{12}} I \qquad \text{on } T_{p_{12}} M \tag{54.16.2}$$

where

$$D_{p_{12}} I : T_{p_{12}} M \to T_0 \mathbb{C}^n = \mathbb{C}^n$$

is the differential of I at p_{12}.

Let $w : (\Sigma, \vec{u}) \to M$ be an element of $\mathcal{M}(\mathfrak{L}, \vec{u}; J)$ with $w(u_{12}) = p_{12}$. We conformally identify (Σ, u_{12}) with $(\mathbb{H} \cup \{\infty\}, 0)$ and consider the composition $I \circ w$ in a neighborhood of 0 in $\mathbb{H}$. We put $\overline{\alpha} = \alpha/\pi$.

Theorem 54.17. *There exists $m \in \mathbb{Z}_{\geq 0}$, $\delta > 0$ and a vector $a = (a_1, \cdots, a_n) \in \mathbb{R}^n \setminus \{0\}$ such that*

$$|(I \circ w)(z) - z^{m+\overline{\alpha}} a| \leq C |z|^{m+\overline{\alpha}+\delta}$$

in a neighborhood of 0.

Here the branch of $z^{m+\overline{\alpha}}$ is taken as

$$z^{m+\overline{\alpha}} = r^{m+\overline{\alpha}} e^{(m+\overline{\alpha})\theta\sqrt{-1}} \tag{54.18}$$

if $z = re^{\theta\sqrt{-1}}$, $\theta \in [0, \pi]$. (Note $z \in \mathbb{H}$.)

Definition 54.19. In the case where Theorem 54.17 holds, we call $m+1$ the *multiplicity* of w at 0 and call the map

$$z \mapsto z^{m+\overline{\alpha}} a$$

the *tangent cone* of w at 0.

The multiplicity one condition in Theorem Z just means that w is asymptotic to $z \mapsto z^{\overline{\alpha}} a$ at 0.

Theorem 54.17 is not new and can be extracted, for example, from the main result in [RoSa01]. For reader's convenience, we give a simple proof thereof under the following additional assumption which will be satisfied for the main examples we consider in this chapter.

Assumption 54.20. Let $I : U \to V \subset \mathbb{C}^n$ be a Darboux chart satisfying (54.16) at $p_{12} \in L_1 \cap L_2$. We assume in addition that $J = I^* J_0$ on a neighborhood U of p_{12}, i.e.,

$$D_p I \circ J = J_0 \circ D_p I \qquad \text{on } T_p M$$

for *every* $p \in U$.

We remark that, compared to (54.16), Assumption 54.20 is much more restrictive. (For example, it implies that J is integrable in a neighborhood of p_{12}.) We put this additional assumption because the analysis of scaled gluing problems entering in our study of metamorphosis of the moduli space under the Lagrangian surgery is much simpler than otherwise and also because this will be enough for the analysis of our main examples in §56 - 57.

Proof of Theorem 54.17 under Assumption 54.20. Consider the map

$$u(z) = z^{-\overline{\alpha}} (I \circ w)(z),$$

on a neighborhood W of 0 in $\mathbb{H}$. (Here the branch of $z^{\overline{\alpha}}$ is taken as in (54.18).) By (54.16.1), we have

$$u(W \cap \partial \mathbb{H}) \subset \mathbb{R}^n. \tag{54.21}$$

We consider the double

$$\widehat{W} = W \cup \{\overline{z} \mid z \in W\} \subset \mathbb{C}.$$

The real boundary condition (54.21) enables us to apply the reflexion principle and extend u to a smooth holomorphic map

$$\widehat{u} : \widehat{W} \to \mathbb{C}^n.$$

We then obtain the conclusion by taking Taylor expansion of $\widehat{u}$ at 0. $\square$

§55. Theorem Z and its generalizations.

We first make the statement of Theorem Z more precise.

Fix a compatible almost complex structure J on M. Let $\mathfrak{L} = (L_0, L_1, L_2)$ be a triple of Lagrangian submanifolds of a symplectic manifold M such that they are mutually transversal. Let $p_{ij} \in L_i \cap L_j$ and assume that

(55.1) The Kähler angles between L_1 and L_2 at p_{12} are all equal to α.

For given three points $u_{01}, u_{12}, u_{20} \in \partial D^2$, we consider the moduli space $\mathcal{M}(\mathfrak{L}, \vec{u}, J)$ of J-holomorphic maps introduced in §54.3. (Here $\vec{u} = (u_{01}, u_{12}, u_{20})$.)

Denote by $w_{\rm tri} \in \mathcal{M}(\mathfrak{L}, \vec{u}, J)$ a J-holomorphic triangle that satisfies the following :

(55.2.1) The multiplicity of $w_{\rm tri}$ at u_{12} is one. (See Definition 54.19.)
(55.2.2) $w_{\rm tri}$ is Fredholm regular. Namely the linearization of the Cauchy-Riemann equation at $w_{\rm tri}$ is surjective. (See §58 for the Fredholm theory of the moduli space $\mathcal{M}(\mathfrak{L}, \vec{u}, J)$.)
(55.2.3) $w_{\rm tri}$ is isolated in $\mathcal{M}(\mathfrak{L}, \vec{u}, J)$.

We then perform Lagrangian surgery at $p_{12} \in L_1 \cap L_2$ and get $L_{\epsilon_1} = L_1 \#_{\epsilon_1} L_2$ as defined in §54.1, §54.2 and consider the set of J-holomorphic 2-gons

$$w : D^2 \to M$$

with the following properties :

(55.3.1) $w(\overline{u_{01}u_{20}}) \subset L_{\epsilon_1}$, $w(\overline{u_{20}u_{01}}) \subset L_0$.
(55.3.2) $w(u_{01}) = p_{01}$, $w(u_{20}) = p_{20}$.

We denote the set of such w's by $\widetilde{\mathcal{M}}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J)$ and its quotient under the action of $\mathrm{Aut}(D^2, (u_{01}, u_{20})) \cong \mathbb{R}$ by $\mathcal{M}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J)$. And we denote by $\mathcal{M}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J; w_{\rm tri}, \epsilon_2)$ the subset of $\mathcal{M}((L_{\epsilon_1}, L_0), (u_{01}u_{20}), J)$ consisting of the elements represented by $w \in \widetilde{\mathcal{M}}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J)$ satisfying

$$\max_{z \in D^2} \mathrm{dist}(w(z), w_{\rm tri}(z)) \le \epsilon_2. \tag{55.4}$$

Theorem 55.5. *Let J and $w_{\rm tri}$ satisfy* (55.1) *and* (55.2) *respectively. We also suppose Assumption* 54.20. *Then for each sufficiently small ϵ_2 and ϵ_1 with $|\epsilon_1| < \epsilon_2^{100}$ we have the following :*

(55.6.1) *If $\epsilon_1 < 0$, then $\mathcal{M}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J; w_{\rm tri}, \epsilon_2)$ consists of one point which is Fredholm regular.*
(55.6.2) *If $\epsilon_1 > 0$, then $\mathcal{M}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J; w_{\rm tri}, \epsilon_2)$ is diffeomorphic to S^{n-2}. Each element of it is Fredholm regular.*

Figure 55.1.

Theorem 55.5 is the precise form of Theorem Z whose proof will be given in §59 - 62.

For some of our applications, we also need to consider the case where $w_{\rm tri}$ appears as a continuous family i.e., where the condition (55.2.3) fails to satisfy. We can generalize Theorem 55.5 to such a situation as Theorem 55.7 below.

Let K be a compact subset of $\mathcal{M}(\mathfrak{L}, \vec{u}, J)$ and U be its relatively compact open neighborhood. Let $\mathcal{M}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J; K, \epsilon_2)$ be the set of the elements in $\mathcal{M}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J)$ represented by $w \in \widetilde{\mathcal{M}}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J)$ satisfying (55.4) for some $w_{\rm tri} \in K$.

Theorem 55.7. *Let J and $w_{\rm tri}$ satisfy* (55.1) *and* (55.2) *respectively. We also suppose Assumption* 54.20*. We assume in addition that any element $w_{\rm tri}$ of U satisfies* (55.2.1) *and* (55.2.2)*.*

Then, for each sufficiently small ϵ_2 and $|\epsilon_1| < \epsilon_2^{100}$, there exists an open neighborhood $\mathcal{M}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J, K, \epsilon_2)^+$ of $\mathcal{M}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J, K, \epsilon_2)$ and a map

$$\pi : \mathcal{M}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J, K, \epsilon_2)^+ \to U$$

with the following properties :

(55.8.1) *Every element of $\mathcal{M}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J, K, \epsilon_2)^+$ is Fredholm regular.*

(55.8.2) *If $[w] \in \mathcal{M}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J, K, \epsilon_2)^+$ and $\pi([w]) = [w_{\rm tri}]$ then we have*

$$\mathrm{dist}(w(z), w_{\rm tri}(z)) \le C\epsilon_2$$

by re-choosing the representative w in the class $[w]$ if necessary.

(55.8.3) *If $\epsilon_1 < 0$ then the restriction $\pi^{-1}(K) \to K$ of π is a diffeomorphism.*

(55.8.4) *If $\epsilon_1 > 0$ then the restriction $\pi^{-1}(K) \to K$ of π is a fiber bundle whose fiber is diffeomorphic to S^{n-2}.*

The proof is a straightforward generalization of that of Theorem 55.5.

We can generalize Theorems 55.5 and 55.7 in a straightforward way to the case where more than three Lagrangian submanifolds are involved. We do not state this generalization here since we do not use it in this book.

We next discuss the Lagrangian surgery of immersed Lagrangian submanifolds. Let $i : L \to M$ be a Lagrangian immersion. Assume that $p \in M$ is the unique double point of $i(L)$ with $i^{-1}(p) = \{p_1, p_2\} \subset L$ and so $i|_{L \setminus \{p_1,p_2\}}$ is an embedding.

We give an ordering for the tangent spaces of the two branches of L at p. It seems that the following terminology of Polterovich [Pol91I] is useful for the further discussion of Lagrangian surgeries.

Definition 55.9. Let $i : L \to M$ be a Lagrangian immersion and $p \in i(L) \subset M$ be a transversal self-intersetion point. We call an *equipment* at p an ordering of the tangent spaces of the two branches of L at the self-intersection p.

With this terminology, one can say that Lagrangian surgery at a self-intersection point depends on the equipment at p.

Again we assume that the Kähler angles between the two branches at p are all the same α and let I be a Darboux chart on a neighborhood of p such that

$$(55.10) \qquad I(i(L)) = \mathbb{R}^n \cup e^{\alpha\sqrt{-1}}\mathbb{R}^n \text{ and } J(p) = (I^*J_0)(p).$$

Let $\widetilde{\mathcal{M}}((L,i),1,J;p)$ be the set of J-holomorphic maps $w : D^2 \to M$ satisfying

$$w(\partial D^2) \subset i(L), \quad w(1) = p.$$

We denote its quotient by the action of $\mathrm{Aut}(D^2,1)$ by $\mathcal{M}((L,i),1,J;p)$.

Given an equipment of L at p, we perform Lagrangian surgery on L at p and obtain L_ϵ for ϵ with sufficiently small $|\epsilon|$. (The discussion of §54.1, 54.2 can be generalized to the case of self intersection in an obvious way.)

Denote the set of J-holomorphic discs $w : (D^2, \partial D^2) \to (M, L)$ by $\widetilde{\mathcal{M}}(L_\epsilon, J)$ and its quotient by the action of $PSL(2;\mathbb{R}) = \mathrm{Aut}(D^2)$ by $\mathcal{M}(L_\epsilon, J)$.

Let K be a compact subset of $\mathcal{M}((L,i),1,J;p)$ and U be its relatively compact open neighborhood. Define $\mathcal{M}(L_{\epsilon_1}, J; K, \epsilon_2)$ to be the set of elements of $\mathcal{M}(L_{\epsilon_1}, J)$ represented by $w \in \widetilde{\mathcal{M}}(L_{\epsilon_1}, J)$ for which there exists $w_0 \in \mathcal{M}((L,i),1,p,J)$ such that

$$\max_{z \in D^2} \mathrm{dist}(w(z), w_0(z)) \le \epsilon_2.$$

Now the following is the analog to Theorem 55.7 for this case.

Theorem 55.11. *Let $i : L \to M$ be a Lagrangian immersion and let $p \in i(L) \subset M$ the unique double point as above. Suppose $J = I^*J_0$ in a neighborhood of $p \in M$, Condition* (55.10) *and that every element of $U \subset \mathcal{M}((L,i),1,J;p)$ is Fredholm regular and of multiplicity* 1 *at* $1 \in \partial D^2$.

Then for each sufficiently small ϵ_2 *and* ϵ_1 *with* $|\epsilon_1| < \epsilon_2^{100}$*, there exists an open neighborhood* $\mathcal{M}(L_{\epsilon_1}, J, K, \epsilon_2)^+$ *of* $\mathcal{M}(L_{\epsilon_1}, J, K, \epsilon_2)$ *and a map*

$$\pi : \mathcal{M}(L_{\epsilon_1}, J, K, \epsilon_2)^+ \to U$$

with the following properties :

(55.12.1) *Every element of* $\mathcal{M}(L_{\epsilon_1}, J, K, \epsilon_2)^+$ *is Fredholm regular.*
(55.12.2) *If* $[w] \in \mathcal{M}(L_{\epsilon_1}, J, K, \epsilon_2)^+$ *and* $\pi([w]) = [w_0]$ *we have*

$$\mathrm{dist}(w(z), w_0(z)) \le C\epsilon_2$$

by changing the representative w *if necessary.*
(55.12.3) *If* $\epsilon_1 < 0$ *then the restriction* $\pi^{-1}(K) \to K$ *of* π *is a diffeomorphism.*
(55.12.4) *If* $\epsilon_1 > 0$ *then the restriction* $\pi^{-1}(K) \to K$ *of* π *is a fiber bundle whose fiber is diffeomorphic to* S^{n-2}*.*

Figure 55.2.

The proof is entirely similar to that of Theorem 55.7.

We also consider the case of a pair of Lagrangian submanifolds L_1 and L_2 intersecting at two points, say p_1, p_2. In this case, after performing Lagrangian surgery at p_2, we obtain an immersed Lagrangian submanifolds L_ϵ which has a self-intersection p_1. Under the assumption similar to those in Theorems 55.7 and 55.11, the moduli space of 2-gones with boundary on $L_1 \cup L_2$ is related to the moduli space $\mathcal{M}((L_\epsilon, i), 1, J; , p_1)$ above. Since we can treat this case in the same way as above we omit its discussion.

We next discuss some homological property of the moduli chain induced by the family of pseudo-holomorphic 2-gons in the fiber of the fiber bundle that appears in (55.6.2) and (55.8.4).

Consider the moduli space $\widetilde{\mathcal{M}}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J)$. Denote by $\overline{u_{01}u_{20}}$ the (open) arc segment of ∂D^2 containing u_{12} among the two connected components of $\partial D^2 \setminus \{u_{01}, u_{20}\}$. We have the natural action of $\mathrm{Aut}(D^2; \{u_{01}, u_{20}\}) \cong \mathbb{R}$ on the product

$$\widetilde{\mathcal{M}}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J) \times \overline{u_{01}u_{20}};\ (w, u) \mapsto (w \circ g^{-1}, g(u)).$$

We have the canonical evaluation map

$$ev : \frac{\widetilde{\mathcal{M}}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J) \times \overline{u_{01}u_{20}}}{\mathrm{Aut}(D^2; \{u_{01}, u_{20}\})} \to L_{\epsilon_1};\ ev(w, u) = w(u). \tag{55.13}$$

Now we assume $\epsilon_1 > 0$ and use the notation from Theorem 55.7. For given $[w_0] \in K$, we parameterize its fiber $\pi^{-1}([w_0])$ by the sphere S^{n-2}. We denote by $[w_x]$ an element corresponding to $x \in S^{n-2}$. Identifying $\mathrm{Aut}(D^2; \{u_{01}, u_{20}\})$ and $\overline{u_{01}u_{20}}$ with $\mathbb{R}$ respectively, the above evaluation map (55.13) restricted to this fiber, which we denote by

$$ev_{[w_0]} : S^{n-2} \times \mathbb{R} \to L_{\epsilon_1},$$

can be written as

$$ev_{[w_0]}(x, t) = w_x(t).$$

We represent the Lagrangian handle of L_{ϵ_1} by

$$S^{n-1} \times (0, 1) \subset L_{\epsilon_1}.$$

(See §54.1.)

The following theorem can be derived from the proof of Theorem Z.

Theorem 55.14. *Let $\epsilon = 2^{-100}$.*

(1) *There exist an interval $(a, b) \subset \mathbb{R}$ and points $x, y \in S^{n-1}$ such that*

$$\mathrm{dist}(ev_{[w_0]}(S^{n-2} \times \{a\}), (x, 0)) \le \epsilon$$
$$\mathrm{dist}(ev_{[w_0]}(S^{n-2} \times \{b\}), (y, 1)) \le \epsilon$$

where $(x, 0), (y, 1)$ are regarded as boundary points of Lagrangian handle $S^{n-1} \times (0, 1) \subset L_{\epsilon_1}$, and the image

$$ev_{[w_0]}(S^{n-2} \times [a, b])$$

is contained in the Lagrangian handle $S^{n-1} \times (0, 1)$.

(2) *Consider the cycle obtained by filling the holes of the image $ev_{[w_0]}(S^{n-2} \times [a, b]) \subset S^{n-1} \times (0, 1)$ around the points $(x, 0)$ and $(y, 1)$ in $S^{n-1} \times (0, 1)$ respectively and its homology class in $H_{n-1}((S^{n-1} \times (0, 1)), \mathbb{Z}) \cong \mathbb{Z}$. Then this homology class is a generator of $H_{n-1}((S^{n-1} \times (0, 1)), \mathbb{Z})$.*

Figure 55.3.

Finally a few remarks about Assumption 54.20 are in order, which we put on (L_1, L_2) and J in Theorems 55.5, 55.7, 55.11, 55.14. This is a rather restrictive assumption since it requires the pair (L_1, L_2) is locally isomorphic to the standard pair of linear Lagrangian submanifolds in $\mathbb{C}^n$ upto a symplectic and biholomorphic isomorphism. This assumption is indeed superfluous and can be removed as we mentioned before.

On the other hand a standard cobordism argument enable us to derive from Theorems 55.5, 55.7, 55.11, 55.14 a similar conclusion in the *homology level* for general transversal pair (L_1, L_2) and for a general compatible almost complex structure. Namely we have the following Corollary 55.15 in general. We only consider the case of Theorem 55.5 and only give a sketch of the proof of Corollary 55.15 since we do not use this in our analysis of the main examples in this chapter.

Corollary 55.15. *Under the same assumption as Theorem* 55.5 *except Assumption* 54.20*. For each sufficiently small* ϵ_2 *and* $|\epsilon_1| < \epsilon_2^{100}$*, the followings hold :*

(55.16.1) $\mathcal{M}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J; w_{\rm tri}, \epsilon_2)$ *is compact and has an oriented Kuranishi structure without boundary. Let* (U, E, s) *be its Kuranishi neighborhood. (The automorphism group* Γ *is trivial in this case.) There is a compact neighborhood* $\mathfrak{U}$ *of* $\mathcal{M}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J, w_{\rm tri}, \epsilon_2)$ *in* U *such that* $s^{-1}(0)$ *is contained in* $\mathfrak{U}$.

(55.16.2) *If* $\epsilon_1 < 0$*, then we may perturb* s *away from* $U \setminus \mathfrak{U}$ *so that the order of* $s^{-1}(0)$ *counted with sign is* 1.

(55.16.3) *If* $\epsilon_1 > 0$*, then we may perturb* s *away from* $U \setminus \mathfrak{U}$ *so that the following holds :* $s^{-1}(0)$ *is a compact oriented* $n-2$ *dimensional manifold without boundary. If we define*

$$ev : s^{-1}(0) \times \mathbb{R} \to L_{\epsilon_1},$$

in the same way as in the situation of Theorem 55.14*, then :*

(55.16.3.1) *There exist an interval* $(a,b) \in \mathbb{R}$ *and points* $x, y \in S^{n-1}$ *such that*

$$\mathrm{dist}(ev(s^{-1}(0) \times \{a\}), (x,0)) \le \epsilon$$
$$\mathrm{dist}(ev(s^{-1}(0) \times \{b\}), (y,1)) \le \epsilon$$

where $(x,0),(y,1)$ *is regarded as boundary points of Lagrangian handle* $S^{n-1} \times (0,1)$ *and*

$$ev(s^{-1}(0) \times [a,b])$$

is contained in the Lagrangian handle $S^{n-1} \times (0,1)$.
(55.16.3.2) *The homology class of the cycle obtained by respectively filling the holes of* $ev_{[w_0]}(s^{-1}(0) \times [a,b]) \subset S^{n-1} \times (0,1)$ *around* $(x,0)$ *and* $(y,1)$ *in the handle* $(S^{n-1} \times (0,1)$ *is a generator of* $H_{n-1}((S^{n-1} \times (0,1))) \cong \mathbb{Z}$.

Sketch of the proof. We sketch how to deduce (55.16.2) from Theorem 55.5. (55.16.1) is easy to show. Consider a smooth path of almost complex structures connecting the given almost complex structure J and J_δ that satisfies Assumption 54.20, i.e., $J_\delta = I^* J_0$ where $J_0 = J(p)$ is the constant almost complex structure on T_pM.

Theorem 55.5 implies $\mathcal{M}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J_\delta; w_{\mathrm{tri}}, \epsilon_2)$ is Fredholm regular and consists of one point. It is cobordant to $\mathcal{M}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J, w_{\mathrm{tri}}, \epsilon_2)$ as the space with Kuranishi structure. (See §A1.) Therefore (55.16.2) follows.

(55.16.3) can be deduced by a similar cobordism argument using Theorem 55.14. We omit the detail. □

An inspection shows that the conclusions of Corollary 55.15 (and other similar conclusions in the above theorems) are enough for the applications we present in §56-57. This is roughly because we only need to study the virtual fundamental chains in the 'homology level' for these applications.

§56. Affine Lagrangian tori in flat symplectic tori

In this section, we will also use the result on the number of holomorphic polygons in the flat symplectic tori. In [Fuk02III], the first named author formulated some axioms that the numbers of holomorphic polygons should satisfy for the case of flat Lagrangian subspacs $\cong \mathbb{R}^n$ in the complex vector space $\mathbb{C}^n$, and did some calculation of the numbers based on the axioms. However, the proof of the axioms was not given in [Fuk02III] at that time. The properties that were assumed as an axiom in [Fuk02III] are now proved in this book, at least for 2 Lagrangians. We did not provide the details of the proof for the cases where there are more than 2 Lagrangian submanifolds. However it would be a minor modification of the argument presented in this book and so omitted. As far as the cases we are interested in, we will provide a self-contained proof in Proposition 56.3. In this and next sections, we use Maslov type indices associated to pseudo-holomorphic polygons. See §58 for the definition and discussion about them.

We will discuss the 4 and 6 dimensional flat tori separately.

56.1. The case of 4 dimensional flat tori.

We represent the flat 4-torus as $\mathbb{C}^2/(\mathbb{Z}[\sqrt{-1}])^2$ and let $z_i = x_i + \sqrt{-1}\, y_i$, $i = 1, 2$ be its coordinate. Let $A = (a_{ij})$ be a symmetric real-valued 2×2 matrix and put

$$\omega_A = \sum a_{ij} dx_i \wedge dy_j.$$

ω_A is nondegenerate and so becomes a symplectic structure on $\mathbb{C}^2/(\mathbb{Z}[\sqrt{-1}])^2$, if A is invertible. The standard complex structure of $\mathbb{C}^2/(\mathbb{Z}[\sqrt{-1}])^2$ is compatible with ω_A if A is positive definite. Hereafter we assume that A is positive definite. We consider three Lagrangian submanifolds defined by

$$\begin{aligned} L_0 &= \{[z_1, z_2] \in \mathbb{C}^2/(\mathbb{Z}[\sqrt{-1}])^2 \mid y_1 = y_2 = 0\}, \\ L_1 &= \{[z_1, z_2] \in \mathbb{C}^2/(\mathbb{Z}[\sqrt{-1}])^2 \mid x_1 = x_2 = 0\}, \\ L_2 &= \{[z_1, z_2] \in \mathbb{C}^2/(\mathbb{Z}[\sqrt{-1}])^2 \mid x_1 = y_1, x_2 = y_2\}. \end{aligned}$$

These are Lagrangian sub-tori of T^4.

Let $v = (v_1, v_2) \in T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and put

$$L_1(v) = \{(z_1, z_2) \mid x_1 = v_1, x_2 = v_2\}.$$

We assume $(v_1, v_2) \neq (0, 0)$. Then we have the pairwise intersections

$$\begin{aligned} L_1(v) \cap L_2 &= \{(v_1(1 + \sqrt{-1}), v_2(1 + \sqrt{-1}))\}, \\ L_2 \cap L_0 &= \{(0, 0)\}, \\ L_0 \cap L_1(v) &= \{(v_1, v_2)\}. \end{aligned}$$

We denote

$$p_{12}(v) = (v_1(1 + \sqrt{-1}), v_2(1 + \sqrt{-1})), \quad p_{20} = (0, 0), \quad p_{01}(v) = (v_1, v_2).$$

We now perform Lagrangian surgery of $L_1(v)$ and L_2 at $p_{12}(v)$ and obtain a one-parameter family $L_\epsilon = L_\epsilon(v)$.

Figure 56.1

Figure 56.2

Note that $\pi_2(T^4, L_i) = 0$ in our case. In particular the Maslov class defined on this group vanishes.

One can also easily see that the class in $\pi_2(T^4, L_\epsilon)$ associated to the vanishing cycle of L_ϵ with the obvious bounding disc has Maslov index zero. Therefore (using the fact $n = 2$) the virtual dimensions of the moduli spaces of the holomorphic discs bounding L_0 or L_ϵ are all $0 + 2 - 3 = -1 < 0$. It follows from Definition 10.6 (or Theorem C) that all the obstruction classes thereof vanish automatically. Hence the Floer cohomology $HF(L_0, L_\epsilon(v))$ is well defined. We remark that in the present case of the pair (L_0, L_ϵ), it follows that $\mathfrak{m}_k = \overline{\mathfrak{m}}_k$ for L_0 and L_ϵ and $b = 0$ is a bounding cochain for both L_0 and L_ϵ. We will omit $b = 0$ from the notation of Floer cohomology in the discussion followed hereafter.

Since the first Chern class of (T^4, ω_A) and the Maslov classes of L_i are trivial, it follows from the index formula that the virtual dimension of the moduli space

$$\mathcal{M}(L_0, L_\epsilon; p_{20}, p_{01}(v)) = \widetilde{\mathcal{M}}(L_0, L_\epsilon; p_{20}, p_{01}(v))/\mathbb{R}$$

consisting of $w : \mathbb{R} \times [0,1] \to M$ satisfying (55.3) does not vary componentwise for given fixed $p_{01}(v)$, p_{20}. A simple Maslov index calculation shows that this is zero for both $\epsilon > 0$ and $\epsilon < 0$. Similarly the virtual dimension of components of $\mathcal{M}(L_0, L_\epsilon; p_{01}(v), p_{20})$ are -2. Furthermore it is not difficult to check that the only nontrivial matrix element of the boundary operator is $\langle \delta[p_{20}], [p_{01}(v)] \rangle$, which we now compute below.

We first consider the set of maps $w : D^2 \to T^4$ satisfying

$$\begin{cases} w(u_{01}) = p_{01}(v),\, w(u_{12}) = p_{12}(v),\, w(u_{20}) = p_{20}, \\ w(\overline{u_{01}u_{12}}) \subset L_1(v),\, w(\overline{u_{12}u_{20}}) \subset L_2,\, w(\overline{u_{20}u_{01}}) \subset L_0. \end{cases} \tag{56.1.1}$$

Lemma 56.2. *The set of homotopy classes of the maps $w : D^2 \to T^4$ satisfying (56.1.1) has one-one correspondence with $\pi^{-1}(v) \cong \mathbb{Z}^2$. And for each $\widetilde{v} \in \pi^{-1}(v)$, the map w corresponding to $\widetilde{v}$ has the symplectic area*

$$\int w^*\omega_A = \frac{1}{2}\widetilde{v}^t A \widetilde{v}.$$

Proof. Let $\widetilde{v} = (\widetilde{v}_1, \widetilde{v}_2) \in \mathbb{R}^2$ be a lift of v. We put

$$\begin{aligned} \widetilde{L}_0 &= \{(z_1, z_2) \in \mathbb{C}^2 \mid y_1 = y_2 = 0\}, \\ \widetilde{L}_1(\widetilde{v}) &= \{(z_1, z_2) \in \mathbb{C}^2 \mid x_1 = \widetilde{v}_1,\, x_2 = \widetilde{v}_2\}, \\ \widetilde{L}_2 &= \{(z_1, z_2) \in \mathbb{C}^2 \mid x_1 = y_1,\, x_2 = y_2\}. \end{aligned}$$

We denote their pairwise intersections by $\widetilde{p}_{01}(\widetilde{v})$, $\widetilde{p}_{12}(\widetilde{v})$ and $\widetilde{p}_{20}$ respectively. Consider the maps $\widetilde{w} : D^2 \to \mathbb{C}^2$ that satisfy

$$\begin{cases} \widetilde{w}(u_{01}) = \widetilde{p}_{01}(v),\, \widetilde{w}(u_{12}) = \widetilde{p}_{12}(v),\, \widetilde{w}(u_{20}) = \widetilde{p}_{20}, \\ \widetilde{w}(\overline{u_{01}u_{12}}) \subset \widetilde{L}_1(v),\, \widetilde{w}(\overline{u_{12}u_{20}}) \subset \widetilde{L}_2,\, \widetilde{w}(\overline{u_{20}u_{01}}) \subset \widetilde{L}_0. \end{cases} \tag{56.1.2}$$

For each map $w : D^2 \to T^4$ satisfying (56.1.1), there exists a unique $\widetilde{v}$ and a unique lift $\widetilde{w}$ of w satisfying (56.1.2). And the maps satisfying (56.1.2) are homotopic to one another. Therefore the set of homotopy classes of w corresponds one to one to that of the lifts of v. The statement on the area is evident. Hence follows the lemma. □

Now we prove the following result which we will use in the later discussion. A weaker version of this proposition was proved in [Fuk02II,02III] using a cobordism argument that is based on the adiabatic degeneration result from [FuOh97].

Here we give a more direct proof by an explicit construction and in fact also prove the uniqueness result, which will be used in our gluing construction for the surgery L_ϵ.

Proposition 56.3. *For any given $\widetilde{v} \in \pi^{-1}(v)$, there exists a unique element w in the moduli space of $\mathcal{M}(\widetilde{L}_0, \widetilde{L}_1(\widetilde{v}), \widetilde{L}_2; \widetilde{p}_{01}(\widetilde{v}), \widetilde{p}_{12}(\widetilde{v}), \widetilde{p}_{20})$. The element w is of multiplicity one at $\widetilde{p}_{12}$ in the sense of Definition* 54.19 *and is Fredholm regular.*

Proof. Consider the triangle Δ formed by the vertices

$$\widetilde{p}_{20} = (0,0), \quad \widetilde{p}_{01}(\widetilde{v}) = (\widetilde{v}_1, \widetilde{v}_2), \quad \widetilde{p}_{12}(v) = (1+\sqrt{-1})(\widetilde{v}_1, \widetilde{v}_2)$$

in $\mathbb{C}^2$. It follows that this is contained in the complex linear subspace $\mathbb{C}\cdot\widetilde{p}_{12}(\widetilde{v})$. Then Riemann mapping theorem gives a unique, modulo holomorphic re-parametrization, holomorphic map $\widetilde{w}$ from the disc whose image becomes this triangle and has multiplicity one. This proves that $\mathcal{M}(\widetilde{L}_0, \widetilde{L}_1(\widetilde{v}), \widetilde{L}_2; \widetilde{p}_{01}(\widetilde{v}), \widetilde{p}_{12}(\widetilde{v}), \widetilde{p}_{20})$ is nonempty.

Now we prove the uniqueness of such a holomorphic triangle. Let $\widetilde{w}' : D^2 \to \mathbb{C}^2$ be any element in $\mathcal{M}(\widetilde{L}_0, \widetilde{L}_1(\widetilde{v}), \widetilde{L}_2; \widetilde{p}_{01}(\widetilde{v}), \widetilde{p}_{12}(\widetilde{v}), \widetilde{p}_{20})$. Since it lies in the same homotopy class as $\widetilde{w}$ and holomorphic, it has the same area as $\widetilde{w}$. Furthermore both maps are area minimizing in their homotopy class. On the other hand, we consider the complex projection

$$\pi_{(\widetilde{v}_1,\widetilde{v}_2)} : \mathbb{C}^2 \to \mathbb{C}\cdot(\widetilde{v}_1, \widetilde{v}_2)$$

along the plane $\mathbb{C}\cdot(-\widetilde{v}_2, \widetilde{v}_1)$. The composition $\pi_{(\widetilde{v}_1,\widetilde{v}_2)} \circ \widetilde{w}'$ is another holomorphic disc satisfying (56.1.2), i.e., also lies in $\mathcal{M}(\widetilde{L}_0, \widetilde{L}_1(\widetilde{v}), \widetilde{L}_2; \widetilde{p}_{01}(\widetilde{v}), \widetilde{p}_{12}(\widetilde{v}), \widetilde{p}_{20})$ and so have the same area as $\widetilde{w}$.

Because the vector $(\widetilde{v}_1, \widetilde{v}_2)$ is a real vector, the complex direct sum

$$\mathbb{C}^2 = \Big(\mathbb{C}\cdot(\widetilde{v}_1, \widetilde{v}_2)\Big) \oplus \Big(\mathbb{C}\cdot(-\widetilde{v}_2, \widetilde{v}_1)\Big)$$

is also Hermitian orthogonal. Therefore the projection $\pi_{(\widetilde{v}_1,\widetilde{v}_2)}$ is a unitary projection.

Furthermore it is easy to check that all three Lagrangian planes $\widetilde{L}_0$, $\widetilde{L}_1(\widetilde{v})$, $\widetilde{L}_2$ are parallel along the projection $\pi_{(\widetilde{v}_1,\widetilde{v}_2)}$ in that the three direct sums

$$\widetilde{L}_0 = (\widetilde{L}_0 \cap \mathbb{C}\cdot(\widetilde{v}_1, \widetilde{v}_2)) \oplus (\widetilde{L}_0 \cap \mathbb{C}\cdot(-\widetilde{v}_2, \widetilde{v}_1))$$

and so on of $\widetilde{L}_0$, $\widetilde{L}_1(\widetilde{v})$ and $\widetilde{L}_2$ are orthogonal. This implies that the image of $\pi_{(\widetilde{v}_1,\widetilde{v}_2)} \circ \widetilde{w}'$ covers the whole triangle Δ. Writing the image of $\pi_{(\widetilde{v}_1,\widetilde{v}_2)} \circ \widetilde{w}$ as a, possibly multi-valued, graph over the plane $\mathbb{C}\cdot(\widetilde{v}_1, \widetilde{v}_2)$, this also implies the inequality

$$\operatorname{Area}(\widetilde{w}') \ge \operatorname{Area}(\pi_{(\widetilde{v}_1,\widetilde{v}_2)} \circ \widetilde{w}') \ge \operatorname{Area}(\Delta) = \operatorname{Area}(\widetilde{w}). \tag{56.4}$$

It follows from these inequalities that the first and the last areas are the same if and only if the images of $\widetilde{w}$ and of $\widetilde{w}'$ coincide with Δ with the same multiplicity 1. In other words, $\widetilde{w}$ and $\widetilde{w}'$ coincide upto the re-parametrization. We can check the Fredholm regularity by the explicit description given above. This finishes the proof. $\square$

Remark 56.5. The proof of Proposition 56.3 implies that all the holomorphic triangles whose edges lie in L_0, $L_1(v)$ and L_2 respectively are flat. This is rather exceptional and is a consequence of our special choice of L_0, $L_1(v)$, L_2. If we replace them by other more general affine Lagrangian submanifolds, the corresponding holomorphic triangles are not necessarily flat in general.

For a generic choice of v, holomorphic triangles in class $\widetilde{v}$ will intersect p_{12} only at the point $u_{12} \in \partial D^2$. Furthermore, the image of the triangle in the universal covering $\mathbb{C}^2$ is uniformly away from the inverse image of the surgery point $\widetilde{p}$ over all $(1+\sqrt{-1})\widetilde{v} = (k,\ell) + \widetilde{v}_0$ with $|k| + |\ell| \le K$ for given constant $K > 0$ except the unique surgery point $\widetilde{w}(u_{12})$. By Proposition 56.3 we can apply Theorem 55.3.

In each of the cases $\epsilon > 0$ or $\epsilon < 0$, the difference between symplectic area of the pseudo-holomorphic triangle and that of peudoholomorphic disc obtained by Theorem 55.5 in this way is proportional to $|\epsilon|$. Namely we have :

Lemma 56.6. *Let w_ϵ be the unique element near w corresponding to the class $\widetilde{v}$. Then we have*

$$\mathrm{Area}(w_\epsilon) - \frac{1}{2}\widetilde{v}^t A \widetilde{v} = C_{\mathrm{sign}\ \epsilon}(S_0, \alpha)|\epsilon|$$

for any $\widetilde{v} \in \pi^{-1}(v)$. Here α is the Kähler angle between L_1 and L_2.

We omit the proof. Here the number $C_\pm(S_0, \alpha) \in \mathbb{R}$ is the area of the domain described in the Figure 56.3 below.

Figure 56.3

We now consider the cases L_ϵ for $\epsilon > 0$, and $\epsilon < 0$, separately. We first consider the case $\epsilon < 0$.

By choosing ϵ sufficiently small, the error appearing in Lemma 56.6 can be made as small as we want for each given class $\widetilde{v} \in \mathbb{Z}^2 + v$. In fact this error can be made small uniformly over all $\widetilde{v}$ with $|k| + |\ell| \le K$ for any given constant $K > 0$.

Thus combining Theorems 55.5, Proposition 56.3 and Lemma 56.6, we have proved the following theorem.

Theorem 56.7. *Let $\pi : \mathbb{C}^2 \to T^4$ be the projection. For any given $R > 0$, we can choose $\epsilon < 0$ with $|\epsilon|$ sufficiently small so that we have*

$$\langle \delta[p_{20}], [p_{01}(v)] \rangle \equiv \sum_{\widetilde{v}:\pi(\widetilde{v})=v} T^{\frac{1}{2}\widetilde{v}^t A \widetilde{v} - C_-(S_0,\pi/4)|\epsilon|} \mod T^R. \tag{56.8}$$

In particular, the Floer cohomology $HF(L_0, L_\epsilon(v); \Lambda^{\mathbb{Z}}_{nov})$ vanishes for $\epsilon < 0$.

The last statement follows because the right hand side of (56.8) is invertible in the universal Novikov ring $\Lambda^{\mathbb{Z}}_{nov}$ or $\Lambda^{\mathbb{Q}}_{nov}$.

Remark 56.9. We may put $T = e^{-1}$ to (56.8) and also consider the "convergent power series version" of the Floer cohomology. The identity (56.8) then will become a theta-type function $\vartheta_A(v)$. This relation of the theta function to Floer cohomology was discovered by M. Kontsevich in [Kon93] in the case of elliptic curve. It was further studied by Polishchuk-Zaslow [PoZa98], and was partially generalized by the first named author [Fuk02II, Fuk02III] to the higher dimension.

Remark 56.10. Remark 56.9 implies that the "convergent power series version of Floer cohomology" is nonzero if and only if $\vartheta_A(v) = 0$. Note that it is not known in general whether the power series expression of $\langle \delta x, y \rangle$ converges or not after we put $T = e^{-1}$. In our case of flat Lagrangian tori L_0, $L_\epsilon(v)$, the convergence follows from Theorem 56.7. However some of the properties of Floer cohomology, especially its invariance under Hamiltonian isotopies, do *not* hold for the convergent power series version of Floer cohomology. In fact, in our example, one can find a Hamiltonian diffeomorphism $\phi_v : T^4 \to T^4$ for each v, such that $L_\epsilon(v) \cap \phi_v(L_0) = \emptyset$, provided $\epsilon < 0$. Namely $HF(L_\epsilon(v), \phi_v(L_0)) = 0$ but $HF(L_\epsilon(v), L_0) \neq 0$ in case $\vartheta_A(v) = 0$ in the convergent version. (The case $A = I$ is illustrated below.)

Figure 56.4

To obtain something invariant under the Hamiltonian isotopy out of the convergent power series version of Floer cohomology, we need to include the 1-loop effect.

Namely we need to include the order of zero and/or infinity of the Floer cohomology analog of Reidemeister torsion (or more precisely the invariant that Hutchings and Lee [HuLe99] constructed in the finite dimensional situation.)

Remark 56.11. The Floer cohomology $HF(L_0, L_\epsilon(v); \Lambda^{\mathbb{Q}}_{0,nov})$ is a torsion. Namely

$$HF(L_0, L_\epsilon(v); \Lambda^{\mathbb{Q}}_{0,nov}) \cong \frac{\Lambda^{\mathbb{Q}}_{0,nov}}{T^{E_\epsilon}\Lambda^{\mathbb{Q}}_{0,nov}}.$$

Here $E_\epsilon = E_0 - C_-(S_0, \pi/4)|\epsilon|$ with

$$E_0 = \min \left\{ \frac{1}{2}\widetilde{v}^t A \widetilde{v} \;\middle|\; \pi(\widetilde{v}) = v \right\}.$$

Note, we can apply Theorem J in this situation.

Next we consider the case when L_ϵ, $\epsilon > 0$. Since we did not check the orientation of the elements in Theorem 55.5, we work over the $\mathbb{Z}_2$ coefficient. (By the argument of Chapter 8, we can work over $\mathbb{Z}_2$ in case the dimension is 2, which implies that all Lagrangian submanifolds are semi-positive.) Then Theorem 55.5 and Proposition 56.3 imply the following.

Theorem 56.12. *Let $\epsilon > 0$ be sufficiently small. For the Floer complex for (L_0, L_ϵ), we have :*

$$\langle \delta[p_{20}], [p_{01}(v)] \rangle \equiv 0 \mod 2.$$

In particular, we have

$$HF(L_0, L_\epsilon(v); \Lambda^{\mathbb{Z}_2}_{nov}) \cong \Lambda^{\mathbb{Z}_2}_{nov} \oplus \Lambda^{\mathbb{Z}_2}_{nov}$$

Using the invariance of Floer cohomology under the Hamiltonian deformation, we immediately obtain

Corollary 56.13. *For $\epsilon > 0$, $L_\epsilon(v) \cap \phi(L_0) \neq \emptyset$ for any Hamiltonian diffeomorphism ϕ.*

We remark that there exists a Hamiltonian diffeomorphism ϕ such that $L_\epsilon(v) \cap \phi(L_0) = \emptyset$ with $\epsilon < 0$. Therefore it follows from Corollary 56.13 that $L_\epsilon(v)$ and $L_{-\epsilon}(v)$ are not Hamiltonian isotopic. It is an interesting problem to check whether they are Lagrangian isotopic.

We will return to the study of the case of 4 dimensional tori in §57.

56.2. The case of 6 dimensional flat tori

We now consider 6 dimensional torus $T^6 = \mathbb{C}^3/(\mathbb{Z}[\sqrt{-1}])^3$. We take a symmetric and positive definite real-valued 3×3 matrix $A = (a_{ij})$ and define $\omega_A = \sum a_{ij} dx_i \wedge dy_j$. We define L_0, $L_1(v)$ and L_2 in the same way as before, i.e.,

$$\begin{aligned}
L_0 &= \{[z_1, z_2, z_3] \in \mathbb{C}^3/(\mathbb{Z}[\sqrt{-1}])^3 \mid y_1 = y_2 = y_3 = 0\}, \\
L_1(v) &= \{[z_1, z_2, z_3] \in \mathbb{C}^3/(\mathbb{Z}[\sqrt{-1}])^3 \mid x_1 = v_1, x_2 = v_2, x_3 = v_3\}, \\
L_2 &= \{[z_1, z_2, z_3] \in \mathbb{C}^3/(\mathbb{Z}[\sqrt{-1}])^3 \mid x_1 = y_1, x_2 = y_2, x_3 = y_3\}.
\end{aligned}$$

(Here $v = (v_1, v_2, v_3)$.)

We perform Lagrangian surgery at all of the pairwise intersection points of the three,

$$p_{12}(v) \in L_1(v) \cap L_2,\ p_{20} \in L_2 \cap L_0, p_{01}(v) \in L_0 \cap L_1(v).$$

We then obtain a Lagrangian submanifold which we denote by $L = L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$. Here $\epsilon_{12}, \epsilon_{20}, \epsilon_{01}$ are the parameters entering in the Lagrangian surgery at these three points, respectively.

In a way similar to the proof of Lemma 56.2, we observe that the set of homotopy classes of maps from $D^2 \to T^6$ satisfying (56.1.1) one-one corresponds to $\mathbb{Z}^3$. We can also identify this with the set

$$\pi^{-1}(v) = \{\widetilde{v} \in \mathbb{R}^3 | \pi(\widetilde{v}) = v\},$$

where $\pi : \mathbb{R}^3 \to \mathbb{R}^3/\mathbb{Z}^3$ is the projection. Similarly as in the proof of Proposition 56.3, we can show that there exists a unique holomorphic triangle in each homotopy class. We denote this unique holomorphic triangle by $w_{\widetilde{v}} : D^2 \to T^6$ for each $\widetilde{v} \in \pi^{-1}(v)$. We can also prove that $w_{\widetilde{v}}$ is of multiplicity one at p_{01}, p_{12}, p_{20} in the sense of Definition 54.19 and is Fredholm regular. Using these observations and Theorem 55.5, we will study the obstruction classes for the well-definedness of Floer cohomology of $L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$. We denote by

$$\beta_{\widetilde{v}} \in H_2(T^6, L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v); \mathbb{Z})$$

the relative homology class represented by a map $w'_{\widetilde{v}} : (D^2, \partial D^2) \to (T^6, L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v))$ close to the triangle $w_{\widetilde{v}}$.

We first consider the case $\epsilon_{12}, \epsilon_{20}, \epsilon_{01} < 0$. Let $\widetilde{v}_0$ be the element of $\pi^{-1}(v)$ for which

$$\int w_{\widetilde{v}_0}^* \omega = \frac{1}{2} \widetilde{v}_0^t A \widetilde{v}_0 = E_0 = \min \left\{ \frac{1}{2} \widetilde{v}^t A \widetilde{v} \,\middle|\, \pi(\widetilde{v}) = v \right\}.$$

We put $\ell_{\widetilde{v}_0} = w'_{\widetilde{v}_0 *}(\partial D^2) \in H_1(L)$. Using the assumption $\epsilon_{12}, \epsilon_{20}, \epsilon_{01} < 0$, we find that the Maslov index of $\ell_{\widetilde{v}_0}$ is 0 and L is oriented. Since all oriented 3-manifolds are spin, we can use the rational coefficient in our construction. In fact, we can even use the $\mathbb{Z}$-coefficients noting that all 3 dimensional Lagrangian submanifolds are semi-positive.

Now in the same way as the proof of Theorem 56.7, we can prove the zero dimensional moduli space $\mathcal{M}(\beta_{\widetilde{v}})$ consists of one point which contributes as ± 1. Note that $L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$ is spin. Although the spin structure is not uniquely determined (Remark 54.8), the moduli space $\mathcal{M}(\beta_{\widetilde{v}})$ consists of a unique element, hence we can compute its contribution to the obstruction cycle up to sign. Combining all these, we have proved the following theorem

Theorem 56.14. *If $\epsilon_{12}, \epsilon_{20}, \epsilon_{01} < 0$, then the first obstruction o_1 to the well definedness of the Floer cohomology of $L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$ is $\pm PD[\ell_{\widetilde{v}_0}] \in H^2(L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v);\mathbb{Z})$.*

We remark that the cohomology class $PD[\ell_{\widetilde{v}_0}]$ does not lie in the image of $H^2(T^6) \to H^2(L)$. This implies that $\mathcal{M}_{\rm def}(L)$ is empty. Moreover since the obstruction lies in $H^2(L)$, it follows that $\mathcal{M}_{\rm def,weak}(L)$ is also empty.

We remark that Theorem 56.14 provides an example mentioned in (1.16.4) (a).

In the same way, we prove that the algebraic order of $\mathcal{M}(L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v);\beta_{\widetilde{v}})$ is one for any $\widetilde{v} \in \mathbb{R}^3$ if $\epsilon_{12}, \epsilon_{20}, \epsilon_{01} < 0$ are sufficiently small. Unfortunately we do not know a correct way of counting the order of the moduli space $\mathcal{M}(L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}; k\beta_{\widetilde{v}})$ for $k \geq 2$.

The naive definition of the order, which would be obtained just by counting the number of elements of $\mathcal{M}(L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v);\beta_{\widetilde{v}}, J)$ for a regular J, will depend on the choice of J's and others unlike the case of closed holomorphic curves. It turns out that to obtain a really symplectic invariant of the pair (M, L) one has to look at the whole system of numbers that are matrix coefficients of the operations. This definition will depend on how we deform the intersection product to an A_∞ algebra. We recall how we deform the cup product into an A_∞ algebra in §30.2 : We choose diffeomorphisms φ_0, $\varphi_1 : L \to L$ and modify the cup product $x \cap y$ of two cochains x, y to $\varphi_0(x) \cap \varphi_1(y)$.

For example, we consider the case $k = 2$ and a simple pseudo-holomorphic disc $u : (D^2, \partial D^2) \to (M, L)$ with $[u] = \beta$. To study the contribution of the double cover of u, we need to take φ_0, φ_1 so that $\varphi_0(u(\partial D^2))$ is transverse to $\varphi_1(u(\partial D^2))$. Such a choice, however, is not unique up to homotopy : Let φ_0' be another diffeomorphism. We connect this to φ_0 by an isotopy φ_0^t such that

$$\varphi_0 = \varphi_0^0, \quad \varphi_0' = \varphi_0^1.$$

Then we note that $\varphi_0^t(u(\partial D^2)) \cap \varphi_1(u(\partial D^2)) = \emptyset$ for generic t except in a codimension one set of $t \in [0,1]$ such that $\varphi_0^t(u(\partial D^2)) \cap \varphi_1(u(\partial D^2))$ is a one point. Thus if we make two different choices of φ_0 the number of the pseudo-holomorphic discs of homology class 2β could be different.

Of course, the A_∞ algebra $(C(L), \mathfrak{m})$ as a whole is independent of such choices up to homotopy equivalence.

Modulo the problem of counting multiple covered discs mentioned above, we can calculate the filtered L_∞ structure on $H(L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v);\Lambda_{0,nov})$. Especially we can determine the structure up to the order of T^{E_0+c} for some $c>0$, which is small, but independent of ϵ. We would like to point out that *this structure is indeed independent of the choices of perturbations of J or of abstract perturbations* and so defines an invariant of the pair (M,L).

Let S^2_{12}, S^2_{20} and S^2_{01} be the vanishing cycles of $p_{12}(v)$, p_{20} and $p_{01}(v)$ respectively. By definition, the supports of these spheres are contained in the neighborhoods $p_{12}(v)$, p_{20} and $p_{01}(v)$ which are the central cross section of the corresponding Lagrangian handles of the Lagrangian surgery.

Figure 56.5.

We now remark that we have the identity

$$[S^2_{12}] = [S^2_{20}] = [S^2_{01}]$$

in homology. We denote

$$\begin{aligned}\mathfrak{a}_1 &= \text{the Poincaré dual to this common homology class}\\ \mathfrak{a}_2 &= PD[\ell_{\widetilde{v}_0}].\end{aligned}$$

Theorem 56.14 bis. *In the situation of Theorem* 56.14 *we have*

$$\langle \mathfrak{a}_1, \mathfrak{l}_k(\mathfrak{a}_1,\cdots,\mathfrak{a}_1)\rangle \equiv \pm\frac{1}{k!}T^{E_0+o(\epsilon)} \mod T^{E_0+c}$$

all other operations $\mathfrak{l}_k$ vanishes modulo T^{E_0+c}.

Here and hereafter $o(\epsilon)$ are real numbers such that $\lim_{\epsilon\to 0} o(\epsilon) = 0$. In fact, $o(\epsilon) = |\epsilon|\,(2C_-(\pi/4) + C_-(\pi/2))$.

Proof. Theorem 56.14 bis is a consequence of Proposition 37.38 and its proof. This is because we know that the algebraic order of $\mathcal{M}(\beta_{\widetilde{v}_0})$ is ± 1. $\square$

To describe the contributions of other $\widetilde{v}$'s we use the superpotential

$$\Psi : H^1(L;\Lambda^+_{0,nov}) \to \Lambda^+_{0,nov}$$

introduced in §11.4 (11.49). We remark that T^6 is a Calabi-Yau 3 fold and the Maslov class of $L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$ is zero if $\epsilon_{12},\epsilon_{20},\epsilon_{01} < 0$. Therefore we are in the situation of §11.4.

For each $R > 0$, we can choose $\epsilon_{ij} < 0$ so small that, for $b \in H^1(L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v);\Lambda_{0,nov})$, we have

$$\Psi(b) \equiv \pm \sum_{\widetilde{v}:\pi(\widetilde{v})=v} e^{b\cap[\partial\widetilde{v}]} T^{\frac{1}{2}\widetilde{v}^t A\widetilde{v}+o(\epsilon)}$$
$$+ \text{(contribution of multiple covered discs)} \mod T^R.$$

Here and hereafter, $o(\epsilon)$ is a real number depending on ϵ_{ij} such that $\lim_{\epsilon_{ij}\to 0} o(\epsilon) = 0$.

We now consider the case where one of the ϵ_{ij}, say ϵ_{12} is positive. In this case the Maslov index of $\ell_{\widetilde{v}_0}$ is 1. In particular L is not oriented. So we use $\mathbb{Z}_2$ coefficient. Theorems 55.5 and 3 dimensional analogue of Proposition 56.3 in this case implies that the moduli space of holomorphic discs $(D^2,\partial D^2) \to (T^6, L)$ of homology class $\beta_{\widetilde{v}}$ is diffeomorphic to S^1. By Theorem 55.14, the fundamental cycle of the evaluation map : $\mathcal{M}_1(\beta_{\widetilde{v}}) = S^1 \times S^1 \to L$ is homologous to the vanishing cycle, the 2 sphere S^2_{12}, which is supported in a neighborhood of $L_1(v)\cap L_2$. (Recall that $\epsilon_{12} > 0$ is the parameter corresponding to the surgery at $L_1(v)\cap L_2$.) Furthermore, the homology class of the vanishing cycles is independent of the choice of the vertices $\widetilde{v}$ with $\pi(\widetilde{v}) = v$. By a dimension counting, we can prove that all obstruction classes are trivial, possibly except the class $2\beta_{\widetilde{v}}$. For the class $2\beta_{\widetilde{v}}$, it defines a top dimensional class, which, however, cannot be surjective from the above description of the fundamental cycle and hence again gives null contribution. Hence we have :

Theorem 56.15. *We consider the case that $\epsilon_{20},\epsilon_{01} < 0 < \epsilon_{12}$. Then for each $R > 0$, we may choose $|\epsilon_{ij}|$ small such that*

$$\mathfrak{m}_0(1) \equiv \sum_{\widetilde{v}:\pi(\widetilde{v})=v} (T^{\frac{1}{2}\widetilde{v}^t A\widetilde{v}+o(\epsilon)} e^{\frac{1}{2}})\ \mathfrak{a}_1 \mod T^R$$

in $H^1(L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v);\Lambda^{\mathbb{Z}_2}_{0,nov})$.

Next we consider the case when $\epsilon_{01} < 0 < \epsilon_{12},\epsilon_{20}$. Then the Maslov index of $[\ell_{\widetilde{v}}]$ is 2. Hence $L = L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$ is oriented. The moduli space $\mathcal{M}(\beta_{\widetilde{v}})$ of holomorphic discs $(D^2,\partial D^2) \to (T^6, L)$ of the homology class $\beta_{\widetilde{v}}$ is diffeomorphic to $S^1 \times S^1$ by Theorem 55.7. (In fact, Theorem 55.7 only asserts the existence of fiber bundle $S^1 \to \mathcal{M}(\beta_{\widetilde{v}}) \to S^1$. We can show that this bundle is trivial by inspecting its proof. We omit the detail since this point is not necessary for our purpose to prove Theorem 56.17.) The image of the evaluation map $S^1 \times S^1 \times S^1 \to L$ (namely $ev : \mathcal{M}_1(\beta_{\widetilde{v}}) \to L$) can be deformed to a subset of $\ell_{\widetilde{v}} \cup S^2_{12} \cup S^2_{20}$. Hence it is homologous to zero. Namely the corresponding obstruction class in $H_{3-2+\mu_L(\ell_{\widetilde{v}})}(L) \cong H^{2-\mu_L(\ell_{\widetilde{v}})}(L) = H^0(L)$ vanishes. We can also show that the other obstruction classes also vanish in this case.

Theorem 56.16. *If $\epsilon_{01} < 0 < \epsilon_{12}, \epsilon_{20}$ or $0 < \epsilon_{12}, \epsilon_{20}, \epsilon_{01}$, then $L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$ is unobstructed. (We use $\mathbb{Z}$ coefficients in the first case and $\mathbb{Z}_2$ coefficients in the second case.)*

Proof. We already discussed the case $\epsilon_{01} < 0 < \epsilon_{12}, \epsilon_{20}$.

In the case $0 < \epsilon_{12}, \epsilon_{20}, \epsilon_{01}$, the Maslov index of $\ell_{\widetilde{v}}$ is 3. Hence $L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$ is unoriented. So we use $\mathbb{Z}_2$ coefficient. We then find that obstruction is in $H_{3-2+\mu_L(\ell_{\widetilde{v}})}(L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v); \mathbb{Z}_2) = H^{2-\mu_L(\ell_{\widetilde{v}})}(L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v); \mathbb{Z}_2) = H^{-1}(L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v); \mathbb{Z}_2) = 0$ and hence vanishes automatically. □

Theorem 56.16 gives an example where the obstruction vanishes while the moduli space of pseudo-holomorphic discs is nonempty.

Let us continue and calculate partially the operators $\mathfrak{m}_1$, $\mathfrak{m}_2$ and $\mathfrak{m}_3$ in the canonical model. We work over $\mathbb{Z}_2$ coefficient in order to avoid the discussion on sign, which is rather delicate.

Theorem 56.17. *In the situation of Theorem* 56.16 *we have*

$$HF(L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v); \Lambda^{\mathbb{Z}_2}_{0,nov}) \cong H(L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v); \Lambda^{\mathbb{Z}_2}_{0,nov}). \tag{56.18}$$

In case $\epsilon_{01} < 0 < \epsilon_{12}, \epsilon_{20}$, we have

$$\mathfrak{m}_2(\mathfrak{a}_2, \mathfrak{a}_2) \equiv (T^{E_0+o(\epsilon)} e)\, \mathfrak{a}_2 \mod T^{E_0+c}, \tag{56.19.1}$$

In case $0 < \epsilon_{12}, \epsilon_{20}, \epsilon_{01}$, we have

$$\mathfrak{m}_2(\mathfrak{a}_2, \mathfrak{a}_2) \equiv (T^{E_0+o(\epsilon)} e^{3/2})\, \mathfrak{a}_1 \mod T^{E_0+c}, \tag{56.19.2}$$

$$\mathfrak{m}_3(\mathfrak{a}_2, \mathfrak{a}_2, \mathfrak{a}_2) \equiv (T^{E_0+o(\epsilon)} e^{3/2})\, \mathfrak{a}_2 \mod T^{E_0+c}. \tag{56.19.3}$$

Note (56.19.1), (56.19.2), (56.19.3) *are equalities in the group* (56.18).

Remark 56.20. Contrary to the situation of Theorems 37.30, 37.32 we can *not* symmetrize $\mathfrak{m}$ to $\mathfrak{l}$ in order to obtain nontrivial product structure, in the situation of Theorem 56.17. This is not only because we work over $\mathbb{Z}_2$ coefficients but also because the product $\mathfrak{l}_2(\mathfrak{a}_2, \mathfrak{a}_2)$ is automatically zero since $\mathfrak{a}_2$ has an odd degree after the degree shifting.

We remark that Theorems 56.16 and 56.17 show that $L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$ provides an example we mentioned in (1.16.4) (c). We also remark that (56.19.1) - (56.19.3) imply an existence theorem of J-holomorphic discs bordered on $L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$ for any compatible almost complex structure J.

Proof of Theorem 56.17. We first prove (56.18). We prove the case $\epsilon_{01} < 0 < \epsilon_{12}, \epsilon_{20}$ in detail and leave the other case to the interested readers. We recall $\mathcal{M}_1(\beta_{\tilde v}) \cong S^1 \times S^1 \times S^1$. The evaluation map $ev : \mathcal{M}_1(\beta_{\tilde v}) \to L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v) = L$ can be deformed so that its image becomes contained in $\ell_{\tilde v} \cup S^2_{12} \cup S^2_{20}$. More precisely we can deform the image to that of the map $h : S^1 \times S^1 \times S^1 \to L$ described as follows. Take a disjoint union of arcs $[a_0, a_1] \cup [b_0, b_1] \subset S^1$. Here and hereafter we put, for $s, t \in S^1$,

$$[s,t] = \{u \mid s, u, t \text{ respects the cyclic order.}\}$$

Recall $\partial\beta_{\tilde v} = [\ell_{\tilde v}]$ in $H_1(L)$. We decompose the loop $\ell_{\tilde v}$ into $\ell_{\tilde v} = \ell_1 \cup \ell_2 \cup \ell_0$ so that

$$\ell_a \cap \ell_b = \ell_{\tilde v} \cap S^2_{ab} (= \text{single point})$$

for pairs of $(a, b) = (1, 2)$ or $(2, 0)$ or $(0, 1)$. (Note ℓ_i almost lies in L_i.) We require h to satisfy the following properties (see Figure 56.6) :

(56.21.1) If $t \in [a_1, b_0] \cup [b_1, a_0]$ then $h(S^1 \times S^1 \times \{t\})$ is one point. We write the common image point as $h(t)$.
(56.21.2) $\{h(t) \mid t \in [a_1, b_0]\} = \ell_2$. $\{h(t) \mid t \in [b_1, a_0]\} = \ell_0 \cup \ell_1$.
(56.21.3) If $t \in [a_0, a_1]$ and $x \in S^1$ then $h(\{x\} \times S^1 \times \{t\})$ is one point. We write it as $h_1(x, t)$.
(56.21.4) $h_1(x, t) \in S^2_{12}$. Moreover $h_1 : S^1 \times [a_0, a_1] \to S^2_{12}$ is of degree one. (We remark that $h_1(S^1 \times \{a_i\})$ are one points. So the degree makes sense.)
(56.21.5) If $t \in [b_0, b_1]$ and $y \in S^1$ then $h(S^1 \times \{y\} \times \{t\})$ is one point. We write it as $h_2(y, t)$.
(56.21.6) $h_2(y, t) \in S^2_{20}$. Moreover $h_2 : S^1 \times [b_0, b_1] \to S^2_{20}$ is of degree one.

Figure 56.6

Figure 56.7

The fact that the image of $ev : \mathcal{M}_1(\beta_{\widetilde{v}}) \to L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$ can be deformed to that of such a map h follows from Theorem 55.14.

Now we calculate the boundary operator $\mathfrak{m}_{1,\beta_{\widetilde{v}}}$. We will mainly study $\mathfrak{m}_{1,\beta_{\widetilde{v}}}(PD[\ell_{\widetilde{v}}])$. (It is easy to see that other contributions are zero. For example the contribution of multiple covered discs vanishes by the degree reason.) We perturb $\ell_{\widetilde{v}}$ to $\ell'_{\widetilde{v}}$ in the same homology class so that $\ell'_{\widetilde{v}} \cap \ell_{\widetilde{v}} = \emptyset$ and $\ell'_{\widetilde{v}}$ intersects transversely to S^2_{12}, S^2_{20} at $h_1(x_0,t_1)$, $h_2(y_0,t_2)$, respectively. We consider the image of the fiber product chain

$$ev_0 : \mathcal{M}_{1+1}(\beta_{\widetilde{v}}) \;{}_{ev_1}\!\times \ell'_{\widetilde{v}} \to L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v). \tag{56.22}$$

We may identify $\mathcal{M}_{1+1}(\beta_{\widetilde{v}}) = \mathcal{M}_1(\beta_{\widetilde{v}}) \times S^1 = (S^1)^4$. So the image of the map (56.22) consists of two copies of $S^1 \times S^1$. Namely we decompose

$$\begin{aligned}\mathcal{M}_{1+1}(\beta_{\widetilde{v}}) \;{}_{ev_1}\!\times \ell'_{\widetilde{v}} &\cong \{(x_0,y,t_1,s) \mid y \in S^1, s \in S^1\} \cup \{(x,y_0,t_2,s) \mid x \in S^1, s \in S^1\} \\ &\overset{\text{def.}}{=} T_1 \sqcup T_2.\end{aligned}$$

See Figure 56.7. It is easy to see that $ev_0(T_1) \sim S^2_{20}$, $ev_0(T_2) \sim S^2_{12}$. (Here $\sim$ means homologous.) Thus

$$\mathfrak{m}_{1,\beta_{\widetilde{v}}}(PD[\ell_{\widetilde{v}}]) = PD\left(ev_0(\mathcal{M}_{1+1}(\beta_{\widetilde{v}}) \;{}_{ev_1}\!\times \ell'_{\widetilde{v}})\right) = PD[S^2_{12}] + PD[S^2_{20}] = 0. \tag{56.23}$$

This implies that $\mathfrak{m}_1 = 0$ and hence (56.18) holds for $\epsilon_{01} < 0 < \epsilon_{12}, \epsilon_{20}$.

Remark 56.24. To prove a result similar to (56.18) for $\mathbb{Q}$ (or $\mathbb{Z}$) coefficient in case $\epsilon_{01} < 0 < \epsilon_{12}, \epsilon_{20}$, we need to check the sign in the above discussion carefully. Especially we need to find whether (56.23) gives $\pm 2\mathfrak{a}_1$ or 0. We are almost sure that it is 0. We can prove it in case $\widetilde{v} = \widetilde{v}_0$ as follows. We consider

$$\langle \mathfrak{l}_{1,\widetilde{v}_0}(\mathfrak{a}_2), \mathfrak{a}_2 \rangle = \langle \mathfrak{m}_{1,\widetilde{v}_0}(\mathfrak{a}_2), \mathfrak{a}_2 \rangle.$$

Since the proof of Proposition 37.38 works at least for the leading term (namely in case $\widetilde{v} = \widetilde{v}_0$) it follows from cyclic symmetry that

$$\langle \mathfrak{l}_{1,\widetilde{v}_0}(\mathfrak{a}_2), \mathfrak{a}_2\rangle = -\langle \mathfrak{l}_{1,\widetilde{v}_0}(\mathfrak{a}_2), \mathfrak{a}_2\rangle = 0.$$

This implies that the sum (56.23) is 0 over $\mathbb{Q}$ coefficient in case of $\widetilde{v} = \widetilde{v}_0$.

The above argument implies that $\mathfrak{m}_1$ vanishes over $\mathbb{Q}$ if cyclic symmetry

$$\langle \mathfrak{m}_1(x), y\rangle = (-1)^{(\deg x+1)(\deg y+1)}\langle \mathfrak{m}_1(y), x\rangle \tag{56.25}$$

holds. (56.25) follows from the argument of §§37.2, 37.3 as far as the term $\mathfrak{m}_{1,\beta_{\widetilde{v}_0}}$ concerns. To prove it in general we need some more arguments which will appear elsewhere. This is the reason why we state (56.18) over $\mathbb{Z}_2$ coefficients.

We next prove (56.19.1). It is easy to see that because we are interested in finding the lowest order term we only need to study $\mathfrak{m}_{2,\beta_{\widetilde{v}_0}}(\mathfrak{a}_2, \mathfrak{a}_2)$. We use the map h again for this purpose. Hereafter we write $\ell = \ell_{\widetilde{v}_0}$ and $\widetilde{v} = \widetilde{v}_0$. Take perturbations $\ell^{(1)}$, $\ell^{(2)}$ of ℓ so that they becomes disjoint from each other and that $\ell^{(j)}$ intersect transversely with S^2_{12}, S^2_{20} at $h_1(x_0^{(j)}, t_1^{(j)})$, $h_2(y_0^{(j)}, t_2^{(j)})$, respectively. ($t_1^{(j)} \in [a_0, a_1]$, $t_2^{(j)} \in [b_0, b_1]$.) We may assume $x_0^{(1)} \ne x_0^{(2)}$, $y_0^{(1)} \ne y_0^{(2)}$. See Figure 56.8.

We regard $\mathcal{M}^{\rm main}_{2+1}(\beta_{\widetilde{v}})$ as a submanifold of $\mathcal{M}_{2+1}(\beta_{\widetilde{v}})$ which is a resolution of $\mathcal{M}_1(\beta_{\widetilde{v}}) \times S^1 \times S^1 \cong (S^1)^5$ along the diagonal. So in particular $\mathcal{M}^{\rm main}_{2+1}(\beta_{\widetilde{v}}) \subset (S^1)^5$. Then the image of the fiber product chain

$$ev_0 : \mathcal{M}^{\rm main}_{2+1}(\beta_{\widetilde{v}}) \ {}_{(ev_1,ev_2)}\times (\ell^{(1)} \times \ell^{(2)}) \to L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v) \tag{56.26}$$

consists of two copies of arcs in $L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$. Namely we have

$$\begin{aligned}\mathcal{M}^{\rm main}_{2+1}(\beta_{\widetilde{v}}) \ {}_{(ev_1,ev_2)}\times (\ell^{(1)} \times \ell^{(2)}) &\cong \{(x_0^{(1)}, y_0^{(2)}, t_1^{(1)}, t_2^{(2)}, s) \mid s \in [t_2^{(2)}, t_1^{(1)}]\} \\ &\qquad \cup \{(x_0^{(2)}, y_0^{(1)}, t_1^{(2)}, t_2^{(1)}, s) \mid s \in [t_1^{(2)}, t_2^{(1)}]\} \\ &\overset{\rm def.}{=} I_1 \sqcup I_2.\end{aligned}$$

It is easy to see that $ev_0(I_1) = \ell_0 \cup \ell_1$, $ev_0(I_2) = \ell_2$. Thus

$$\begin{aligned}\mathfrak{m}_{2,\beta_{\widetilde{v}_0}}(\mathfrak{a}_2, \mathfrak{a}_2) &= ev_0(\mathcal{M}^{\rm main}_{2+1}(\beta_{\widetilde{v}}) \ {}_{(ev_1,ev_2)}\times (\ell^{(1)} \times \ell^{(2)})) \\ &= PD[\ell_0 \cup \ell_1 \cup \ell_2] = PD[\ell] = \mathfrak{a}_2.\end{aligned}$$

(56.19.1) is proved.

Figure 56.8

We next prove (56.19.2). Namely we consider the case $0 < \epsilon_{12}, \epsilon_{20}, \epsilon_{01}$. In this case $\mathcal{M}_1(\beta_{\tilde v}) \cong S^1 \times S^1 \times S^1 \times S^1$ and the evaluation map $ev : \mathcal{M}_1(\beta_{\tilde v}) \to L$ can be deformed to h with the following properties. (This fact follows from Theorems 55.11 and 55.13.) Let $a_0, a_1, b_0, b_1, c_0, c_1 \in S^1$ which respect counter clockwise cyclic order of S^1. See Figure 56.9.

(56.27.1) If $t \in [a_1, b_0] \cup [b_1, c_0] \cup [c_1, a_0]$ then $h(S^1 \times S^1 \times \{t\})$ is one point. We write it as $h(t)$.

(56.27.2) $\{h(t) \mid t \in [a_1, b_0]\} = \ell_2$. $\{h(t) \mid t \in [b_1, c_0]\} = \ell_0$. $\{h(t) \mid t \in [c_1, a_0]\} = \ell_1$.

(56.27.3) If $t \in [a_0, a_1]$ and $x \in S^1$ then $h(\{x\} \times S^1 \times S^1 \times \{t\})$ is one point. We write it as $h_1(x, t)$. If $t \in [b_0, b_1]$ and $y \in S^1$ then $h(S^1 \times \{y\} \times S^1 \times \{t\})$ is one point. We write it as $h_2(y, t)$. If $t \in [c_0, c_1]$ and $z \in S^1$ then $h(S^1 \times S^1 \times \{z\} \times \{t\})$ is one point. We write it as $h_3(z, t)$.

(56.27.4) h_i $(i = 1, 2, 3)$ define maps $h_1 : S^1 \times [a_0, a_1] \to S^2_{12}$, $h_2 : S^1 \times [b_0, b_1] \to S^2_{20}$, $h_3 : S^1 \times [c_0, c_1] \to S^2_{01}$, of degree one.

Figure 56.9

Figure 56.10

We take perturbations, $\ell^{(1)}$, $\ell^{(2)}$ of ℓ such that

$$\ell \cap \ell^{(1)} = \ell \cap \ell^{(2)} = \ell^{(1)} \cap \ell^{(2)} = \emptyset,$$

and that $\ell^{(j)}$ intersect with S^2_{12}, S^2_{20}, S^2_{01} at one point $(x^{(j)}, t_1^{(j)})$, $(y^{(j)}, t_2^{(j)})$, $(z^{(j)}, t_3^{(j)})$, respectively. We remark that $t_1^{(j)} \in [a_0, a_1]$, $t_2^{(j)} \in [b_0, b_1]$, $t_3^{(j)} \in [c_0, c_1]$.

We now consider the fiber product

$$\mathcal{M}^{\text{main}}_{2+1}(\beta_{\widetilde{v}}) \ {}_{(ev_1, ev_2)}\times (\ell^{(1)} \times \ell^{(2)}). \tag{56.28}$$

It consists of 6 copies of $S^1\times$ interval. Namely

(56.29.1) $\{x^{(1)}\} \times \{y^{(2)}\} \times S^1 \times [t_2^{(2)}, t_1^{(1)}]$,
(56.29.2) $\{x^{(1)}\} \times S^1 \times \{z^{(2)}\} \times [t_3^{(2)}, t_1^{(1)}]$,

(56.29.3) $\{x^{(2)}\} \times \{y^{(1)}\} \times S^1 \times [t_1^{(2)}, t_2^{(1)}]$,
(56.29.4) $\{x^{(2)}\} \times S^1 \times \{z^{(1)}\} \times [t_1^{(2)}, t_3^{(1)}]$,
(56.29.5) $S^1 \times \{y^{(1)}\} \times \{z^{(2)}\} \times [t_3^{(2)}, t_2^{(1)}]$,
(56.29.6) $S^1 \times \{y^{(2)}\} \times \{z^{(1)}\} \times [t_2^{(2)}, t_3^{(1)}]$.

The ev_0 image of (56.29.1) is homologous to S^2_{01} since $[c_0, c_1] \subseteq [t_2^{(2)}, t_1^{(1)}]$. We find that ev_0 image of (56.29.2) is 0 since $[b_0, b_1] \cap [t_3^{(2)}, t_1^{(1)}] = \emptyset$. In a similar way, we find that ev_0 images of (56.29.3), (56.29.4), (56.29.5), (56.29.6) are 0, S^2_{20}, S^2_{12}, 0, respectively.

Therefore

$$\begin{aligned}\mathfrak{m}_{2,\beta_{\widetilde{v}}}(\mathfrak{a}_2, \mathfrak{a}_2) &= ev_{0*}(\mathcal{M}^{\text{main}}_{2+1}(\beta_{\widetilde{v}})\ {}_{(ev_1,ev_2)} \times (\ell^{(1)} \times \ell^{(2)})) \\ &= [S^2_{01}] + [S^2_{12}] + [S^2_{20}] = \mathfrak{a}_1.\end{aligned}$$

(56.19.2) is proved. The proof of (56.19.3) is similar and is left to interested readers. □

Remark 56.30. If we were able to lift (56.19.3) to $\mathbb{Z}$ then the cyclic symmetry would imply

$$\langle \mathfrak{a}_2, \mathfrak{m}_{3,\beta_{\widetilde{v}}}(\mathfrak{a}_2, \mathfrak{a}_2, \mathfrak{a}_2)\rangle = -\langle \mathfrak{a}_2, \mathfrak{m}_{3,\beta_{\widetilde{v}}}(\mathfrak{a}_2, \mathfrak{a}_2, \mathfrak{a}_2)\rangle = 0,$$

and would be inconsistent with (56.19.3). We remark however that $L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$ is not oriented and hence Poincaré duality $\langle\ \rangle$ in the above formula is defined only over $\mathbb{Z}_2$.

We next construct an example mentioned in (1.16.3) (b), that is, an example where obstruction vanishes but $\mathfrak{m}_1 \neq \overline{\mathfrak{m}}_1$. We recall that the cancellation (56.23) occurs because $[S^2_{20}] = [S^2_{12}]$. So we need to modify the Lagrangian torus so that this equality fails to hold.

We put

$$T^{6\prime} = \mathbb{C}^3/((2\mathbb{Z} + 3\sqrt{-1}\mathbb{Z}) \oplus (\mathbb{Z} + \sqrt{-1}\mathbb{Z})^2).$$

We take affine Lagrangian subspaces $\widetilde{L}_0$, $\widetilde{L}_1(\widetilde{v})$, $\widetilde{L}_2$ of $\mathbb{C}^3$ and let L'_0, $L'_1(v)$, L'_2 be the Lagrangian subtorus of $T^{6\prime}$ induced by them.

We note $L'_0 \cap L'_2$ consists of 2 points, which we denote by

$$L'_0 \cap L'_2 = \{p^j_{20} \mid j = 1, 2\}.$$

And $L'_1(v) \cap L'_2$ consists of 3 points, denoted by

$$L'_1(v) \cap L'_2 = \{p^m_{12}(v) \mid m = 1, 2, 3\}.$$

Finally $L'_0 \cap L'_1(v)$ consists of a single point

$$L'_0 \cap L'_1(v) = \{p_{01}(v)\}.$$

Figure 56.11

We perform surgery to $L'_0 \cup L'_1(v) \cup L'_2$ at those 6 points. More precisely, we perform the surgery for $\epsilon > 0$ at the 5 points p^j_{20}, $p^m_{12}(v)$ respectively, and the surgery for $\epsilon < 0$ at $p_{01}(v)$. (In other words, we consider $\epsilon_{01} < 0 < \epsilon_{12}, \epsilon_{20}$.) We denote by $L'_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v) = L'$ the Lagrangian submanifolds obtained in this way. In a way similar to the proof of Theorem 56.16 we can prove that $L'_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$ is unobstructed.

There are $6 = 2 \times 3 \times 1$ choices of the triple $(p^j_{20}, p_{01}(v), p^m_{12}(v))$. By taking a vector v appropriately, among the triangles whose vertices are $p^1_{20}, p_{01}(v), p^1_{12}(v)$ and whose edges are in L'_0, $L'_1(v)$, L'_2, we may assume that there exists a unique choice of $(p^1_{20}, p_{01}(v), p^1_{12}(v))$ and then that of the homotopy class $\widetilde{v}_0$ with the smallest possible area, respectively. Let E_0 be this area of $\beta_{\widetilde{v}_0}$. We put $[\partial\widetilde{v}_0] = \mathfrak{a}'_2$. Let $S^2_{p^1_{20}}$, $S^2_{p^1_{12}(v)}$ be the 2 spheres, i.e. vanishing cycles, in $L'_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$ which lie in a neighborhood of $p^1_{20}, p^1_{12}(v)$ respectively. Now the proof of Theorem 56.17 (especially Formula (56.23)) implies

$$\mathfrak{m}_{1,\widetilde{v}_0}(\mathfrak{a}'_2) = \pm PD([S^2_{p^1_{20}}]) \pm PD([S^2_{p^1_{12}(v)}]). \tag{56.31}$$

The right hand side of (56.31) is nonzero since $[S^2_{p^1_{20}}] \neq \pm[S^2_{p^1_{12}(v)}]$.

We remark that the fundamental cycle $[L'_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)]$ is nonzero in Floer cohomology. This follows from the spectral sequence in Theorem D by using the fact that the union of $ev(\mathcal{M}(\beta_{\widetilde{v}}))$ for various $\widetilde{v}$ does not cover $L'_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$. Thus we have :

Theorem 56.32. *$L'_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$ is unobstructed. Moreover we have :*

$$0 \neq HF(L'_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v); \Lambda^R_{0,nov}) \neq H(L'_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v); \Lambda^R_{0,nov})$$

for any field R.

Thus $L'_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$ provides an example we mentioned in (1.16.4) (b).

We next construct an example mentioned in (1.16.4) (d), that is a pair of Lagrangian submanifolds such that the boundary operator of the Floer cohomology is trivial but the bimodule structure is nontrivial. Consider T^6 and $L_0, L_1(v), L_2$ as before. We choose another $L_1(v')$ by taking different $v' \in T^3$. Let $L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$ be a Lagrangian submanifold obtained by Lagrangian surgery as before. We note $L_0 \cap L_1(v')$ and $L_2 \cap L_1(v')$ consist of a single point respectively. We denote them by

$$x = (v'_1, v'_2, v'_3) \in L_0 \cap L_1(v'),$$
$$y = (v'_1(1+\sqrt{-1}), v'_2(1+\sqrt{-1}), v'_3(1+\sqrt{-1})) \in L_2 \cap L_1(v').$$

Then it follows that x, y give rise to the intersection points of $L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$ and $L_1(v')$. We denote them by the same letter

$$x, y \in L_1(v') \cap L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v).$$

By suitably choosing $\widetilde{v}'$, we may assume that

$$\frac{1}{2}\widetilde{v}' \cdot A\widetilde{v}' = E_1 < E_0 = \frac{1}{2}\widetilde{v} \cdot A\widetilde{v}. \tag{56.33}$$

where E_0 is the minimal area of pseudo-holomorphic triangle whose edges lie on L_0, $L_1(v)$, L_2, which is close to the minimal area of pseudo-holomorphic discs bordered on $L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$, and E_1 is the minimal area of pseudo-holomorphic triangle whose edges lie on L_0, $L_1(v')$, L_2, respectively.

Figure 56.12

By Theorem F we have a filtered A_∞ $\Big(H(L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v); \Lambda^{\mathbb{Q}}_{0,nov}), H(L'_1(\widetilde{v}'); \Lambda^{\mathbb{Q}}_{0,nov})\Big)$ bimodule structure $\mathfrak{n}_{k_1,k_2}$ on $\Lambda^{\mathbb{Q}}_{0,nov}[x] \oplus \Lambda^{\mathbb{Q}}_{0,nov}[y]$. The following theorem provides some partial information on this bimodule structure.

Theorem 56.34. *If* $\epsilon_{20} < 0 < \epsilon_{12}, \epsilon_{01}$*, then*

$$\mathfrak{n}_{0,0}([x]) \equiv \pm(T^{E_1+o(\epsilon)}) \cdot [y] \mod T^{E_1+c}. \tag{56.35}$$

If $\epsilon_{01} < 0 < \epsilon_{12}, \epsilon_{20}$ *then*

$$\mathfrak{n}_{0,0}([x]) = 0, \tag{56.36.1}$$

and

$$\mathfrak{n}_{1,0}(\mathfrak{a}_2, [x]) \equiv \pm(T^{E_1+o(\epsilon)}e) \cdot [y] \mod T^{E_1+c}. \tag{56.36.2}$$

Proof. By Proposition 56.3, the order, counted with sign, of the moduli space $\mathcal{M}(L_0, L_1(v'), L_2; p_{20}, x, y)$ is ± 1. (Note that the orientation of the moduli space of holomorphic discs depends on the spin structure. The spin structure is not uniquely determined, cf. Remark 54.8.) For the case $\epsilon_{20} < 0 < \epsilon_{12}, \epsilon_{01}$, the element of the moduli space $\mathcal{M}(L_0, L_1'(\widetilde{v}'), L_2; p_{20}, x, y)$ gives a unique pseudo-holomorphic 2-gon, which contributes to the boundary operator $\langle \mathfrak{m}_{1,\beta_{\widetilde{v}}}([x]), [y] \rangle$. This implies (56.35).

For the case $\epsilon_{01} < 0 < \epsilon_{12}, \epsilon_{20}$, the index difference between x and y is 2 and hence we have (56.36.1). To show (56.36.2), we remark that each element of $\mathcal{M}(L_0, L_1'(\widetilde{v}'), L_2; p_{20}, x, y)$ defines an S^1 parameterized family of holomorphic 2-gons. Namely $\mathcal{M}(L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v), L_1(v'); x, y; \beta)$ is diffeomorphic to S^1 where β is an appropriate homotopy class.

Consider the moduli space $\mathcal{M}_{1,0}(L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v), L_1(v'); x, y; \beta)$, consisting of the elements of $\mathcal{M}(L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v), L_1(v'); x, y; \beta)$ together with one marked point on the boundary $\mathbb{R} \times \{0\}$. We may assume that it is $S^1 \times \mathbb{R}$, although we do not specify its orientation here (cf. Remark 54.8) and the evaluation map

$$ev : \mathcal{M}_{1,0}(L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v), L_1(v'); \beta) \to L_{\epsilon_{12},\epsilon_{20},\epsilon_{01}}(v)$$

represents the class $\pm[S^2_{20}]$. (Theorem 55.14.) (56.36.2) follows. □

It seems to be interesting to study the collection of the Lagrangian submanifolds obtained by Lagrangian surgeries, starting from the flat Lagrangian tori and carried out at more complicated configurations of intersections in a similar way. (We refer to the last section of [Fuk02III] for some related study.) This is a subject of the future research.

Remark 56.37. In this section we have used suitable chains representing given cohomology classes in our calculations of various operations.

The discussion presented in §30 proves that we can use *any* choice of chains, as long as the transversality condition is satisfied, for the calculation up to the level (n, K) we want to calculate. Then the whole discussion in §30 is designed to verify that the $A_{n,K}$ structure we compute using the chosen particular choice can be extended to an A_∞ structure and the resulting A_∞ structure is independent of such choices up to homotopy equivalence.

In this section we illustrated analysis of contributions of the moduli spaces of the *lowest energy level.* Therefore we can safely work with homology classes rather than cycles because the relevant moduli spaces have no boundary.

We like to compare the machinery developed in §30 with that of usual singular homology theory : This plays a role similar to the way how the general theory of the standard singular homology theory works in the homology theory. Recall, for example, that when we compute an intersection pairing of two homology classes in the usual singular homology theory, we take appropriate representatives satisfying some relevant transversality conditions for the computation. The general algebraic and geometric machinery of the singular homology theory ensures that such calculation is independent of the choice of representatives used. We recall that while actual calculations using the transversal representatives look rather *ad hoc and simple*, this general singular (or any kind of) homology theory needed for justification is rather *heavy* and *not simple* at all. By the same token, we emphasize that our rather ad-hoc looking calculations carried out in this section are completely rigorous which are justified by the general heavy machinery developed in §30.

More specifically speaking, in actual calculation of the A_∞ (or $A_{n,K}$) structures, the general theory established in §30 and others makes it unnecessary to go back to the details of proofs carried out in §30 : To obtain the structure constants of the A_∞ (or $A_{n,K}$) structures, we have only to analyze the moduli spaces of pseudo-holomorphic discs up to the order that we want to know. They can then be calculated by taking appropriate chains that satisfy the relevant transversality conditions, and then taking the fiber products among them and etc.

§57. Wall crossing and monodromy.

In this section we consider the case $n = 2$ and $\omega = \sum dx_i \wedge dy_i$. (In this section we only consider the case when the Kähler angle α between two Lagrangian submanifolds we consider is $\pi/2$.) Then, $\mathbb{C}^2$ has a family of automorphisms realising hyper-Kähler rotation : for each $\theta \in (-\pi, \pi]$, we define a diffeomorphism Rot_θ : $\mathbb{C}^2 \to \mathbb{C}^2$ by

$$\begin{aligned}&\mathrm{Rot}_\theta(x_1 + \sqrt{-1}\,y_1, x_2 + \sqrt{-1}\,y_2)\\ &= (x_1 + \sqrt{-1}\,(y_1\cos\theta + y_2\sin\theta), x_2 + \sqrt{-1}\,(-y_1\sin\theta + y_2\cos\theta)).\end{aligned} \tag{57.1}$$

Although it is not a symplectic diffeomorphism (with respect to standard symplectic structure), a straightforward calculation shows that this rotation preserves the

Lagrangian property of the particular Lagrangian submanifold H_ϵ, defined in (54.4). Namely all of

$$H_{|\epsilon|e^{\sqrt{-1}\theta}} = \mathrm{Rot}_\theta(H_{|\epsilon|})$$

stay Lagrangian and so we have S^1-family of Lagrangian submanifolds each of which is asymptotic to $\mathbb{R}^2 \cup \sqrt{-1}\,\mathbb{R}^2$. We would like to use this S^1-family to define an S^1-family of Lagrangian surgery at each double point of an immersed Lagrangian surface. However this rotation does not preserve the Liouville class. (See below.) In particular, H_ϵ is not *exact* unless ϵ is real. This makes it impossible to interpolate it, by a Lagrangian surface, to $\mathbb{R}^2 \cup \sqrt{-1}\,\mathbb{R}^2$ which is exact. So unlike the case where ϵ is real and so H_ϵ is exact, we *cannot localize* the other surgery for $\theta \neq 0, \pi$, but need to use a more global construction for them.

We first analyze how the Liouville class of H_ϵ changes over $\epsilon = |\epsilon|e^{\sqrt{-1}\theta}$ with $\theta \in (-\pi, \pi]$. When ϵ is positive real (i.e., when $\theta = 0$), we consider the circle $\nu_0 : S^1 \to \mathbb{C}^2 \cong T^*\mathbb{R}^2$ defined by

$$\nu_0(\phi) = df_r(\sqrt{|\epsilon|}(\cos\phi, \sin\phi))$$

For $\epsilon = |\epsilon|e^{\sqrt{-1}\theta}, \theta \in (-\pi, \pi]$, we put

$$\nu_\epsilon = \nu_\theta := \mathrm{Rot}_\theta \circ \nu_0 : S^1 \to H_\epsilon,$$

where $\mathrm{Rot}_\theta : \mathbb{C}^2 \to \mathbb{C}^2$ is the diffeomorphism defined in (57.1). We denote by $[\nu_\epsilon]$ the homotopy classs $H_2(\mathbb{C}^2, H_\epsilon)$ through the natural isomorphism $H_2(\mathbb{C}^2, H_\epsilon) \cong H_1(H_\epsilon) \cong \mathbb{Z}$. This generates $H_2(\mathbb{C}^2, H_\epsilon)$. Then a simple computation shows that

$$\nu_\theta^*(-(y_1dx_1 + y_2dx_2)) = -|\epsilon|\sin\theta d\phi$$

where $-(y_1dx_1 + y_2dx_2)$ is the Liouvile form. It follows from this that

$$\omega[\nu_\epsilon] = -2\pi|\epsilon|\sin\theta \begin{cases} = 0 & \text{for } \theta = 0, \pi, \\ < 0 & \text{for } \theta \in (0, \pi), \\ > 0 & \text{for } \theta \in (-\pi, 0). \end{cases} \tag{57.2}$$

This in particular shows that H_ϵ is not exact, unless ϵ is real, i.e, $\theta = 0, \pi$.

Before proceeding further, we study the moduli space $\mathcal{M}(\mathbb{C}^2, H_\epsilon; [\nu_\epsilon])$, $\epsilon = |\epsilon|e^{\sqrt{-1}\theta}$ for each $\theta \in (-\pi, \pi]$.

Proposition 57.3. *$\mathcal{M}(\mathbb{C}^2, H_\epsilon; [\nu_\epsilon])$ consists of one point if $\theta = -\pi/2$ and is empty otherwise. Similarly $\mathcal{M}(\mathbb{C}^2, H_\epsilon; [-\nu_\epsilon])$ consists of one point if $\theta = \pi/2$ and is empty otherwise.*

Proof. We first note that all H_ϵ are congruent because they are the images of $H_{|\epsilon|}$ under the map Rot_θ which are all isometries. Furthermore, for the case of $H_{|\epsilon|}$, the flat disc with boundary $w_{|\epsilon|} : (D^2, \partial D^2) \to (\mathbb{C}^2, H_{|\epsilon|})$ defined by

$$w_{|\epsilon|}(z) = \sqrt{|\epsilon|}(z, z)$$

is not only a minimal disc (in the Riemannian geometric sense) but also area minimizing among the maps representing the homotopy class $[\partial w_{|\epsilon|}] = [\nu_0] \in \pi_2(\mathbb{C}^2, H_{|\epsilon|}) \cong \pi_1(H_{|\epsilon|})$. In fact if $w : (D^2, \partial D^2) \to (\mathbb{C}^2, H_\epsilon)$ is any map homotopic to $w_{|\epsilon|}$ then

$$\mathrm{Area}(w(D^2)) \geq \int w^*\omega = \int w_{|\epsilon|}^*\omega = \mathrm{Area}(w_{|\epsilon|}(D^2)).$$

Obviously the image of this disc under the rotations which lies in H_ϵ is again area-minimizing in the corresponding class $[\nu_\theta] \in \pi_1(H_\epsilon) \cong \pi_2(\mathbb{C}^2, H_\epsilon)$. Denote this disc by $w_\epsilon : (D^2, \partial D^2) \to (\mathbb{C}^2, H_\epsilon)$. Therefore if there exists a holomorphic map representing the class ν_ϵ, the map w_ϵ must be holomorphic. However it is easy to see that w_ϵ can be holomorphic only if $\theta = -\pi/2$ and is anti-holomorphic only if $\theta = \pi/2$. The proof of Proposition 57.3 follows. □

Now we want to implant the S^1-family of local models H_ϵ into a given compact symplectic manifold at each double point of a Lagrangian immersion.

Let $\Psi : \Sigma \to M$ be a Lagrangian immersion into a symplectic 4-manifold. We assume that Ψ is an embedding on $\Sigma \backslash \{p, q\}$ and has an ordinary double point at $x = \Psi(p) = \Psi(q)$. As in §54, we choose a small neighborhood U of x and take a symplectic diffeomorphism $I : U \to D^4 \subset \mathbb{C}^2$ where D^4 is the unit ball in $\mathbb{C}^2$. I maps the two branches of the immersion to $(\mathbb{R}^2 \cup \sqrt{-1}\,\mathbb{R}^2) \cap D^4$. We put $A = D^4 \backslash \frac{1}{2} D^4$. Here $\frac{1}{2}D^4$ is the four ball of radius $\frac{1}{2}$ centered at the origin. Let $A_r = A \cap \mathbb{R}^2$, $A_i = A \cap \sqrt{-1}\,\mathbb{R}^2$. (Here r and i stand for 'real' and 'imaginary', respectively.) We may identify the neighborhoods U_r, U_i of A_r, A_i in $\mathbb{C}^2$ with neighborhoods of the zero section of the cotangent bundle of A_r, A_i respectively.

We consider the Lagrangian submanifolds H_ϵ in $\mathbb{C}^2$ for $\epsilon = |\epsilon| e^{\sqrt{-1}\theta}$ defined in (54.4) for $\theta \in (-\pi, \pi]$. Since $H_\epsilon \cap A$ converge to $(\mathbb{R}^2 \cup \sqrt{-1}\,\mathbb{R}^2) \cap A$ as $|\epsilon| \to 0$ and so are contained as a Lagrangian graph in the above Darboux neighborhood of $A_r \cup A_i$, there exist closed one forms u_r, u_i on A_r, A_i such that $H_\epsilon \cap A$ can be identified with the graphs of u_r, u_i respectively on the cotangent bundle. In fact, with respect to the polar coordinates (t_r, ϕ_r) of $A_r \cong [\frac{1}{2}, 1] \times S^1 \subset \mathbb{R}^2_{x_1, x_2}$, we have

$$u_r = -|\epsilon| \sin\theta d\phi_r + |\epsilon| \cos\theta \frac{dt_r}{t_r}. \tag{57.4.1}$$

Similarly due to the symmetry of H_ϵ along the diagonal Δ, we also have

$$u_i = -|\epsilon| \sin\theta d\phi_i + |\epsilon| \cos\theta \frac{dt_i}{t_i} \tag{57.4.2}$$

on $A_i \cong [\frac{1}{2}, 1] \times S^1 \subset \mathbb{R}^2_{y_2, y_1}$. Here (t_i, ϕ_i) is the polar coordinates on the (y_2, y_1)-plane.

We denote

$$\Sigma_{\rm out} = \Sigma\backslash\Psi^{-1}\Big(\frac{1}{2}D^4\Big) \cong \Psi(\Sigma)\backslash\frac{1}{2}D^4$$

It is easy to see that we can implant the local model $H_\epsilon \cap D^4$ at the double point to produce the required Lagrangian surgery L_ϵ, if we can find a closed one form u on $\Sigma_{\rm out}$ such that the restrictions of u to A_r, A_i are u_r, u_i respectively. By considering the exact sequence

$$\longrightarrow H^1(\Sigma_{\rm out};\mathbb{R}) \longrightarrow H^1(\partial\Sigma_{\rm out};\mathbb{R}) \overset{\delta}{\longrightarrow} H^2(\Sigma_{\rm out},\partial\Sigma_{\rm out};\mathbb{R}) \to$$

we can find such a closed one form if $\delta([u_r]\oplus[u_i]) = 0$.

Recall that, if $\Sigma_{\rm out}$ is orientable,

$$H^2(\Sigma_{\rm out},\partial\Sigma_{\rm out};\mathbb{R}) \cong (H_0(\Sigma_{\rm out};\mathbb{R}))^*$$

by the Lefschetz duality. Therefore if we assume in addition that Σ is connected, we have $H^2(\Sigma_{\rm out},\partial\Sigma_{\rm out};\mathbb{R}) \cong \mathbb{R}$ and so in this case, $\delta([u_r]\oplus[u_i])$ is characterized by the real number

$$\delta([u_r]\oplus[u_i])[\Sigma_{\rm out}] = ([u_r]\oplus[u_i])(\partial\Sigma_{\rm out}).$$

This number vanishes by (57.4), provided Σ is oriented and the self-intersection point is positive in the sense of §54.1. Therefore $[u_r]\oplus[u_i] \in H^1(A_r\cup A_i;\mathbb{R})$ is always in the image of $H^1(\Sigma;\mathbb{R})$. This proves the following proposition.

Proposition 57.5. *Suppose that Σ is connected and oriented so that the self-intersection point is positive in the sense of* §54.1. *Then for each $\epsilon = |\epsilon|e^{\sqrt{-1}\theta}$ with $|\epsilon|$ sufficiently small there exists a Lagrangian submanifold $\Sigma_\epsilon \subseteq M$ such that :*

(57.6.1) $\Sigma_\epsilon \cap (M\backslash U)$ *will converge to* $\Psi(\Sigma)\cap(M\backslash U)$ *in* C^∞ *topology as* ϵ *goes to zero.*
(57.6.2) $I(\Sigma_\epsilon\cap V) = (\frac{1}{2}D^4\cap H_\epsilon)$ *where* $x\in V\subset\overline{V}\subset U$.

Remark 57.7. (1) If Σ is connected and $\Sigma_{\rm out}$ is non-orientable, $H^2(\Sigma_{\rm out},\partial\Sigma_{\rm out};\mathbb{R})$ vanishes. Hence there is no obstruction to extending u_r, u_i to closed one-forms on $\Sigma_{\rm out}$. If Σ is connected and oriented such that the self-intersection point is negative in the sense of §54.1, we cannot perform the Lagrangian surgery unless ϵ is *real.* (Note that to make the parametrization of $\partial\Sigma_{\rm out}$ consistent with the induced boundary orientation of $\partial\Sigma_{\rm out}$, we should parameterize the two components $\partial\Sigma_{\rm out}\cap A_r$ and $\partial\Sigma_{\rm out}\cap A_i$ in the opposite directions.)

(2) When p belongs to a component different from that of q in Σ, the above argument shows that $[u_r]\oplus[u_i]$ does not lie in the image

$$\mathrm{Im}(H^1(\Sigma,\mathbb{R}) \to H^1(A_r\cup A_i;\mathbb{R})) \subset H^1(A_r\cup A_i;\mathbb{R})$$

unless ϵ is real. Therefore in this case, we cannot find L_ϵ satisfying (57.6) in Proposition 57.5. This is the case, for example, when $M = T^4$, $\Psi(\Sigma) = T^2 \cup T^2$ where the two tori intersect transversely at one point.

(3) By generalizing the above argument to a configuration of several Lagrangian submanifolds, one can derive a sufficient condition for a simultaneous surgery at several intersections points in terms of the data at the intersection points and of the topology of the configuration. Since we do not need such a study in this paper, we do not discuss this point further here.

Example 57.8. Let us consider the case of T^4 studied in §56.1 and consider the symplectic form $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. We take $L_0, L_1(v) \subset T^4$ as in §56.1. We put

$$L'_2 = \{(z_1, z_2) \mid x_1 = y_1, x_2 = -y_2\}.$$

We first perform a surgery of their union $L_0 \cup L_1(v) \cup L'_2$ at two of their intersection points, say p'_{20} and $p'_{12}(v)$. (Here $p'_{20} \in L_0 \cap L'_2$, $p'_{12}(v) \in L_1(v) \cap L'_2$.) We then obtain $L'_{\epsilon_{20},\epsilon_{12}}$ for sufficiently small *real* numbers $\epsilon_{20}, \epsilon_{12}$.

We remark that $L'_{\epsilon_{20},\epsilon_{12}}$ is an immersed Lagrangian submanifold with self intersection at $p_{01}(v)$. (See Figure 57.1 below.) It follows *from the two dimensionality* of the Lagrangian immersions that the surgery $L'_{\epsilon_{20},\epsilon_{12}}$ gives rise to an *oriented* submanifold with either *equipment* at $p_{01}(v)$ in the sense of Definition 55.9. This in particular implies that the self-intersection point $p_{01}(v)$ is positive in the sense of §54.1.

We remark that Kähler angle of our Lagrangian submanifolds at $p_{01}(v)$ is $\pi/2$.

Therefore $\Sigma = L'_{\epsilon_{20},\epsilon_{12}}$ satisfies the assumption of Proposition 57.5 and hence we can construct a Lagrangian submanifold $\Sigma_\epsilon = (L'_{\epsilon_{20},\epsilon_{12}})_\epsilon$ for each *complex* number ϵ (which is sufficiently close to 0).

Figure 57.1

We remark that we can not use $L_2 = \{(z_1, z_2) \mid x_1 = y_1, x_2 = y_2\}$ in place of L'_2 in the above construction. In fact then the self intersection at $p_{01}(v)$ will be negative.

Example 57.9. We also have the following example where the assumption of Proposition 57.5 is satisfied. Let M be an elliptic K3 surface and $\pi : M \to B$ be the associated fibration. Recall that K3 surface is hyper-Kähler and carries a S^2-parameterized family of symplectic structures, which are all Kähler with respect to the given complex structure J on M. We take the one, among those symplectic structures, with respect to which the fibers of π become (special) Lagrangian submanifolds. If M has a type-I singular fiber $F = \pi^{-1}(b_0)$ in the sense of Kodaira classification, F is the image of a Lagrangian immersion of S^2. In this case, the Lagrangian submanifold Σ_ϵ appearing in Proposition 57.5 can be taken as the smooth fiber $\pi^{-1}(b)$ where b is a base point near the critical value b_0.

Now we continue our discussion from §54, §56.1. Consider the Lagrangian submanifold Σ_ϵ given as in Proposition 57.5 and the second Lagrangian submanifold L. More specifically, we consider the following pair : Let $\Sigma_\epsilon = (L'_{\epsilon_{20},\epsilon_{12}})_\epsilon$ be as in Example 57.8. Here we assume $\epsilon_{20} = \epsilon_{12} < 0$ We put $\epsilon' = -\epsilon_{01}$. Take $\widehat{v} \in T^2$ different from but close to v and consider

$$L_2(\widehat{v}) = \{(z_1, z_2) \mid y_1 = x_1 + \widehat{v}_1, y_2 = x_2 + \widehat{v}_2\}$$

and put

$$x = (\widehat{v}_1, \widehat{v}_2), \qquad y = (\widehat{v}_1, \widehat{v}_2) + ((1+\sqrt{-1})(v_1 - \widehat{v}_1), (1+\sqrt{-1})(v_2 - \widehat{v}_2)).$$

Figure 57.2

We assume $\widehat{v}_1 \neq v_1$. We then have $L_2(\widehat{v}) \cap \Sigma_\epsilon = \{x, y\}$ since $L'_2 \cap L_2(\widehat{v}) = \emptyset$. We will now study Floer cohomology

$$HF(\Sigma_\epsilon, L_2(\widehat{v})).$$

We use $\mathbb{Z}_2$ coefficients. For this purpose, we study the moduli space $\mathcal{M}(\Sigma_\epsilon, L_2(\widehat{v}); x, y)$ used to define the boundary operator of the Floer cohomology. We assume ϵ' is sufficiently small compared to $|\epsilon|$.

We put

$$q(v_1, v_2) = v_1^2 - v_2^2$$

and

$$E_0 = \inf \left\{ \frac{1}{2} q(\widetilde{v}) \;\middle|\; \pi(\widetilde{v}) = v, \quad q(\widetilde{v}) > 0 \right\}.$$

We remark that $\frac{1}{2}q(\widetilde{v})$ is the symplectic area of the triangle whose vertices are $\widetilde{p}'_{20}$, $\widetilde{p}'_{01}(\widetilde{v})$, $\widetilde{p}'_{12}(\widetilde{v})$. (Here $\widetilde{p}'_{20}$, $\widetilde{p}'_{01}(\widetilde{v})$, $\widetilde{p}'_{12}(\widetilde{v})$ are appropriate lifts of p'_{20}, $p'_{01}(v)$, $p'_{12}(v)$, respectively.)

By taking $v \in \mathbb{Q}^2$, for example, we may assume that $E_0 > 0$.

We then assume that there exists a lift $\widetilde{\widehat{v}} \in \mathbb{R}^2$ of $\widehat{v}$ such that

$$\frac{1}{2}(\widetilde{v} - \widetilde{\widehat{v}}) \cdot (\widetilde{v} - \widetilde{\widehat{v}}) = E_1 < E_0.$$

We remark that Σ_ϵ and $L_2(\widehat{v})$ are unobstructed with 0 as a bounding chain by the degree reason. In the following theorem, we work with $\mathbb{Z}_2$-coefficients.

Theorem 57.10. *If* $\operatorname{Re}\epsilon < 0$ *then we have*

$$\langle \delta[x], [y] \rangle \equiv T^{E_1 + h_1(\epsilon) + h_2(\epsilon')} \mod T^{E_1 + c}. \tag{57.11}$$

Here $\lim_{\epsilon \to 0} h_1(\epsilon) = 0$. $\lim_{\epsilon' \to 0} h_2(\epsilon') = 0$ *and* $c > 0$ *is independent of* ϵ *and* ϵ'.

If $\operatorname{Re}\epsilon > 0$ *then we have*

$$\langle \delta[x], [y] \rangle \equiv T^{E_1 + h_1(\epsilon) + h_2(\epsilon')} + T^{E_1 + h_1(\epsilon) + 2\pi|\operatorname{Im}\epsilon| + h_2(\epsilon')} \mod T^{E_1 + c}. \tag{57.12}$$

Here $\lim_{\epsilon \to 0} h_1(\epsilon) = 0$. $\lim_{\epsilon' \to 0} h_2(\epsilon') = 0$ *and* $c > 0$ *is independent of* ϵ *and* ϵ'.

Proof. By Proposition 56.3, the moduli space $\mathcal{M}(L_0, L_2(\widehat{v}), L_1(v); p_{01}(v), x, y; \beta)$ consists of a single point w_0 whose symplectic area is E_1. (Here β is the homotopy class of 'small' holomorphic triangle as drawn in Figure 57.3 below.) Moreover w_0 is of multiplicity one at $p_{01}(v)$ and is Fredholm regular.

Figure 57.3

Then we can apply Theorems 55.5 to characterize $\mathcal{M}(\Sigma_\epsilon, L_2(\widehat{v}); x, y)$ when ϵ is real. In this way, we obtain (57.11), (57.12) in that case.

Precisely speaking, we provide an equipment (see Definition 55.9) at the self-intersection point $p_{01}(v)$ of $\Sigma = L'_{\epsilon_{20},\epsilon_{12}}$ and obtain the associated surgery Σ_ϵ. With respect to one of the two possible equipments at $p_{01}(v)$, we will have a unique pseudo-holomorphic 2-gon for the case $\epsilon \in \mathbb{R}_{<0}$ and two distinct holomorphic 2-gons for the case $\epsilon \in \mathbb{R}_{>0}$ in a given Hausdorff neighborhood of w_0. For the latter case $\epsilon \in \mathbb{R}_{>0}$, if we denote the homology class of one of the J-holomorphic discs by, say ξ_θ, then the other's homology class will be $\xi_\theta + [\nu_\epsilon]$. Here $[\nu_\epsilon]$ is as in Proposition 57.3.

If we choose the other equipment at $p_{01}(v)$, the sign of ϵ will be reversed in the above discussion.

We now turn to the case when $\epsilon = |\epsilon| e^{\sqrt{-1}\,\theta}$ is not necessarily real. We may choose $|\epsilon|$ arbitrarily small. Then we only need to study pseudo-holomorphic 2-gons sufficiently close to the pseudo-holomorphic triangle w_0.

We start from $\epsilon \in \mathbb{R}$, i.e, $\theta = 0$ and vary θ towards 2π. A standard cobordism argument proves that the order, counted with sign, of the moduli space $\mathcal{M}(\Sigma_\epsilon, L_2(\widehat{v}); x, y)$ does not change as long as there occurs no bubbling. Observe that the bubbling is possible only when there exists a pseudo-holomorphic disc bordered on Σ_ϵ with its symplectic area $\le E_1$. Using $E_1 < E_0$, it is easy to see that the image of such a disc is necessarily supported in a small neighborhood of $p_{01}(v)$. Proposition 57.3 then implies that such a disc exists if and only if $\theta = \pm\pi/2$.

Therefore if $\operatorname{Re}\epsilon < 0$, there is a unique pseudo-holomorphic 2-gon from which (57.11) follows.

If $\operatorname{Re}\epsilon > 0$, there are two pseudo-holomorphic 2-gons (in a neighborhood of w_0) with one in homology class, say ξ_θ, the other $\xi_\theta + [\nu_\epsilon]$. We then obtain (57.12) from (57.2). □

We immediately obtain the following non-vanishing result (Corollary 57.13) of the Floer cohomology. Recall the decomposition

$$HF(\Sigma_\epsilon, L_2(\widehat{v}); \Lambda_{0,nov}^{\mathbb{Z}_2}) \cong \left(\Lambda_{0,nov}^{\mathbb{Z}_2}\right)^{\oplus a} \oplus \bigoplus_{i=1}^{b} \frac{\Lambda_{0,nov}^{\mathbb{Z}_2}}{T^{\lambda_i}\Lambda_{0,nov}^{\mathbb{Z}_2}}$$

from (28.32). We note that

$$T^E HF(\Sigma_\epsilon, L_2(\widehat{v}); \Lambda_{0,nov}^{\mathbb{Z}_2}) = 0$$

if and only if

$$a = 0 \qquad \text{and} \qquad E \ge \lambda_i \text{ (all } i).$$

Corollary 57.13. *There exists $c > 0$ such that if $|\epsilon|, |\epsilon'|$ are sufficiently small compared with c, then the following are equivalent :*

(1) $T^{E_1+c} HF(\Sigma_\epsilon, L_2(\widehat{v}); \Lambda_{0,nov}^{\mathbb{Z}_2}) \neq 0$

(2) *ϵ is a real number and positive.*

Proof. The part of non-vanishing follows from Theorem 57.10 and the 'only if' part of the second statement follows immediately from Theorem 57.10. On the other hand, the 'if' part of the latter can be proved in the same way as the proof of Theorem 56.7. □

If we continuously trace the homotopy class ξ_θ as θ varies from $-\pi$ to π, the class $\xi_{2\pi}$ will become $\xi_0 + [\nu_0]$ because of the presence of non-trivial monodromy. Note that $\epsilon = |\epsilon| e^{\sqrt{-1}\,\theta} = |\epsilon| e^{\sqrt{-1}\,(\theta+2\pi)}$. From these observations, we find that $\mathcal{M}(L_\epsilon, L_2(\widehat{v}); x, y)$ pictorially looks like Figure 57.4 below.

Figure 57.4

Theorem 57.10 and Figure 57.4 show how a wall crossing phenomenon of Floer cohomology occurs at $\theta = \pm\pi/2$.

We remark that Theorem 57.10 does not apply to the cases $\theta = \pm\pi/2$: This is because the moduli space $\mathcal{M}(\Sigma_\epsilon, L_2(\widehat{v}); x, y)$ is not transversal for the cases, as can be seen from Figure 57.4. For these cases, the moduli space $\mathcal{M}(\Sigma_\epsilon; [\nu_\epsilon])$ has its virtual dimension -1 but is nonempty. To make the relevant moduli spaces transversal and to define Floer cohomology, we need to perturb the moduli space by a suitable perturbation. Depending on the way how we perturb it, the moduli space $\mathcal{M}(\Sigma_\epsilon, L_2(\widehat{v}); x, y; \beta)$ may consist of either two points or of a single point.

Let $\mathfrak{n}_{0,0}^{(1)}$ and $\mathfrak{n}_{0,0}^{(2)}$ be the operations obtained by the two different perturbations, respectively. They will have the expression

$$\mathfrak{n}_{0,0}^{(1)}([x]) = T^{E_1+h_1(\epsilon)+h_2(\epsilon')}[y] \mod T^{E_1+c} \tag{57.14.1}$$

(57.14.2) $\mathfrak{n}^{(2)}_{0,0}([x]) \equiv (T^{E_1+h_1(\epsilon)+h_2(\epsilon')} + T^{E_1+h_1(\epsilon)+2\pi|\operatorname{Im}\epsilon|+h_2(\epsilon')})[y] \mod T^{E_1+c}.$

We can also easily see that

$$\mathfrak{n}^{(1)}_{1,0}(\partial[\nu_\epsilon]\otimes[x]) \equiv T^{E_1+h_1(\epsilon)+h_2(\epsilon')}[y] \mod T^{E_1+c}. \tag{57.14.3}$$

The discrepancy between the two operations is measured by a filtered A_∞ homomorphism

$$\mathfrak{f}_* : (H(\Sigma_\epsilon;\Lambda_{0,nov}),\mathfrak{m}^{(2)}) \to (H(\Sigma_\epsilon;\Lambda_{0,nov}),\mathfrak{m}^{(1)}).$$

Here we observe

$$\mathfrak{f}_0(1) \equiv T^{2\pi|\operatorname{Im}\epsilon|}PD(\partial[\nu_\epsilon]) \mod T^c. \tag{57.15}$$

The isomorphism between the Floer cohomologies stated in Theorem 14.5 is given by the identity

$$\mathfrak{n}^{(1)}_{*,*}(e^{\mathfrak{f}_*(0)},[x],e^0) = \mathfrak{n}^{(2)}_{*,*}(e^0,[x],e^0) \tag{57.16}$$

for the current example. (We remark that $\omega(\nu_\epsilon) = 2\pi\operatorname{Im}\epsilon$ by (57.2).) This formula is consistent with the ones given in (57.14) and (57.15).

We also remark that the conclusion of Theorem 57.10 is based on our specific choice of J. The wall crossing line itself is not well-defined in that it depends on the choices of almost complex structures J and of multi-sections of the Kuranishi structure. However "the homology class" of the wall crossing line will have some invariant meaning. For example, the number of the times at which bifurcations of the moduli space occur will be at least two for any choice of J as we move along from $\theta = 0$ to $\theta = 2\pi$. (A similar phenomenon was exploited in [Fuk02III] for some calculation.)

Here in Theorem 57.10 we restrict ourselves to the case of Lagrangian submanifold of the special type provided in Example 57.8. However the same kind of the bifurcation picture as in Figure 57.4 can be shown to occur for the more general case where the Lagrangian submanifold Σ_ϵ is the one as in Proposition 57.5. Especially it holds for the example of type-I singular fiber in K3 surface. (See Example 57.9.) This case seems to have an important implication in relation to the mirror symmetry as we remark below.

Remark 57.17. In the paper [KoSo04] of the year 2004, Kontsevich-Soibelman discussed a complex structure of the K3 surface $M^\dagger$ appearing as the mirror of another K3 surface M which forms a fibration over $B = S^2$ whose singular fibers are of type I. They stated an axiom which the quantum effect on the complex structure of the K3 surface $M^\dagger$ are supposed to satisfy. This quantum effect conjecturally occurs along the locus consisting of the points $x \in B$ at which the Lagrangian fiber $\pi^{-1}(x)$ admits some nontrivial pseudo-holomorphic discs. (Such a phenomenon had been observed in some physics literature before, and also by the first-named author in [Fuk05I].) One of the axioms, Axiom 1 in §9.2, stated in [KoSo04] reads that at a point x_0 where $\pi^{-1}(x_0)$ is of type I singular fiber, the locus of non-trivial quantum effect consist of two lines flowing out. This axiom exactly coincides with the discussion in this section (Proposition 57.3) and that of §29 of 2000 version of this book ([FOOO00]).

§58. Fredholm theory of pseudo-holomorphic polygons

In this section, we review the Fredholm theory of the moduli space $\mathcal{M}(\mathcal{L}, \vec{u}; J)$ of pseudo-holomorphic polygons $w : D^2 \to M$ satisfying (54.15), where $\mathcal{L}$ is the Lagrangian chain intersecting pairwise transversely without triple intersections. (See the beginning of §54.3.) We have already used this Fredholm theory in the previous sections in order to define an appropriate notion of Fredholm regularity of pseudo-holomorphic polygons, for example. The discussion of this section is not new and has been known among the experts.

We can generalize the story to the case where Lagrangian submanifolds are of clean intersections. Then for the case where $\mathcal{L} = (L, \cdots, L)$ of 'total collapse' the discussion here goes back to the one provided in §29.

We fix a sufficiently small closed neighborhood $U_i \subset D^2$ of $u_{i(i+1)}$ for $i = 0, 1, \cdots, k$ so that they are disjoint and fix a conformal isomorphism

$$\varphi_i : U_i \setminus \{u_{i(i+1)}\} \to (-\infty, 0] \times [0, 1] \quad \text{for } i = 0, \cdots, k-1$$

and

$$\varphi_k : U_k \setminus \{u_{k0}\} \to [0, \infty) \times [0, 1].$$

In other words, we regard the punctures $u_{01}, \cdots, u_{(k-1)k}$ as *incoming ends* and u_{k0} as the *outgoing end*.

We denote by $\tau + \sqrt{-1}t$ the standard complex coordinate of $\mathbb{R} \times [0, 1]$, and fix a metric h on each $\operatorname{Int} D^2$ such that

$$h = \pi(dt^2 + d\tau^2)$$

on U_i near the puncture $u_{i(i+1)}$.

Note in §29 we used a parametrization of the moduli space of marked disc over the moduli space of metric ribbon trees. Based on this parametrization we gave a canonical way to find such a coordinate in a neighborhood of each $u_{i(i+1)}$ together with a metric given as above. When we do not move but fix the point $u_{i(i+1)}$, making such a canonical choice is not an essential matter. In this chapter we are mainly interested in the case of three marked points for which we can fix $u_{i(i+1)}$ after reparametrization.

We fix a Riemannian metric g on M so that L_i are totally geodesic near the the intersections $L_i \cap L_j$ for all $i \neq j$, and $\exp_p : T_pM \to M$ the associated exponential map at $p \in L_i \cap L_j$. Then we can choose a neighborhood V_p of the zero of T_pM such that

$$\exp_p(T_pL_i \cap V_p) \subset L_i, \quad \exp_p(0) = p$$

at each intersection $p \in L_i \cap L_j$. We also fix such neighborhoods V_p for each $p \in L_i \cap L_j$.

Now for given $p_{ij} \in L_i \cap L_j$, we define

$$\alpha_{p_{ij}} = \min\{\alpha \in (0,\pi) \mid \alpha \text{ are Kähler angles between } L_i \text{ and } L_j \text{ at } p_{ij}\}.$$

By the hypothesis that L_i's are pairwise transversal, we have $\alpha_{p_{ij}} > 0$. Now we fix a constant δ so that

$$0 < \delta < \alpha_{p_{ij}}.$$

For the simplicity of notation, we denote

$$\dot{D}^2 = D^2 \setminus \{u_{01}, \cdots, u_{(k-1)k}\}.$$

For a given positive constant $p > 2$, we consider the maps

$$w : \dot{D}^2 \to M$$

satisfying (54.15), (54.16) and

(58.1.1) $\qquad w \in W^{1,p}_{loc}$

(58.1.2)

$$\int e^{\delta|\tau|} \left(\left|(w \circ \varphi_i^{-1})(\tau,t)\right|^p + \left|(\nabla w \circ \varphi_i^{-1})(\tau,t)\right|^p \right) d\tau dt < \infty$$

for each i. Here integration is taken on $(-\infty,0]\times[0,1]$ for $i \neq 0$ and on $[0,+\infty)\times[0,1]$ for $i = 0$.

The condition (58.1.1) implies that w is continuous thanks to the choice $p > 2$. The condition (58.1.2) then implies that the map w converges to a point in $L_i \cap L_{i+1}$ as $u \to u_{i(i+1)}$ because it provides an exponential decay as $u \to u_{i(i+1)}$ in the coordinates $\tau + \sqrt{-1}t$ on U_i. Therefore we can impose the following asymptotic condition

(58.1.3) $\qquad w(u_{i(i+1)}) \in p_{i(i+1)} \in L_i \cap L_{i+1}$ for such maps w.

Now we define the set

$$W^{1,p}_\delta(\dot D^2,\mathcal{L};\vec p),\quad \vec p=\{p_{01},p_{12},\cdots,p_{k0}\}$$

to be the set of all $w:\dot D^2\to M$ satisfying (54.15) and (58.1). Any such map defines a continuous map $w:D^2\to M$ satisfying the condition (54.15) and

$$w(u_{i(i+1)})=p_{i(i+1)}. \tag{58.2}$$

We denote by

$$\pi_2(\mathcal{L};\vec p)$$

the set of homotopy classes of such continuous maps and by B an element from $\pi_2(\mathcal{L};\vec p)$. Finally we define the set

$$W^{1,p}_\delta(\dot D^2,\mathcal{L};\vec p;B)=\{w\in W^{1,p}_\delta(\dot D^2,\mathcal{L};\vec p)\mid [w]=B\}. \tag{58.3}$$

By the exponential decay property of $w\in\mathcal{M}(\mathcal{L};\vec p;B)$ with its decay rate being at least $e^{-\alpha_{p_{ij}}|\tau|}$ at p_{ij} and by the choice of δ satisfying $0<\delta<\alpha_{p_{ij}}$, we have

$$\mathcal{M}(\mathcal{L};\vec p;B)\subset W^{1,p}_\delta(\dot\Sigma,\mathcal{L};\vec p;B).$$

We write

$$C^0(w)=T_wW^{1,p}_\delta(\dot D^2,\mathcal{L};\vec p;B)$$

and

$$C^1(w)=L^p_\delta(\Lambda^{(0,1)}(w^*TM)).$$

Then the formal linearization of the Cauchy-Riemann operator $\overline\partial$ defines a linear Fredholm operator

$$D_w\overline\partial:C^0(w)\to C^1(w)$$

in a standard way.

Remark 58.4. Since L_i and L_j are transversal, the operator

$$D_w\overline\partial:T_wW^{1,p}(\dot D^2,\mathcal{L};\vec p)\to L^p(\Lambda^{(0,1)}(w^*TM))$$

is actually Fredholm. In other words, we do *not* need to use *weighted* Sobolev space for Fredholm theory here. We put weight $e^{\delta|\tau|}$ here in order only for the boundary value $w(u_{i(i+1)})$ to be well defined for w which may not be pseudo-holomorphic.

On the othe hand, when we use the cylindrical metric on the target space M also (as we do in §60 - 62), we do need to use weighted Sobolev space. This is because if we use cylindrical metric on the target space M, the linearization of Cauchy-Riemann equation is degenerate at the end.

§59. Local model of holomorphic discs in $\mathbb{C}^n$ I: construction of local models

59.1. Statement of the result of §59 and §60.

In this section, we consider the pair

$$\mathbb{R}^n, \quad e^{\sqrt{-1}\alpha}\mathbb{R}^n = \Lambda \subset \mathbb{C}^n$$

of Lagrangian subspaces with the common Kähler angle α.

In terms of the labelling

$$L_1 = \mathbb{R}^n, \quad L_2 = e^{\sqrt{-1}\alpha}\mathbb{R}^n = \Lambda$$

we denote the relevant surgery by $L_\epsilon = \mathbb{R}^n \#_\epsilon \Lambda$. We divide our discussion into two different cases : one is the case for $\epsilon > 0$ and the other for $\epsilon < 0$. We assume

$$0 < \alpha < \pi.$$

We will obtain all the proper holomorphic curves bordered on H^α_ϵ with appropriate asymptotic conditions : the solutions will be explicit or *algebraic* for the case

$$\epsilon > 0, \qquad \text{or} \quad \epsilon < 0,\, \alpha = \frac{\pi}{2}.$$

For the remaining cases, $\epsilon < 0$, $\epsilon \neq \frac{\pi}{2}$ the solutions will be more *transcendental* and so not be given explicitly.

In this section we study the case $\epsilon > 0$ or $\alpha = \pi/2$. The other case will be studied in the next section.

Consider a holomorphic map $w : \operatorname{Int}\mathbb{H} \to \mathbb{C}^n$ such that :

(59.1.1) w extends continuously to $\mathbb{H} \to \mathbb{C}^n$.
(59.1.2) $w(\partial\mathbb{H}) \subset H^\alpha_\epsilon$.
(59.1.3) There exist τ_0 and $c, C > 0$ such that

$$e^{-\alpha\tau}\left|w(e^{\pi(\tau+\sqrt{-1}t)}) - (e^{\alpha(\tau-\tau_0+\sqrt{-1}t)}, 0, \cdots, 0)\right| \leq Ce^{-c\tau}$$

for $\tau > 0$.

We remark that Condition (59.1.3) implies that $w(z)$ is asymptotic to

$$E^\alpha = \left\{(z, 0, \cdots, 0) \in \mathbb{C}^n \,\middle|\, z = re^{\sqrt{-1}\theta},\ 0 < \theta < \alpha\right\}.$$

and corresponds to the requirement of multiplicity 1 in Theorem 55.5.

Denote by $\operatorname{Aut}(\mathbb{H})$ (or $\operatorname{Aut}(\mathbb{H}, \{\infty\})$) the group of affine transformations $z \mapsto az + b$, $(a, b \in \mathbb{R},\ a > 0)$. We remark that $\operatorname{Aut}(\mathbb{H})$ is the group of biholomorphic automorphism of $\mathbb{H}$ which fix ∞.

The following is the main result of this section and the next.

Theorem 59.2. *Let $0 < \alpha < \pi$ and consider the holomorphic maps*

$$w = (w_1, \cdots, w_n) : \mathbb{H} \to \mathbb{C}^n$$

satisfying (59.1).

(59.3.1) *If $\epsilon > 0$, such w is unique modulo the action of* $\mathrm{Aut}(\mathbb{H})$. *Moreover $w_2 = \cdots = w_n = 0$.*
(59.3.2) *If $\epsilon < 0$, then the set of such w's, modulo the action of* $\mathrm{Aut}(\mathbb{H})$, *is parameterized by S^{n-2}.*

We remark that the sign of ϵ in (59.3) appears in a different way from Theorem 55.3. See Remark 61.23 about this point.

59.2. The case $\epsilon > 0$.

In this subsection, we consider the case $\epsilon > 0$ and prove (59.3.1).

The proof will be carried out in a series of lemmata. We define the (double) cones as follows :

$$\begin{aligned} WC(\alpha) &= \{z \in \mathbb{C} \mid 0 \leq \arg z \leq \alpha \ \text{ or } \ \pi \leq \arg z \leq \pi + \alpha\} \cup \{0\}, \\ C_+(\alpha) &= \{z \in \mathbb{C} \mid 0 \leq \arg z \leq \alpha\} \cup \{0\}, \\ C_-(\alpha) &= \{z \in \mathbb{C} \mid \pi \leq \arg z \leq \pi + \alpha\} \cup \{0\}. \end{aligned}$$

Note that the projection of H^α_ϵ to each factor is contained in $WC(\alpha)$.

Lemma 59.4. *Let $w = (w_1, \cdots, w_n) : \mathbb{H} \to \mathbb{C}^n$ be a holomorphic map satisfying* (59.1) *for H^α_ϵ. Then the followings hold :*

(59.5.1) *The image of w_1 is contained in $C_+(\alpha)$. Moreover $0 \notin w_1(\mathrm{Int}\,\mathbb{H})$.*
(59.5.2) *For $i = 2, \cdots, n$, if w_i is not constant then the image of w_i is contained in either $C_+(\alpha)$ or $C_-(\alpha)$. Moreover $0 \notin w_i(\mathrm{Int}\,\mathbb{H})$ if w_i is not constant.*
(59.5.3) $w^{-1}(0, \cdots, 0) = \emptyset$.

Proof. Denote by $\pi_i : \mathbb{C}^n \to \mathbb{C}$ the coordinate projection to the i-th coordinate plane for $i = 1, \cdots, n$.

Recall that H^α_ϵ is defined by $H^\alpha_\epsilon = \gamma^\alpha_\epsilon \cdot S^{n-1}$ (see (54.12.3)) and hence it follows that the image $\pi_i(H^\alpha_\epsilon)$ is contained in the double cone $WC(\alpha)$. By the boundary condition (59.1.2), we have $w_i(\partial\mathbb{H}) \subset WC(\alpha)$. And (59.1.3) implies that

(59.6.1) $$w_1(\partial\mathbb{H}) \cap C_+(\alpha) \neq \emptyset.$$

We will now prove

(59.6.2) $$w_i(\mathbb{H}) \subset WC(\alpha) \quad \text{for all } i.$$

Suppose to the contrary that (59.6.2) does not hold for some i. Then it follows from (59.1.3) that there exists an interior point $z \in \mathbb{H}$ such that $w_i(z) \notin WC(\alpha)$ and that $w_i(z)$ is in the boundary of $w_i(\mathbb{H})$. Since z is an interior point of $\mathbb{H}$ this contradicts to the maximum principle and hence follows (59.6.2).

Now if w_i is non-constant and $0 \in w_i(\mathrm{Int}\,\mathbb{H})$, $w_i(\mathrm{Int}\,\mathbb{H})$ must contain a neighborhood of 0 which will contradict to (59.6.2). This and (59.6.2) imply (59.5.2). (59.5.1) can be proved similarly using (59.6.1), and (59.6.2).

Finally for the proof of (59.5.3), we note that (59.5.1) implies that $w(u)$ can be $(0, \cdots, 0)$ only at a point $u \in \partial\mathbb{H}$. However this is impossible by the boundary condition (59.1.2) which is $w(u) \in H_\epsilon^\alpha$ for $u \in \partial\mathbb{H}$. This finishes the proof. □

Motivated by (59.5.3), we consider the holomorphic map

$$g : \mathbb{H} \to \mathbb{C}P^{n-1}$$

defined by

$$g(u) = [w_1(u) : \cdots : w_n(u)].$$

Lemma 59.8. *The boundary condition* (59.1.2) *implies*

$$g(\partial\mathbb{H}) \subset \mathbb{R}P^{n-1}. \tag{59.9}$$

Proof. Let $(z_1, \cdots, z_n) = (x_1 + \sqrt{-1}y_1, \cdots, x_n + \sqrt{-1}y_n) \in H_\epsilon^\alpha$. By the definition of H_ϵ^α (see (54.12)), we have

$$z_i = re^{\sqrt{-1}\theta}a_i,$$

where

$$(2\epsilon)^{\frac{\pi}{2\alpha}} = r^{\frac{\pi}{\alpha}} \sin\left(\frac{\pi\theta}{\alpha}\right), \quad \sum a_i^2 = 1.$$

Then it follows that we have

$$[z_1 : \cdots : z_n] = [re^{\sqrt{-1}\theta}a_1 : \cdots : re^{\sqrt{-1}\theta}a_n] = [a_1 : \cdots : a_n] \in \mathbb{R}P^{n-1}$$

which proves (59.9). □

The condition (59.1.3) enables us to extend g to a continuous map $g : \mathbb{H} \cup \{\infty\} \to \mathbb{C}P^{n-1}$ by setting

$$g(\infty) = [1 : 0 \cdots : 0]. \tag{59.10}$$

We now prove the following :

Lemma 59.11. *In the above situation, g must be the constant map having the value $[1 : \cdots : 0] \in \mathbb{C}P^{n-1}$. In particular, we have*

$$w_2 = \cdots = w_n \equiv 0. \tag{59.12}$$

Proof. Using Lemma 59.8, we apply the reflection principle to $g : \mathbb{H} \to \mathbb{C}P^{n-1}$ and obtain a real holomorphic map $\widehat{g} : \mathbb{C}P^1 \to \mathbb{C}P^{n-1}$ as follows :

$$\widehat{g}(u) = \begin{cases} g(u) & \text{if } u \in \mathbb{H} \\ \overline{g(\overline{u})} & \text{if } u \in \mathbb{C} \setminus \mathbb{H}. \end{cases}$$

To finish the proof, it is enough to prove that $\widehat{g}$ is a constant map $\widehat{g} \equiv [1 : 0 : \cdots : 0]$.

Suppose to the contrary that $\widehat{g}$ is non-constant and so has non-zero degree. Then the image of $\widehat{g}$ must intersect the hyperplane $\{[z_1 : \cdots : z_n] \in \mathbb{C}P^{n-1} \mid z_1 = 0\}$ at a finite number of points. Considering $A \cdot w$ for a matrix contained the isotropy group $SO(n-1) \subset SO(n)$ of $(1, 0, \cdots, 0)$ if necessary, we may assume that $w_2(u) \neq 0$ whenever $w_1(u) = 0$.

We then can define $g_2 : \mathbb{H} \to \mathbb{C}P^1$ by $g_2(u) = [w_1(u) : w_2(u)]$. Then g_2 continuously extends to $\mathbb{H} \cup \{\infty\}$ in the same way as above. We denote its double by $\widehat{g}_2 : \mathbb{C}P^1 \to \mathbb{C}P^1$. It follows from (59.5.2) that either $w_2(\mathbb{H}) \subset C_+(\alpha)$ or $w_2(\mathbb{H}) \subset C_-(\alpha)$.

We first consider the case $w_2(\mathbb{H}) \subset C_+(\alpha)$. We obtain

$$-\alpha \leq \arg\left(\frac{w_2}{w_1}\right) \leq \alpha \quad \text{and} \quad -\alpha \leq \arg\left(\frac{\overline{w}_2}{\overline{w}_1}\right) \leq \alpha. \tag{59.13.1}$$

This implies that the image of $\widehat{g}_2$ does not contain $[1 : -1]$. This is a contradiction.

In the case $w_2(\mathbb{H}) \subset C_-(\alpha)$ we obtain

$$\pi - \alpha \leq \arg\left(\frac{w_2}{w_1}\right) \leq \pi + \alpha \quad \text{and} \quad \pi - \alpha \leq \arg\left(\frac{\overline{w}_2}{\overline{w}_1}\right) \leq \pi + \alpha. \tag{59.13.2}$$

This implies that the image of $\widehat{g}_2$ does not contain $[1 : 1]$. This is a contradiction.

Therefore $\widehat{g}$ must be the constant map $\widehat{g} \equiv [1 : 0 : \cdots : 0]$. Hence we derive $w_2 = w_3 = \cdots = w_n = 0$ which proves the lemma ☐

Now Theorem 59.2 for $\epsilon > 0$ immediately follows from Lemma 59.11. ☐

59.3. The case $\epsilon < 0$, $\alpha = \pi/2$.

In this subsection, we consider the case $\epsilon < 0$, $\alpha = \pi/2$ and prove (59.3.2) in case $\alpha = \pi/2$. We recall in case $\epsilon < 0$, our Lagrangian submanifold $H_\epsilon^{\pi/2}$

$$H_\epsilon^{\pi/2} = \gamma_\epsilon \cdot S_{\mathbb{R}^n}^{n-1}$$

where the curve $\gamma_\epsilon \subset \mathbb{C}$ is defined by

$$\gamma_\epsilon = \{re^{\sqrt{-1}\theta} \mid 2\epsilon = r^2 \sin 2\theta,\ \frac{\pi}{2} \leq \theta \leq \pi\}.$$

This case is much more subtle to deal with than the previous case of $\epsilon > 0$.

It turns out that an explicit description of the solution of Cauchy-Riemann equation with boundary condition (59.1) for $\epsilon < 0$, does not seem to be easy, except the case of $\alpha = \frac{\pi}{2}$. In this subsection, we provide an explicit description of the case with $\alpha = \frac{\pi}{2}$. In the next section, we will study the case $\alpha \neq \frac{\pi}{2}$ in a more indirect and transcendental way.

Proposition 59.14. *Let $\epsilon < 0$ and $\alpha = \frac{\pi}{2}$. Consider the holomorphic maps $w = (w_1, \cdots, w_n) : \mathbb{H} \to \mathbb{C}^n$ satisfying* (59.1). *Then the set of such w's, modulo the action of* $\operatorname{Aut}(\mathbb{H})$, *is parameterized by S^{n-2}.*

Proof. We define

$$f(u) = \sum_{i=1}^{n} w_i(u)^2 \tag{59.15}$$

In case $\alpha = \pi/2$, the condition $z = re^{\theta\sqrt{-1}} \in H_\epsilon^\alpha$ is equivalent to $r^2 \sin 2\theta = 2\epsilon$. It follows that $\operatorname{Im} z^2 = 2\epsilon$. Therefore the boundary condition of w implies that

$$f(\partial\mathbb{H}) \subset \mathbb{R} + 2\epsilon\sqrt{-1}. \tag{59.16}$$

Lemma 59.17. *f is a biholomorphic map between $\mathbb{H}$ and*

$$\mathbb{H} + 2\epsilon\sqrt{-1} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 2\epsilon\}.$$

Namely, by composing with an element of $\operatorname{Aut}(\mathbb{H})$, *we may assume :*

$$f(u) = u + 2\epsilon\sqrt{-1}.$$

Proof. We define $\widehat{f} : \mathbb{C}P^1 \to \mathbb{C}P^1$ by

$$\widehat{f}(z) = \begin{cases} f(z) & \text{if } z \in \mathbb{H} \\ \overline{f(\overline{z}) - 2\epsilon\sqrt{-1}} + 2\epsilon\sqrt{-1} & \text{if } z \in \mathbb{C} \setminus \mathbb{H}. \end{cases}$$

(59.16) implies that $\widehat{f}$ is continuous and holomorphic. (59.1.3) implies that

$$\widehat{f}^{-1}(\{\infty\}) = \{\infty\}$$

and the multiplicity is 1. Therefore $\widehat{f}$ is biholomorphic. (59.1.3) implies

$$\lim_{\mathrm{Im}\, u\to\infty} \mathrm{Im}\, f(u) = \infty.$$

Hence the lemma. □

Lemma 59.17 implies that f has a unique zero $u_0 = -2\epsilon\sqrt{-1}$ and $f'(u_0) \neq 0$. In particular we have $w(u) \neq (0,\cdots,0)$ unless $u \neq u_0$. On the other hand, since $f'(u_0) \neq 0$, it also follows that $w(u_0) \neq (0,\cdots,0)$. Therefore $w^{-1}(0,\cdots,0) = \emptyset$ and we can define the projectivization of w denoted by

$$g(u) = [w_1(u) : \cdots : w_n(u)] \in \mathbb{C}P^{n-1}. \tag{59.18}$$

We can smoothly extend g to $\widehat{g} : \mathbb{C}P^1 \to \mathbb{C}P^{n-1}$ by putting

$$\widehat{g}(\overline{u}) = \overline{g(u)}.$$

for $u \in \mathbb{C}P^1 \setminus \mathbb{H}$.

Lemma 59.19. *$\widehat{g} : \mathbb{C}P^1 \to \mathbb{C}P^{n-1}$ is of degree one.*

Proof. We put

$$X = \left\{[z_1 : \cdots : z_n] \in \mathbb{C}P^{n-1} \,\Big|\, \sum z_i^2 = 0\right\}. \tag{59.20}$$

X is a hypersurface of degree 2. Since u_0 is a unique zero of f on $\mathbb{H}$ and $f'(u_0) \neq 0$ it follows that $\widehat{g}(\mathbb{H})$ intersects with X at one (interior) point transversally. Since $\widehat{g}$ on $\mathbb{H}$ is obtained by reflection and X is invariant of reflection (complex conjugation), it follows that the intersection number between $\widehat{g}(\mathbb{C}P^1)$ and X is 2. Therefore $\widehat{g}$ is of degree one as required. □

Now we are in the position to prove Proposition 59.14. Note that $u_0 = -2\epsilon\sqrt{-1}$ is the unique point with $f(u_0) = 0$.

Lemma 59.21. *Let $\epsilon < 0$, $\alpha = \pi/2$, $f(u) = u + 2\epsilon\sqrt{-1}$, $u_0 = -2\epsilon\sqrt{-1}$. Let X be as in* (59.20)*.*

Then there exists a one-one correspondence via the equation (55.18) *of w and g between the following two sets of w and g :*

(59.22.1) $w = (w_1,\cdots,w_n) : \mathbb{H} \to \mathbb{C}^n$ *satisfies* (59.1) *and* (59.15).
(59.22.2) *The double $\widehat{g} : \mathbb{C}P^1 \to \mathbb{C}P^{n-1}$ of g is of degree one, and satisfies the formulas $\widehat{g}(\infty) = [1 : 0 : \cdots : 0]$, $\widehat{g}(u_0) \in X$.*

Proof. Construction of g out of w has been already carried out in Lemma 59.19 and the discussions above. Therefore we will focus on the other direction of the lemma.

Let f and $\widehat{g}$ be given as in the hypothesis. We would like to construct the lifting of $g = \widehat{g}|_{\mathbb{H}}$ to a map $w = (w_1, w_2, \cdots, w_n) : \mathbb{H} \to \mathbb{C}^n$ so that it satisfies (59.22.1). Since $\widehat{g}$ has degree one, there is exactly one $u = u_1 \in \mathbb{C}P^1$ such that

$$g(u) \in \{[z_1 : \cdots : z_n] \mid z_2 = 0\} \subset \mathbb{C}P^{n-1}$$

whose intersection multiplicity is 1. (59.22.2) implies that $u_1 = \infty$. In other words, there exists $g_j : \mathbb{H} \to \mathbb{C}$, $(j \neq 2)$ such that

$$\widehat{g}(u) = [g_1(u) : 1 : g_3(u) : \cdots : g_n(u)].$$

We define w_2 by the equation

$$w_2(u)^2 = \frac{f(u)}{\left(1 + \sum_{j\neq 2} g_j^2(u)\right)}. \tag{59.23}$$

By assumption, we have

$$f(u) = 0 \Longleftrightarrow u = u_0 \Longleftrightarrow 1 + \sum_{j\neq 2} g_j^2(u) = 0$$

and u_0 is a simple zero of both equations. Hence (59.23) defines

$$w_2 : \operatorname{Int}\mathbb{H} \to \mathbb{C} \setminus \{0\}.$$

(59.23) determines w_2 uniquely in terms of $\widehat{g}$ and f upto the multiple of ± 1. We take one of the two choices of w_2 uniquely so that

$$w_1(u) = g_1(u) w_2(u)$$

satisfies (59.1.3).

We then put

$$w_j(u) = g_j(u) w_2(u)$$

for $j \neq 1, 2$. (59.16) and (59.22) then immediately follow. The proof of Lemma 59.21 is complete. □

We recall that the set of degree one rational curve Σ on $\mathbb{C}P^{n-1}$ that is defined over $\mathbb{R}$ and $[1 : 0 : \cdots : 0] \in \Sigma$ is parameterized by $\mathbb{R}P^{n-2}$.

For each such Σ and given u_0, there exist two choices of $\widehat{g} : \mathbb{C}P^1 \to \Sigma \subset \mathbb{C}P^{n-1}$ such that

$$\widehat{g}(\infty) = [1 : 0 : \cdots : 0], \quad \widehat{g}(u_0) \in X, \tag{59.24}$$

since $\Sigma \cap X$ consists of two points.

Therefore, by Lemma 59.21, the set of w has one-one correspondence with the non-trivial double cover of $\mathbb{R}P^{n-2}$ which is precisely diffeomorphic with S^{n-2}. The proof of Proposition 59.14 is now complete. $\square$

For the case $n = 2$, we have $X = \{(\pm\sqrt{-1}, 1)\} \subset \mathbb{C}P^1$. It follows that $\widehat{g}(u) = [\pm\frac{u}{2\epsilon} : 1]$. So

$$w_2(u)^2 = \frac{u + 2\epsilon\sqrt{-1}}{1 + u^2/(2\epsilon)^2} = \frac{(2\epsilon)^2}{u - 2\epsilon\sqrt{-1}}.$$

Therefore we have

$$\begin{cases} w_1(u) = u(u - 2\epsilon\sqrt{-1})^{-1/2}, \\ w_2(u) = \pm 2\epsilon(u - 2\epsilon\sqrt{-1})^{-1/2}. \end{cases} \tag{59.25}$$

Similarly for the dimension $n \geq 3$, we can find an explicit solution for each given degree one curve $\widehat{g} : \mathbb{C}P^1 \to \mathbb{C}P^n$, i.e., complex line satisfying (59.24) in a similar way.

From this explicit expression of the model solution for $\alpha = \frac{\pi}{2}$, we obtain the following corollary.

Corollary 59.26. *Let $\alpha = \frac{\pi}{2}$. Then all the model solution satisfy*

$$\min_{z \in \partial\mathbb{H}} |w(z)| = \sqrt{2|\epsilon|} \tag{1}$$

and the minimum is realized at the unique point $z = (0, 0) \in \partial\mathbb{H}$. We have

$$0 < \min_{z \in \mathbb{H}} |w(z)| < \sqrt{2|\epsilon|}. \tag{2}$$

Proof. (1) follows from a straightforward calculation. One can also easily see (2) by computing the normal derivative

$$\frac{\partial|w|}{\partial y}(0, 0) < 0.$$

We omit the detail since we do not use this in the rest of this book. $\square$

Remark 59.27. The readers might find that our analysis of the model solutions using the coordinate transformation $\mathbb{C}^n \setminus \{0\} \to \mathbb{C} \times \mathbb{C}P^{n-1}$ given by

$$(z_1, \cdots, z_n) \mapsto (z_1^2 + \cdots + z_n^2, [z_1 : z_2 : \cdots : z_n])$$

looks rather ad hoc. This coordinate transformation can be shed some light on in terms of the standard *Lefschetz fibration*

$$q : \mathbb{C}^n \to \mathbb{C}\,;\; q(z_1, \cdots, z_n) = z_1^2 + \cdots + z_n^2.$$

In terms of this singular fibration, we have just shown that

$$q(H_\epsilon^\alpha) \subset \mathbb{H} + \sqrt{-1} \cdot [2\epsilon, \infty) \subset \mathbb{C}.$$

As $\epsilon \to 0$ i.e., when the base of the fiber approaches the branch point of q, the corresponding Lagrangian submanifold degenerates to the union of two affine Lagrangian planes $\mathbb{R}^n \cup \sqrt{-1}\mathbb{R}^n$.

§60. Local model of holomorphic discs in $\mathbb{C}^n$ II : Fredholm regularity of the local models

The purpose of this section is to study the case $\epsilon < 0$, $\alpha \neq \pi/2$ and complete the proof of Theorem 59.2 by proving (59.3.2). The discussion of §59 is rather elementary and we obtain explicit description of the local models there. Since we do not know such explicit description of the local model in the case $\epsilon < 0$, $\alpha \neq \pi/2$ our proof in this section is rather indirect. Namely we start with the case $\alpha = \pi/2$ and will prove that the moduli space of local model does not change when we vary α in $(0, \pi)$.

We will prove this by showing the smoothness of the moduli space of local model for any $\alpha \in (0, \pi)$. (See Theorem 60.26.) For this purpose, we set up an appropriate Fredholm theory and study the linearized operator of the Cauchy-Riemann equation defined on an appropriate function spaces.

In §60 - §62, we use the Fredholm theory of the moduli space of pseudo-holomorphic discs in a symplectic manifold with cylindrical ends. We study Fredholm theory and gluing argument in the Bott-Morse situation and so follows the line of ideas which works in general for various similar cases, and in particular those we gave in §29. The main difference is that in §60 - §62, we use cylindrical coordinates not only for the domain but also for the target, while in §29 we used cylindrical coordinate for the domain but not for the target.

In the current setting the target becomes noncompact and so we need to use an idea going back to Hofer [Hof93] for the relevant analysis. We will adapt various arguments used in the existing literature to our current relative setting and provide full details of the proofs for completeness' sake. In the case of 3 dimensions, the basic references are a series of papers [HWZ96I], [HWZ95], [HWZ96II], [HWZ99] by Hofer-Wysocki-Zehnder. Hereafter, in the rest of this chapter, we will just quote them as [HWZ] unless we need to specify a particular one to quote. There is also a paper [Abb04] by Abbas which deals with the case of pseudo-holomorphic strip in the symplectization of 3 dimensional contact manifold and the Legendrian boundary condition. Some of those results are generalized to higher dimensions in [BEHWZ03], [HWZ02]. (See also Remark 61.48 for [Bou02].)

The main novelty of this section is the proof of transversality of the model solutions stated in Theorem 60.26, where we use the $O(n)$-invariance of the relevant boundary value problem in the Bott-Morse setting in an essential way.

60.1. Cylindrical coordinates of the end of the target $\mathbb{C}^n$.

We first review the description of symplectic structure on the target space $\mathbb{C}^n$ in the cylindrical coordinate and the relevant Lagrangian submanifold H_ϵ^α. We identify

$$\mathbb{C}^n \setminus \{0\} \cong \mathbb{R} \times S^{2n-1}(1)$$

via its cylindrical coordinates (s, Θ) where $r = e^s$ and the standard symplectic form on $\mathbb{C}^n \setminus \{0\}$

$$\omega_0 = d(e^{2s}\Theta^*\lambda).$$

We always equip the end of $\mathbb{C}^n \setminus \{0\}$ with the cylindrical metric $ds^2 + g_{S^{2n-1}}$ on $\mathbb{R} \times S^{2n-1}$ where $g_{S^{2n-1}}$ is the standard metric on the unit sphere $S^{2n-1} = S^{2n-1}(1)$. In the standard polar coordinates (r, Θ), this metric is translated to

$$\frac{dr^2}{r^2} + g_{S^{2n-1}} = \frac{1}{r^2} g_{\mathbb{C}^n}$$

where $g_{\mathbb{C}^n}$ is the standard Euclidean metric on $\mathbb{C}^n$.

We note that the unit sphere S^{2n-1} has the standard contact form given by

$$\lambda = \frac{1}{2}\sum_{i=1}^n (x_i dy_i - y_i dx_i)$$

and the associated Reeb vector field by

$$X_\lambda = \sum_{i=1} \left(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \right).$$

Let H_ϵ^α ($\epsilon \neq 0$, $\alpha \in (0, \pi)$) be the Lagrangian submanifold H_ϵ^α. As $s \to \infty$ the spherical part $\Theta(H_\epsilon^\alpha)$ thereof is asymptotic to the union of

$$S_{\mathbb{R}^n}^{n-1} := S^{2n-1}(1) \cap \mathbb{R}^n, \quad S_\Lambda^{n-1} := S^{2n-1}(1) \cap \Lambda$$

where $\Lambda = e^{\alpha\sqrt{-1}}\mathbb{R}^n \subset \mathbb{C}^n$. These are Legendrian submanifolds of the contact manifold $(S^{2n-1}(1), \lambda)$.

Let

$$w : \mathbb{R} \times [0,1] \to \mathbb{R} \times S^{2n-1}$$

be a J_0-holomorphic strip with the boundary condition

$$w(\tau, 0) \in \mathbb{R}^n \quad w(\tau, 1) \in \Lambda.$$

It is well-known (see [HWZ]) that the spherical projection

$$\Theta \circ w : \mathbb{R} \times [0,1] \to S^{2n-1}$$

can be regarded (at least intuitively) as a gradient line of the action functional

$$\gamma \mapsto \int_\gamma \lambda \tag{60.1}$$

defined on the path space :

$$\{\gamma : [0,1] \to S^{2n-1}(1) \mid \gamma(0) \in S^{2n-1}(1), \gamma(1) \in S^{2n-1}_\Lambda(1)\}.$$

The critical point of (60.1) is known to be an integral curve of the Reeb vector field. An integral curve of Reeb vector field connecting two (possibly the same) Legendrian submanifolds is called a *Reeb chord*. We will call a *chord* any curve, not necessarily an integral curve, in the above path space.

It follows from the expression of the Reeb vector field X_λ that the *minimal* Reeb chords (that is the Reeb chord for which the value of (60.1) is minimal) of the pair $(S^{n-1}_{\mathbb{R}^n}, S^{n-1}_\Lambda)$ are given by the curves $\gamma^\alpha_a = \gamma_a : [0,1] \to \mathbb{C}^n$ satisfying

$$\gamma^\alpha_a(t) = \gamma_a(t) = e^{\sqrt{-1}\alpha t} a, \quad a \in S^{n-1}_{\mathbb{R}^n}. \tag{60.2}$$

We note that, for the pair $(\mathbb{R}^n, \Lambda)$, all such Reeb chords have the same periods and are nondegenerate in the Bott-Morse sense. (Namely the set of Reeb chords of the form (60.2) is a nondegenerate critical submanifold of the Bott-Morse function (60.1).)

60.2. Fredholm formulation in symplectization.

In this subsection, we set up an appropriate Fredholm theory for moduli space of pseudo-holomorphic maps satisfying (59.1). The weighted Sobolov space we will use for this purpose is similar to one we used in §29.

We take a cylindrical coordinate $(\tau, t) \in \mathbb{R} \times [0,1] = \mathbb{H} \setminus \{0\}$ of $\mathbb{H}$. Namely we put

$$z = x + \sqrt{-1}y = e^{\pi(\tau + \sqrt{-1}t)}. \tag{60.3}$$

We identify $\mathbb{H} \setminus \{0\} \cong \mathbb{R} \times [0,1]$ by this isomorphism. We define a (cylindrical) metric $h_{\mathbb{H}}$ that has the form

$$h_{\mathbb{H}} = \pi(d\tau^2 + dt^2)$$

for τ large. Our metric $h_{\mathbb{H}}$ is conformal to the standard Euclidean metric $|dz|^2$ on $\mathbb{H}$ such that

$$h_{\mathbb{H}} = (|z|')^{-2}|dz|^2$$

where $|z|' : \mathbb{H} \to \mathbb{R}$ is a positive radial function (namely $|z|'$ depends only on $|z|$) such that

$$|z|' = |z| \tag{60.4}$$

when $|z|$ sufficiently large. We also equip a metric

$$g'_{\mathbb{C}^n} = \mu(r) g_{\mathbb{C}^n}, \quad \mu > 0 \tag{60.5.1}$$

on $\mathbb{C}^n$ such that $\mu : \mathbb{C}^n \to \mathbb{R}$ is a positive radial function and $g'_{\mathbb{C}^n}$ becomes the cylindrical metric

$$ds^2 + g_{S^{2n-1}} = \frac{1}{r^2} g_{\mathbb{C}^n}$$

when $r = \sum_{i=1}^n |z_i|^2$ is sufficiently large, i.e.,

$$\mu(r) = \frac{1}{r^2}. \tag{60.5.2}$$

Lemma 60.6. *Let $w : \mathbb{H} \to \mathbb{C}^n$ be a holomorphic map satisfying* (59.1)*. Then there exists c_k, C_k, R_0, s_0 such that*

$$|\nabla^k (w - w^{\rm flat}_{a_0,s_0})|(\tau, t) < C_k e^{-c_k \tau} \tag{60.7}$$

for $\tau > R_0$. Here

$$w^{\rm flat}_{a_0,s_0}(\tau, t) = (\alpha\tau + s_0, e^{\sqrt{-1}\alpha t} a_0), \qquad a_0 = (1, 0, \cdots, 0).$$

We use the metrics $h_{\mathbb{H}}$ and $g'_{\mathbb{C}^n}$ in (60.7)*.*

Proof. By rewriting (59.1.3) with cylindrical coordinates, we obtain (60.7) for $k = 0$. Lemma 60.6 then follows from elliptic regularity. □

Lemma 60.6 dictates the adequate function space for the proper Fredholm theory of the pseudo-holomorphic curves in our problem, which we now explain. Let $\delta < \alpha$ be a positive number and $p > 2$.

With respect to these metrics on the domain and the target, we now define the space $W^{1,p}_\delta(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon, a, \tau_0)$ for each fixed $a \in S^{n-1}$ and $\tau_0 \in \mathbb{R}$ as follows.

Definition 60.8. $W^{1,p}_\delta(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon,a,\tau_0)$ is the set of all w such that

(60.9.1) $w \in W^{1,p}_{loc}$
(60.9.2) Using the coordinates (τ,t) as in (60.3), w satisfies

$$e^{\frac{\delta|\tau|}{p}}\|w(\tau,t)-e^{\alpha(\tau-\tau_0+\sqrt{-1}t)}\gamma^\alpha_a(t)\| \in W^{1,p}([0,\infty)\times[0,1],\mathbb{R})$$

where we use the metrics $h_{\mathbb{H}}$ to define $W^{1,p}$ and the metric $g'_{\mathbb{C}^n}$ of $\mathbb{C}^n$ to define $\| \quad \|$. (γ^α_a) is as in (60.2).)

We like to remind the readers that the metrics $h_{\mathbb{H}}$ and $g'_{\mathbb{C}^n}$ are of product type on the ends of $\mathbb{H}\subset\mathbb{C}$ and on $\mathbb{C}^n$ respectively.

Lemma 60.6 implies that any holomorphic map $w:\mathbb{H}\to\mathbb{C}^n$ satisfying (59.1) is contained in

$$W^{1,p}_\delta(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon,a_0,\tau_0)$$

for some τ_0. Define

$$W^{1,p}_\delta(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon) = \bigcup_{a\in S^{n-1}}\bigcup_{\tau_0\in\mathbb{R}} W^{1,p}_\delta(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon,a,\tau_0).$$

Definition 60.10. Using the metric $g'_{\mathbb{C}^n}$ and $h_{\mathbb{H}}$ on $\mathbb{C}^n$, $\mathbb{H}$ respectively, we define :

(60.11.1)
$$\widetilde{\mathcal{M}}(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon) = \{w\in W^{1,p}_\delta(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon) \mid w \text{ is holomorphic}\}.$$

(60.11.2)
$$\widetilde{\mathcal{M}}(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon,a) = \widetilde{\mathcal{M}}(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon)\cap W^{1,p}_\delta(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon,a).$$

(60.11.3)
$$\widetilde{\mathcal{M}}(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon,a,\tau_0) = \widetilde{\mathcal{M}}(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon)\cap W^{1,p}_\delta(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon,a,\tau_0).$$

$\mathrm{Aut}(\mathbb{H})$ acts on $\widetilde{\mathcal{M}}(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon,a)$ and $\widetilde{\mathcal{M}}(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon,a)$. Its subgroup $\mathbb{R}\subset\mathrm{Aut}(\mathbb{H})$ consisting of translation $z\mapsto z+v$ acts on $\mathcal{M}(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon,a,\tau_0)$. We put

(60.12.1)
$$\mathcal{M}(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon) = \frac{\widetilde{\mathcal{M}}(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon)}{\mathrm{Aut}(\mathbb{H})}$$

(60.12.2)
$$\begin{aligned}\mathcal{M}(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon,a) &= \frac{\widetilde{\mathcal{M}}(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon,a)}{\mathrm{Aut}(\mathbb{H})}\\ &\cong \frac{\widetilde{\mathcal{M}}(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon,a,\tau_0)}{\mathbb{R}}\end{aligned}$$

We remark that the moduli spaces appeared in Theorem 59.2 is $\mathcal{M}(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon,a_0)$.

Lemma 60.13. $W^{1,p}_\delta(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon)$ *has structure of Banach manifold such that the obvious projection*

$$W^{1,p}_\delta(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon) \to \mathbb{R}\times S^{n-1} \tag{60.14}$$

is a locally trivial fiber bundle.

Proof. The tangent space of this Banach manifold is constructed in a similar way to Lemma 29.5 as follows. We take a function $\chi : [0,\infty) \to [0,1]$ such that $\chi(\tau) = 1$ for τ larg and $\chi(\tau) = 0$ for $\tau < 1$.

Let $w \in W^{1,p}_\delta(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon,a,\tau_0)$.

We consider the set of all triple $(W,V_{S^{n-1}},V_\mathbb{R})$ such that

(60.15.1) $V_{S^{n-1}} \in T_aS^{n-1}$, $V_\mathbb{R} \in \mathbb{R} \cong T_{\tau_0}\mathbb{R}$.
(60.15.2) $W \in W^{1,p}_{loc}(\mathbb{H};w^*T\mathbb{C}^n)$.
(60.15.3)

$$e^{\frac{\delta|\tau|'}{p}}\|W(\tau,t) - \chi(\tau)\alpha V_\mathbb{R} w(\tau,t) - \chi(\tau)e^{\alpha((\tau-\tau_0)+\sqrt{-1}t)}V_{S^{n-1}}\|_{g'_{\mathbb{C}^n}} \in W^{1,p}(\mathbb{H},\mathbb{R})$$

Here we regard $V_{S^{n-1}}$ as a vector normal to a in $\mathbb{R}^n$ and then as an element of $\mathbb{C}^n$. $g'_{\mathbb{C}^n}$ is as in (60.5.1). $|\tau|' = |\tau|$ for $\tau \ge 2$ and $|\tau|' = 1$ for $\tau \le 1$.

Let $C^0(w)$ be the set of all such triples. It becomes a Banach space with norm

$$\begin{aligned}&\|(W,V_{S^{n-1}},V_\mathbb{R})\|^p\\ &= \left\| e^{\frac{\delta|\tau|}{p}}\left|W(\tau,t) - \chi(\tau)\alpha V_\mathbb{R} w(\tau,t) - \chi(\tau)e^{\alpha((\tau-\tau_0)+\sqrt{-1}t)}V_{S^{n-1}}\right|_{g'_{\mathbb{C}^n}}\right\|^p_{W^{1,p}}\\ &\quad + \|V_{S^{n-1}}\|^p + \|V_\mathbb{R}\|^p.\end{aligned}$$

We remark that $V_{S^{n-1}}, V_\mathbb{R}$ are determined from W in case $\|(W,V_{S^{n-1}},V_\mathbb{R})\|$ is finite.

It is standard to check that $W^{1,p}_\delta(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon)$ is a Banach manifold and

$$C^0(w) = T_wW^{1,p}_\delta(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon).$$

To show that (60.14) is a locally trivial fiber bundle we use the $O(n)$ action as a biholomorphic isometry on $\mathbb{C}^n$ which preserves H^α_ϵ. It induces an $O(n)$ action on $W^{1,p}_\delta(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon)$. Then the map (60.14) is $O(n)$-equivariant. (Here the $O(n)$ action on S^{n-1} is an obvious one.)

On the other hand the group $\mathbb{R} \cong \mathrm{Aut}(\mathbb{H},\{0\}) \cong \mathrm{Aut}(D^2;\{\pm 1\})$ acts on our space $W^{1,p}_\delta(\mathbb{H},\mathbb{C}^n;H^\alpha_\epsilon)$ as the automorphism of the domain. Then (60.11) is $\mathbb{R}$-equivariant. (Here the $\mathbb{R}$-action on $\mathbb{R}$ is the one given by translations.)

The local triviality (60.14) follows from this equivariance and the fact that $\mathbb{R}\times S^{n-1}$ is a homogeneous of $\mathbb{R}\times O(n)$ action. □

We next put
$$C^1(w) = L^p_\delta(\mathbb{H}, \Lambda^{(0,1)}(w^*T\mathbb{C}^n)),$$
where we use the metri $h_{\mathbb{H}}$ on $\mathbb{H}$ and $g'_{\mathbb{C}^n}$ on $\mathbb{C}^n$. Then there exists an infinite dimensional vector bundle over $W^{1,p}_\delta(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon)$ whose fiber at w is $C^1(w)$.

The formal linearization of the Cauchy-Riemann operator $\overline{\partial}$ defines an operator

$$D_w\overline{\partial} : C^0(w) \to C^1(w). \tag{60.16}$$

We apply $D_w\overline{\partial}$ only the first component W. Using the fact that
$$(D_w\overline{\partial})\left(\alpha V_{\mathbb{R}} w(\tau,t) - e^{\alpha((\tau-\tau_0)+\sqrt{-1}t)}V_{S^{n-1}}\right)$$
goes to zero in the exponential order as $\tau \to \infty$, we can show that $(D_w\overline{\partial})(W)$ is contained in $C^1(w)$.

Lemma 60.17. ([HWZ],[Bou92]) *The operator* (60.16) *is Fredholm.*

Proof. Using the Bott-Morse property of the Reeb chords γ^α_a in our interest, the proof of Lemma 60.17 is standard by now. We recall the proof for completeness.

We first rewrite the equation $\overline{\partial} w = 0$ near $\infty \in \mathbb{H} \cup \{\infty\} \cong D^2$ with respect to the cylindrical coordinates (τ, t) on $\mathbb{H}$ and the polar coordinates
$$(s, \Theta) : \mathbb{C}^n \setminus \{0\} \to \mathbb{R} \times S^{2n-1}; \quad x = e^s\Theta \in \mathbb{C}^n \setminus \{0\}.$$
Denote by $\operatorname{Ker}\lambda$ the standard CR-structure (or contact structure) on S^{2n-1} and then we have
$$TS^{2n-1} = \mathbb{R}\cdot X_\lambda \oplus \operatorname{Ker}\lambda.$$
Here X_λ is the Reeb vector field. Then we have the decomposition
$$T_{(r,\Theta)}\mathbb{C}^n = \mathbb{R}\cdot \frac{\partial}{\partial r} \oplus \mathbb{R}\cdot X_\lambda \oplus \operatorname{Ker}\lambda$$
Let Π be the projection to the third factor. We note $\partial/\partial s = r\partial/\partial r$. ($r = e^s$.) Now we have the formula
$$\begin{aligned} dw &= d(s\circ w)\otimes \frac{\partial}{\partial s} + d(\Theta\circ w) \\ &= d(s\circ w)\otimes \frac{\partial}{\partial s} + \lambda(d(\Theta\circ w))X_\lambda + \Pi\circ d(\Theta\circ w) \end{aligned}$$
where $d(\Theta\circ w) : T\mathbb{H} \to TS^{2n-1}$ is the derivative of the composition $\Theta\circ w : \mathbb{H} \to S^{2n-1}$, and $\lambda(d(\Theta\circ w)) : T\mathbb{H} \to \mathbb{R}$ is the one-form on $\mathbb{H}$ that measures the coefficient of X_λ-component of the derivative $d(\Theta\circ w)$. It follows
$$\begin{aligned} \overline{\partial} w = &\left(\frac{d(s\circ w) - \lambda\circ d(\Theta\circ w)\circ j}{2}\right)\frac{\partial}{\partial s} \\ &+ \left(\frac{\lambda\circ d(\Theta\circ w) + d(s\circ w)\circ j}{2}\right) X_\lambda + (\Pi d(\Theta\circ w))^{(0,1)}. \end{aligned}$$

where $(\Pi d(\Theta \circ w))^{(0,1)}$ is the $(0,1)$-component of $\Pi d(\Theta \circ w)$ which is given by

$$(\Pi d(\Theta \circ w))^{(0,1)} = \frac{\Pi d\Theta \circ w + J \circ (\Pi d\Theta \circ w) \circ j}{2}.$$

Therefore the equation $\overline{\partial} w = 0$ can be written into

$$d(s \circ w) - (\Theta \circ w)^* \lambda \circ j = 0, \tag{60.18.1}$$

$$\Pi d\Theta \circ w + J \circ (\Pi d\Theta \circ w) \circ j = 0. \tag{60.18.2}$$

In the cylindrical coordinates (τ, t) near $\infty \in \mathbb{H} \cup \{\infty\}$, (60.18) can be also written as

$$s_\tau = \lambda \left(\frac{\partial \Theta}{\partial t} \right), \ s_t = -\lambda \left(\frac{\partial \Theta}{\partial \tau} \right) \tag{60.19.1}$$

$$\Pi \left(\frac{\partial \Theta}{\partial \tau} \right) + (J \circ \Pi) \left(\frac{\partial \Theta}{\partial t} \right) = 0 \tag{60.19.2}$$

with respect to the coordinates $(s, \Theta) : \mathbb{C}^n \setminus \{0\} \to \mathbb{R} \times S^{2n-1}$ and the cylindrical coordinates (τ, t) near $\infty \in \mathbb{H} \cup \{\infty\}$.

Now the Fredholm property of the operator (60.16) is a consequence of the general theory of elliptic operators on the spaces with cylindrical ends (see [LoMc85] for example). The index is independent of the choice of the constant δ as long as

$$0 < \delta < \lambda_{min} = \alpha$$

where λ_{min} is the smallest eigenvalue of the asymptotic operator

$$-J \left(\frac{\partial}{\partial t} - (DX_\lambda)(\gamma_a^\alpha) \right).$$

acting on the space of $W^{1,p}$ sections b on $[0,1]$ of $(\gamma_a^\alpha)^* TS^{2n-1}$ with $b(0) \in T_{\gamma_a^\alpha(0)} S^{n-1}_{\mathbb{R}^n}$, $b(1) \in T_{\gamma_a^\alpha(1)} S^{n-1}_\Lambda$. The number λ_{min} can be explicitly calculated, which is precisely α in the current case of our interest. $\square$

We next review the computation of the index of $D_w \overline{\partial}$, in terms of another Maslov-type index that is assigned to each map $w \in W^{1,p}_\delta(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon)$. (This calculation is *not* used in the other part of this book.)

Let $w : \mathbb{H} \to \mathbb{C}^n$ lying in $W^{1,p}_\delta(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon, \gamma^\alpha_a, \tau_0)$. To each such map w, we can assign a loop λ_w in the Lagrangian Grassmannian $\Lambda(n)$ (see beginning of §2.1) in the following way : First, we consider the Gauss map

$$\partial \mathbb{H} \to \Lambda(n) \, ; \, \theta \mapsto T_{w(\theta)} H^\alpha_\epsilon. \tag{60.20}$$

We compactify $\partial\mathbb{H} = \mathbb{R}$ to $S^1 = \mathbb{R} \cup \{\infty\}$. The map (60.20) is not continuous at ∞. But using the asymptotic condition

$$w(\infty, t) = \gamma_a^\alpha(t),$$

we can connect $\lim_{x\to+\infty} T_{w(x)} H_\epsilon^\alpha = \mathbb{R}^n$ to $\lim_{x\to-\infty} T_{w(x)} H_\epsilon^\alpha = \Lambda_\alpha$ by a path

$$\theta \mapsto e^{i\theta} \cdot \mathbb{R}^n; \quad \theta \in [0, \alpha]. \tag{60.21}$$

This is the kind of path described in Proposition 2.3. Then the concatenation of (60.20) and (60.21) provides a loop of Lagrangian subspaces to which we can assign the Maslov index given in [Arn67].

Definition 60.22. Let $w : \mathbb{H} \to \mathbb{C}^n$ be a map lying in $W^{1,p}_\delta(\mathbb{H}, \mathbb{C}^n; H_\epsilon^\alpha)$ with its asymptotic Reeb chord given by γ_a^α. We consider the loop λ_w of Lagrangian subspace obtained by concatenating (60.20) and (60.21). We denote its Maslov index $\mu(\lambda_w)$ by

$$\mu(w; H_\epsilon^\alpha).$$

Because in both cases of $H_\epsilon^\alpha \subset \mathbb{C}^n$ of our current interest, all the disc maps w with the given asymptotic Reeb chord are homotopic to each other, this index in fact depends only on the pair $(\gamma_a, H_\epsilon^\alpha)$.

Proposition 60.23. *We have*

$$\mu(w; H_\epsilon^\alpha) = \begin{cases} 1 & \text{if } \epsilon > 0, \\ n-1 & \text{if } \epsilon < 0. \end{cases}$$

Proof. For the case $\epsilon > 0$, this immediately follows by an explicit calculation of the Maslov index considering the unique solution w obtained in §59 and using the fact that its image is contained in the coordinate plane $\mathbb{C} \cdot a$: Consider the model w on the plane $\mathbb{C} \cdot a_0 \cong \mathbb{C}$ and the Lagrangian loop $\alpha_{w,\gamma_{a_0}}$ which is the Gauss map of w

$$\theta \mapsto T_{w(\theta)} H_\epsilon^\alpha = T_{w(\theta)} \gamma_\epsilon \oplus w(\theta) \cdot \mathbb{R}^{n-1} \subset \mathbb{C}^n$$

followed by the path $\theta \in [0, \alpha] \mapsto e^{i\theta} \cdot \mathbb{R}^n$. From this expression, we obtain $\mu(\gamma_a; H_\epsilon^\alpha) = 1$: the normal contribution to $\mu(\alpha_{w,\gamma_{a_0}})$ from the $\{0\} \oplus \mathbb{C}^{n-1}$-component is zero while the contribution from $\mathbb{C} \oplus \{0\}^{n-1}$ is 1.

On the other hand, for the case $\epsilon < 0$, one can prove this either by direct calculation or by analyzing the change of Maslov index under the Lagrangian surgery from H_ϵ^α. We refer readers to [Proposition 8, Pol91] for this latter study. ☐

The next theorem is a consequence of the standard result of the index theory of elliptic operators with product type end. (See [EGH00].)

We can also derive it from the explicit calculation of the index in the next subsection. (See the end of §60.3.)

Theorem 60.24. *Let* $w \in W^{1,p}_\delta(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon)$ *with its asymptotic Reeb chord given by* γ^α_a. *Then we have*

$$\text{Index}\, D_w\overline{\partial} = \mu(w; H^\alpha_\epsilon) + n. \tag{60.25}$$

60.3. Surjectivity of the linearization.

The main result of this section is the following surjectivity of the linearization operator

$$D_w\overline{\partial} : C^0(w) \to C^1(w).$$

We note that since the almost complex structure on $\mathbb{C}^n$ is integrable, we have

$$D_w\overline{\partial} = \text{the standard Dolbeault operator.}$$

Theorem 60.26. *Let* w *be a pseudo-holomorphic disc in* $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon)$ *with the asymptotic data* (a, τ_0). *Then the operator*

$$D_w\overline{\partial} \oplus D\pi : C^0(w) \to C^1(w) \oplus (T_{\tau_0}\mathbb{R} \oplus T_a S^{n-1}) \tag{60.27}$$

is surjective.

(We remark that $\pi : W^{1,p}_\delta(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon) \to \mathbb{R} \times S^{n-1}$ is as in (60.14).)

Remark 60.28. We remark that Theorem 60.26 still holds when we replace H^α_ϵ by $(H^\alpha_\epsilon)'$. (The proof given below equally works without change.)

Proof. By the $O(n)$-invariance of the equation, it suffices to consider the case when $a_0 = (1, 0, \cdots, 0)$ in (60.27). We first recall that we have a splitting :

$$C^0(w) = T_w W^{1,p}_\delta(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon) = \mathbb{R}^n(w) \oplus W^{1,p}_\delta(w^*T\mathbb{C}^n; w^*TH^\alpha_\epsilon). \tag{60.29}$$

In fact $O(n)$ acts on $\mathbb{C}^n$ with $O(n-1)$ as the isotropy subgroup of the vector $a_0 = (1, 0, \cdots, 0)$. We identify $o(n)/o(n-1)$ with $\mathbb{R}^{n-1}$ and find an embedding $\mathbb{R}^{n-1} \hookrightarrow o(n)$ such that

$$\mathbb{R}^{n-1} \oplus o(n-1) = o(n).$$

We also take a generator X of $\mathbb{R} \cong \text{aut}(\mathbb{H}, \{0\})$, which corresponds to $X = \frac{1}{r}\frac{\partial}{\partial r}$ in the standard coordinates $z = x + \sqrt{-1}y$ on $\mathbb{H}$ with $r = |z|$. Then the assignments

$$A \in o(n)/o(n-1) \mapsto A \cdot w, \quad X \mapsto \mathcal{L}_X w$$

defines an embedding

$$\mathbb{R}^n \cong \mathbb{R} \oplus \mathbb{R}^{n-1} \to C^0(w) = T_w W^{1,p}_\delta(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon) \tag{60.30}$$

whose composition with the projection

$$D\pi : T_w W^{1,p}_\delta(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon) \to T_{(\tau_0, a)}(\mathbb{R} \times S^{n-1})$$

is an isomorphism : See the proof of Lemma 60.13. Here π is the projection defined in (60.14). Since $W^{1,p}_\delta(w^*T\mathbb{C}^n; w^*TH^\alpha_\epsilon)$ is the kernel of $D\pi$ from definition, we obtain the decomposition (60.29) if we set $\mathbb{R}^n(w)$ as the image of the embedding (60.30). More explicitly, we can write

$$\mathbb{R}^n(w) = \mathbb{R} \cdot \mathcal{L}_X w \oplus (o(n)/o(n-1)) \cdot w$$

where $(o(n)/o(n-1)) \cdot w$ is realized as the span of the variations given by

$$z \mapsto \frac{d}{d\epsilon}\Big|_{\epsilon=0} (\exp \epsilon A_j \cdot w(z))$$

for a set of basis $A_j \in o(n)$ that induces a basis of $o(n)/o(n-1)$.

Because each element of $O(n)$ and of $\mathrm{Aut}(\mathbb{H})$ acts as a biholomorphic map, preserves the boundary condition and moves each minimal Reeb chord to another, we have

$$\mathbb{R}^n(w) \subset \mathrm{Ker}\, D_w\overline{\partial}. \tag{60.31}$$

We derive from this that

$$\dim \mathrm{Ker}\, D_w\overline{\partial} \geq n.$$

Thanks to (60.31), to prove Theorem 60.26, it suffices to prove the surjectivity of the map

$$E(w) := D_w\overline{\partial}\Big|_{W^{1,p}_\delta(w^*T\mathbb{C}^n, w^*TH^\alpha_\epsilon)} : W^{1,p}_\delta(w^*T\mathbb{C}^n, w^*TH^\alpha_\epsilon) \to C^1(w).$$

By the integrability of the standard complex structure of $\mathbb{C}^n$, the operator $E(w)$ becomes the standard Dolbeaut operator

$$\overline{\partial} : W^{1,p}_\delta(w^*T\mathbb{C}^n, w^*TH^\alpha_\epsilon) \to L^p_\delta(\Lambda^{(0,1)} w^*T\mathbb{C}^n).$$

We denote

$$\eta = \beta\left(\frac{\partial}{\partial z}\right).$$

Then the one-form $\beta = \eta dz$ has its norm given by the canonical norm $\|\beta\|$ induced by the metric $h_{\mathbb{H}}$ on $\mathbb{H}$ and $g'_{\mathbb{C}^n}$ on $\mathbb{C}^n$. More explicitly, we have

$$\|\beta\| = |z|'\|\eta\| = |z|'\sqrt{\mu(w)}|\eta|_{\mathbb{C}^n}$$

where the norm $\|\eta\|$ is the norm associated to the metric $g'_{\mathbb{C}^n}$ and $|\eta|_{\mathbb{C}^n}$ to the standard Euclidean norm on $\mathbb{C}^n$. Asymptotically at the infinity of $\mathbb{C}^n$, we have

$$\|\beta\| \sim |z|'\|\eta\| \sim \frac{|z||\eta|_{\mathbb{C}^n}}{|w|_{\mathbb{C}^n}}. \tag{60.32}$$

Now we consider the adjoint operator

$$E(w)^* : (L^p_\delta(\Lambda^{(0,1)} w^*T\mathbb{C}^n))^* \to (W^{1,p}_\delta(w^*T\mathbb{C}^n, w^*TH^\alpha_\epsilon))^*.$$

Note we use the metric $h_{\mathbb{H}} = |z|'^{-2}|dz|^2$ on the domain and the metric $g'_{\mathbb{C}^n}$ on the target to define the weighted Sobolev space above.

We define hermitian metric $h'_{\mathbb{C}^n}$ on $\mathbb{C}^n$ by

$$h'_{\mathbb{C}^n} = \mu(r) h_{\mathbb{C}^n}$$

where $h_{\mathbb{C}^n}$ is the standard Herminitan metric on $\mathbb{C}^n$. Then $h'_{\mathbb{C}^n}$ induces Riemannian metric $g'_{\mathbb{C}^n}$ by complex structure J_0. (We remark that $(h'_{\mathbb{C}^n}, J_0)$ is *not* Kähler.)

By the nondegenerate pairing

$$(\cdot,\cdot) = \mathrm{Re}\langle\cdot,\cdot\rangle : L^p_\delta(\Lambda^{(0,1)} w^*T\mathbb{C}^n) \times L^q_{-\delta}(\Lambda^{(1,0)} w^*T\mathbb{C}^n) \to \mathbb{R},$$

(here we use $h_{\mathbb{H}}$ and $h'_{\mathbb{C}^n}$ to define the pairing $\langle\ ,\ \rangle$), we identify $(L^p_\delta(\Lambda^{(0,1)} w^*T\mathbb{C}^n))^*$ with $L^q_{-\delta}(\Lambda^{(1,0)} w^*T\mathbb{C}^n)$ for q satisfying

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and $E(w)^*$ with the adjoint

$$E(w)^\dagger : L^q_{-\delta}(\Lambda^{(1,0)} w^*T\mathbb{C}^n) \to (W^{1,p}_\delta(w^*T\mathbb{C}^n, w^*TH^\alpha_\epsilon))^*.$$

(We remark that we take $2 < p < \infty$ and hence $1 < q < 2$.) Then an element $\beta \in \mathrm{Ker}\, E(w)^\dagger$ is characterized by the equation

$$\int_{\mathbb{H}} \mathrm{Re}\langle\beta, \overline{\partial}\xi\rangle = \mathrm{Re}\int_{\mathbb{H}} \langle\beta, \overline{\partial}\xi\rangle = 0 \quad \text{for all } \xi \in C^0(w). \tag{60.33}$$

By the standard elliptic regularity of the Cauchy-Riemann operator with totally real boundary condition, any solution $\eta = \beta(\partial/\partial z)$ of (60.33) is smooth up to the boundary and satisfies the conjugate boundary condition. (See [McSa04] P548, Theorem C2.3 (ii).) We now use this fact and integration by parts to prove the following lemma

Lemma 60.34. *Let β is a solution of* (60.33) *and $\beta = \eta dz$ for $\eta \in \operatorname{coker} D_w\overline{\partial} \subset L^q_{-\delta}(w^*T\mathbb{C}^n)$. Then β is characterized by the equation*

$$\begin{cases} \overline{\partial}^*\beta = 0 \\ \eta(x,0) \in T_{w(x,0)}H^\alpha_\epsilon \end{cases} \tag{60.35}$$

where $\overline{\partial}^*$ is the formal adjoint of $\overline{\partial}$ defined in (60.37).

Proof. We use the standard complex coordinates z of $\mathbb{H}$ and $(w_1, \cdots, w_n)$ of $\mathbb{C}^n$, and denote the metrics $h_{\mathbb{H}}$ on $\mathbb{H}$ and $g'_{\mathbb{C}^n}$ on $\mathbb{C}^n$ with cylindrical ends chosen before by

$$h_{\mathbb{H}} = h_{z\overline{z}} dz d\overline{z}$$
$$g'_{\mathbb{C}^n} = \sum_{i,j} g'_{i\overline{j}} dw_i d\overline{w}_j$$

in coordinates. Then we have

$$h_{z\overline{z}} = |z|'^{-2}, \quad g'_{i\overline{j}} = \delta_{i\overline{j}}\mu(w) \sim \frac{\delta_{i\overline{j}}}{|w|^2}.$$

We denote

$$\beta = \eta dz = \sum_i \eta_i \frac{\partial}{\partial w_i} \otimes dz$$

and

$$\overline{\partial}\xi = \sum_j \frac{\partial \xi_j}{\partial \overline{z}} \frac{\partial}{\partial w_j} \otimes d\overline{z}.$$

We obtain

$$\langle \beta, \overline{\partial}\xi \rangle = \sum_{i,j} \eta_i \left(\overline{\frac{\partial \xi_j}{\partial \overline{z}}} \right) \delta_{i\overline{j}} |z|'^2 \mu(w) = \sum_i \eta_i \left(\overline{\frac{\partial \xi_i}{\partial \overline{z}}} \right) |z|'^2 \mu(w)$$

and hence

$$\begin{aligned}
\langle \beta, \overline{\partial}\xi \rangle \, dA_h &= \langle \beta, \overline{\partial}\xi \rangle \frac{\sqrt{-1} dz \wedge d\overline{z}}{2|z|'^2} \\
&= \sum_i \eta_i \left(\overline{\frac{\partial \xi_i}{\partial \overline{z}}} \right) \mu(w) \frac{\sqrt{-1} dz \wedge d\overline{z}}{2} \\
&= \sum_i \eta_i \left(\overline{\frac{\partial \xi_i}{\partial \overline{z}}} \right) \mu(w) \frac{\sqrt{-1} dz \wedge d\overline{z}}{2} \\
&= d\left(\sum_i \eta_i \overline{\xi}_i \frac{\mu(w)\sqrt{-1}}{2} d\overline{z} \right) - \sum_i \overline{\xi}_i \frac{\partial}{\partial z} (\mu(w)\eta_i) \frac{\sqrt{-1} dz \wedge d\overline{z}}{2}.
\end{aligned}$$

(Here $dA_{h_{\mathbb{H}}}$ is the volume form of the metric $h_{\mathbb{H}}$.) By Stokes' formula, we have derived

$$\begin{aligned}\int_{\mathbb{H}} \langle \beta, \overline{\partial}\xi \rangle \, dA_{h_{\mathbb{H}}} = &\int_{\partial \mathbb{H}} \sum_i \eta_i \overline{\xi}_i \frac{\mu(w)\sqrt{-1}}{2} d\overline{z} \\ &- \int_{\mathbb{H}} \sum_i \overline{\xi}_i \frac{\partial}{\partial z} \left(\mu(w)\eta_i\right) \frac{\sqrt{-1} dz \wedge d\overline{z}}{2}.\end{aligned}$$

Therefore if $\beta = \eta \, dz$ satisfies $E(w)^\dagger(\beta) = 0$, then we derive the equation

$$\begin{cases} \operatorname{Re}\left(\int_{\mathbb{H}} \sum_i \overline{\xi}_i \frac{\partial}{\partial z} \left(\mu(w)\eta_i\right) dx dy \right) = 0 \\ \operatorname{Re}\left(\int_{\mathbb{R}} \sum_i \eta_i \overline{\xi}_i \frac{\mu(w)\sqrt{-1}}{2} dx \right) = 0 \end{cases} \tag{60.36}$$

for all ξ satisfying $\xi(x) \in T_{w(x)} H_\epsilon^\alpha$ for $x \in \partial \mathbb{H}$. Noting that

$$\begin{aligned}\operatorname{Re}\left(\sum_i \eta_i \overline{\xi}_i \frac{\mu(w)\sqrt{-1}}{2} \right) &= -\operatorname{Im}\left(\sum_i \eta_i \overline{\xi}_i \frac{\mu(w)}{2} \right) \\ &= -\operatorname{Im}\left(\sum_i \eta_i \overline{\xi}_i \right) \frac{\mu(w)}{2} = \frac{\mu(w)\omega_0(\eta, \xi)}{2},\end{aligned}$$

we derive that the second equality of (60.36) becomes

$$\eta(x) \in T_{w(x)} H_\epsilon^\alpha$$

and the interior equation is nothing but $\overline{\partial}^* \beta = 0$ in coordinates since we have

$$\overline{\partial}^* \beta = \sum_i |z|'^2 \frac{\partial}{\partial z} \left(\mu(w)\eta_i\right) \frac{\partial}{\partial w_i}. \tag{60.37}$$

This finishes the proof. □

Formula (60.37) also provides the coordinate expression of the operator $\overline{\partial}^*$ whose symbol coincides with ∂ and so (60.35) is an elliptic first order boundary value problem with scalar symbol. In particular, we can apply Aronzjasin's unique continuation theorem [Aro57] to the system (60.35).

Now we would like to show that the only solution of (60.35) is $\beta = 0$. For this purpose, we use the $O(n)$-invariance of the problem. More precisely, we consider a vector field ξ along w give by

$$\xi = A \cdot w \quad \text{or} \quad \mathcal{L}_X w$$

for any Lie algebra element $A \in o(n)/o(n-1)$ and $A \cdot w$ is the vector field along w generated by the infinitesimal action of A and $X = r\partial/\partial r$.

To analyze a solution η of (60.35), we will use the fact that $\mathcal{L}_X w$ and $A \cdot w$ provide elements $\operatorname{Ker} D_w \overline{\partial}$ and their asymptotic values span the tangent space $T_{w(\infty,t)}\mathbb{C}^n$ point-wise at each $w(\infty,t) = \gamma_{a_0}$ along the asymptotic chord γ_{a_0}. ($a_0 = (1, 0, \cdots, 0)$.)

Let ξ be any of w' or $A \cdot w$. Then we compute

$$\begin{aligned}\frac{\partial}{\partial \overline{z}}\langle \xi, \eta\rangle = \frac{\partial}{\partial \overline{z}}\left(\xi_i \overline{\eta}_i \mu(w)\right) &= \xi_i \frac{\partial}{\partial \overline{z}}\left(\mu(w)\overline{\eta}_i\right) + \frac{\partial \xi_i}{\partial \overline{z}} \overline{\eta}_i \mu(w) \\ &= \xi_i \overline{\frac{\partial}{\partial z}\left(\eta_i \mu(w)\right)} + \frac{\partial \xi_i}{\partial \overline{z}} \overline{\eta}_i \mu(w) = 0\end{aligned}$$

using the equation $\overline{\partial}\xi = 0$ and (60.37). This shows that the natural Hermitian inner product

$$\langle \xi, \eta \rangle$$

associated to the given metrics $h_{\mathbb{H}}$ and $h'_{\mathbb{C}^n}$ is a holomorphic function that satisfies the boundary condition

$$\operatorname{Im}(\langle \xi, \eta\rangle(x)) = 0, \qquad \text{for } x \in \partial \mathbb{H}.$$

The latter boundary condition follows from the fact that both $\eta(x)$ and $\xi(x)$ lie in the same Lagrangian subspace $T_{w(x,0)}H^\alpha_\epsilon$. Therefore the reflection principle produces an entire function on $\mathbb{C}$.

Next we will prove that $\langle \xi, \eta \rangle$ converges to zero when $\operatorname{Im} z \to \infty$ so must vanish by Liouville's theorem. We recall

$$\beta = \eta dz \in L^q_{-\delta}(\Lambda^{(1,0)} w^* T\mathbb{C}^n)$$

with respect to the metric $h_{\mathbb{H}}$ and $g'_{\mathbb{C}^n}$ which is equivalent to saying that $|z|\|\eta\| \sim \|\beta\|$ lies in $L^q_{-\delta}(w^*T\mathbb{C}^n)$. We regard η as a $\mathbb{C}^n$ valued function on $\mathbb{H}$. And we have

$$\mathcal{L}_X w \sim z^{\frac{\alpha}{\pi}} a_0, \quad A \cdot w \sim z^{\frac{\alpha}{\pi}} A a_0. \tag{60.38}$$

On the other hand (60.37) implies that $\mu(w)\eta =: \zeta$ is an anti-holomorphic vector function on $\mathbb{H}$, and ζ satisfies the boundary conditions

$$\zeta(\tau, t) \in T_{w(x,t)} H^\alpha_\epsilon \tag{60.39}$$

for $t = 0, 1$. We also remark that w is asymptotic to $z^{\frac{\alpha}{\pi}}$.

Since we denote by $\|\beta\|$ and by $\|\eta\|$ the norms of $\beta = \eta dz$ and of η with respect to the metric $h_{\mathbb{H}} = |z|'^{-2}|dz|^2$ and $g'_{\mathbb{C}^n} = \mu(w) g_{\mathbb{C}^n}$, and by $|\eta|_{\mathbb{C}^n}$ the standard norm of η as a vector in $\mathbb{C}^n$, we have

$$\|\beta\| = \|\eta\| |z| = \frac{|\eta|_{\mathbb{C}^n} |z|}{|w|}$$

on

$$\mathbb{H}_{|z|>R} = \{z \in \mathbb{H} \mid |z| > R\},$$

for some sufficiently large $R > 0$. We put

$$f(z) = (\mu(w)\eta(z))\overline{z}^{1+\alpha/\pi-\delta/q}.$$

We remark

$$\partial f = 0. \tag{60.40}$$

Since $\beta \in L^q_{-\delta}$ implies that $\|\beta\||z|^{-\frac{\delta}{q}}$ lies in L^q with respect to the metric $h_{\mathbb{H}} = \frac{|dz|^2}{|z|'^2}$, $|w(z)| \sim |z|^{\alpha/\pi}$ and hence

$$\begin{aligned} |f(z)| &\sim \frac{\eta(z)}{|w(z)|^2}\overline{z}^{1+\alpha/\pi-\delta/q} \\ &\sim \left(\frac{|\eta|_{\mathbb{C}^n}|z|}{|w|}\right)|z|^{-\frac{\delta}{q}} \sim \|\beta\||z|^{-\frac{\delta}{q}} \end{aligned}$$

it follows that $f(z)$ is of L^q class associated to the metric $h_{\mathbb{H}}$ on $\mathbb{H}$ and the metric $g'_{\mathbb{C}^n}$ on $\mathbb{C}^n$. We also write $f(\tau,t) = f(e^{\pi(\tau+\sqrt{-1}t)})$. Then f lies in L^q in the standard metric $d\tau^2 + dt^2$ on $[0,\infty)\times[0,1]$, and

$$\lim_{\tau\to\infty}\frac{f(\tau,0)}{|f(\tau,0)|} \in \mathbb{R}^n \cap S^{2n-1}, \quad \lim_{\tau\to\infty}\frac{f(\tau,1)}{|f(\tau,1)|} \in e^{\pi\sqrt{-1}\delta/q}\mathbb{R}^n \cap S^{2n-1}$$

Here to show the second equality we calculate

$$\lim_{\tau\to\infty}\arg f_i(\tau,1) = \pi + \alpha - \alpha + \pi\delta/q = \pi + \pi\delta/q.$$

Here $f = (f_1,\cdots,f_i,\cdots,f_n)$ and we use (60.39).

More precisely there exists $\gamma_0(\tau) = (\gamma_{0,1}(\tau),\cdots,\gamma_{0,n}(\tau)) \in \mathbb{R}^n$ and $\gamma_1(\tau) = (\gamma_{1,1}(\tau),\cdots,\gamma_{1,n}(\tau)) \in \mathbb{R}^n$ such that

$$\arg f_i(\tau,0) = \gamma_{0,i}(\tau), \tag{60.41.1}$$

$$\arg f_i(\tau,1) = \gamma_{1,i}(\tau), \tag{60.41.2}$$

$$\left|\frac{d^k}{d\tau^k}(\gamma_{0,i} - 0)\right| < C_k e^{-c_k\tau}, \quad k = 0,1,2,\cdots, \tag{60.41.3}$$

$$\left|\frac{d^k}{d\tau^k}(\gamma_{1,i} - \delta_1/q)\right| < C_k e^{-c_k\tau}, \quad k = 0,1,2,\cdots. \tag{60.41.4}$$

Moreover f was shown to be in L^q and anti-holomorphic.

Lemma 60.42. *Let f be as above. Then $f \in L^\infty$, i.e., there exists $C > 0$*

$$\|f\|_{L^\infty} < C$$

where $\|\cdot\|_{L^\infty}$ is taken in the standard norm on $\mathbb{C}^n$.

Proof. This lemma seems to be well-known. We will however give a proof below for completeness, since elliptic estimate of boundary valued problem in terms of $W^{1,q}$ norm with $2 > q > 1$ is not so standard.

We use the cylindrical coordinates $\mathbb{H} \setminus \{0\} \cong \mathbb{R} \times [0,1]$. Since the interior bound will be easier to prove, we will focus on the bound at points in the boundary $\mathbb{R} \times \{0\}$. The boundary points on $\mathbb{R} \times \{1\}$ can be handled similarly.

Let $\mathfrak{I}_\tau = (\mathfrak{I}_{\tau,1}, \cdots, \mathfrak{I}_{\tau,n}) : \mathbb{C}^n \to \mathbb{C}^n$, be a linear anti-holomorphic involution such that

$$\mathfrak{I}_{\tau,i}\big|_{e^{\sqrt{-1}\gamma_{0,i}(\tau)}} = id, \quad i = 1, \cdots, n.$$

Denote by $\mathfrak{I}_0 : \mathbb{C}^n \to \mathbb{C}^n$ the complex conjugation. It follows from (60.41.3) that we have

$$\left\| \frac{d^k}{d\tau^k}(\mathfrak{I}_\tau - \mathfrak{I}_0) \right\|_{C^k} < o_k(\tau_0), \quad k = 0, 1, 2, \cdots \tag{60.43}$$

Consider the open disc

$$D_\rho(\tau_0, 0) = \{(\tau, t) \in \mathbb{R}^2 \mid (\tau - \tau_0)^2 + t^2 < \rho^2\}$$

and the semi-disc

$$D_\rho^{\geq 0}(\tau_0, t_0) = \{(\tau, t) \in D_\rho(\tau_0, t_0) \mid t \geq 0\}.$$

For a given function $F = (F_1, \cdots, F_n) : D_\rho^{\geq 0}(\tau_0, t_0) \to \mathbb{C}^n$ satisfying

$$\arg F_i(\tau, 0) = \gamma_{0,i}(\tau)$$

we define its double $\operatorname{Ref} F : D_\rho(\tau_0, t_0) \to \mathbb{C}^n$ by the formula

$$\operatorname{Ref} F(\tau, t) = \begin{cases} F(\tau, t) & \text{if } t \geq 0 \\ \mathfrak{I}_\tau(F(\tau, -t)) & \text{if } t \leq 0 \end{cases}$$

By construction, we have

$$\operatorname{Ref}(\partial F) = \partial(\operatorname{Ref}(F)) + P(\operatorname{Ref}(F))$$

for a differential operator P of first order whose coefficients are continuous and smaller than $o(\tau_0)$ pointwise. (Here the function $o(\tau_0)$ denotes any function satisfying $\lim_{\tau_0 \to \infty} o(\tau_0) = 0$.) Now we choose a cut function $\chi : D_\rho^{\geq 0}(\tau_0, t_0) \to 1$ which is

1 on $D^{\ge 0}_{\rho/2}(\tau_0,t_0)$ and is zero on $D^{\ge 0}_{\rho}(\tau_0,t_0)\setminus D^{\ge 0}_{2\rho/3}(\tau_0,t_0)$. By applying the above equality we have

$$\partial(\mathrm{Ref}(\chi f)) = \mathrm{Ref}(\partial(\chi f)) - P(\mathrm{Ref}(\chi f)). \tag{60.44}$$

Using $\partial f = 0$ and (60.44) we have

$$\|\partial(\mathrm{Ref}(\chi f))\|_{L^q} \le C\|f\|_{L^q\left(D^{\ge 0}_{\rho}(\tau_0,t_0)\right)} + o(\tau_0)\|\mathrm{Ref}(\chi f)\|_{W^{1,q}}. \tag{60.45}$$

Since $\mathrm{Ref}(\chi f)$ is of compact support in $D_{\rho/2}(\tau_0,t_0)$ we have

$$\|\mathrm{Ref}(\chi f)\|_{W^{1,q}} \le C(\|\mathrm{Ref}(\chi f)\|_{L^q} + \|\partial(\mathrm{Ref}(\chi f))\|_{L^q}) : \tag{60.46}$$

This inequality follows from the fact $\frac{1}{\pi\bar z\sqrt{-1}}$ is the fundamental solution of ∂ on $\mathbb{R}^2 \cong \mathbb{C}$.

Combining (60.45), (60.46) we obtain

$$\|\mathrm{Ref}(\chi f)\|_{W^{1,q}} \le C\|f\|_{L^q\left(D^{\ge 0}_{\rho}(\tau_0,t_0)\right)},$$

as long as τ_0 is sufficiently large, say $|\tau_0| \ge R_1$. It follows that

$$\|f\|_{W^{1,q}\left(D^{\ge 0}_{\rho/2}(\tau_0,t_0)\right)} \le C\|f\|_{L^q\left(D^{\ge 0}_{\rho}(\tau_0,t_0)\right)}.$$

Therefore from the Sobolev embedding $W^{1,q} \hookrightarrow L^{2q/2-q}$ we obtain

$$\|f\|_{L^{2q/(2-q)}\left(D^{\ge 0}_{\rho/2}(\tau_0,t_0)\right)} \le C\|f\|_{L^q\left(D^{\ge 0}_{\rho}(\tau_0,t_0)\right)}.$$

By repeating the same argument twice (namely using Moser's iteration) and using Sobolev inequality again, we obtain

$$\begin{aligned}\| f\|_{C^0\left(D^{\ge 0}_{2^{-4}\rho}(\tau_0,t_0)\right)} &\le C\|f\|_{W^{1,2}\left(D^{\ge 0}_{2^{-3}\rho}(\tau_0,t_0)\right)}\\ &\le C^4\|f\|_{L^q\left(D^{\ge 0}_{\rho}(\tau_0,t_0)\right)} \le C^4\|f\|_{L^q(\mathbb{H})} < \infty\end{aligned}$$

for all (τ_0,t_0) with $t_0 = 0$ and $|\tau_0| \ge R_1$. We note that the constant C appearing above can be chosen independent of τ_0. Easier argument gives rise to the same pointwise bound at an interior point of $\mathbb{H}$. Redefining C to be $C^4\|f\|_{L^q(\mathbb{H})}$ we have finished the proof. □

Let $\xi = A\cdot w$. Then we have

$$\|\xi\| = \|A\cdot w\| \sim \|w\| \sim \|z^{\frac{\alpha}{\pi}}a_0\| = |z|^{\frac{\alpha}{\pi}}\frac{1}{|w|} = |z|^{\frac{\alpha}{\pi}}\cdot\frac{1}{|z|^{\frac{\alpha}{\pi}}} = 1. \tag{60.47}$$

Therefore it follows from the definition of f (see right above (60.40)) and Lemma 60.42 that the holomorphic function $\langle \xi, \eta \rangle$ satisfies

$$\lim_{\tau \to \infty} |\langle \xi, \eta \rangle(\tau, t)| \leq \lim_{\tau \to \infty} \left| \frac{\eta(\tau, t)}{w(\tau, t)} \right| \leq \lim_{\tau \to \infty} C|z|^{\delta/q - 1} = 0.$$

Here the last identity follows if we choose δ so small that $0 < \delta < q$.

From this, we derive

$$\langle \xi, \eta \rangle \equiv 0,$$

everywhere in $\mathbb{H}$, as the double of $\langle \xi, \eta \rangle$ is an entire function on $\mathbb{C}$ that converges to zero as $|z| \to \infty$.

Using the property $\mathcal{L}_X w \sim z^{\frac{\alpha}{\pi}}$, whose order is the same as that of $A \cdot w$, similar computation gives

$$\langle \mathcal{L}_X w, \eta \rangle \equiv 0,$$

everywhere in $\mathbb{H}$. By considering the asymptotic values of η and ξ at infinity,

$$\langle \xi, \eta \rangle = \langle \mathcal{L}_X w, \eta \rangle = 0$$

for all $A \in o(n)/o(n-1)$ in a neighborhood of $1 \in D^2$ where w is embedded. Since the set

$$\{\mathcal{L}_X w\} \cup \{A \cdot w \mid A \in o(n)\}$$

complex linearly spans $w^*T\mathbb{C}^n$ near γ_{a_0} at each point of z in a neighborhood of the infinity, we have derived that η must vanish on the neighborhood and so everywhere by the unique continuation. This finishes the proof of Theorem 60.26. $\square$

Since we have established surjectivity of the linearization operator $D_w\overline{\partial}$, the moduli space $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; H_\epsilon^\alpha)$ becomes a smooth manifold whose tangent space can be identified with the kernel of the operator $D_w\overline{\partial} : C^0(w) \to C^1(w)$.

Therefore Theorem 59.2 which was already proved in §59 for the case $\epsilon > 0$ or $\epsilon < 0$, $\alpha = \pi/2$ immediately proves the following theorem. This in particular computes the index of $D_w\overline{\partial}$, when combined with the surjectivity proven in the previous subsection.

Theorem 60.48. *Assume* $0 < \delta_1 < \lambda_{min}$. *Let* w *be the pseudo-holomorphic disc constructed in* §57 *associated to the Reeb chord* γ_a^α *for a given* $a \in S_{\mathbb{R}^n}^{n-1}$. *Then we have*

$$\dim \operatorname{Ker} D_w\overline{\partial} = \begin{cases} n+1 & \text{for } \epsilon > 0 \\ 2n-1 & \text{for } \epsilon < 0 \end{cases}$$

Proof. We remark that it suffices to consider the case when $\alpha = \pi/2$, since index is invariant under the continuous deformation of Fredholm operators.

Theorem 59.2 implies that in case $\epsilon < 0$ the set of holomorphic map w satisfying (59.1) is $n-2$ dimensional modulo $\operatorname{Aut}(\mathbb{H})$. We note $\operatorname{Aut}(\mathbb{H})$ is two dimensional and one of them corresponds to the $\mathbb{R}$ factor of $S^{n-2} \times \mathbb{R}$ in (60.14). Hence Theorem 60.26 implies

$$\operatorname{Ker} D_w\overline{\partial} = n - 2 + 2 + n - 1 = 2n - 1.$$

The case $\epsilon > 0$ is similar. □

We would like to separately state the following obvious corollary of Theorem 60.28 and 60.48.

Corollary 60.49. *We have*

$$\operatorname{Index} D_w\overline{\partial} = \begin{cases} n+1 & \text{for } \epsilon > 0 \\ 2n-1 & \text{for } \epsilon < 0. \end{cases}$$

Proof of Theorem 60.24. This is an immediate consequence of Corollary 60.49 and Proposition 60.23. □

60.4. Proof of Theorem 59.2.

In this subsection we complete the proof of Theorem 59.2 assuming the following theorem whose proof will be postponed until §62.8.

Theorem 60.50. *The map*

$$\bigcup_{\alpha\in(0,\pi)} \mathcal{M}(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon) \to (0, \pi) \tag{60.51.1}$$

is proper. Here (60.51.1) *maps elements in* $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon)$ *to* α.

The topology on $\bigcup_{\alpha\in(0,\pi)} \mathcal{M}(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon)$ is the induced topology from the topology of $\bigcup_{\alpha\in(0,\pi)} W^{1,p}_\delta(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon)$, which will be defined in §62.8.

Theorem 60.26 implies that

$$\bigcup_{\alpha\in(0,\pi)} \mathcal{M}(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon, a_0) \to (0, \pi) \tag{60.51.2}$$

is proper, where $\epsilon < 0$ and $a_0 = (1, 0, \cdots, 0)$. Theorem 60.26 implies that (60.51.2) is a submersion. Therefore (60.51.2) is a locally trivial fiber bundle. In particular the diffeomorphism classes of the fibers $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon, a_0)$ are independent of α. We already proved in §59.3 that

$$\mathcal{M}(\mathbb{H}, \mathbb{C}^n; H^{\pi/2}_\epsilon, a_0) \cong S^{n-2}$$

Thus we have proved (59.3.2). The proof of Theorem 59.2 is complete. □

We also have the following symmetry statement for our moduli space. For this we need some notations.

Let $\mathrm{Ref}_{\frac{\alpha}{2}} : \mathbb{C} \to \mathbb{C}$ is the reflection about the line $\arg z = \frac{\alpha}{2}$. We denote

$$(\mathrm{Ref}_{\frac{\alpha}{2}}, \cdots, \mathrm{Ref}_{\frac{\alpha}{2}}) : \mathbb{C}^n \to \mathbb{C}^n$$

by $\mathrm{Ref}_{\frac{\alpha}{2}}$ also. We can check easily that

$$\mathrm{Ref}_{\frac{\alpha}{2}}(H^\alpha_\epsilon) = H^\alpha_\epsilon.$$

We also define an action of $O(n-1)$ on $\mathbb{C}^n$ by

$$(z_1, \cdots, z_n) = (z_1, z^{n-1}) \mapsto (z_1, Az^{n-1}).$$

We first note that the Lagrangian submanifold H^α_ϵ and the Reeb chord γ_{a_0} of H^α_ϵ are invariant under the reflection and the action of $O(n-1)$. We note that the action of $O(n-1)$ is holomorphic and the reflection is anti-holomorphic. Therefore the $O(n-1)$-action on $\mathbb{C}^n$ naturally induces an action on the moduli space $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon, a_0)$ and $\mathrm{Ref}_{\frac{\alpha}{2}}$ induces an involution $(\mathrm{Ref}_{\frac{\alpha}{2}})_*$ on $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon, a_0)$ by

$$(\mathrm{Ref}_{\frac{\alpha}{2}})_*([w]) = [\mathrm{Ref}_{\frac{\alpha}{2}} \circ w \circ *]$$

where $* : \mathbb{H} \to \mathbb{H}$ is defined by $*z = -\overline{z}$.

Proposition 60.52.

(60.53.1) *The action of $O(n-1)$ on $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon, a_0)$ is transitive.*

(60.53.2) *The action of $(\mathrm{Ref}_{\frac{\alpha}{2}})_*$ on $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon, a_0)$ is trivial : For any element of $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon, a_0)$ we have a representative w such that*

$$w(-\overline{z}) = \mathrm{Ref}_{\frac{\alpha}{2}}(w(z)) \tag{60.54}$$

Proof. We can prove that the action of $O(n-1)$ is transitive and the action of $(\mathrm{Ref}_{\frac{\alpha}{2}})_*$ is trivial for $\alpha = \pi/2$ by its explicit description given in (59.25). Since the transitivity and triviality of the action of compact groups are preserved under the deformation of the actions, it follows from the proof of Theorem 60.50 that both hold for all $\alpha \in (0, \pi)$.

Now we prove existence of a representative $w \in \widetilde{\mathcal{M}}(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon, a_0)$ satisfying (60.54). It follows from the triviality of the action of $(\mathrm{Ref}_{\frac{\alpha}{2}})_*$ on $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; H^\alpha_\epsilon, a_0)$ that any representative w satisfies

$$\mathrm{Ref}_{\frac{\alpha}{2}}(w(-\overline{z})) = w(az + b)$$

for some $a > 0$ and $b \in \mathbb{R}$. Since $\mathrm{Ref}_{\frac{\alpha}{2}}(w(-\overline{z}))$ satisfies (59.1.3) for the same τ_0 as w, it follows that $a = 1$. Replacing now w by $z \mapsto w(z - b/2)$, we find w satisfying (60.54) which finishes the proof. □

60.5. Local models do not hit the origin.

In this subsection, we will prove that the local models do not hit the origin. This property will be used later in our gluing arguments in section §61 and §62.

We first examine H^α_ϵ for $\epsilon < 0$ more closely. We put

$$S^{n-1}_c = e^{c\sqrt{-1}}\mathbb{R}^n \cap S^{2n-1}, \tag{60.55}$$

for $c \in \mathbb{R}$. By definition, for $s > \log\sqrt{2|\epsilon|}$, we have

$$H^\alpha_\epsilon \cap (\{s\} \times S^{2n-1}) = \{s\} \times (S^{n-1}_{h_1(s)} \cup S^{n-1}_{h_2(s)}), \tag{60.56}$$

where

$$h_1(s) < 0 < \alpha < h_2(s)$$

and

$$\begin{cases} \lim_{s\to\infty} h_1(s) = 0, & \lim_{s\to\log\sqrt{2|\epsilon|}} h_1(s) = \dfrac{\alpha}{2} - \dfrac{\pi}{2}, \quad h_1'(s) > 0, \\ \lim_{s\to\infty} h_2(s) = \alpha, & \lim_{s\to\log\sqrt{2|\epsilon|}} h_2(s) = \dfrac{\alpha}{2} + \dfrac{\pi}{2}, \quad h_2'(s) < 0. \end{cases} \tag{60.57}$$

These follow from the definition of γ_ϵ for $\epsilon < 0$ and from (54.12.3).

Figure 60.1

We have :

$$\lambda|_{H^\alpha_\epsilon} = \begin{cases} -\frac{dh_1}{ds}ds & \text{on } \bigcup_s(\{s\} \times S^{n-1}_{h_1(s)}) \\ +\frac{dh_2}{ds}ds & \text{on } \bigcup_s(\{s\} \times S^{n-1}_{h_2(s)}) \end{cases} \tag{60.58.1}$$

It implies

$$d\lambda|_{H^\alpha_\epsilon} = 0, \tag{60.58.2}$$

$$\lambda|_{H^\alpha_\epsilon \cap (\{s\}\times S^{2n-1})} = 0 \quad \text{for all } s. \tag{60.58.3}$$

With this preparation, we prove

Proposition 60.59. *Let $w : \mathbb{H} \to \mathbb{C}^n$ be a holomorphic map satisfying* (59.1)*. Then $0 = (0, \cdots, 0) \in \mathbb{C}^n$ is not in the image of w.*

Proof. In case $\epsilon > 0$, this was proved in Lemma 59.4. We consider the case $\epsilon < 0$ now. Recall that we write both the contact form on S^{2n-1}

$$\lambda = \frac{1}{2}\left(\sum_i x_i dy_i - y_i dx_i\right)\Bigg|_{S^{2n-1}},$$

and its pull-back to $\mathbb{R} \times S^{2n-1}$ by the same letter λ. Via the diffeomorphism $(s, \Theta) : \mathbb{C}^n \setminus \{0\} \cong \mathbb{R} \times S^{2n-1}$ we also regard λ as a form on $\mathbb{C}^n \setminus \{0\}$. We alert readers that λ is *not* the Liouville one-form which is given by

$$\frac{1}{2}\left(\sum_i x_i dy_i - y_i dx_i\right)$$

on $\mathbb{C}^n$.

Lemma 60.60. *If $\mathbb{C} \subset T_p(\mathbb{C}^n \setminus \{0\})$ is a one dimensional complex linear subspace, then $d\lambda|_{\mathbb{C}} = c dx \wedge dy$ with $c \geq 0$.*

Proof. We recall that $\mathrm{Ker}\lambda = \xi$ is a J_0-invariant linear subspace of $TS^{2n-1} \subset T\mathbb{C}^n$. (In fact it gives the standard CR structure on S^{2n-1}.)

Considering the translational invariant distribution, also denoted by ξ, on $\mathbb{R} \times S^{2n-1} \cong \mathbb{C}^n$ we have the decomposition

$$T(\mathbb{R} \times S^{2n-1}) = \mathrm{span}_{\mathbb{R}}\left\{\frac{\partial}{\partial s}, J_0\left(\frac{\partial}{\partial s}\right)\right\} \oplus \xi.$$

(Here J_0 is the standard complex structure of $\mathbb{C}^n$ and s is a coordinate of $\mathbb{R}$. See §60.1.) Moreover

$$(X, Y) \mapsto (d\lambda)(X, J_0 Y)$$

defines a positive definite symmetric bilinear form on ξ and vanish on $\mathrm{span}_{\mathbb{R}}\left\{\frac{\partial}{\partial s}, J_0\left(\frac{\partial}{\partial s}\right)\right\}$. The lemma immediately follows from these facts. □

Let

$$w^{-1}(0) = \{z_1, \cdots, z_m\} \subset \mathrm{Int}\,\mathbb{H}$$

(We would like to show this set is empty.) Writing

$$w_i(z) = (z - z_i)^{n_i} g_i(z), \quad n_i \geq 1$$

with $g_i(z_i) \neq 0$, we easily obtain

$$\lim_{\delta \to 0} \sum \int_{\partial B_{z_i}(\delta)} w^* \lambda = 2\pi \sum_{i=1}^{m} n_i \geq 2\pi m. \tag{60.61}$$

Here $B_{z_i}(\delta) = \{z \in \mathbb{C} \mid |z - z_i| \leq \delta\}$. We next prove

Lema 60.62.

$$\int_{\partial \mathbb{H}} w^*\lambda + \lim_{R\to\infty} \int_{\partial B_0(R)\cap \mathbb{H}} w^*\lambda = \pi.$$

Proof. It follows from (59.1.3) that the set

$$\{x \in \partial\mathbb{H} = \mathbb{R} \mid w(x) \in \{s\} \times S^{2n-1}\}$$

consists of two points $x_1(s) < 0 < x_2(s)$ for s sufficiently large. Then (60.58) implies that

$$\int_{x_1(s)}^{x_2(s)} w^*\lambda$$

depends only on the homology class

$$w_*([x_1(s), x_2(s)], \partial[x_1(s), x_2(s)]) \\ \in H_1(H^\alpha_\epsilon \cap ([-\infty, s] \times S^{2n-1}), H^\alpha_\epsilon \cap (\{s\} \times S^{2n-1})).$$

We use this fact and (60.57) to find

$$\int_{x_1(s)}^{x_2(s)} w^*\lambda = h_1(s) - h_2(s) + \pi. \tag{60.63}$$

In fact we can deform the arc $w([x_1(s), x_2(s)])$ to the union of the following three paths (here we put $a_0 = (1, 0 \cdots, 0)$) :

(I) $t \mapsto (s - t, e^{\sqrt{-1}h_2(s-t)}a_0)$, for $t \in [0, s - \log\sqrt{2|\epsilon|}]$.

(II) $t \mapsto (\log\sqrt{2|\epsilon|}, e^{\sqrt{-1}(\frac{\alpha}{2}+\frac{\pi}{2})}(\cos t, \sin t, 0, \cdots, 0))$, for $t \in [0, \pi]$:

(III) $t \mapsto (t + \log\sqrt{2|\epsilon|}, e^{\sqrt{-1}h_1(t+\log\sqrt{2|\epsilon|})}a_0)$, for $t \in [0, s - \log\sqrt{2|\epsilon|}]$

Figure 60.2.

Since $h_2(\log\sqrt{2|\epsilon|}) = \frac{\alpha}{2}+\frac{\pi}{2}$, it follows from (60.58.1) that the integral on the part (I) is $\frac{\alpha}{2}+\frac{\pi}{2} - h_2(s)$.

By (60.58.3), the integral on the part (II) is 0.

Since $h_1(\log\sqrt{2|\epsilon|}) = \frac{\alpha}{2}-\frac{\pi}{2}$, it follows from (60.58.1) that the integral on the part (III) is $h_1(s) - \frac{\alpha}{2}+\frac{\pi}{2}$. (60.63) then follows.

By (60.63) and (60.57) we have

$$\int_{\partial\mathbb{H}} w^*\omega = \lim_{s\to\infty}\int_{x_1(s)}^{x_2(s)} w^*\lambda = \lim_{s\to\infty}(h_1(s)-h_2(s)) + \pi = \pi - \alpha. \tag{60.64}$$

On the other hand, by (59.1.3) we have

$$\lim_{R\to\infty}\int_{\partial B_0(R)\cap\mathbb{H}} w^*\lambda = \alpha.$$

This finishes the proof of Lemma 60.62. □

(60.61) and Lemma 60.62 imply

$$\int_{\mathbb{H}} w^*d\lambda \le \pi - 2\pi m.$$

But Lemma 60.60 implies that the left hand side is nonnegative. Hence $m = 0$, as required. This finishes the proof. □

§61. Proof of Theorem Z, I : Gluing

In this section and the next we will complete the proof of Theorem Z which is Theorem 55.5. We associate a pseudo-holomorphic strip (resp. a family of pseudo-holomorphic strips parameterized by S^{n-2}) between L_0 and $L_{-\epsilon_1}$ (resp. L_0 and $L_{+\epsilon_1}$) to each pseudo-holomorphic triangle w. In the next section, we will show that the family we will construct in this section are all the pseudo-holomorphic strips near the given pseudo-holomorphic triangle w. We assume w is isolated and Fredholm regular.

61.1. Cylindrical models.

We start with studying symplectic and almost complex structures in the cylindrical coordinate. Let $(\mathbb{C}^n,\omega_0,J_0)$ be the standard linear complex space endowed with the standard Kähler structure. We denote by (r,Θ) the polar coordinates

$$\mathbb{C}^n\setminus\{0\}\to(0,\infty)\times S^{2n-1}(1)$$

of $\mathbb{C}^n \setminus \{0\}$. By putting $r = e^{2s}$, where $s \in \mathbb{R}$, we have a diffeomorphism

$$(s, \Theta) : \mathbb{C}^n \setminus \{0\} \to \mathbb{R} \times S^{2n-1}(1). \tag{61.1.1}$$

We denote by

$$\varphi : \mathbb{R} \times S^{2n-1}(1) \to \mathbb{C}^n \setminus \{0\} \tag{61.1.2}$$

the inverse of (61.1.1).

We denote also by ω_0 and J_0 the induced sympletic and complex structures on $(0, \infty) \times S^{2n-1}(1)$ or on $\mathbb{R} \times S^{2n-1}(1)$. We denote by λ the canonical contact form given by

$$\frac{1}{2}\left(\sum_i x_i dy_i - y_i dx_i\right)\Bigg|_{S^{2n-1}}$$

on the unit sphere S^{2n-1}, or the corresponding scale invariant one form on $\mathbb{C}^n \setminus \{0\} \cong \mathbb{R} \times S^{2n-1}$. (We like to alert the readers that this form is *not* the Liouville one form

$$\frac{1}{2}\left(\sum_i x_i dy_i - y_i dx_i\right)$$

on $\mathbb{C}^n$.) Then we have

$$\omega_0 = 2r dr \wedge \lambda + r^2 d\lambda = e^{2s}(2ds \wedge \lambda + d\lambda) = d(e^{2s}\lambda).$$

Here $r = e^s$. We denote by $X_\lambda = J_0 \frac{\partial}{\partial r}$ the Reeb vector field on S^{2n-1}.

We fix a positive number $\epsilon_0 > 0$ such that the Darboux chart

$$\exp^I_{p_{12}} := I^{-1} : B^{2n}(\epsilon_0) \to M$$

chosen as in (54.16) induces a diffeomorphism onto its image. Here $B^{2n}(\epsilon_0) \subset \mathbb{C}^n \cong (T_{p_{12}}M, J_{p_{12}})$ is the ball of radius ϵ_0 centered at origin. We denote the corresponding image by $B(p_{12}; \epsilon_0) \subset M$. (54.16.1) implies

$$\exp^I_{p_{12}}(\mathbb{R}^n \cap B^{2n}(\epsilon_0)) \subset L_1, \quad \exp^I_{p_{12}}(\Lambda \cap B^{2n}(\epsilon_0)) \subset L_2.$$

Recall that ϵ_0 depends only on the size of the Darboux chart at p_{12}, which can be chosen depending only on (M, ω).

We compose $\exp^I_{p_{12}}$ with the diffeomorphism (61.1.2) and obtain

$$\exp^I_{p_{12}} \circ \varphi : (-\infty, \log \epsilon_0) \times S^{2n-1} \to B(p_{12}; \epsilon_0).$$

By our choice we have

$$
\begin{aligned}
\varphi^{-1}(L_1 \setminus (L_1 \cap L_2)) &= (-\infty, \log \epsilon_0) \times S^{n-1}_{\mathbb{R}}, \\
\varphi^{-1}(L_2 \setminus (L_1 \cap L_2)) &= (-\infty, \log \epsilon_0) \times S^{n-1}_{\Lambda}.
\end{aligned} \tag{61.2}
$$

The next proposition states that the pull back of the complex structures of M to $T_{p_{12}}M$ is asymptotic to the standard one as $s \to -\infty$. In the next lemma and hereafter we use the *product metric* $g_{\mathbb{R}\times S^{n-1}}$ on $\mathbb{R} \times S^{n-1}$ to define the norms of tensors on it.

We may also choose the Darboux chart I so that the differential of $d_{p_{12}}I$: $(T_{p_{12}}M, J_{p_{12}}) \to \mathbb{C}^n$ becomes a unitary transformation. Using these facts we easily obtain the following :

Lemma 61.3. *There exists c_k, C_k independent of ϵ_1, ϵ_0 such that the following holds for $k = 0, 1, \cdots$, $s \le \log \epsilon_0$:*

$$
|(\nabla^k((\exp^I_{p_{12}} \circ \varphi)^*\omega - \omega_0)(s,x)|_{g_{\mathbb{R}\times S^{2n-1}}} < C_k e^{c_k s}, \tag{61.4.1}
$$

$$
|\nabla^k((\exp^I_{p_{12}} \circ \varphi)^*J - J_0)(s,x)|_{g_{\mathbb{R}\times S^{2n-1}}} < C_k e^{c_k s}. \tag{61.4.2}
$$

Proof. (61.4.1) follows by standard exponential estimates starting from

$$
|\exp^I_{p_{12}} \circ \varphi(s,t)|,\ |d\exp^I_{p_{12}} \circ d\,\varphi(s,t)| \sim \text{const } e^s
$$

One can then easily derive (61.4.2) from this. We omit the details of these derivations. □

61.2. Description of L_{ϵ_1} in cylindrical coordinates.

In this subsection, we review the construction of the Lagrangian surgery and rewrite it in terms of the cylindrical coordinate. Recall we have

$$
H^\alpha_{\epsilon_1} = \mathbb{R}^n \#_{\epsilon_1} \Lambda = \gamma^\alpha_{\epsilon_1} \cdot S^{n-1}_{\mathbb{R}^n} \tag{61.5}
$$

where $\gamma^\alpha_{\epsilon_1}$ is given by (54.12).

In (54.14), we modified $H^\alpha_{\epsilon_1}$ in the domain $B(2S_0\sqrt{|\epsilon_1|}) \setminus B(S_0\sqrt{|\epsilon_1|})$ and glue it with $(\mathbb{R}^n \cup \Lambda) \setminus B(2S_0\sqrt{|\epsilon_1|})$ to obtain $(H^\alpha_{\epsilon_1})'$. Namely we have :

$$
(H^\alpha_{\epsilon_1})' \cap B(S_0\sqrt{|\epsilon_1|}) = H^\alpha_{\epsilon_1} \cap B(S_0\sqrt{|\epsilon_1|}). \tag{61.6.1}
$$

$$
(H^\alpha_{\epsilon_1})' \setminus B(2S_0\sqrt{|\epsilon_1|}) = (\mathbb{R}^n \cup \Lambda) \setminus B(2S_0\sqrt{|\epsilon_1|}). \tag{61.6.2}
$$

By construction of $L_1 \#_{\epsilon_1} L_2$ we have

$$(L_1 \#_{\epsilon_1} L_2) \cap B(p_{12}; \epsilon_0) = \exp^I_{p_{12}}((H^\alpha_{\epsilon_1})') \cap B(p_{12}; \epsilon_0)$$

where I is the Darboux chart defined on a neighborhood U of p_{12} so that $I(U) = B^{2n}(2\epsilon_0) \subset \mathbb{C}^n$ and so

$$\exp^I_{p_{12}} : B^{2n}(2\epsilon_0) \to U, \qquad B(p_{12}; \epsilon_0) \subset U$$

and satisfies (54.16.1) i.e.,

$$(\exp^I_{p_{12}})(\mathbb{R}^n \cap B^{2n}(2\epsilon_0)) \subset L_1, \ (\exp^I_{p_{12}})^{-1}(\Lambda \cap B^{2n}(2\epsilon_0)) \subset L_2. \tag{61.7}$$

61.3. Implanting the local model.

We next implant the local model of §59 - 60 into a neighborhood of p_{12} and smooth off the corner at p_{12} of the given pseudo-holomorphic triangle.

Consider the set of holomorphic maps $w : \mathbb{H} \to \mathbb{C}^n$ satisfying (59.1.1),

$$w(\partial \mathbb{H}) \subset (H^\alpha_{\epsilon_1})' \tag{59.1.2$'$}$$

and

$$e^{-\alpha\tau} \left| w(e^{\pi(\tau + \sqrt{-1}t)}) - e^{\alpha(\tau - \tau_0 + \sqrt{-1}t)} a \right|_{\mathbb{C}^n} \leq C e^{-c\tau} \tag{59.1.3$'$.a}$$

for some $a \in S^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}^n$, $\tau_0 \in \mathbb{R}$ and $c, C > 0$. (Here the norm in the left hand side is the standard Euclidean norm.)

We denote the set of such w's by $\widetilde{\mathcal{M}}(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{\epsilon_1})', a)$ and denote

$$\begin{cases} \widetilde{\mathcal{M}}(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{\epsilon_1})') = \bigcup_{a \in S^{n-1}} \widetilde{\mathcal{M}}(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{\epsilon_1})', a), \\ \mathcal{M}(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{\epsilon_1})') = \bigcup_{a \in S^{n-1}} \mathcal{M}(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{\epsilon_1})', a), \\ \mathcal{M}(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{\epsilon_1})', a) = \widetilde{\mathcal{M}}(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{\epsilon_1})', a)/\mathrm{Aut}(\mathbb{H}, \{\infty\}), \\ \mathcal{M}(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{\epsilon_1})') = \widetilde{\mathcal{M}}(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{\epsilon_1})')/\mathrm{Aut}(\mathbb{H}, \{\infty\}). \end{cases} \tag{61.8}$$

Proposition 61.9. *There exists a constant $S_0(\alpha)$ independent of ϵ_1 with the following properties : Let $0 < \alpha < \pi$ and $S_0 \ge S_0(\alpha)$. Then $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{\epsilon_1})', a)$ is diffeomorphic to $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; H^\alpha_{\epsilon_1}, a)$.*

Proof. We prove the case $\epsilon_1 > 0$. The case $\epsilon_1 < 0$ is similar. Consider

$$\epsilon_1^{-1/2}(H^\alpha_{\epsilon_1})' = \{\epsilon_1^{-1/2} z \mid z \in (H^\alpha_{\epsilon_1})'\}.$$

By definition of $(H^\alpha_{\epsilon_1})'$, we have

$$\epsilon_1^{-1/2}(H^\alpha_{\epsilon_1})' \cap B(S_0) = H^\alpha_1 \cap B(S_0).$$

In particular the left hand side is independent of ϵ_1.

Moreover the difference between $\epsilon_1^{-1/2}(H^\alpha_{\epsilon_1})'$ and H^α_1 is estimated by a number depending only on S_0 and converging to zero as $S_0 \to \infty$. More precisely we have a family of diffeomorphisms

$$\psi_{S_0} : \mathbb{C}^n \to \mathbb{C}^n$$

such that the following holds for $k = 0, 1, \cdots$, $e^s = |x|$.

$$|\nabla^k(\psi_{S_0} - id)| < \min\{C_k e^{-c_k S_0}, e^{-c_k|s|}\} \tag{61.10.1}$$

$$\psi_{S_0}(H^\alpha_1) = \epsilon_1^{-1/2}(H^\alpha_{\epsilon_1})'. \tag{61.10.2}$$

For each $w \in \widetilde{\mathcal{M}}(\mathbb{H}, \mathbb{C}^n; H^\alpha_1, a_0)$, (61.10.1) implies

$$|\overline{\partial}(\psi_{S_0} \circ w)|(\tau, t) < \min\{Ce^{-cS_0}, Ce^{-c|\tau|}\} :$$

Here the constants C, c are not uniform over w but may depend on w.

Using (61.10) and the Fredholm transversality, we can apply the implicit function theorem to $\psi_{S_0} \circ w$ to obtain an element of $\widetilde{\mathcal{M}}(\mathbb{H}, \mathbb{C}^n; \epsilon_1^{-1/2}(H^\alpha_{\epsilon_1})', a)$ for a sufficiently large S_0.

Conversely for each element $w' \in \widetilde{\mathcal{M}}(\mathbb{H}, \mathbb{C}^n; \epsilon_1^{-1/2}(H^\alpha_{\epsilon_1})', a)$ we find an element of $\widetilde{\mathcal{M}}(\mathbb{H}, \mathbb{C}^n; H^\alpha_1, a_0)$ in a neighborhood of $\psi_{S_0}^{-1} \circ w'$.

Since the rescaling $\epsilon_1^{-1/2} \times : \mathbb{C}^n \to \mathbb{C}^n$ induces a obvious one-one correspondence

$$\widetilde{\mathcal{M}}(\mathbb{H}, \mathbb{C}^n; \epsilon_1^{-1/2}(H^\alpha_{\epsilon_1})', a) \cong \widetilde{\mathcal{M}}(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{\epsilon_1})', a) \tag{61.11}$$

which is equivariant under the action of $\mathrm{Aut}(\mathbb{H}, \{\infty\})$. Proposition 61.9 follows. □

We next determine a good slice of $\widetilde{\mathcal{M}}(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{\epsilon_1})', a)$ for the action $\mathrm{Aut}(\mathbb{H}, \{\infty\})$ so that we have some uniform decay estimates for the representatives in the slice over the elements in the quotient $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{\epsilon_1})', a)$.

Since $\mathrm{Ref}_{\alpha/2}((H^\alpha_{\epsilon_1})') = (H^\alpha_{\epsilon_1})'$ and $\mathrm{Ref}_{\alpha/2}(\mathbb{C} \cdot a) = \mathbb{C} \cdot a$ for $a \in S^{n-1}_{\mathbb{R}^n}$, it follows that $\mathrm{Ref}_{\alpha/2}$ acts on $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{\epsilon_1})', a)$. Moreover, since the action of $\mathrm{Ref}_{\alpha/2}$ on $\mathbb{C}^n$ commutes with $SO(n)$ action, it follows from Proposition 60.52 the following lemma :

Lemma 61.12. *On each* $\mathrm{Aut}(\mathbb{H},\{\infty\})$*-orbit of* $\widetilde{\mathcal{M}}(\mathbb{H},\mathbb{C}^n;(H^\alpha_{\epsilon_1})',a)$*, there exists exactly one element* w *that satisfies*

$$\mathrm{Ref}_{\alpha/2}(w(z)) = w(z^*), \tag{61.13.1}$$

$$e^{-\alpha\tau}\left|w(e^{\pi(\tau+\sqrt{-1}t)}) - e^{\alpha(\tau+\sqrt{-1}t)}a\right|_{\mathbb{C}^n} \leq Ce^{-c\tau}. \tag{61.13.2}$$

for $\tau \geq 0$*. (Note the norm in* (61.13.2) *is the standard Euclidean norm.)*

We remark (61.13.2) means that $(59.1.3'.a)$ holds with $\tau_0 = 0$.

Proof. By Proposition 60.52, we can find a representative that satisfies (61.13.1). Such a representative is unique up to the action of $\mathbb{R}_+ \subset \mathrm{Aut}(\mathbb{H},\{\infty\})$. Here $\mathbb{R}_+$ acts by $v \cdot z = vz$. Then the condition (61.13.2) fixes the unique representative. □

Definition 61.14. We denote by $\widetilde{\mathcal{M}}_0(\mathbb{H},\mathbb{C}^n;(H^\alpha_{\epsilon_1})',a)$ the set of elements $w \in \mathcal{M}(\mathbb{H},\mathbb{C}^n;(H^\alpha_{\epsilon_1})',a)$ satisfying (61.13). We denote any element therein by w_{lmd}. (Here 'lmd' stands for 'local model'.) Then we form the union

$$\widetilde{\mathcal{M}}_0(\mathbb{H},\mathbb{C}^n;(H^\alpha_{\epsilon_1})') = \bigcup_{a\in S^{n-1}} \widetilde{\mathcal{M}}_0(\mathbb{H},\mathbb{C}^n;(H^\alpha_{\epsilon_1})',a)$$

and call any element therein a *normalized* local model.

We remark that $\widetilde{\mathcal{M}}_0(\mathbb{H},\mathbb{C}^n;(H^\alpha_{\epsilon_1})') \cong \mathcal{M}(\mathbb{H},\mathbb{C}^n;(H^\alpha_{\epsilon_1})')$ and $SO(n)$ acts on $\widetilde{\mathcal{M}}_0(\mathbb{H},\mathbb{C}^n;(H^\alpha_{\epsilon_1})')$ via the standard action of $U(n) \supset SO(n)$ on the target $\mathbb{C}^n$. The projection

$$(\pi_1,\pi_2) : W^{1,p}_\delta(\mathbb{H},\mathbb{C}^n;(H^\alpha_\epsilon)') \to S^{n-1}\times\mathbb{R}$$

is defined in an obvious way and induces a projection

$$\pi : \widetilde{\mathcal{M}}_0(\mathbb{H},\mathbb{C}^n;(H^\alpha_{\epsilon_1})') \to S^{n-1}$$

which is equivariant under this $SO(n)$-action.

Lemma 61.15. *Let* $a \in S^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}^n$ *and consider the action of the isotropy group*

$$SO(n)^a := \{g \in SO(n) \mid ga = a\} \cong SO(n-1)$$

on the fiber $\pi^{-1}(a)$*.*
(1) *If* $\epsilon_1 > 0$*, this action is trivial.*
(2) *If* $\epsilon_1 < 0$*, the action has the isotropy group isomorphic to* $SO(n-2)$ *at each* $w \in \pi^{-1}(a)$ *and so induces a diffeomorphism*

$$\pi^{-1}(a) \cong SO(n-1)/SO(n-2) \cong S^{n-2}.$$

Proof. (1) is obvious from construction and from $\#(\widetilde{\mathcal{M}}_0(\mathbb{H},\mathbb{C}^n;(H^\alpha_{\epsilon_1})') = 1$. (2) follows from Theorem 59.2, Propositions 61.9 and 60.52. (We remark that the process to go from $H^\alpha_{\epsilon_1}$ to $(H^\alpha_{\epsilon_1})'$ does not change the symmetry at all.) □

Denote

$$T_{\epsilon_1} = -\frac{1}{\alpha}\log(\sqrt{|\epsilon_1|}S_0) = -\frac{1}{\alpha}\left(\frac{1}{2}\log|\epsilon_1| + \log S_0\right).$$

Lemma 61.16. *Let* $w \in \pi^{-1}(a) \subset \widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{\epsilon_1})')$. *Then we have the following inequality* (61.17) *for* $k = 0, 1, \cdots$, *and for* $\tau \geq 0$.

$$\left|\nabla^k(w - w^{\rm flat}_{a,-\alpha T_{\epsilon_1}})\right|(\tau, t) \leq C_k e^{-c_k(\tau - T_{\epsilon_1})}, \tag{61.17}$$

where

$$w^{\rm flat}_{a,-\alpha T_{\epsilon_1}}(\tau, t) = (\alpha(\tau - T_{\epsilon_1}), e^{\alpha\sqrt{-1}t}a)$$

and C_k, c_k *is independent of* w, a, ϵ_1.

Proof. This is a restatement of Lemma 60.6 except the uniformity of the constants. The uniformity of the estimate (that is independence of C_k, c_k of w, a, ϵ_1) can be proved as follows.

We consider $w \in \widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{\epsilon_1})')$. Then

$$(\tau, t) \mapsto |\epsilon_1|^{-1/2} w(\tau, t) = \widetilde{w}(\tau, t) \tag{61.18}$$

is an element of $\widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H^\alpha_1)')$. (61.17) is equivalent to

$$\left|\nabla^k(\widetilde{w} - w^{\rm flat}_{a,-\alpha T_1})\right|(\tau, t) \leq C_k e^{-c_k(\tau - \alpha T_1)}.$$

(Note $T_1 = -\alpha^{-1}\log S_0$ and the scaling $\epsilon_1^{-1/2}\times : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^n \setminus \{0\}$ corresponds to the translations $(\tau, x) \mapsto (\tau - \frac{1}{2}\log|\epsilon_1|, x)$ in cylindrical coordinate.)

We use Fredholm theory to show that the constant C_k, c_k can be taken uniformly as far as w lies in a small open set of our moduli space $\widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H^\alpha_1)')$. Then, since w, a run on compact space, the uniformity of the estimate follows. □

Let $\overline{\partial}_{J'}$ be the $\overline{\partial}$ operator on $\mathbb{R} \times S^{2n-1}$ with respect to the pull back almost complex structure $(\varphi \circ \exp_{p^I_{12}})^* J =: J'$, where φ is as in (61.1.2).

Lemma 61.19. *We have*

$$w_{\rm lmd}(\partial\mathbb{H}) \subset (H^\alpha_{\epsilon_1})', \tag{61.20.1}$$

$$|\overline{\partial}_{J'}\widetilde{w}_{\rm lmd}(\tau, t)|_{g'_{\mathbb{R}\times S^{2n-1}}} < Ce^{-c\tau} \quad \text{for } \tau \geq 0. \tag{61.20.2}$$

Proof. (61.20.1) is an immediate consequence of the construction. By definition $\overline{\partial}_{J_0}\widetilde{w}_{\rm lmd} = 0$. (61.20.2) then follows from (61.8.2). □

61.4. Pseudo-holomorphic triangle in cylindrical coordinate.

We identify $\operatorname{Int} D^2 = \operatorname{Int} \mathbb{H}$ and put

$$u_{20} = u_{02} = -1, \quad u_{21} = u_{12} = 0, \quad u_{10} = u_{01} = +1.$$

Let

$$w_{\rm tri} = w : D^2 = \mathbb{H} \cup \{\infty\} \to M$$

be an element of

$$\mathcal{M}((L_0, L_2, L_1); (u_{02}, u_{21}, u_{10}); J),$$

defined in §54.3. (Here 'tri' stands for 'triangle'.) In other words

$$w_{\rm tri}(0) = p_{12}, \quad w_{\rm tri}(1) = p_{01}, \quad w_{\rm tri}(-1) = p_{20}. \tag{61.21}$$

and $w_{\rm tri}$ is a pseudo-holomorphic map

$$w_{\rm tri} : \mathbb{H} \cup \{\infty\} \to M$$

such that

$$w_{\rm tri}([-1, 0]) \subset L_2 \tag{61.22.1}$$

$$w_{\rm tri}([0, +1]) \subset L_1 \tag{61.22.2}$$

$$w_{\rm tri}((-\infty, -1] \cup [+1, \infty)) \subset L_0. \tag{61.22.3}$$

Remark 61.23. We here remark one rather confusing point of our notation. In Theorem 55.3 we start with an element of

$$\mathcal{M}((L_0, L_1, L_2); (u_{01}, u_{12}, u_{20}); J),$$

and then Theorem 55.3 asserts that we can find an element of (resp. S^{n-2} parametrized family of elements of)

$$\mathcal{M}((L_1 \#_{\epsilon_1} L_2, L_0), (u_{01}, u_{20}), J; w_{\rm tri}, \epsilon_2)$$

for $\epsilon_1 < 0$ (resp. $\epsilon_1 > 0$).

In this section, we start with an element of

$$\mathcal{M}((L_0, L_2, L_1); (u_{02}, u_{21}, u_{10}); J).$$

So to prove Theorem 55.3 it suffices to find an element of (resp. S^{n-2} parametrized family of element of)

$$\mathcal{M}((L_1 \#_{\epsilon_1} L_2, L_0), (u_{12}, u_{20}), J; w_{\rm tri}, \epsilon_2),$$

for $\epsilon_1 > 0$ (resp. $\epsilon_1 < 0$). In §61 and §62, we prove this statement. In fact, we have

$$L_1 \#_{\epsilon_1} L_2 \cong L_2 \#_{-\epsilon_1} L_1.$$

In other words, the signs of ϵ_1 appear in an opposite way in §61, §62 and in §55. See Figures 61.1 and 61.2.

Figure 61.1.

Figure 61.2.

We put

$$\mathbb{H}_{|z|<o} = \{z \in \mathbb{H} \mid |z| < o\}.$$

We may choose a positive number o so that

$$w_{\rm tri}(\mathbb{H}_{|z|<o}) \subset B(p_{12}; \epsilon_0).$$

We may choose Darboux chart φ at p_{12} such that the tangent cone (Definition 54.19) of $w_{\rm tri}$ at $u_{12} = 0$ is

$$z \mapsto z^{\alpha/\pi} a_0, \qquad a_0 = (1, 0, \cdots, 0).$$

Furthermore the multiplicity one condition at $0 \in \mathbb{H}$ on $w_{\rm tri}$ implies that w is embedded on $\mathbb{H}_{|z|<o}$ for a sufficiently small $o > 0$. In particular, we have

$$w_{\rm tri}(\mathbb{H}_{0<|z|<o}) \subset B(p_{12}; \epsilon_0) \setminus \{p_{12}\}$$

which enables us to define

$$\widetilde{w}_{\rm tri} : \mathbb{H}_{0<|z|<o} \to \mathbb{R} \times S^{2n-1}$$

by

$$\widetilde{w}_{\rm tri} = \varphi^{-1} \circ w_{\rm tri}.$$

Lemma 61.24. *There exist* $\tau_{\rm tri}, C_k > 0, c_k > 0$ *such that*

$$\left|\nabla^k(\widetilde{w}_{\rm tri} - w^{\rm flat}_{a_0,-\alpha\tau_{\rm tri}})\right|(\tau, t) \le C_k e^{c_k\tau}, \tag{61.25}$$

for $\tau \le 0$. *Here*

$$w^{\rm flat}_{a,-\alpha\tau_{\rm tri}}(\tau, t) = (\alpha(\tau - \tau_{\rm tri}), e^{\alpha\sqrt{-1}t} a_0).$$

We remark that the sign $c_k\tau$ is opposite to those appearing in Lemma 61.16 etc. This is because we study the asymptotic behavior as $\tau \to -\infty$ here but we do as $\tau \to +\infty$ in Lemma 61.16. Lemma 61.24 follows from Theorem 54.17 and the multiplicity one assumption we put on $w_{\rm tri}$.

61.5. Pregluing.

Note that, in §61.1, we took a constant ϵ_0 that is the size of the Darboux neighborhood of p_{12} in M. ϵ_0 depends only on M. We take the constant S_0 appeared in §54.1 so that $S_0 \ge S_0(\alpha)$ where $S_0(\alpha)$ is as in Proposition 61.9. It is large and can be taken independent of ϵ_1. The *positive* number ϵ_1 parametrizes the way how we perform the Lagrangian surgery to obtain L_{ϵ_1}. The number ϵ_1 may depend on ϵ_0 and S_0.

We define $R_{\epsilon_1} > 0$ so that

$$-T_{\epsilon_1} + R_{\epsilon_1} + 1 = \tau_{\rm tri} + (2\alpha)^{-1}\log\epsilon_0. \tag{61.26}$$

We remark that by taking ϵ_1 small compared to ϵ_0 and e^{-S_0} we may assume $R_{\epsilon_1} > 0$. Note that $R_{\epsilon_1} \to \infty$ as $\epsilon_1 \to 0$ by the definition and $T_{\epsilon_1} = -\alpha^{-1}(\frac{1}{2}\log\epsilon_1 + \log S_0) > 0$. From now one we take ϵ_1 sufficiently small. (We do not change ϵ_0, S_0.)

We consider a normalized local model

$$w_{\rm lmd}^{\epsilon_1} \in \widetilde{\mathcal M}_0(\mathbb H, \mathbb C^n; (H^{\alpha}_{\epsilon_1})').$$

We denote the right hand side of (61.26)

$$\tau'_{\rm tri} = \tau_{\rm tri} + (2\alpha)^{-1}\log \epsilon_0.$$

We recall from Proposition 60.59, that the image of $w_{\rm lmd}^{\epsilon_1}$ is away from the origin $0 \in \mathbb C^n$. Therefore we can define

$$\widetilde w_{\rm lmd}^{\epsilon_1} = \varphi^{-1} \circ w_{\rm lmd}^{\epsilon_1} : \mathbb H \to \mathbb R \times S^{2n-1}.$$

We denote the annuli domain

$$A(\epsilon_1) = \{z = e^{\pi(\tau + \sqrt{-1}t)} \mid -T_{\epsilon_1} + R_{\epsilon_1}/2 < \tau < -T_{\epsilon_1} + R_{\epsilon_1}/2 + 1, t \in [0,1]\}. \tag{61.27}$$

It follows from Lemmata 61.19 and 61.24 that we can write

$$\widetilde w_{\rm tri}(\tau,t) = \exp_{\widetilde w_{\rm lmd}^{\epsilon_1}(\tau,t)}(X(\tau,t))$$

on $A(\epsilon_1)$ for

$$X(\tau,t) \in T_{\widetilde w_{\rm lmd}^{\epsilon_1}(\tau,t)}(\mathbb R \times S^{2n-1})$$

if ϵ_1 is sufficiently small. Here exp is the exponential map of the Riemannian manifold $\mathbb R \times S^{2n-1}$. We also have

$$|(\nabla^k X)(\tau,t)| \le C_k e^{-c_k R_{\epsilon_1}} \tag{61.28}$$

on $A(\epsilon_1)$. We take a cut-off function

$$\chi_{T_{\epsilon_1},R_{\epsilon_1}} : \mathbb R \to [0,1]$$

such that

$$\chi_{T_{\epsilon_1},R_{\epsilon_1}}(\tau) = \begin{cases} 0 & \tau < -T_{\epsilon_1} + R_{\epsilon_1}/2 \\ 1 & \tau > -T_{\epsilon_1} + R_{\epsilon_1}/2 + 1. \end{cases} \tag{61.29}$$

Using the cylindrical coordinates $\mathbb{C}^n \setminus \{0\} \cong \mathbb{R} \times S^{2n-1}$, we now glue $w^{\epsilon_1}_{\rm lmd}$ and $w_{\rm tri}$ on the annuli domain $A(\epsilon_1)$. See Figure 61.3.

Figure 61.3.

Definition 61.30. We define

$$w_{\rm app} = w^{\epsilon_1}_{\rm lmd} \# w_{\rm tri} : \mathbb{H} \to M$$

as follows (Here 'app' stands for approximate solution.) :

$$w_{\rm app}(z) = \begin{cases} \varphi \circ \exp_{\widetilde{w}^{\epsilon_1}_{\rm lmd}(\tau,t)}(\chi_{T_{\epsilon_1},R_{\epsilon_1}}(\tau) X(\tau,t)), & z = e^{\pi(\tau+\sqrt{-1}t)} \in A(\epsilon_1), \\ w_{\rm tri}(z), & |z| > e^{-T_{\epsilon_1}+R_{\epsilon_1}/2+1}, \\ w^{\epsilon_1}_{\rm lmd}(z), & |z| < e^{-T_{\epsilon_1}+R_{\epsilon_1}/2}. \end{cases}$$

Lemma 61.31. *$w_{\rm app}$ has the following properties.*

(1) $w_{\rm app}([-1,1]) \subset L_{\epsilon_1}$.
(2) $w_{\rm app}((-\infty,-1] \cup [1,\infty)) \subset L_0$.
(3) *On* $(-\infty, \alpha^{-1}\log\epsilon_0] \times [0,1]$ *we have*

$$|(\overline{\partial}_J w_{\rm app})(\tau,t)| < Ce^{-cR_{\epsilon_1}}. \tag{61.32}$$

(4) $w_{\rm app}$ *is J-holomorphic outside the domain* $(-\infty, \alpha^{-1}\log\epsilon_0] \times [0,1]$.

Proof. (1),(2),(4) are obvious from construction. (Note that R_{ϵ_1} is sufficiently large.) Let us prove (61.32). By construction $w_{\rm app}$ is equal to $w_{\rm tri}$ outside $(-\infty, -T_{\epsilon_1} + R_{\epsilon_1}/2+1) \times [0,1]$ and hence is J holomorphic there. On $(-\infty, -T_{\epsilon_1} + R_{\epsilon_1}/2+1) \times [0,1]$, we note

$$\mathrm{dist}_{\mathbb{C}^n}(p_{12}, w_{\rm app}(\tau,t)) < Ce^{c(-T_{\epsilon_1}+R_{\epsilon_1}/2+1)}$$

which in turn implies

$$|J - I^* J_0| < Ce^{-cR_{\epsilon_1}}$$

since $-T_{\epsilon_1} + R_{\epsilon_1}/2 + 1 < -R_{\epsilon_1}/2 + \tau_{\rm tri}$. (Here C may depend on $\tau_{\rm tri}$.) Therefore it does not matter whether we use J or $I^* J_0$ to prove (61.32). The inequality (61.32) then follows from (61.20.2) and (61.28). □

Lemma 61.31 implies that $w_{\rm app}$ provides a good approximate solution of the equation we want to solve.

61.6. Weighted Sobolev norm and a right inverse.

Let o_k be a sequence of positive numbers such that $\lim_{k\to\infty} o_k = 0$. We denote by C_k a sequence of positive numbers which are independent of ϵ_1. We consider the set of maps of the form

$$w : (\mathbb{H}, \partial\mathbb{H}; -1, 1) \to (M, L_{\epsilon_1} \cup L_0; p_{20}, p_{01})$$

such that

$$w(\tau, t) = \exp_{w_{\rm app}}(Y(\tau, t))$$

with pointwise bounds

$$|(\nabla^k Y)(\tau, t)| \le o_k. \tag{61.33}$$

Here in (61.33) we use the following metric g'_M on M. We decompose

$$M = B(p_{12}; \epsilon_0) \cup (M \setminus B(p_{12}; \epsilon_0))$$

and

$$B(p_{12}; \epsilon_0) = B(p_{12}; S_0\sqrt{|\epsilon_1|}) \cup (B(p_{12}; \epsilon_0) \setminus B(p_{12}; S_0\sqrt{|\epsilon_1|})).$$

Let g_M, $g_{\mathbb{R}\times S^{2n-1}}$, $g_{\mathbb{C}^n}$ be the metric on M, standard metrics on $\mathbb{R} \times S^{2n-1}$ and on $\mathbb{C}^n$, respectively. We equip a metric g'_M adapted to this decomposition by

$$g'_M = \begin{cases} (\epsilon_0)^{-1/2} g_M & \text{on } M \setminus B(p_{12}; \epsilon_0) \\ \varphi_* g_{\mathbb{R}\times S^{2n-1}} & \text{on } (B(p_{12}; \epsilon_0) \setminus B(p_{12}; S_0\sqrt{|\epsilon_1|})) \\ (S_0\sqrt{|\epsilon_1|})^{-1/2} I^* g_{\mathbb{C}^n} & \text{on } B(p_{12}; S_0\sqrt{|\epsilon_1|}) \end{cases}$$

with a suitable smoothing along the gluing hypersurfaces : Here we note that the restrictions of the metrics $(\epsilon_0)^{-1/2} I_* g_M$ and $(S_0\sqrt{|\epsilon_1|})^{-1/2} g_{\mathbb{C}^n}$ on their boundaries provide a family of metrics uniformly quasi-isometric to S^{2n-1} with the standard metric over all $\epsilon_0 > 0$ smaller than a constant depending only on M, and ϵ_1 satisfying

$0 < S_0\sqrt{|\epsilon_1|} < \epsilon_0$. Therefore we will use the metric g'_M in our derivation of the required uniform estimates for w independent of ϵ_1.

Figure 61.4.

We take a smooth function $\rho : (-\infty, \tau'_{\rm tri}] \to \mathbb{R}_{>0}$ such that

$$\rho(\tau) = \begin{cases} 1 & \tau \le -T_{\epsilon_1}, \\ e^{\delta|\tau + T_{\epsilon_1}|} & -T_{\epsilon_1} \le \tau \le -T_{\epsilon_1} + R_{\epsilon_1}/2, \\ e^{\delta R_{\epsilon_1}/2} & -T_{\epsilon_1} + R_{\epsilon_1}/2 \le \tau \le -T_{\epsilon_1} + R_{\epsilon_1}/2 + 1, \\ e^{\delta|\tau - \tau'_{\rm tri}|} & -T_{\epsilon_1} + R_{\epsilon_1}/2 + 1 \le \tau \le \tau'_{\rm tri}. \end{cases} \tag{61.34}$$

See Figure 61.5. (Note $\tau'_{\rm tri} = \tau_{\rm tri} + (2\alpha)^{-1}\log\epsilon_0 = -T_{\epsilon_1} + R_{\epsilon_1} + 1$.)

Figure 61.5.

We use this metric g'_M and weight function ρ to define a weighted Sobolev space $W^{1,p}_\rho(w^*TM; w^*(L_{\epsilon_1}))$ in the following way.

Let V be a section of w^*TM (defined on $\mathbb{H}$) of locally $W^{1,p}$ class. We define its $W^{1,p}_\rho$ norm by

$$\begin{aligned}
&\|V\|^p_{1,p,\rho}\\
&= |V(-T_{\epsilon_1} + R_{\epsilon_1}/2 + 1/2, 1/2)|^p + \int_{z\in\mathbb{H}, |z|>e^{\pi\tau'_{\text{tri}}}} \left(|\nabla V|^p_g + |V|^p_g\right) dz \\
&\quad + \int_{(-\infty,\tau'_{\text{tri}}]\times[0,1]} \rho(\tau)\left(|\nabla(V - V_0)|^p_{g_{\mathbb{R}\times S^{2n-1}}} + |(V - V_0)|^p_{g_{\mathbb{R}\times S^{2n-1}}}\right) dA_{h_{\mathbb{H}}},
\end{aligned} \tag{61.35}$$

where V_0 will be defined by (61.39) below. Here $dA_{h_{\mathbb{H}}}$ is the volume element of the metric $h_{\mathbb{H}} = (|z|')^{-2}|dz|^2$. We consider the decomposition

$$T_{w(\tau,t)}(\mathbb{R}^1 \times S^{2n-1}) \cong T_{s(\tau,t)}\mathbb{R}^1 \oplus T_{\Theta(\tau,t)}S^{2n-1}. \tag{61.36}$$

(Here we write $w(\tau, t) = (s(\tau, t), \Theta(\tau, t)) \in \mathbb{R} \times S^{2n-1}$.) We decompose

$$V(-T_{\epsilon_1} + R_{\epsilon_1}/2 + 1/2, 1/2) = V_{0,s} \oplus V_{00,\Theta} \tag{61.37}$$

according to (61.36).

We next consider $so(n) = so(n-1) \oplus \mathbb{R}^{n-1}$ where $so(n-1)$ is the isotropy group of $a = (1, 0, \cdots, 0)$. We fix complement $\mathbb{R}^{n-1}$ of this direct sum decomposition, and let $A_1, \cdots, A_{n-1}$ be a basis of it. We then take the orthonormal decomposition

$$T_{\Theta(\tau,t)}S^{2n-1} = \mathbb{R}^{n-1} \oplus (\mathbb{R}^{n-1})^\perp \tag{61.38}$$

where the first component $\mathbb{R}^{n-1}$ is spanned by

$$A_i(\Theta(\tau, t)), \qquad i = 1, 2, \cdots.$$

(Note element of $so(n) \subset u(n) \subset so(2n)$ induces a vector field on S^{2n-1}. In the above formula, we denote by A_i the vector field induced by A_i.) Let

$$V_{0,\Theta}(-T_{\epsilon_1} + R_{\epsilon_1}/2 + 1/2, 1/2) = \sum_{i=1}^{n-1} a_i A_i(\Theta(-T_{\epsilon_1} + R_{\epsilon_1}/2 + 1/2, 1/2))$$

be the projection of $V_{00,\Theta}$ to the first component of (61.38).

Next take a smooth function $\chi : \mathbb{R} \to [0, 1]$ such that

$$\chi(\tau) = \begin{cases} 1 & \tau < \tau'_{\text{tri}} - 1, \\ 0 & \tau > \tau'_{\text{tri}} \end{cases}$$

and define

$$V_0(\tau, t) = \chi(\tau)V_{0,s} \oplus \chi(\tau)\sum a_i A_i(\Theta(\tau, t)). \tag{61.39}$$

Here we identify $T_{s(\tau,t)}\mathbb{R} \cong T_{s(-T_{\epsilon_1}+R_{\epsilon_1}/2+1/2,1/2)}\mathbb{R}$ in an obvious way. This finishes the description of the norm $\|V\|_{1,p,\rho}$.

Definition 61.40. We define $W^{1,p}_{\rho}(w^*TM;w^*T(L_{\epsilon_1}))$ as the set of all sections V of w^*TM which satisfy the following conditions.

(61.41.1) V is locally of $W^{1,p}$ class.
(61.41.2) $\|V\|_{1,p,\rho} < \infty$.
(61.41.3) $V(z) \in T_{w(z)}(L_{\epsilon_1})$ if $z \in \partial\mathbb{H}$.

Note that $V_0(\tau,t)$ satisfies the boundary condition (61.41.3) and so the boundary condition is consistent with the norm $\|V\|_{1,p,\rho}$. And since $W^{1,p}$ section is continuous, it makes sense to put boundary condition (61.41.3).

We remark that the definition here is similar to that of the norm $\|(V,\vec{v})\|_{1,p,\alpha}$ appearing right before (29.26) in §29.

The space $W^{1,p}_{\rho}(w^*TM;w^*T(L_{\epsilon_1}))$ is a Banach space with norm $\|V\|_{1,p,\rho}$.

We next define $L^p_{\rho}(\Lambda^{0,1}(w^*TM\otimes))$.

Definition 61.42. $L^p_{\rho}(\Lambda^{0,1}(w^*TM\otimes))$ is the set of all sections V of $\Lambda^{0,1}(w^*TM\otimes)$ on $\mathbb{H}$ locally of L^p class such that

$$\begin{aligned}\|V\|^p_{p,\rho} = &\int_{z\in\mathbb{H},|z|>e^{\pi\tau'_{\rm tri}}} |V|^p_{g'_M,h_{\mathbb{H}}} dz\\ &+ \int_{(-\infty,\tau'_{\rm tri}]\times[0,1]} \rho(\tau)|V|^p_{g_{\mathbb{R}\times S^{2n-1}}} dA_{h_{\mathbb{H}}} < \infty,\end{aligned}$$

where $dA_{h_{\mathbb{H}}}$ is as in (61.35).

Lemma 61.43. *If the constant δ in* (61.34) *is smaller than a positive constant (which is independent of ϵ_1) then the operator*

$$D_w\overline{\partial}_J : W^{1,p}_{\rho}(w^*TM;w^*TL_{\epsilon_1}) \to L^p_{\rho}(\Lambda^{0,1}(w^*TM\otimes))$$

is Fredholm and is bounded by a constant independent of small constants ϵ_1.

Proof. By choosing δ sufficiently small in the definition of the weight function ρ, we obtain

$$\|D_w\overline{\partial}_J(V_0)\|_{p,\rho} < Ce^{-cR_{\epsilon_1}}|V(-T_{\epsilon_1}+R_{\epsilon_1}/2+1/2,1/2)|.$$

Uniform bound of $D_w\overline{\partial}_J$ follows from this fact. The rest of the proof is by now standard and omitted. □

Proposition 61.44. *If the constant δ in* (61.34) *is smaller than a positive constant (which is independent of ϵ_1) then there exists*

$$Q_w : L^p_{\rho}(\Lambda^{0,1}(w^*TM\otimes)) \to W^{1,p}_{\rho}(w^*TM;w^*T(L_{\epsilon_1}))$$

such that

$$D_w\overline{\partial}_J \circ Q_w = \textit{identity}$$

and that the operator norm of Q_w is bounded by a constant independent of small constants ϵ_1.

Proof. This proposition is a consequence of Theorem 60.26 and the same argument as the proof of Proposition 29.27. In fact Theorem 60.26 implies that the operator

$$C^0(w_{\rm lmd}) \to C^1(w_{\rm lmd}) \oplus T_{(\tau_0,a)}(\mathbb{R} \times S^{n-1}) \tag{61.45}$$

is surjective. On the other hand, we assumed that the solution $w_{\rm tri}$ of $\overline{\partial}_J$ equation is Fredholm transversal i.e., its linearized operator is surjective. Our operator $D_w\overline{\partial}_J$ is obtained by gluing these two operators in the same way as in §29. Hence the construction of its right inverse Q_w is the same as the proof of Proposition 29.27. We remark that including the second factor $T_{(\tau_0,a)}(\mathbb{R} \times S^{n-1})$ to the surjectivity of (61.45) is crucial here for the surjectivity of the glued operator $D_w\overline{\partial}_J$: This corresponds to the transversality of the evaluation maps which appeared in §29 and played a crucial role in the proof of Lemma 29.20 there. □

Now we are in the position to complete the first half of the proof of Theorem Z. Let c be the constant in (61.32). Taking a δ smaller than $c/3$, we derive the error bound

$$\|(\overline{\partial}_J w_{\rm app})\|_{p,\rho} < Ce^{-cR_{\epsilon_1}/2}$$

from (61.32). Combining Proposition 61.44 and (61.32), we can perturb $w_{\rm app}$ to find a J-holomorphic curve w of the form

$$w(\tau,t) = \exp_{w_{\rm app}}(Y(\tau,t))$$

with $\|Y\| < Ce^{-cR_{\epsilon_1}/2}$. The argument of this step is by now standard and omitted : it has been carried out in many literature in various contexts starting from Taubes' celebrated work on the existence of anti-self-dual connections on 4 manifolds. For the case of the pseudo-holomorphic curve, a similar argument can be found, for example, in [MaSa94].

So far we have discussed the case of $\epsilon_1 > 0$. The case of $\epsilon_1 < 0$ can be treated by the same way, except that we start with the S^{n-2}-family of $w_{\rm lmd}$'s in place of a single $w_{\rm lmd}$. (Note $\widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{-\epsilon_1})', a_0) \cong S^{n-2}$. This provides representative of each element of $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{-\epsilon_1})', a_0) \cong S^{n-2}$.)

Now we summarize the result of this section as follows (see also Remark 61.23) :

Theorem 61.46. *Let J and $w_{\rm tri}$ satisfy* (55.1) *and* (61.21), (61.22), (55.2) *respectively. Then for each sufficiently small ϵ_2 and ϵ_1 with $|\epsilon_1| < \epsilon_2^{100}$ we have the following :*

(1) *If $\epsilon_1 > 0$, then $\mathcal{M}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J; w_{\rm tri}, \epsilon_2)$ contains an element which is Fredholm regular.*

(2) *If $\epsilon_1 < 0$, then $\mathcal{M}((L_{\epsilon_1}, L_0), (u_{01}, u_{20}), J; w_{\rm tri}, \epsilon_2)$ contains an S^{n-2} parametrized family of elements. Each element of it is Fredholm regular.*

Remark 61.47. Bourgeois [Bou02] previously studied a gluing problem similar to that of this section in the context without Lagrangian boundary condition. More precisely speaking, he looked at the case of pseudo-holomorphic annuli in the symplectization of a contact manifold where the moduli space of closed Reeb orbits is not isolated but forms a Bott-Morse family. Both the gluing analysis in [Bou02] (that is, §5.3.3 thereof) and the one in this section are somewhat similar to the one given in [FOOO00] §18 or in [Fuk96II]. This kind of gluing analysis in the context that is degenerate at infinity, has its origin in Mrowka's thesis [Mrow89]. However the decay estimate carried out in §62 (or in [Bou02]), which is crucial for the proof of compactness etc. in the study of proper pseudo-holomorphic curves in the symplectization of a contact manifold, uses ideas different from those needed in §18 [FOOO00] or in [Fuk96II]. This, especially the idea of using the λ-energy, is due to Hofer [Hof93]. (See the next section.) We also like to mention that there are other references such as [Abb04] closely related to the content of this section.

Remark 61.48. We take this opportunity to point out that it is safe to say that only the parts of gluing analysis and decay estimate from [Bou02] are salient enough. This is because there are some essential drawbacks in other parts of [Bou02]. It seems to us that many points that we carefully discussed in this book or in [FOOO00] should appear in a similar way in the context of [Bou02].

More specific concerns of ours lie in the following points (1),(2),(3) concerning the reference [Bou02] :

(1) The *statements* of Propositions 6.4 and 6.5 [Bou02] do not make much sense as they are :

(1.a) The notion of *relative* virtual cycles is not defined.

(1.b) The *isotopy* class of virtual fundamental chain (or cycle) will depend on the choice of perturbations in general.

As far as we see, there seems to be no reasonable way to make sense out of the statements of Propositions 6.4 and 6.5 [Bou02]. This is because there is no natural way of stating the well-definedness of virtual fundamental chains/cycles in terms of a *single* moduli space, e.g., if we fix a homology class of the relevant pseudo-holomorphic maps. In general Floer theory in which bubbling phenomena are present, the matrix coefficients of the boundary operator do depend on the choice of perturbations even in the simplest case. What is well-defined is the chain homotopy class of a chain complex. This invariant encodes characteristics of the virtual fundamental chains/cycles of *many* moduli spaces. (This point has been mentioned many times throughout this book.) It turns out that a relevant homological algebra should be developed in order to formulate a correct statement on this kind of well-definedness. In relation to this, we have developed full details of this homological algebra in many parts of Chapters 3-5 of this book.

(2) In order to be able to apply virtual fundamental chain techniques to the problem of studying the *adiabatic limit* $\epsilon \to 0$ as in the context of §11 of [BEHWZ03],

one needs to construct a Kuranishi neighborhood (or 'virtual neighborhood') which contains both the limiting moduli space and the one near to the limit. Here the limiting moduli space involves both Morse gradient trajectories and pseudo-holomorphic maps. Defining such a Kuranishi neighborhood is a highly non-trivial problem, which has not been carried out in the existing literature yet.

In [Bou02], there is some discussion on the compactness statement on this limiting problem in the context where one perturbs the moduli space of proper pseudo-holomorphic curves with cylindrical ends, by a Morse function defined on the Morse-Bott critical manifold consisting of Reeb orbits. However [Bou02] lacks the relevant Fredholm theory which is crucial for construction of a Kuranishi neighborhood.

(3) In page 80 [Bou02], it is casually stated that one uses an induction on energy and etc. to construct a coherent system of multi-sections. However as we demonstrated in §30.2 of this book, this induction does not seem possible with the induction over the energy alone, when one needs to use *fiber* products of various moduli spaces.

As a consequence it is *very difficult* to achieve transversality via perturbations of the critical submanifold with a *single* Morse function, if possible at all : This makes hardly convincing the author's claim in [Bou02] that this can be done.

On the the other hand, several calculations involving the contact homology carried out in [Bou02] are very interesting. One needs to resolve this transversality matter in order to justify his calculation. Various techniques laid out in §30 are developed to achieve this kind of transversality via the framework of singular homology instead of the analytically much harder framework of taking the adiabatic degeneration of perturbations of small Morse functions. Alternatively we can apply the method of continuous family of perturbations (see §33) using the de Rham theory.

§62. Proof of Theorem Z, II : No other solutions

62.1. Statement of the results and outline of its proofs.

In this section we prove that the pseudo-holomorphic strips between L_0 and L_ϵ we produced in Theorem 61.46 exhausts all the solutions nearby the given pseudo-holomorphic triangle, and complete the proof of Theorem Z. A more detailed description on what is achieved in this section is in order.

The proof of similar 'surjectivity' is one of the essential components of the study of moduli spaces of pseudo-holomorphic curves in non-compact symplectic manifolds with cylindrical ends. There are many announced results related to various gluing formulae in the literature based on some degeneration and compactness arguments but without treating this surjectivity problem in detail. (See [EGH00] and others, for example.) The proof of 'surjectivity' is closely related to but is harder to study than that of 'compactness'. However we hardly find a proof of this kind of 'surjectivity' in the literature that is applicable to the case we study in this book. Because of this we will give a complete self-contained proof of this surjectivity in our context.

Many of the methods we use in this section can be used in a more general situation. Since it is not our main purpose in this book to study pseudo-holomorphic curve in noncompact symplectic manifold, we will restrict ourselves to the case directly relevant to prove Theorem Z. We do not attempt to discuss the general case of pseudo-holomorphic maps from a bordered Riemann surface to non-compact symplectic manifold with Lagrangian boundary conditions with cylindrical ends. Instead we take short cuts in several places exploiting the special feature of our situation.

We state our result of this section only for the harder case of $L_{-\epsilon_1}$ ($\epsilon_1 > 0$). The case of $L_{+\epsilon_1}$ is similar and easier to deal with. Let $w_{\rm tri}$ be the pseudo-holomorphic triangle regarded it as a map $w_{\rm tri} : \mathbb{H} \to M$ satisfying the boundary condition

$$w_{\rm tri}([-1,0]) \subset L_2,\ w_{\rm tri}([0,1]) \subset L_1,\ w_{\rm tri}(\mathbb{R} \setminus [-1,1]) \subset L_0.$$

(See Remark 61.23.) We assume (55.2). For the notational convenience, we will just denote $\mathbb{H}$ for $\mathbb{H} \cup \{\infty\}$ from now on, as long as there is no danger of confusion.

Consider the family of solutions

$$w_b : \mathbb{H} \to M$$

constructed by Theorem 61.46, which is parameterized by $b \in S^{n-2}$. These have the following properties :

(62.1.1) w_b is pseudo-holomorphic.
(62.1.2) $w_b(z) \in L_{-\epsilon_1}$ for $z \in [-1,1] \subset \mathbb{R} = \partial\mathbb{H} \setminus \{\infty\}$.
(62.1.3) $w_b(z) \in L_0$ for $z \in \mathbb{R} \setminus [-1,1] \subset \mathbb{R} = \partial\mathbb{H} \setminus \{\infty\}$.

By construction, w_b is C^0-close to $w_{\rm tri}$.

Theorem 62.2. *If ϵ_1, ϵ_2 be sufficiently small positive numbers with $\epsilon_1 < \epsilon_2^{100}$, then, for any $w : \mathbb{H} \to M$ such that*

(62.3.1) $\max_{z\in\mathbb{H}} {\rm dist}_{g_M}(w_{\rm tri}(z), w(z)) < \epsilon_2$,
(62.3.2) $w(z) \in L_{-\epsilon_1}$ *for* $z \in [-1,1] \subset \mathbb{R} = \partial\mathbb{H} \setminus \{\infty\}$,
(62.3.3) $w(z) \in L_0$ *for* $z \in \mathbb{R} \setminus [-1,1] \subset \mathbb{R} = \partial\mathbb{H} \setminus \{\infty\}$,

there exists a biholomorphic map $\psi : \mathbb{H} \to \mathbb{H}$ with $\psi(\pm 1) = \pm 1$ and $b \in S^{n-2}$ such that

(62.4) $$w = w_b \circ \psi.$$

Theorem 62.2 together Theorem 61.46 completes the proof of Theorem Z.

The strategy of the proof of Theorem 62.2 is similar to the one, which is originally due to Donaldson [Don83] and §9 [FrUh84]. This proceeds as follows :

(62.5.1) By an index calculation we find that the S^{n-2}-parameterized family of the pseudo-holomorphic curves in Theorem 61.46 is of 'correct dimension'. In other words, the virtual dimension of the pseudo-holomorphic maps w satisfying (62.3) is $n-2$ (modulo the action of $\mathrm{Aut}(\mathbb{H};\{\pm 1\})$).
(62.5.2) If $\epsilon_2 > 0$ is sufficiently small, for any general solution w satisfying (62.3), we find a path $w(r)$ such that $w(0) = w$ and $w(1) = w_b$ for some $b \in S^{n-2}$.
(62.5.3) Using the implicit function theorem and Fredholm regularity of w_b's, we modify the path $w(r)$ to $w'(r)$ so that each element of the path $w'(r)$ is pseudo-holomorphic and satisfies (62.3) and $w'(0) = w$, $w'(1) = w_b$.
(62.5.4) Now (62.5.1) and (62.5.3) imply that $w = w_{b'}$ for some b' (modulo the action of $\mathrm{Aut}(\mathbb{H},\{\pm 1\})$). This completes the proof of Theorem 62.2.

To carry out the strategy laid out here, we need to employ several new ingredients that are not needed in the works such as in [Don83] and [FrUh84]. We highlight a few main points below. (The most essential one is (62.6.3) among them.)

(62.6.1) We need to consider the singular degeneration as $\epsilon_1 \to 0$ where the Lagrangian submanifolds $L_{-\epsilon_1}$ becomes singular in the limit. Therefore to carry out (62.5.2) and (62.5.3), we first need to improve the estimate from (62.3.1) to a much sharper one. (Note a similar situation appeared in [FuOh97].)
(62.6.2) To handle the singular degeneration problem mentioned in (62.6.1), we need to use a carefully chosen weighted Sobolev norm described in §61.6.
(62.6.3) To obtain the uniform estimate for the weighted Sobolev norm mentioned in (62.6.2), we start with certain energy estimates. Such an energy estimate at the 'neck region' is far from being standard. This is because we need to blow up the metric of the domain and the target simultaneously. As a consequence the boundedness of the usual symplectic area $\int w^*\omega$ (which follows from (62.3.1)) does *not* provide the energy estimate we need. To overcome this subtlety we use the idea of λ-energy due to Hofer [Hof93].
(62.6.4) We also need to carefully choose the domain coordinates and the target metrics for the estimates.

We remark that points (62.6.3), (62.6.4) do not appear when we prove a similar 'surjectivity' results in the situation of §29.

62.2. Statements of the main estimates : Beginning of the proof.

Let $\epsilon_0 > 0$ be the constant given in §61. Recall that this constant depends only on the size of Darboux chart.

Consider arbitrary sequences of $\epsilon_{1,i}$, $\epsilon_{2,i} > 0$ and w_i such that

(62.7.1) $\lim_{i\to\infty} \epsilon_{1,i} = \lim_{i\to\infty} \epsilon_{2,i} = 0$,
(62.7.2) $w_i : \mathbb{H} \to M$ is a pseudo-holomorphic map,
(62.7.3) $w_i(z) \in L_{-\epsilon_{1,i}}$, for $z \in [-1,1] \subset \mathbb{R}$, $w_i(z) \in L_0$, for $z \in \mathbb{R}\cup\{\infty\}\setminus[-1,1]$,
(62.7.4) $\mathrm{dist}_{g_M}(w_{\rm tri}(z), w_i(z)) < \epsilon_{2,i}$,
(62.7.5) $\epsilon_{1,i} < \epsilon_{2,i}^{100}$,

In the rest of the section, we will prove that for any given such sequences, there exists $b_i \in S^{n-2}$ such that $w_i = w_{b_i} \circ g_i$ for some $g_i \in \mathrm{Aut}(\mathbb{H}; \{\pm 1\})$ for all sufficiently large i's, after choosing a subsequence of i if necessary.

Once we have proved this, the proof of Theorem 62.2 will be finished by contradiction : If we assume the contrary to Theorem 62.2, we can select the above sequences satisfying all the above conditions together with the additional condition

(62.7.6) $$w_i \neq w_b \circ g \text{ for any } b \in S^{n-2} \text{ and } g \in \mathrm{Aut}(\mathbb{H}; \{\pm 1\}).$$

This obviously contradicts to the above statement and will finish the proof.

We start with the following standard lemma

Lemma 62.8. *Let $|\cdot|_{g_M}$ be the norm in terms of the given metric g_M on M. For any given $\epsilon > 0$ and $k = 0, 1, 2, \cdots$, we have*

$$\lim_{i\to\infty} \sup_{z\in\mathbb{H}, |z|\ge\epsilon} \left|\nabla^k w_{\rm tri} - \nabla^k w_i\right|_{g_M}(z) = 0.$$

Proof. This is a consequence of standard elliptic regularity estimate and (62.7.4). □

We like to mention that for this estimate we do not need to use rescaled metric near the neighborhood of p_{12}. However Lemma 62.8 is not strong enough to carry out the details of the scheme laid out in (62.5). This is because we need to carefully study the fine behavior of w_i in a neighborhood of $p_{12} = w_{\rm tri}(0)$. For this purpose, we need to use a rescaled metric around p_{12}.

We denote

$$\overline{\varphi} = \exp^I_{p_{12}} \circ \varphi : (-\infty, \log \epsilon_0] \times S^{2n-1} \to M \setminus \{p_{12}\}$$

which was defined at the beginning of §61.1 and

$$\Sigma_i = w_i^{-1}(B(p_{12}; \epsilon_0)) \subseteq \mathbb{H}.$$

We consider the sphere

$$S^{n-1} = S^{n-1}_{\mathbb{R}^n} = \{z \in \mathbb{R}^n \mid |z| = 1\} \subset \mathbb{R}^n \subset \mathbb{C}^n.$$

For each $a \in S^{n-1}$, we have the Reeb chord between $S^{n-1}_{\mathbb{R}^n}$ and S^{n-1}_{Λ} in the contact manifold S^{2n-1}

$$\gamma_a : [0,1] \to S^{2n-1}, \qquad \gamma_a(t) = e^{\sqrt{-1}\alpha t} a$$

tangent to the vector $J_0 a \perp T_a S^{n-1}_{\mathbb{R}^n} \subset T_a S^{2n-1}$. γ_a is nothing but a part of great circle in S^{2n-1}. We also consider the corresponding curve

$$\gamma_{\text{out},a} : [0,1] \to \mathbb{R} \times S^{2n-1}, \qquad \gamma_{\text{out},a}(t) = (\log \epsilon_0, \gamma_a(t))$$

in $\{\log \epsilon_0\} \times S^{2n-1} \subset \mathbb{R} \times S^{2n-1}$ regarded as lying on the sphere $S^{2n-1}(\epsilon_0) \subset \mathbb{C}^n$. Here we regard $S^{2n-1}(\epsilon_0)$ as a subset of $\mathbb{C}^n$.

Lemma 62.8 and Theorem 54.17 (for $m = 1$) now imply

Corollary 62.9. *There exists a sequence of curves*

$$\widehat{\gamma}_{i,\text{out}} : [0,1] \to \mathbb{H}$$

with

$$\widehat{\gamma}_{i,\text{out}}(0) \in \mathbb{R}_+, \quad \widehat{\gamma}_{i,\text{out}}(1) \in \mathbb{R}_-,$$

and a constant $S_1 \in \mathbb{R}$ *such that*

$$\partial \Sigma_i = [\widehat{\gamma}_{i,\text{out}}(1), \widehat{\gamma}_{i,\text{out}}(0)] \cup \widehat{\gamma}_{i,\text{out}}([0,1]), \tag{62.10.1}$$

$$\lim_{i\to\infty} \left|\nabla^k(\overline{\varphi}^{-1} \circ w_i \circ \widehat{\gamma}_{i,\text{out}} - \gamma_{\text{out},a})\right| = 0, \tag{62.10.2}$$

$$\lim_{i\to\infty} \left|\nabla^k(\widehat{\gamma}_{i,\text{out}} - \widehat{\gamma}_{S_1})\right| = 0, \tag{62.10.3}$$

where $\widehat{\gamma}_{S_1}(t) = e^{\pi(S_1+\sqrt{-1}t)} \in \mathbb{H} \setminus \{0\}$, *and we use the product metric on the target* $\mathbb{R} \times S^{2n-1}$ *to estimate the norm of tensors in* (62.10.2).

From now on for the clarity of exposition, we will put a 'hat' for the curves in the domain $\mathbb{H}$ or $\mathbb{R} \times [0,1] \cong \mathbb{H} \setminus \{0\}$ in order to distinguish them from the curves in the target $\mathbb{C}^n$, or $\mathbb{R} \times S^{2n-1} \cong \mathbb{C}^n \setminus \{0\}$.

We also omit $\overline{\varphi}$ and write $w_i \circ \widehat{\gamma}_{i,\text{out}}$ etc. in place of $\overline{\varphi}^{-1} \circ w_i \circ \widehat{\gamma}_{i,\text{out}}$ regarding it as a map to $\mathbb{R} \times S^{2n-1}$, whenever there is no danger of confusion.

Figure 62.1.

Now the main part of the proof of Theorem 62.2 is Theorem 62.13 below. We use the identification

$$\mathbb{R} \times [0,1] \cong \mathbb{H} \setminus \{0\}, \quad (\tau, t) \mapsto e^{\pi(\tau + \sqrt{-1}t)}$$

to regard w_i as a map defined on $\mathbb{R} \times [0,1]$ and $\Sigma_i \setminus \{0\}$ as a subset of $\mathbb{R} \times [0,1]$.

We recall from §61.3 that we defined a moduli space $\widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H^{\alpha}_{-\epsilon_{1,i}})')$ together with the fiber bundle

$$\pi : \widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H^{\alpha}_{-\epsilon_{1,i}})') \to S^{n-1} \tag{62.11}$$

whose fiber is diffeomorphic to S^{n-2}. (In fact in §61.3 we mainly discussed the case of $(H^{\alpha}_{+\epsilon_{1,i}})'$. In that case (62.11) is a diffeomorphism. We can discuss the present case in the same way.)

Note for $w \in \widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H^{\alpha}_{-\epsilon_{1,i}})')$ with $\pi(w) = a$ we have

$$\Theta(w(e^{\pi(\tau+\sqrt{-1}t)})) \sim \gamma_a(t), \quad e^{\pi(\tau+\sqrt{-1}t)} \in \mathbb{H}$$

where $\sim$ means that the left hand side will converge to the right hand side as $\tau \to +\infty$. (Here Θ is as in (61.1.1).)

Denote $a_0 = (1, 0, \cdots, 0) \in S^{n-1}$ and fix a trivialization of (62.11) in a neighborhood of a_0. For each $a \in S^{n-1}$ close to a_0 and $b \in S^{n-2} \cong \pi^{-1}(a)$ let

$$w_{a,b} : \mathbb{H} \to \mathbb{C}^n$$

be the element of $\widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H^{\alpha}_{1})')$ corresponding to (a, b). We recall

$$e^{-\alpha\tau} |w_{a,b}(\tau, t) - e^{\alpha(\tau+\sqrt{-1}t)} a|_{\mathbb{C}^n} \le C e^{-c\tau}$$

(for $\tau \ge 0$) by definition of $\widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H^{\alpha}_{-1})')$.

Like the definition T_{ϵ_1} given in §61, we put

$$T_i = -\alpha^{-1}\left(\frac{1}{2}\log \epsilon_{1,i} + \log S_0\right) \in \mathbb{R}_+. \tag{62.12}$$

Theorem 62.13. *Let w_i satisfy* (62.7). *There exist a_i, b and $\delta_{k,i} > 0$ (which appears in* (62.15.3.1) *and* (62.15.3.2)*), such that* $\lim_{i\to\infty} a_i = a_0$, $\lim_{i\to\infty} \delta_{k,i} = 0$ *and w_i satisfies the following properties* (62.14) *and* (62.15) *:*

(62.14) *There exists an open subset $\mathcal{U}_{i,\mathrm{out}} \subset \mathbb{H}$ containing $\mathbb{H} \setminus \Sigma_i$, a bounded sequences of numbers $C_{1,i}$, $C_{2,i}$ and a biholomorphic embedding*

$$\psi_{i,\mathrm{neck}} : (-T_i + C_{1,i}, S_1 + C_{2,i}) \times [0,1] \to \mathcal{U}_{i,\mathrm{out}}$$

(where S_1 is as in (62.10.3)*) such that*

(62.14.1) *the image* $\mathcal{U}_{i,\text{neck}}$ *of* $\psi_{i,\text{neck}}$ *contains* $\mathcal{U}_{i,\text{out}} \cap \Sigma_i$ *and satisfies*

(62.14.2) $$|\nabla^k((w_i \circ \psi_{i,\text{neck}}) - w^{\text{flat}}_{a_i,0}))|(\tau,t) < C_k e^{-c_k \min(|\tau|,|\tau+T_i|)}.$$

Here c_k, C_k *are independent of* i *and we put*

$$w^{\text{flat}}_{a_i,0}(\tau,t) = (\alpha\tau, \gamma_{a_i}(t))$$

and use the product metrics for both the domain and the target in (62.14.2).

(62.14.3) $$\begin{aligned}\psi_{i,\text{neck}}(\tau,0) &\in \mathbb{R} = \partial\mathbb{H},\\ \psi_{i,\text{neck}}(\tau,1) &\in \mathbb{R} = \partial\mathbb{H}.\end{aligned}$$

Figure 62.2.

(62.15) *There exist a sequence* $R_i \to \infty$, *open sets* $\mathcal{U}_{i,\text{int}} \subset \mathbb{H}$ *and a biholomorphic map*

$$\psi_{i,\text{int}} : [-\infty, R_i) \times [0,1] \to \mathcal{U}_{i,\text{int}}$$

with the following properties :

(62.15.1) $\mathcal{U}_{i,\text{int}} \cup \mathcal{U}_{i,\text{out}} = \mathbb{H}$.
(62.15.2) $\mathcal{U}_{i,\text{int}} \cap \mathcal{U}_{i,\text{out}} = \text{Im}(\psi_{i,\text{int}}) \cap \text{Im}(\psi_{i,\text{neck}})$.

(62.15.3) $$\lim_{i\to\infty} \text{dist}_{C^k}(w_i, w_{a_i,b}) = 0.$$

We now explain the precise meaning of (62.15.3). Divide

$$[-\infty, R_i] \times [0,1] = \mathbb{H}_{|z|\le 1} \cup ([0,R_i] \times [0,1]),$$

where (τ, t) is the coordinates $\mathbb{R}\times[0,1]$ and denote $z = e^{\pi(\tau+\sqrt{-1}t)} \in \mathbb{H}$. We collapse $\{-\infty\}\times[0,1]$ to $\{0\}\in\mathbb{H}$ by an abuse of notation in this decomposition.

We first describe the precise meaning of the convergence on $\mathbb{H}_{|z|\leq 1}$ in (62.15.3). We define

$$\widetilde{w_{i,\rm int}} : \mathbb{H}_{|z|\leq 1} \to \mathbb{C}^n$$

by

$$\widetilde{w_{i,\rm int}}(z) = \epsilon_{1,i}^{-1/2}\left((w_i \circ \psi_{i,\rm int})(z)\right). \tag{62.16}$$

Then (62.15.3) on $\mathbb{H}_{|z|\leq 1}$ means the following :

$$\sup_{z\in\mathbb{H}_{|z|\leq 1}} \left|\nabla^k(\widetilde{w_{i,\rm int}} - w_{a_i,b})\right|(z) \leq \delta_{k,i} \tag{62.15.3.1}$$

in the *standard* metrics of $\mathbb{H}$ and $\mathbb{C}^n$.

Next we consider (62.15.3) on $[0,R_i]\times[0,1]$. Here the convergence means the inequality

$$|\nabla^k(\epsilon_{1,i}^{-1/2}w_i - w_{a_i,b})|(\tau,t) \leq \min\left(\delta_{k,i}, C_k e^{-c_k|\tau - R_i/2|}\right) \tag{62.15.3.2}$$

in the *product* metrics on both the domain and the target.

Once Theorem 62.13 is established the rest of the proof of Theorem 62.2 proceeds in the same way as in [Don83], [FrUh84] using the function spaces similar to those introduced in §61.6. Namely the strategy of the proof (62.5) safely applies. We will carry this out in §62.7. The proof of Theorem 62.13 will occupy §62.3 - 6.

62.3. Energies and their estimates.

In this subsection, following Hofer [Hof93], we introduce two different energies and derive some basic estimates on them.

We denote by $M_{\rm neck}\subset M$ the image of the composition

$$\left[\frac{1}{2}\log\epsilon_{1,i} + \log S_0, \log\epsilon_0\right]\times S^{2n-1} \overset{\overline{\varphi}}{\longrightarrow} \mathbb{C}^n \cong T_{p_{12}}M \overset{\exp^I_{p_{12}}=I^{-1}}{\longrightarrow} M \tag{62.17}$$

and identify

$$M_{\rm neck} = \left[\frac{1}{2}\log\epsilon_{1,i} + \log S_0, \log\epsilon_0\right]\times S^{2n-1}. \tag{62.18}$$

Under this identification, we have

$$L_{-\epsilon_{1,i}} \cap M_{\text{neck}} \cong \left[\frac{1}{2}\log\epsilon_{1,i} + \log S_0, \log\epsilon_0\right] \times (S^{n-1}_{\mathbb{R}^n} \cup S^{n-1}_{\Lambda}). \tag{62.19}$$

From now on, we put Assumption 54.20 on the almost complex structure J, i.e.,

$$J = I^* J_0 \text{ on a neighborhood of } p_{12}. \tag{62.20}$$

In particular, J is also assumed to be invariant under the translation of $\mathbb{R}$-direction on $M_{\text{neck}} \cong [\frac{1}{2}\log\epsilon_{1,i} + \log S_0, \log\epsilon_0] \times S^{2n-1}$ as ω, λ, L_1 and L_2 are so in a neighborhood of p_{12} in the Darboux chart I. We also remark that

$$\lambda|_{S^{n-1}_{\mathbb{R}^n}} = \lambda|_{S^{n-1}_{\Lambda}} = 0 \tag{62.21}$$

i.e., both $S^{n-1}_{\mathbb{R}^n}$ and S^{n-1}_{Λ} are Legendrian submanifolds of the contact manifold (S^{2n-1}, λ).

Remark 62.22. We remark that we did not use (62.20) or Assumption 54.20 in §61. We put it to simplify the exposition of the analysis carried out in this section. As we mentioned before, this assumption can be removed with some additional analytic underpinning of the complications arising from non-integrability and lack of translational invariance (in the cylindrical coordinates) of the almost complex structure J on the chosen Darboux neighborhood.

Let Σ be a bordered Riemann surface. (We do not assume that Σ is compact.) In our circumstance, Σ will be an open subset of $\mathbb{H} \setminus \{0\}$.

We decompose $\partial\Sigma$ into two parts

$$\partial\Sigma = \partial_0\Sigma \cup \partial_1\Sigma \tag{62.23}$$

and assume that

$$w : \Sigma \to M_{\text{neck}}$$

satisfies the following properties :

(62.24.1) The set $\Sigma_0 := \{z \in \Sigma \mid \operatorname{dist}(w(z), \partial M_{\text{neck}}) \geq 1\}$ is compact.

(62.24.2) $w(\partial_0\Sigma) \subset [\frac{1}{2}\log\epsilon_{1,i} + \log S_0, \log\epsilon_0] \times S^{2n-1}_{\mathbb{R}^n}$.

(62.24.3) $w(\partial_1\Sigma) \subset [\frac{1}{2}\log\epsilon_{1,i} + \log S_0, \log\epsilon_0] \times S^{2n-1}_{\Lambda}$.

For such w, we introduce the following

Definition 62.25. We define the *$d\lambda$-energy*, denoted by $E_{d\lambda}$ by

$$E_{d\lambda}(w) = \int_{\Sigma_0} w^* d\lambda.$$

Remark 62.26. Lemma 62.31 below implies that the integrand is a nonnegative form for a J-holomorphic map w for J compatible to ω. Therefore $E_{d\lambda}(w) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ is defined.

We also use another energy, the so called λ-*energy* [Hof93] denoted by E_λ. Let $\mathcal{C}$ be the set of smooth functions

$$\rho : \left(\frac{1}{2}\log \epsilon_{1,i} + \log S_0, \log \epsilon_0\right) \to \mathbb{R}_{\geq 0}$$

such that

(62.27.1) ρ is of compact support,
(62.27.2) $\int \rho(s) ds = 1$.

Composing ρ with the projection to the $\mathbb{R}$-direction, we regard ρ as a function on $M_{\rm neck}$.

Definition 62.28. We define $E_\lambda(w)$ and $E(w)$ by :

$$\begin{aligned} E_\lambda(w) &= \sup_{\rho \in \mathcal{C}} \int_{\Sigma_0} w^*(\rho\, ds \wedge \lambda), \\ E(w) &= E_\lambda(w) + E_{d\lambda}(w). \end{aligned}$$

Let w_i be as in (62.7). We choose $\Sigma_{i,0,+}$ so that $(\Sigma_{i,0,+}, w_i|_{\Sigma_{i,0,+}})$ satisfies (62.24) and

$$\Sigma_{i,0,+} \supseteq \{z \in \Sigma \mid \mathrm{dist}(w_i(z), \partial M_{\rm neck}) \geq 1\} =: \Sigma_{i,0}. \tag{62.29}$$

We now prove the following

Proposition 62.30. *We still denote $w_i = w_i|_{\Sigma_{i,0}}$. Then $E_\lambda(w_i)$, $E_{d\lambda}(w_i)$, $E(w_i)$ are uniformly bounded from above.*

The proof of Proposition 62.30 will be carried out by a sequence of lemmas.

Lemma 62.31. *Let J_0 and $\omega_0 = d(r^2\lambda)$ be the standard complex and symplectic structures on $\mathbb{C}^n$. If $w : \Sigma \to M_{\rm neck} \subset \mathbb{C}^n$ is J_0-holomorphic, then we have*

$$w^* d\lambda \geq 0, \tag{62.32.1}$$

$$w^*(ds \wedge \lambda + d\lambda) > 0 \tag{62.32.2}$$

as two forms on Σ.

Note the inequalities (62.32) mean that the left hand sides are nonnegative (positive) functions times a given area form of the complex orientation on (Σ, j).

Proof. (62.32.1) is Lemma 60.60. The proof of (62.32.2) is similar. $\square$

Remark 62.33. If we consider a J-holomorphic map w for general J which is sufficiently close to J_0 but not equal to J_0, then we will still have (62.32.2) but not (62.32.1) in general : this is because $ds \wedge \lambda + d\lambda$ is strictly positive on J_0-linear planes while $d\lambda$ is only semi-positive.

This lack of positivity of (62.32.1) for J-holomorphic maps w for general J is *the* reason why we assumed $J = I^* J_0$ in Assumption 54.20 and in this section.

We recall that any contact hypersurface (N, ξ) of a symplectic manifold (M, ω) has the canonical *co-orientation* [Wei79]. If a smooth map $w : \Sigma \to M$ from an oriented surface Σ is transversal to a contact hypersurface $N \subset M$, then the preimage $w^{-1}(N)$ has a natural orientation induced by the co-orientation of $N \subset M$. Call this the induced orientation on $w^{-1}(N)$ and denote o_{ind}.

When Σ is given a complex structure j, it carries the complex orientation on it and its boundary $\partial\Sigma$ has the boundary orientation o_{bdy} defined by the convention

$$\vec{n} \oplus o_{bdy} = o_\Sigma$$

where $\vec{n}$ is the unit normal outward to Σ on the boundary.

Now assume that Σ is oriented and $\partial\Sigma = \coprod_j \partial_j\Sigma$ where each $\partial_i\Sigma$ denotes a connected component of $\partial\Sigma$. If $w : \Sigma \to M$ is transversal to a contact hypersurfaces $N_j \subset M$ and $w^{-1}(N_j) = \partial_j\Sigma$, then $\partial_j\Sigma$ carries two orientations o_{ind} and o_{bdy}.

Definition 62.34. Let (w, Σ) as above. We say that a component $\partial_i\Sigma$ is an *outside boundary* if $o_{ind} = o_{bdy}$, and an *inside boundary* if $o_{ind} = -o_{bdy}$. We denote by $\partial_{out}\Sigma$ the union of outside boundaries and by $\partial_{in}\Sigma$ the union of inside boundaries.

Now we go back to the proof of Proposition 62.30.

Definition 62.35. $s_0 \in [\frac{1}{2}\log\epsilon_{1,i} + \log S_0, \log\epsilon_0]$ is said to be a *regular level* if w_i is transversal to $\{s_0\} \times S^{2n-1}$. For a regular level s_0, we put

$$\widehat{\gamma}_{i,s_0} = w_i^{-1}(\{s_0\} \times S^{2n-1}) \subset \partial\mathbb{H}$$

and let γ_{i,s_0} be the restriction of w_i to $\widehat{\gamma}_{i,s_0}$.

We remark that the set of all regular levels is of full measure by Sard's theorem. The proof of the following lemma follows immediately from Stokes' theorem and (62.32.1).

Lemma 62.36. *Let $w : \Sigma \to \mathbb{R} \times S^{2n-1}$ be transversal to $\{s_1\} \times S^{2n-1}$ and $\{s_2\} \times S^{2n-1}$. Consider the submanifold $\Sigma' := w^{-1}([s_1, s_2] \times S^{2n-1})$. We give the induced orientation o_{ind} on $w^{-1}(s_i)$, $i = 1, 2$.*

Suppose that the submanifold $\Sigma' \subset \Sigma$ has its boundary decomposed into

$$\partial\Sigma' = (\Sigma' \cap \partial_0\Sigma) \cup (\Sigma' \cap \partial_1\Sigma) \cup \partial_{\rm out}\Sigma' \cup \partial_{\rm in}\Sigma'$$

where $\partial_0 L \subset \mathbb{R}^n$ *and* $\partial_1 L \subset \Lambda$ *and* $w^{-1}(\{s_1\} \times S^{2n-1}) \cup w^{-1}(\{s_2\} \times S^{2n-1}) = \partial_{out}\Sigma' \cup \partial_{in}\Sigma'$ *is the decomposition according to Definition* 62.34. *Then we have*

$$\int_{\partial_{\rm out}\Sigma'} w^*\lambda - \int_{\partial_{\rm in}\Sigma'} w^*\lambda = \int_{\Sigma'} w^* d\lambda.$$

The following lemma is the key lemma for the proof of Proposition 62.30.

Lemma 62.37.

$$\int_{\Sigma_{i,0}} w_i^* d\lambda \le \int \gamma_{i,\rm out}^*\lambda + C(S_0),$$

where $\gamma_{i,\rm out} = w_i|\widehat\gamma_{i,\rm out}$ *and* $\widehat\gamma_{i,\rm out}$ *is as in Corollary* 62.9.

Here and afterwards $C(S_0)$ denotes a number depending only on S_0, which may vary during the proof.

Proof. Suppose $s_0 \in [\frac{1}{2}\log\epsilon_{1,i} + \log S_0, \log\epsilon_0]$ is a regular level. We take a subdomain $\Sigma_{i,s_0} \subset \Sigma_{i,0}$ such that its boundary is decomposed into

$$\partial\Sigma_{i,s_0} = \widehat\gamma_{i,s_0} \cup \widehat\gamma_{i,\rm out} \cup (\Sigma_{i,s_0} \cap \partial\mathbb{H}).$$

(See Figure 62.3.) Lemma 62.36 then implies

$$\int_{\Sigma_{i,s_0}} w_i^* d\lambda = \int \gamma_{i,\rm out}^*\lambda - \int \gamma_{i,s_0}^*\lambda.$$

Therefore it remains to show that there exists a constant $C(S_0)$ depending only on S_0 such that the inequality

$$\int \gamma_{i,s_0}^*\lambda \ge -C(S_0) \tag{62.38}$$

holds for s_0 sufficiently close to $\frac{1}{2}\log\epsilon_{1,i} + \log S_0$.

To prove (62.38) we take a subdomain $\Sigma_{i,s_0,\rm int} \subset \mathbb{H}$ such that

$$\partial\Sigma_{i,s_0,\rm int} = (\Sigma_{i,s_0,\rm int} \cap \partial\mathbb{H}) \cup \widehat\gamma_{i,s_0}, \quad \Sigma_{i,s_0,\rm int} \cap \Sigma_{i,s_0} = \widehat\gamma_{i,s_0}.$$

(See Figure 62.3.) We denote $\mu_{i,s_0} = w_i|_{\Sigma_{i,s_0,\rm int}\cap\partial\mathbb{H}}$ which defines a curve

$$\mu_{i,s_0} : \Sigma_{i,s_0,\rm int} \cap \partial\mathbb{H} \to (H^\alpha_{-\epsilon_{1,i}})' \subset \mathbb{R} \times S^{2n-1}.$$

Figure 62.3.

Consider the canonical symplectic form

$$\omega_0 = d(e^{2s}\lambda)$$

on $\mathbb{C}^n$ regarded as a form on $\mathbb{R} \times S^{2n-1}$ via the diffeomorphism $(s, \Theta) : \mathbb{C}^n \setminus \{0\} \to \mathbb{R} \times S^{2n-1}$.

We recall that $(H^{\alpha}_{-\epsilon_{1,i}})'$ is Lagrangian, i.e.,

$$\omega_0|_{(H^{\alpha}_{-\epsilon_{1,i}})'} = 0 \tag{62.39}$$

by definition of $(H^{\alpha}_{-\epsilon_{1,i}})'$. By Stokes' theorem we derive

$$\begin{aligned} 0 \leq \int_{\Sigma_{i,s_0,\mathrm{int}}} w_i^* \omega_0 &= \int_{\partial\Sigma_{i,s_0,\mathrm{int}}} w_i^*(e^{2s}\lambda) \\ &= \int \gamma_{i,s_0}^*(e^{2s}\lambda) - \int \mu_{i,s_0}^*(e^{2s}\lambda) \end{aligned}$$

and hence

$$\int \gamma_{i,s_0}^*(e^{2s}\lambda) \geq \int \mu_{i,s_0}^*(e^{2s}\lambda). \tag{62.40}$$

We put

$$(H^{\alpha,s_0}_{-\epsilon_{1,i}})' = (H^{\alpha}_{-\epsilon_{1,i}})' \cap \left((-\infty, s_0] \times S^{2n-1}\right).$$

(See Figure 62.4.) Consider the relative homology classes

$$[\mu_{i,s_0}, \partial\mu_{i,s_0}] \in H_1((H^{\alpha,s_0}_{-\epsilon_{1,i}})', \partial(H^{\alpha,s_0}_{-\epsilon_{1,i}})') \tag{62.41}$$

and

$$(62.42)\quad [\gamma_{i,s_0}, \partial\gamma_{i,s_0}] \in H_1(\{s_0\} \times S^{2n-1}, \partial(H^{\alpha,s_0}_{-\epsilon_{1,i}})') \cong H_1(S^{2n-1}, S^{n-1}_{\mathbb{R}^n} \cup S^{n-1}_{\Lambda}).$$

Here we note that there is a canonical isomorphism between $H_1(\{s_0\}\times S^{2n-1}, \partial(H^{\alpha,s_0}_{-\epsilon_{1,i}})')$ and $H_1(S^{2n-1}, S^{n-1}_{\mathbb{R}^n} \cup S^{n-1}_{\Lambda})$.

Figure 62.4.

Sublemma 62.43. *We regard* $(\mu_{i,s_0}, \partial\mu_{i,s_0})$ *as a relative one-cycle for the pair*

$$((H^{\alpha,s_0}_{-\epsilon_{1,i}})', \partial(H^{\alpha,s_0}_{-\epsilon_{1,i}})').$$

Then the integral

$$\int \mu^*_{i,s_0}(e^{2s}\lambda)$$

depends only on the relative homology class $[\mu_{i,s_0}, \partial\mu_{i,s_0}] \in H_1((H^{\alpha,s_0}_{-\epsilon_{1,i}})', \partial(H^{\alpha,s_0}_{-\epsilon_{1,i}})')$.

Proof. First (62.39) implies the Liouville one-form $e^{2s}\lambda$ is closed on $(H^{\alpha}_{-\epsilon_{1,i}})'$. On the other hand, by definition of $(H^{\alpha}_{-\epsilon_{1,i}})'$, we have

$$\partial(H^{\alpha,s_0}_{-\epsilon_{1,i}})' = S^{2n-1}(s_0) \cap (\mathbb{R}^n \cup \Lambda)$$

and so

$$(62.44)\qquad \lambda\Big|_{\partial(H^{\alpha,s_0}_{-\epsilon_{1,i}})'} = 0.$$

This implies the one-form $e^{2s}\lambda$ vanishes on the boundary $\partial(H^{\alpha,s_0}_{-\epsilon_{1,i}})'$. (However we remark that λ does not vanish everywhere on the Lagrangian submanifold $(H^{\alpha,s_0}_{-\epsilon_{1,i}})'$.)

Now Stokes' formula finishes the proof. $\square$

We remark that $((H^{\alpha,s_0}_{-\epsilon_{1,i}})', \partial(H^{\alpha,s_0}_{-\epsilon_{1,i}})')$ is homeomorphic to $(S^{n-1}\times[0,1], S^{n-1}\times\{0,1\})$ for all i and s_0 given above and has the canonical isomorphism

$$H_1((H^{\alpha,s_0}_{-\epsilon_{1,i}})', \partial(H^{\alpha,s_0}_{-\epsilon_{1,i}})') \cong H_1(S^{n-1}\times[0,1], S^{n-1}\times\{0,1\}) \cong \mathbb{Z}.$$

Therefore the boundary map

$$\partial : H_1((H^{\alpha,s_0}_{-\epsilon_{1,i}})', \partial(H^{\alpha,s_0}_{-\epsilon_{1,i}})') \to \widetilde{H}_0(\partial(H^{\alpha,s_0}_{-\epsilon_{1,i}})') \cong \mathbb{Z} \tag{62.45}$$

is an isomorphism. (Here $\widetilde{H}_0$ is the reduced homology.) This is trivial for $n \neq 2$ since $H_1((H^{\alpha,s_0}_{-\epsilon_{1,i}})', \mathbb{Z}) = \{0\}$ for $n \neq 2$. On the other hand for $n = 2$, this follows from the tautological exact sequence because the canonical homomorphism

$$H_1((H^{\alpha,s_0}_{-\epsilon_{1,i}})', \mathbb{Z}) \to H_1((H^{\alpha,s_0}_{-\epsilon_{1,i}})', \partial(H^{\alpha,s_0}_{-\epsilon_{1,i}})')$$

is trivial for the pair $\Big((H^{\alpha,s_0}_{-\epsilon_{1,i}})', \partial(H^{\alpha,s_0}_{-\epsilon_{1,i}})'\Big)$.

Sublemma 62.46. *Let* $s_0 \in [\frac{1}{2}\log\epsilon_{1,i} + \log S_0, \log\epsilon_0]$ *be any regular level. Then* (62.41) *is mapped to* $\pm 1 \in \mathbb{Z}$ *by the isomorphism* (62.45).

Proof. We remark

$$\partial : H_1(S^{2n-1}, S^{n-1}_{\mathbb{R}^n} \cup S^{n-1}_{\Lambda}) \to \widetilde{H}_0(S^{n-1}_{\mathbb{R}^n} \cup S^{n-1}_{\Lambda}) \cong \mathbb{Z} \tag{62.47}$$

is an isomorphism and $\partial\mu_{i,s_0} = \partial\gamma_{i,s_0}$. Hence it suffices to calculate the homology class $[\gamma_{i,s_0}, \partial\gamma_{i,s_0}]$ given in (62.42). Since

$$\partial\left(\Sigma_{i,s_0',\mathrm{int}} \setminus \Sigma_{i,s_0,\mathrm{int}}\right) = [\gamma_{i,s_0'}, \partial\gamma_{i,s_0'}] - [\gamma_{i,s_0}, \partial\gamma_{i,s_0}]$$

holds in the relative singular chain complex

$$C_*([s_0, s_0'] \times S^{2n-1}, [s_0, s_0'] \times (S^{n-1}_{\mathbb{R}^n} \cup S^{n-1}_{\Lambda}))$$

for $s_0 \le s_0'$, it follows that the homology class (62.42) in $H_1(S^{2n-1}, S^{n-1}_{\mathbb{R}^n} \cup S^{n-1}_{\Lambda})$ is independent of the choice of s_0 for all sufficiently large i's. Moreover when s_0 is sufficiently close to $\log\epsilon_0$, (62.42) goes to ± 1 by the isomorphism (62.47). (This is a consequence of (62.10.2).) Hence the sublemma. $\square$

Sublemma 62.48. *Let* $\frac{1}{2}\log\epsilon_{1,i} + \log S_0 < s_0 < \frac{1}{2}\log\epsilon_{1,i} + \log 2S_0$. *Then we have*

$$\int \mu_{i,s_0}^*(e^{2s}\lambda) = \epsilon_{1,i} C(S_0)$$

where $C(S_0)$ *is independent of* i *and depends only on* S_0.

Proof. We define a diffeomorphism

$$\mathfrak{T}_{\frac{1}{2}\log\epsilon_{1,i}} : \mathbb{R}\times S^{2n-1} \to \mathbb{R}\times S^{2n-1}$$

by

$$\mathfrak{T}_{\frac{1}{2}\log\epsilon_{1,i}}(s,x) = (s - \frac{1}{2}\log\epsilon_{1,i}, x).$$

Then $\mathfrak{T}_{\frac{1}{2}\log\epsilon_{1,i}}$ maps $[\frac{1}{2}\log\epsilon_{1,i} + \log S_0, \frac{1}{2}\log\epsilon_{1,i} + \log 2S_0]$ to $[\log S_0, \log 2S_0]$ and satisfies

$$(\mathfrak{T}_{\frac{1}{2}\log\epsilon_{1,i}}^{-1})^*(e^{2s}\lambda) = \epsilon_{1,i}(e^{2s}\lambda).$$

Moreover

$$\mathfrak{T}_{\frac{1}{2}\log\epsilon_{1,i}}^{-1}((H_{-\epsilon_{1,i}}^{\alpha})') = (H_{-1}^{\alpha})'$$

and so $\mathfrak{T}_{\frac{1}{2}\log\epsilon_{1,i}}^{-1}((H_{-\epsilon_{1,i}}^{\alpha})')$ is independent of i. Therefore we derive

$$\begin{aligned}\int \mu_{i,s_0}^*(e^{2s}\lambda) &= \int (\mathfrak{T}_{\frac{1}{2}\log\epsilon_{1,i}} \circ \mu_{i,s_0})^*(\mathfrak{T}_{\frac{1}{2}\log\epsilon_{1,i}}^{-1})^*(e^{2s}\lambda)\\ &= \epsilon_{1,i}\int (\mathfrak{T}_{\frac{1}{2}\log\epsilon_{1,i}} \circ \mu_{i,s_0})^*(e^{2s}\lambda).\end{aligned}$$

On the other hand, the family of curves

$$\mathfrak{T}_{\frac{1}{2}\log\epsilon_{1,i}} \circ \mu_{i,s_0}$$

lie on $(H_{-1}^{\alpha})'$ with their boundaries contained in the region with $s \in [\log S_0, \log 2S_0]$ for all i.

We now set

$$C(S_0) = -(e^{2s}\lambda)[\beta]$$

for β being a generator of $H_1((H_{-1}^{\alpha})', \partial(H_{-1}^{\alpha})')$. (We like to recall that the form $e^{2s}\lambda$ is the Liouville one-form on $\mathbb{C}^n$.)

Then Sublemma 62.46 implies

$$\int (\mathfrak{T}_{\frac{1}{2}\log\epsilon_{1,i}}\mu_{i,s_0})^*(e^{2s}\lambda) = \pm C(S_0)$$

for all i and for $s_0 \in [\frac{1}{2}\log\epsilon_{1,i} + \log S_0, \frac{1}{2}\log\epsilon_{1,i} + \log 2S_0]$. This finishes the proof. $\square$

Therefore, by choosing s_0 sufficiently close to $\frac{1}{2}\log\epsilon_{1,i} + \log S_0$, Sublemma 62.48 and (62.40) imply

$$\int \gamma_{i,s_0}^*(\lambda) \geq e^{-2s_0} \int \mu_{i,s_0}^*(e^{2s}\lambda) \cong \frac{1}{\epsilon_{1,i}\cdot S_0^2}(\epsilon_{1,i}C(S_0)) = C(S_0)/S_0^2.$$

(Note $s = s_0$ on γ_{i,s_0}.) Redefining $C(S_0)$, we have proved $\int \gamma_{i,s_0}^*(\lambda) \geq -C(S_0)$. (Note the sign of $C(S_0)$ does not matter in Lemma 62.37.) This finishes the proof of (62.38). The proof of Lemma 62.37 is now complete. $\square$

Remark 62.49. We can also give a slightly different proof of Lemma 62.37 based on a similar idea as in §60.5.

Lemma 62.37 and (62.10.3) prove the bound

$$E_{d\lambda}(w_i) \leq C(S_0). \tag{62.50}$$

We also use the following bound for the symplectic area of w_i on $\Sigma_{i,s_0,\mathrm{int}}$: Recall

$$\omega_0 = d(e^{2s}\lambda).$$

Proposition 62.51. *Let* $s_0 = \frac{1}{2}\log\epsilon_{1,i} + \log 2S_0$. *Then*

$$\int_{\Sigma_{i,s_0,\mathrm{int}}} w_i^*\omega_0 \leq \epsilon_{i,1}C(S_0).$$

Proof. By Stokes' theorem, we have

$$\int_{\Sigma_{i,s_0,\mathrm{int}}} w_i^*(d(e^{2s}\lambda)) \leq 4S_0^2\epsilon_{1,i}\int \gamma_{i,s_0}^*(\lambda) - \int \mu_{i,s_0}^*(e^{2s}\lambda). \tag{62.52}$$

On the other hand,

$$\int \gamma_{i,s_0}^*(\lambda) = \int \gamma_{i,\mathrm{out}}^*(\lambda) - \int_{\Sigma_{i,s_0}} w_i^* d\lambda \leq \int \gamma_{i,\mathrm{out}}^*(\lambda) \leq \alpha + 1. \tag{62.53}$$

where the first identity is by Stokes' and the second inequality by the positivity (62.32.1), and the last by the convergence $\Theta(\gamma_{i,s_0}) \to \Theta(\gamma_a)$ as $i \to \infty$. (See (62.10.2).) Now Sublemma 62.48, (62.52) and (62.53) finish the proof. $\square$

Lemma 62.54. *Denote $w_i = w_i|_{\Sigma_{i,0}}$ be as before. There exists a constant $C > 0$ independent of i such that*

$$E_\lambda(w_i) \le C.$$

Proof. Let $\rho \in \mathcal{C}$. We put

$$\widetilde{\rho}(s) = \int_{-T_i}^{s} \rho(s) ds$$

We use Stokes' theorem to show

$$\int_{\Sigma_{i,0}} d(w_i^*(\widetilde{\rho}\lambda)) = \int \gamma_{i,\mathrm{out}}^*\lambda \le \alpha + 1. \tag{62.55}$$

On the other hand, we also have

$$\int_{\Sigma_{i,0}} d(w_i^*(\widetilde{\rho}\lambda)) = \int_{\Sigma_{i,0}} w_i^*(\rho\; ds \wedge \lambda) + \int_{\Sigma_{i,0}} w_i^*(\widetilde{\rho}\; d\lambda).$$

Since the second term is non-negative by (62.32.1), it follows that

$$\int_{\Sigma_{i,0}} w_i^*(\rho\; ds \wedge \lambda) \le \alpha + 1.$$

Hence Lemma 62.54. □

Remark 62.56. We remark that the term $\int_{\Sigma_{i,0}} w_i^*(\widetilde{\rho}\; d\lambda)$ would be non-negative upto an exponentially small error, if we consider J-holomorphic maps w_i for J_0-holomorphic ones for non-integrable J. A precise control of this term would be needed to study J-holomorphic maps for J not satisfying Assumption 54.20.

Now Proposition 62.30 follows by combining (62.50) and Lemma 62.55. □

62.4. C^∞ convergence on the neck region.

In this subsection and the next, using the energy estimates obtained in the last subsection we prove that w_i converges to the cylindrical map

$$(\tau, t) \mapsto (\alpha\tau + \mathrm{const}, \gamma_a(t))$$

for $a \in S^{n-1}$ on M_{neck}. This will prove (62.14), which is the first half of Theorem 62.13.

One difficult point of the proof is a choice of parametrization of the maps. We recall that there is no canonical coordinates on the domain for the maps that we are considering. A general strategy laid out in §11 and the appendix of [FuOn99II] to handle this problem is to add auxiliary marked points to the domain so that the corresponding sequence of the domain marked Riemann surfaces forms a sequence of *stable curves*. The same strategy has been applied in Lemma 10.7 [BEHWZ03].

Our current circumstance is regarded as a sub-case of the relative version of [BEHWZ03]. To work out the analytic details needed to carry out this proof for general J in all details, we need do all the ingredients necessary to establish the analytic foundation of symplectic field theory advocated in [EGH00]. This is too much for the purpose of this section.

Instead, in this subsection, we will exploit various special features that are present in the case of our study. We list those points here for the readers' convenience :

(1) J satisfies Assumption 54.20 and in particular is integrable around p_{12} and so in the neck region $M_{\rm neck}$.
(2) The ends of the Lagrangian submanifolds $(H^{\alpha}_{-\epsilon_{1,i}})'$ are cylindrical, not just asymptotically cylindrical as $H^{\alpha}_{-\epsilon_{1,i}}$.
(3) The chord $\gamma_{{\rm out},i}$ is C^∞-close to the Reeb chord $\gamma_{{\rm out},a}$ and $\gamma_{{\rm out},a}$ is the Reeb chord of the minimal λ-length.
(4) We study pseudo-holomorphic curves in an exact symplectic manifold with exact Lagrangian submanifolds as boundary conditions.

The integrability of J and the exactness above remove various difficult points to handle the general case. This enables us to exploit ideas coming from the study of one dimensional complex analytic varieties and positive currents, and the Hausdorff convergence of analytic varieties in some compactness arguments. We also note that for analytic varieties Hausdorff topology is much weaker and easier to use than the stable map topology introduced in [FuOn99II]. However the study of pseudo-holomorphic maps for general J would require to use the stable map topology. Our current argument here is somewhat similar to [Pan94] in spirit.

For given s_0, we define the translation map

$$\mathfrak{T}_{s_0} : \mathbb{R} \times S^{2n-1} \to \mathbb{R} \times S^{2n-1}$$

by

$$\mathfrak{T}_{s_0}(s,x) = (s - s_0, x).$$

Let $s_i \in \mathbb{R}$ be a sequence satisfying

$$\lim_{i\to\infty} s_i = -\infty \tag{62.57.1}$$

$$\lim_{i\to\infty} s_i - \frac{1}{2}\log 2\epsilon_{1,i} = +\infty. \tag{62.57.2}$$

In terms of the standard polar coordinates (r,Θ) of $\mathbb{C}^n$ with $r = e^s$, the condition (62.57) corresponds to

$$\lim_{i\to\infty} r_i = 0, \quad \lim_{i\to\infty} \frac{r_i}{\sqrt{2\epsilon_{1,i}}} = \infty.$$

Let X be a closed subset of $\mathbb{R} \times S^{2n-1} \cong \mathbb{C}^n \setminus \{0\}$ and A_i be a sequence of closed subsets thereof such that

$$A_i \subset (-\infty, \log \epsilon_0] \times S^{2n-1}.$$

Definition 62.58. We say $\mathfrak{T}_{s_i}(A_i)$ converges to X in *compact Hausdorff topology* if for each $R > 0$ the sequence of sets

$$([-R, R] \times S^{2n-1}) \cap \mathfrak{T}_{s_i}(A_i)$$

converges to

$$([-R, R] \times S^{2n-1}) \cap X$$

in Hausdorff topology.

Definition 62.59. (1) Here the Hausdorff topology stands for the convergence with respect to the Hausdorff distance on the set of all closed subsets of the metric space $[-R, R] \times S^{2n-1}$. See, e.g., [Grom99] for the definition of the Hausdorff distance.

(2) We note that (62.57) implies

$$\mathfrak{T}_{s_i}^{-1}([-R, R] \times S^{2n-1}) \subset [\frac{1}{2} \log \epsilon_{1,i} + \log S_0, \log \epsilon_0] \times S^{2n-1}$$

for sufficiently large i for each given R.

Lemma 62.60. *For each given sequence s_i satisfying* (62.57) *there exists a subsequence, still denoted by s_i, such that*

$$\mathfrak{T}_{s_i}(w_i(\Sigma_{i,0}))$$

converges to a closed set $X(\{s_i\})$ in compact Hausdorff topology.

See (62.24.1) for $\Sigma_{i,0}$.

Proof. This is immediate from the fact that the set of closed subsets of a given compact metric space is compact in Hausdorff topology. □

We now use the energy estimate given in Proposition 62.30 and prove the following :

Lemma 62.61. *Consider R_1, R_2 with $\frac{1}{2} \log \epsilon_{1,i} + \log S_0 < R_1 < R_2 < \log \epsilon_0$ and denote $R_2 - R_1 = R$. Assume $R > 1$. Then we have*

$$\int_{\mathfrak{T}_{s_i}(w_i(\Sigma_{i,0})) \cap ([R_1, R_2] \times S^{2n-1})} (ds \wedge \lambda + d\lambda) \leq CR$$

for some constant $C > 0$ independent of i and R.

Proof. By the translational invariance of $d\lambda$, we have

$$\int_{\mathfrak{T}_{s_i}(w_i(\Sigma_{i,0}))\cap([R_1,R_2]\times S^{2n-1})} d\lambda = \int_{w_i(\Sigma_{i,0})\cap([R_1+s_i,R_2+s_i]\times S^{2n-1})} d\lambda.$$

Since (62.57) implies $[R_1+s_i, R_2+s_i]\times S^{2n-1} \subset [\frac{1}{2}\log\epsilon_{1,i} + \log S_0, \log\epsilon_0]\times S^{2n-1}$ for all sufficiently large i and $w_i^* d\lambda \geq 0$, the inequality

$$\int_{w_i(\Sigma_{i,0})\cap([R_1+s_i,R_2+s_i]\times S^{2n-1})} d\lambda \leq E_{d\lambda}(w_i) \leq C \tag{62.62}$$

follows from Proposition 62.30.

To estimate the integral of the first integrand above, we first note

$$w_i^*(ds\wedge\lambda) \geq 0. \tag{62.63}$$

We then take a function $\rho \in \mathcal{C}$ such that $\rho \equiv 1/(2R)$ on $[R_1, R_2]$. By the translational invariance of $ds\wedge\lambda$, we have

$$\int_{\mathfrak{T}_{s_i}(w_i(\Sigma_{i,0}))\cap([R_1,R_2]\times S^{2n-1})} ds\wedge\lambda = \int_{w_i(\Sigma_{i,0})\cap([R_1+s_i,R_2+s_i]\times S^{2n-1})} ds\wedge\lambda.$$

Now Proposition 62.30 and (62.63) imply

$$\begin{aligned}\int_{\mathfrak{T}_{s_i}(w_i(\Sigma_{i,0}))\cap([R_1,R_2]\times S^{2n-1})} ds\wedge\lambda &\leq 2R\int_{\Sigma_{i,0}} w_i^*(\rho\, ds\wedge\lambda)\\ &\leq 2RE_\lambda(w_i) \leq CR.\end{aligned} \tag{62.64}$$

Adding (62.62) and (62.64) and redefining C, we have finished the proof. □

Now we quote the following well established facts in several complex variable theory. (See Chapter 2 §11 [Chi89], for example.) For readers' convenience, we recall the main arguments of their proofs.

Proposition 62.65. *Let $U \subset \mathbb{C}^n$ be an open set and $A_i \subset \mathbb{C}^n$ be a sequence of* 1 *dimensional complex subvarieties. We assume that*

$$\int_{A_i\cap U}\omega_0$$

is uniformly bounded over i. Let

$$X = \lim_{i\to\infty}(A_i\cap U)$$

be the limit with respect to the compact Hausdorff convergence. Then we have the following :

(62.66.1) *X is a one dimensional complex subvariety.*

(62.66.2) *There exists a positive integer valued function, called the multiplicity, $m : X_{\rm reg} \to \mathbb{Z}_{>0}$ on the regular point set $X_{\rm reg}$ of X such that m is locally constant and that $A_i \cap U$ converges to mX as an integral current on any compact subsets of U, after taking a subsequence if necessary.*

(62.66.3) *Suppose X is smooth and $m \equiv 1$ in addition. Then any compact subset of U has its open neighborhood V for which $A_i \cap V$ is smooth for all sufficiently large i and is diffeomorphic to $X \cap V$. Moreover $A_i \cap V$ converges to $X \cap V$ in C^∞-topology of complex submanifolds.*

Proof. (62.66.1) is a theorem by Bishop [Bis64]. We prove (62.66.2), (62.66.3) below for completeness.

Let $p \in X$ be a regular point. We choose a neighborhood W of p and after changing the coordinates on W we may assume

$$W \cap X = (\mathbb{C} \times \{0\}) \cap W.$$

Let ϵ be a small positive number. Then for $z_0 \in \mathbb{C}$ sufficiently close to 0 and for any sufficiently large i, we have

$$(\{z_0\} \times \partial B_\epsilon(0, \mathbb{C}^{n-1})) \cap A_i = \emptyset.$$

Here $B_\epsilon(0, \mathbb{C}^{n-1}) \subset \mathbb{C}^{n-1}$ is the ball centered at 0 and of radius ϵ. For such z_0 and i the local intersection number

$$[\{z_0\} \times B_\epsilon(0, \mathbb{C}^{n-1})] \cap [A_i] \in \mathbb{Z}_{>0}$$

is well-defined and independent of z_0. This is the definition of *multiplicity* $m_i(p)$ of the analytic variety A_i at p. (See [Chi89].) If follows from the uniform bound for the area $\int_{A_i \cap U} \omega_0$ that $m_i(p)$ is finite. (See Figure 62.5.)

Therefore by taking a subsequence if necessary, we may assume that $m_i(p)$ is independent of i. In this way we obtain a locally constant function m. It is now easy to see that $A \cap U_i$ converges to mX as integral currents on any compact subsets of U. This finishes the proof of (62.66.2).

We next assume $m = 1$. Then by positivity of local intersection numbers, it follows that $\{z_0\} \times B_\epsilon(0, \mathbb{C}^{n-1})$ intersects transversally with A_i at exactly one point for any $z_0 \in \mathbb{C} \times \{0\}$. We can use this fact to find a local holomorphic parametrization of A_i that converge in C^∞-topology. This finishes the proof of (62.66.3). $\square$

Figure 62.5.

Lemma 62.67. *The compact Hausdorff limit*

$$X(\{s_i\}) = \lim_{i\to\infty} \mathfrak{T}_{s_i}(w_i(\Sigma_{i,0}))$$

obtained in Lemma 62.60 *has the structure of one dimensional complex analytic set with boundary lying in* $\mathbb{R}\times(S^{n-1}_{\mathbb{R}^n}\cup S^{n-1}_{\Lambda})$ *in the following sense :*

(62.68.1) *If* $p\in\mathbb{R}\times(S^{2n-1}\setminus(S^{n-1}_{\mathbb{R}^n}\cup S^{n-1}_{\Lambda}))$*, then there exists a neighborhood* U *of* p *such that* $U\cap X(\{s_i\})$ *is a one dimensional complex analytic set.*
(62.68.2) *If* $p\in\mathbb{R}\times S^{n-1}_{\mathbb{R}^n}$*, we can choose a neighborhood* U *of* p *and an anti-holomorphic isometric involution* $\mathrm{Inv}: U\to U$ *such that its fixed point set is* $(\mathbb{R}\times S^{n-1}_{\mathbb{R}^n})\cap U$ *and the double* $(X(\{s_i\})\cap U)\cup\mathrm{Inv}(X(\{s_i\})\cap U)$ *has the structure of one dimensional complex analytic set. The case* $p\in\mathbb{R}\times S^{n-1}_{\Lambda}$ *is similar.*

Moreover, after a taking subsequence, $\mathfrak{T}_{s_i}(w_i(\Sigma_{i,0}))$ *converges to* $mX(\{s_i\})$ *on compact subsets as currents. Here* $m: X(\{s_i\})_{\rm reg}\to\mathbb{Z}_{>0}$ *is the multiplicity function defined on the set of regular points of* $X(\{s_i\})$.

Proof. (62.68.1) is a consequence of Lemma 62.61 and (62.66.1). We can use the reflection principle to reduce (62.68.2) to (62.68.1). (Note $S^{n-1}_{\mathbb{R}^n}$ and S^{n-1}_{Λ} are the intersections with S^{2n-1} of the fixed point sets of some involutions defined on $\mathbb{C}^n$ that are isometric and anti-holomorphic.)

The last statement is a consequence of (62.66.2). □

For a one dimensional complex analytic subset $X\subset\mathbb{R}\times S^{2n-1}\cong\mathbb{C}^n\setminus\{0\}$, we define the analogs to the energies given in Proposition 62.30 in a similar way : We first define

$$E_{d\lambda}(X) := \int_X d\lambda$$

regarding X as a holomorphic 1-current on $\mathbb{C}^n$.

We next let $\mathcal{C}$ be the set of all smooth functions $\rho : \mathbb{R} \to \mathbb{R}_{\geq 0}$ of compact support such that $\int \rho = 1$. We then set

$$E_\lambda(X) = \sup_{\rho \in \mathcal{C}} \int_X \rho(s)(ds \wedge \lambda).$$

We finally put

$$E(X) = E_{d\lambda}(X) + E_\lambda(X).$$

We remark that the two forms $d\lambda$ and $ds \wedge \lambda$ are nonnegative on one dimensional complex analytic set X. Then, by a version of Fatou's lemma, we have the following

Lemma 62.69.

$$E_\lambda(X(\{s_i\})) \leq \liminf_{i \to \infty} E_\lambda(w_i).$$

The same holds for $E_{d\lambda}$ *and* E.

Proof. The lemma follows easily from (62.66.2). □

An immediate corollary of this lemma and Proposition 62.30 is the following finiteness of energies.

Corollary 62.70. $E_\lambda(X(\{s_i\}))$, $E_{d\lambda}(X(\{s_i\}))$, $E(X(\{s_i\}))$ *are all finite.*

We next take a sequence t_j^+ (resp. t_j^-) converging $+\infty$ (resp. $-\infty$). We will just write t_j below for either of t_j^+ or t_j^-.

Lemma 62.71. *There exists a subsequence of* t_j*, again denoted by* t_j*, such that*

$$\lim_{j \to \infty} \mathfrak{T}_{t_j}(X(\{s_i\}))$$

converges both in compact Hausdorff topology and in the weak topology of integral currents. Furthermore the limit has the structure of one dimensional complex analytic set with boundary contained $\mathbb{R} \times (S^{n-1}_{\mathbb{R}^n} \cup S^{n-1}_\Lambda)$*, which has finite multiplicity.*

Proof. Once we have the finiteness of energies in Corollary 62.70, the proof is the same as that of Lemma 62.60 and of Lemma 62.67 and so omitted. □

Now translational invariance of $d\lambda$, finiteness of $E_{d\lambda}(X(\{s_i\}))$ and Lemma 62.71 immediately give rise to the following vanishing result for the energy $E_{d\lambda}$.

Lemma 62.72.

$$E_{d\lambda}\left(\lim_{j \to +\infty} \mathfrak{T}_{t_j}(X(\{s_i\}))\right) = 0.$$

This then gives rise to

Corollary 62.73. *There exist a finitely many Reeb chords $\gamma_1, \dots, \gamma_K$ with*

$$\gamma_i(0), \gamma_i(1) \in S^{n-1}_{\mathbb{R}^n} \cup S^{n-1}_{\Lambda}$$

such that

$$\lim_{j\to+\infty} \mathfrak{T}_{t_j}(X(\{s_i\})) = \bigcup_{k=1}^{K} m_k(\mathbb{R}\times\gamma_k).$$

Here m_k are the multiplicities of γ_k.

Proposition 62.74. *$K = 1$ in Corollary 62.73. Moreover γ_1 is a minimal Reeb chord joining $S^{n-1}_{\mathbb{R}^n}$ to S^{n-1}_{Λ}. Furthermore $m_1 = 1$ and $\mathfrak{T}_{t_j}(X(\{s_i\}))$ converges to $\mathbb{R}\times\gamma_1$ in compact C^∞ topology.*

Proof. First by taking a diagonal subsequence we can choose an increasing map $j \mapsto i(j)$ such that

$$\lim_{j\to\infty} \mathfrak{T}_{t_j+s_{i(j)}}(w_{i(j)}(\Sigma_{i(j),0})) = \lim_{j\to+\infty} \mathfrak{T}_{t_j}(X(\{s_i\})). \tag{62.75}$$

And we may assume that $t_j + s_{i(j)}$ is a regular level. We consider the chord

$$\gamma_{t_j+s_{i(j)},i(j)} = w_{i(j)}(\Sigma_{i(j),0}) \cap (\{t_j + s_{i(j)}\}\times S^{2n-1})$$

connecting $S^{n-1}_{\mathbb{R}^n} \cup S^{n-1}_{\Lambda}$ to $S^{n-1}_{\mathbb{R}^n} \cup S^{n-1}_{\Lambda}$. From (62.75) and Corollary 62.73 we obtain

$$\lim_{j\to\infty} \int_{\gamma_{t_j+s_{i(j)},i(j)}} \lambda = \sum_k m_k \int_{\gamma_k} \lambda.$$

We recall from the remark after (62.47) that

$$[\gamma_{t_j+s_{i(j)},i(j)}, \partial\gamma_{t_j+s_{i(j)},i(j)}]$$

represents the same homology class as $\gamma_{i,\mathrm{out}} \in H_1(S^{2n-1}, (S^{n-1}_{\mathbb{R}^n} \cup S^{n-1}_{\Lambda}))$. Therefore there exists at least one of its connected components joining $S^{n-1}_{\mathbb{R}^n}$ to S^{n-1}_{Λ}. Let γ_1 be the limit of that component. We remark that $\int_{\gamma_k}\lambda$ are nonnegative for the Reeb chords γ_k. Combining all these, we obtain

$$\alpha \le \int_{\gamma_1} \lambda \le \sum_k m_k \int_{\gamma_k} \lambda = \lim_{j\to\infty} \int_{\gamma_{t_j+s_{i(j)},i(j)}} \lambda. \tag{62.76.1}$$

On the other hand, Lemma 62.31 and (62.10.2) imply

$$\lim_{j\to\infty} \int_{\gamma_{t_j+s_{i(j)},i(j)}} \lambda \le \lim_{j\to\infty} \int_{\gamma_{i(j),\mathrm{out}}} \lambda = \alpha. \tag{62.76.2}$$

Therefore all the inequalities in (62.76.1) and (62.76.2) become the equality. This implies $K = 1$, $m_1 = 1$ and γ_1 is a minimal Reeb chord. Now $m_1 = 1$ and (62.66.3) imply that $\mathfrak{T}_{t_j}(X(\{s_i\}))$ converges to $\mathbb{R} \times \gamma_1$ in compact C^∞ topology. This finishes the proof. □

Let s_i be a sequence that consist of regular levels and that satisfy (62.57). We put

$$\Sigma_{i,\geq s_i} = \{z \in \Sigma_{i,0} \mid s(w_i(z)) \in [s_i, \log \epsilon_0]\}.$$

Here $s : \mathbb{R} \times S^{2n-1} \to \mathbb{R}$ is the projection. $\Sigma_{i,\geq s_i}$ is a manifold with smooth boundary and corner.

Lemma 62.77.

$$\lim_{i\to\infty} \int_{\Sigma_{i,\geq s_i}} w_i^* d\lambda = 0.$$

Proof. Suppose that the lemma does not hold for s_i. Then by taking a subsequence, we may assume

$$\int_{\Sigma_{i,\geq s_i}} w_i^* d\lambda > c > 0.$$

Moreover we can take a subsequence and t_j^- so that (62.75) holds. Then the equality holds in (62.76.2) in the same way as that of Lemma 62.74. On the other hand, Stokes' theorem and Lemma 62.31 give rise to

$$\int \gamma^*_{i(j),\mathrm{out}} \lambda - \int \gamma^*_{t_j^- + s_{i(j)}, i(j)} \lambda \geq \int_{\Sigma_{i(j),s\geq s_i}} w_i^* d\lambda \geq c > 0.$$

This is a contradiction to the equality for (62.75). □

Corollary 62.78. *For each $\epsilon > 0$ there exists S' and I_0 such that if $i > I_0$ then*

$$\int_{\Sigma_{i,\geq \frac{1}{2}\log \epsilon_{1,i} + S'}} w_i^* d\lambda \leq \epsilon.$$

Proof. If Corollary 62.78 is false there exists a sequence $i_k \geq k$ such that

$$\int_{\Sigma_{i,\geq \frac{1}{2}\log \epsilon_{1,i_k} + k}} w_{i_k}^* d\lambda \geq \epsilon.$$

We may assume that

$$\lim_{k\to\infty} \sup_{j\geq k} \left(\frac{1}{2} \log \epsilon_{1,i_j} + k \right) = -\infty$$

by taking a subsequence of i_k if necessary. We then obtain a contradiction by applying Lemma 62.77 to w_{i_k} and $s_k = \frac{1}{2}\log \epsilon_{1,i_k} + k$. □

Now we are going to prove the main result of this subsection.

Proposition 62.79. *For each given k, there exist I_0, R_0 and constants $o(i, R_0 \mid k)$ with*

$$\lim_{i\to\infty} \lim_{R_0\to\infty} o(i, R_0 \mid k) = 0$$

for each k, such that for all $i \geq I_0$ and for all $s > \frac{1}{2}\log \epsilon_{1,i} + \log S_0 + R_0$ the followings hold :

(62.80.1) *s is a regular level. The curve $w_i(\Sigma_i) \cap (\{s\} \times S^{2n-1})$ is parameterized by an arc $\gamma_{i,s} : [0,1] \to \{s\} \times S^{2n-1}$ for which there exists $a \in S^{n-1}$ such that*

$$|\nabla^k(\gamma_a - \gamma_{i,s})| < o(i, R_0 \mid k). \tag{62.81.1}$$

(62.80.2) *Moreover, for $s_1 \in [\frac{1}{2}\log \epsilon_{1,i} + \log S_0 + R_0, \log \epsilon_0 - R_0]$ the set*

$$\Sigma_{i,s_1-1\leq s\leq s_1+1} = w_i(\Sigma_i) \cap ([s_1 - 1, s_1 + 1] \times S^{2n-1})$$

has a parametrization

$$w_{i,s_1-1\leq s\leq s_1+1} : [-1/\alpha, 1/\alpha] \times [0,1] \to \Sigma_{i,s_1-1\leq s\leq s_1+1}$$

for which we have

$$|\nabla^k(w_{i,s_1-1\leq s\leq s_1+1} - w^{\text{flat}}_{a,s_1})| < o(i, R_0 \mid k). \tag{62.81.2}$$

Here we put

$$w^{\text{flat}}_{a,s_1}(\tau, t) = (\alpha\tau + s_1, \gamma_a(t)).$$

Proof. The proof will be given by contradiction. Suppose to the contrary. Then we may assume, by taking a subsequence if necessary, that there exists a sequence s_i with

$$s_i - \frac{1}{2}\log 2\epsilon_{1,i} \to \infty$$

such that one of the following must hold :

(62.82.1) s_i is not a regular level or $w_i^{-1}(\{s_i\} \times S^{2n-1})$ is disconnected.
(62.82.2) s_i is a regular level. There exist k and $c > 0$ such that

$$|\nabla^k(\gamma_a - \gamma_{i,s_i})| > c$$

for any a and parametrization γ_{i,s_i} of $w_i(\Sigma_i) \cap (\{s_i\} \times S^{2n-1})$.
(62.82.3) There exist k and $c > 0$ such that

$$|\nabla^k(w_{i,s_i-1\leq s\leq s_i+1} - w^{\text{flat}}_{a,s_i})| > c$$

for any parametrization $w_{i,s_i-1\leq s\leq s_i+1}$ of $w_i(\Sigma_i) \cap ([s_1 - 1, s_1 + 1] \times S^{2n-1})$.

Note by the choice s_i satisfies (62.57.2). Lemma 62.8 also allows us to assume (62.57.1). By the uniform area bound for $\mathfrak{T}_{s_i}(w_i(\Sigma_{i,0}))$ we obtain in Lemma 62.61, Proposition 62.65 (62.66.3) implies that $\lim_{i\to\infty}\mathfrak{T}_{s_i}(w_i(\Sigma_{i,0}))$ converges to a one dimensional complex analytic set $X(\{s_i\})$. Lemma 62.77 then implies

$$E_{d\lambda}(X(\{s_i\}))=0$$

and so the conclusion of Corollary 62.73 holds for $X(\{s_i\})$. By the same way as Proposition 62.74 we prove

$$X(\{s_i\})=\mathbb{R}\times\gamma_a$$

for some a. In particular $X(\{s_i\})$ is smooth and has multiplicity one. Therefore (62.66.3) implies that $\mathfrak{T}_{s_i}(w_i(\Sigma_{i,0}))$ converges to $\mathbb{R}\times\gamma_a$ as a smooth manifold in compact C^∞ topology. It follows from this that none of (62.82.1) - (62.82.3) can occur, a contradiction. The proof of Proposition 62.79 is now finished. $\square$

62.5. Exponential decay in the neck region.

In this subsection, we prove (62.14), the exponential convergence to Reeb chords. The main tool for such a convergence result is the characterization of the asymptotics of J_0-holomorphic maps with small $d\lambda$-energy $E_{d\lambda}$, Theorem 60.85 below. This is a minor variation of Theorem 1.3 [HWZ02] in our relative context.

Let $R>0$ be given and

$$w:[-R,R]\times[0,1]\to\mathbb{R}\times S^{2n-1}$$

be a J_0-holomorphic map satisfying the boundary condition

$$w(\tau,0)\in\mathbb{R}\times S^{n-1}_{\mathbb{R}^n},\quad w(\tau,1)\in\mathbb{R}\times S^{n-1}_{\Lambda}. \tag{62.83}$$

As before we consider the energies

$$E_{d\lambda}(w)=\int_{[-R,R]\times[0,1]}w^*d\lambda, \tag{62.84.1}$$

$$E_{\lambda}(w)=\sup_{\rho\in\mathcal{C}}\int_{[-R,R]\times[0,1]}w^*(\rho\,ds\wedge\lambda), \tag{62.84.2}$$

$$E(w)=E_{d\lambda}(w)+E_{\lambda}(w), \tag{62.84.3}$$

where $\mathcal{C}$ is as in (62.27).

Theorem 62.85. *For each $E_0 > 0$ and k there exist positive constants e_0, R_0, c_k, and C_k with the following properties. Let $w : [-R, R] \times [0,1] \to \mathbb{R} \times S^{2n-1}$ be a J_0-holomorphic map that satisfy* (62.83). *We assume :*

(62.86.0) $E(w) \leq E_0$,
(62.86.1) $R \geq R_0$,
(62.86.2) $E_{d\lambda}(w) \leq e_0$,
(62.86.3) *the chord* $w_0(t) := w(0,t)$ *satisfies*

$$\int w_0^*\lambda \leq \frac{3\alpha}{2}.$$

Then, we can find $a \in S^{n-1}$ and $s_1 \in \mathbb{R}$ for which we have

$$|\nabla^k(w - w^{\rm flat}_{a,s_1})|(\tau,t) \leq C_k e^{-c_k(R-|\tau|)} \tag{62.87}$$

on $(\tau,t) \in [-R+10, R-10] \times [0,1]$ where $w^{\rm flat}_{a,s_1}$ is the cylindrical strip defined by $w^{\rm flat}_{a,s_1}(\tau,t) = (s_1 + \alpha\tau, \gamma_a(t))$ as before.

For the case of w without boundary, that is, for the case of pseudo-holomorphic map w from the annulus $[-R,R] \times S^1$, the analog of Theorem 62.85 was proved in [HWZ02] for the case of non-degenerate isolated Reeb orbits and for its Bott-Morse version in [Bou02] and [BEHWZ03] respectively.

For the completeness's sake, we give the proof of Theorem 62.85 below. The first step is to prove the following version of the ε-regularity result.

Proposition 62.88. (ϵ-regularity) *Let E_0, R and k be given and consider J_0-holomorphic maps $w : [-R,R] \times [0,1] \to \mathbb{R} \times S^{2n-1}$ satisfying* (62.83) *and $E(w) \leq E_0$. Then there exists a sufficiently small $e_1 > 0$ such that we have*

$$|\nabla^k w| < C_k$$

on $[-R+10, R-10] \times [0,1]$ for all w satisfying $E_{d\lambda}(w) \leq e_1$. Here $C_k > 0$ is independent of R and of such w's.

Proof. As in the scheme of [Hof93], we start with the following lemma.

Lemma 62.89. *There exists a constant $C_0 > 0$ with the following properties : Let $w : [-R,R] \times [0,1] \to \mathbb{R} \times S^{2n-1}$ be a C^1 map and assume*

$$\sup_{(\tau,t)\in[-R+10,R-10]\times[0,1]} |\nabla w|(\tau,t) \geq C_0. \tag{62.89.1}$$

Then there exists $(\tau_0,t_0) \in [-R+5, R-5]$ such that

$$|\nabla w|(\tau_0,t_0) = C \geq C_0, \tag{62.90.1}$$

$$|\nabla w|(\tau_0',t_0') \leq 2C \text{ if } \operatorname{dist}((\tau_0',t_0'),(\tau_0,t_0)) \leq C^{-1/2}. \tag{62.90.2}$$

Proof. We will prove that any choice of $C_0 > 0$ satisfying

$$\frac{\sqrt{2}C_0^{-1/2}}{\sqrt{2}-1} < 1$$

will do our purpose. This choice of C_0 will be justified in the course of the proof.

Let w satisfy (62.89.1). We will prove the lemma by contradiction. Suppose to the contrary that for any choice of $(\tau, t) \in [-R+5, R-5]$, (62.90) fails to hold. Take $(\tau_1, t_1) \in [-R+10, R-10] \times [0,1]$ for which we have

$$|\nabla w|(\tau_1, t_1) = C_1 \geq C_0.$$

We will inductively construct a sequence (τ_k, t_k) satisfying

(62.91.1) $\quad |\nabla w|(\tau_k, t_k) = C_k \geq 2^{k-1} C_1,$

(62.91.2) $\quad \mathrm{dist}((\tau_k, t_k), (\tau_{k-1}, t_{k-1})) \leq C_{k-1}^{-1/2}.$

Suppose such a sequence (τ_i, t_i) has been chosen for $i < k$. We obtain (62.92)

$$\mathrm{dist}((\tau_1, t_1), (\tau_{k-1}, t_{k-1})) \leq \sum_{i=1}^{k-2} C_i^{-1/2} \leq \left(\sum_{i=1}^{\infty} 2^{(1-i)/2}\right) C_0^{-1/2} = \frac{\sqrt{2}C_0^{-1/2}}{\sqrt{2}-1}$$

Therefore by the choice of C_0 in the beginning of the proof, we have

$$\mathrm{dist}((\tau_1, t_1), (\tau_{k-1}, t_{k-1})) \leq \frac{\sqrt{2}C_0^{-1/2}}{\sqrt{2}-1} < 1.$$

Then clearly $(\tau_{k-1}, t_{k-1}) \in [-R+5, R-5]$ and (τ_{k-1}, t_{k-1}) satisfies (62.90.1) by the induction hypothesis (62.91.1). By the standing hypothesis in the beginning of the proof, (60.90.2) must fail to hold for $(\tau_0, t_0) = (\tau_{k-1}, t_{k-1})$. In other words, there must exist $(\tau_0', t_0') = (\tau_k, t_k)$ for which (62.91.2) and (62.91.1) hold.

Now since $C_k^{-1} \leq \frac{1}{2^{k-1}C_0}$ by the choice of C_k in (62.91.1), (62.91.2) implies

$$\lim_{k_0 \to \infty} \sum_{k \geq k_0} \mathrm{dist}((\tau_k, t_k), (\tau_{k-1}, t_{k-1})) = 0$$

and hence (τ_k, t_k) is a Cauchy sequence. Denote its limit by (τ_∞, t_∞).

Since w is assumed to be C^1, we have

$$|\nabla w(\tau_\infty, t_\infty)| := C < \infty \quad \text{and } |\nabla w(\tau_k, t_k) - \nabla w(\tau_\infty, t_\infty)| \to 0$$

and in particular we have

$$|\nabla w(\tau_k, t_k)| \leq C + 1 < \infty$$

for all sufficiently large k. On the other hand, (62.91.1) implies

$$\lim_{k\to\infty} |\nabla w|(\tau_k, t_k) = \infty,$$

a contradiction. This finishes the proof. $\square$

Obviously the particular choice of the pair $5 < 10$ in the lemma is nothing special but can be replaced by any pair of positive numbers $K_1 < K_2$.

We continue the proof of Proposition 62.88. By the standard elliptic regularity of pseudo-holomorphic maps, it suffices to show the derivative bound

$$|\nabla w| < C \qquad \text{on } [-R+7, R-7] \times [0,1] \tag{62.93}$$

where $(-R+7, R-7) \times [0,1] \supset [-R+10, R-10] \times [0,1]$.

We will prove (62.93) by contradiction. We determine e_1 later in the proof. (This e_1 in fact can be chosen independently of E_0.) Suppose that there exists no C such that (62.93) holds for any $w : [-R, R] \times [0,1] \to \mathbb{R} \times S^{2n-1}$ satisfying (62.83) and $E(w) \le E_0$, $E_{d\lambda}(w) \le e_1$. Then there exists $w_i : [-R_i, R_i] \times [0,1] \to \mathbb{R} \times S^{2n-1}$, $(\tau_i, t_i) \in [-R_i + 7, R_i - 7] \times [0,1]$ such that

$$\lim_{i\to\infty} |\nabla w_i|(\tau_i, t_i) = \infty.$$

We now apply Lemma 62.89 to find $(\tau_i', t_i') \in [-R_i + 5, R_i - 5] \times [0,1]$ such that the following holds.

(62.94.1) $|\nabla w_i|(\tau_i', t_i') = C_{1,i} \to \infty$.

(62.94.2) If $\mathrm{dist}((\tau, t), (\tau_i', t_i')) \le C_{1,i}^{-1/2}$ then

$$|\nabla w_i|(\tau, t) \le 2C_{1,i}.$$

We now deduce contradiction from (62.94) by a blowing argument. We put

$$D_i = \left\{ x + \sqrt{-1}y \in \mathbb{C} \;\middle|\; \begin{array}{l} \mathrm{dist}((xC_{1,i}^{-1} + \tau_i', yC_{1,i}^{-1} + t_i'), (\tau_i', t_i')) \le C_{1,i}^{-1/2} \\ (xC_{1,i}^{-1} + \tau_i', yC_{1,i}^{-1} + t_i') \in \mathbb{R} \times [0,1] \end{array} \right\}$$

and define

$$\widetilde{w}_i : D_i \to \mathbb{R} \times S^{2n-1}$$

by

$$\widetilde{w}_i(x + \sqrt{-1}y) = w_i(xC_{1,i}^{-1} + \tau_i', yC_{1,i}^{-1} + t_i').$$

By taking subsequence, we may assume one of (62.95.1)-(62.95.3) below hold :

(62.95.1) $\lim_{i\to\infty} D_i = D_\infty = \mathbb{C}$.

(62.95.2) There exists $c \geq 0$ such that

$$\lim_{i\to\infty} D_i = \{z \in \mathbb{C} \mid \mathrm{Im} z \geq -c\} = D_\infty = \mathbb{H} - c\sqrt{-1}.$$

(62.95.3) There exists $c \geq 0$ such that

$$\lim_{i\to\infty} D_i = \{z \in \mathbb{C} \mid \mathrm{Im} z \leq c\} = D_\infty = -\mathbb{H} + c\sqrt{-1}.$$

(62.94.2) implies that

$$|\nabla \widetilde{w}_i| \leq 2. \tag{62.96}$$

Since $\widetilde{w}_i$ is holomorphic, we can use (62.96) to find a subsequence such that $\widetilde{w}_i$ converges to

$$\widetilde{w}_\infty : D_\infty \to \mathbb{R} \times S^{2n-1}$$

in compact C^∞ topology.

For the case (62.95.2), we have

$$\widetilde{w}_\infty(\partial D_\infty) \subseteq \mathbb{R} \times S^{n-1}_{\mathbb{R}^n}. \tag{62.97.1}$$

For the case (62.95.3), we have

$$\widetilde{w}_\infty(\partial D_\infty) \subseteq \mathbb{R} \times S^{n-1}_{\Lambda}. \tag{62.97.2}$$

(62.94.1) implies that

$$|\nabla \widetilde{w}_\infty|(0) = 1. \tag{62.98}$$

Now consider the energies of $\widetilde{w}_\infty$

$$E_{d\lambda}(\widetilde{w}_\infty) = \int_{D_\infty} \widetilde{w}_\infty^* d\lambda, \tag{62.99.1}$$

$$E_\lambda(\widetilde{w}_\infty) = \sup_{\rho\in\mathcal{C}} \int_{D_\infty} \widetilde{w}_\infty^*(\rho\, ds \wedge \lambda), \tag{62.99.2}$$

$$E(\widetilde{w}_\infty) = E_{d\lambda}(\widetilde{w}_\infty) + E_\lambda(\widetilde{w}_\infty), \tag{62.99.3}$$

where $\mathcal{C}$ is as in (62.27). The following easily follows from the hypothesis $E(w_i) \leq E_0$, $E_{d\lambda}(w_i) \leq e_1$ and non-negativity of the forms $d\lambda$ and $ds \wedge \lambda$ whose proof is omitted.

Lemma 62.100. $E(\widetilde{w}_\infty) \leq E_0$, $E_{d\lambda}(\widetilde{w}_\infty) \leq e_1$.

We next define a holomorphic map

$$\overline{w}_\infty : D_\infty \to \mathbb{C}P^{n-1} \tag{62.101}$$

as the composition of $\widetilde{w}_\infty$ and the projection

$$\pi : \mathbb{R} \times S^{2n-1} \to S^{2n-1} \to \mathbb{C}P^{n-1}.$$

Here $S^{2n-1} \to \mathbb{C}P^{n-1}$ is the Hopf fibration whose fibers are the Reeb orbits of the contact form λ. We note that $d\lambda$ is a pull back of the Kähler form $\omega_{\mathbb{C}P^{n-1}}$ of $\mathbb{C}P^{n-1}$ under the projection. Hence Lemma 62.100 implies

$$\int_{D_\infty} \overline{w}_\infty^* \omega_{\mathbb{C}P^{n-1}} \leq e_1 < \infty. \tag{62.102}$$

We are now ready to wrap up the proof of Proposition 62.88. We discuss the two cases (62.95.1), (62.95.2) separately. ((62.95.3) is obviously similar to (62.95.2).)

Case (62.95.1) : We use (62.102) and the removable singularity theorem to obtain

$$\overline{w}_\infty^+ : S^2 \to \mathbb{C}P^{n-1}. \tag{62.103.1}$$

We finally fix $e_1 > 0$ to be

$$e_1 = \frac{1}{3} w_{\mathbb{C}P^n}[L] \tag{62.103.2}$$

where $[L]$ is the homology class of the projective line in $\mathbb{C}P^n$. Then we have

$$\int_{S^2} \overline{w}^* \omega_{\mathbb{C}P^{n-1}} > e_1$$

for any nonconstant holomorphic map w. Therefore we derive from (62.102) that $\overline{w}_\infty$ must be constant.

What this means is that there exists a Reeb orbit $\gamma : S^1 \to S^{2n-1}$ such that

$$\widetilde{w}_\infty(\mathbb{C}) \subset \mathbb{R} \times \gamma(S^1) \subset \mathbb{R} \times S^{2n-1}.$$

Since $\mathbb{R} \times \gamma(S^1)$ is a cylinder and the map

$$\widetilde{w}_\infty : \mathbb{C} \to \mathbb{R} \times \gamma(S^1)$$

is nonconstant, its image must be dense : This is because $\widetilde{w}_\infty$ lifts to a nonconstant holomorphic map $\mathbb{C} \to \mathbb{C}$ whose image is dense by Picard's theorem. This implies

$$E(\widetilde{w}_\infty) = \infty$$

which is a contradiction to Lemma 62.100. Therefore this case cannot occur.

Case (62.95.2) : We use (62.102) and the removable singularity theorem to obtain

$$\overline{w}^+_\infty : (D^2, \partial D^2) \to (\mathbb{C}P^{n-1}, \mathbb{R}P^{n-1}).$$

The choice $e_1 < \frac{1}{2}\omega_{\mathbb{C}P^n}[L]$ made in (62.103.2) implies that $\overline{w}^+_\infty$ must be constant. Similarly as in the case of (62.95.1), this implies that there exists a Reeb orbit $\gamma : S^1 \to S^{2n-1}$ such that

$$\widetilde{w}_\infty(\mathbb{H} - c\sqrt{-1}) \subset \mathbb{R} \times \gamma(S^1).$$

Moreover there is a point $p \in \gamma(S^1)$ such that

$$\widetilde{w}_\infty(\partial\mathbb{H} - c\sqrt{-1}) \subset \mathbb{R} \times \{p\}.$$

Applying the reflection principle to $\widetilde{w}_\infty$ we obtain a nonconstant holomorphic map

$$\widehat{w}_\infty : \mathbb{C} \to \mathbb{R} \times \gamma(S^1)$$

which has an infinite energy, a contradiction. Hence this case cannot occur either.

In conclusion, (62.93) must hold and hence the proof of Proposition 62.88. □

Remark 62.104. The argument in the last step of the proof of Proposition 62.88 is rather ad hoc since we exploit the fact that the set of Reeb orbits of the contact structure on S^{2n-1} consists of the fibers of the Hopf fibration. (See also [LiRu01].) Because of this, the above proof seems to be simpler than other proofs such as the ones given in [HWZ02], [Bou2], [BEHWZ03]. Since study of this particular case is enough for our purpose in this book, we refrain ourselves from taking the more general route that works for arbitrary Bott-Morse case, including the case of non-integrable J's and with possibly non-cylindrical but only asymptotically cylindrical ends.

Now let us continue the proof of Theorem 62.85. For each given Reeb orbit γ, we denote $w^{\text{flat}}_{\gamma,s_1}$ by

$$w^{\text{flat}}_{\gamma,s_1}(\tau, t) = (\alpha_\gamma \tau + s_1, \gamma(t)),$$

where α_γ is given by

$$\alpha_\gamma = \int_{S^1} \gamma^* \lambda.$$

We call α_γ the *action* of γ. For the notational convenience, we will regard a constant curve $\gamma \equiv (s_1, \Theta_1)$ as a 'Reeb orbit' with zero action and still denote by

$$w^{\text{flat}}_{\Theta_1,s_1} \equiv (s_1, \Theta_1)$$

in Lemma 62.105 and Lemma 62.159 below.

Lemma 62.105. *Let ϵ, E_0 and k_0 be given. Then there exist e_2, R_2 with the following property : For any J_0-holomorphic map $w : [-R,R] \times [0,1] \to \mathbb{R} \times S^{2n-1}$ with $R \geq R_2$, and $E_{d\lambda}(w) < e_2$, and for any $\tau_1 \in [-R+3, R-3]$, there exist s_1 and a Reeb chord $\gamma : [0,1] \to S^{2n-1}$ joining $S^{n-1}_{\mathbb{R}^n}$ to S^{n-1}_{Λ} such that*

$$|\nabla^k (w - w^{\text{flat}}_{\gamma,s_1})|(\tau,t) \leq \epsilon \tag{62.106}$$

on $[\tau_1 - 1, \tau_1 + 1] \times [0,1]$ for $k \leq k_0$.

We note that γ may vary depending on τ_1 in this lemma.

Proof. Suppose to the contrary. Then, after taking a subsequence, we can find constants $\epsilon > 0$ and sequences of $e_{2,i} \to 0$, $R_i \to \infty$, $(\tau_{1,i}, t_i) \in [-R_i+3, R_i-3]\times[0,1]$ and pseudo-holomorphic maps

$$w_i : [-R_i, R_i] \times [0,1] \to \mathbb{R} \times S^{2n-1}$$

such that they satisfy $E(w_i) \leq E_0$, $E_{d\lambda}(w_i) < e_{2,i}$, and such that for some $k \leq k_0$ we have

$$\sup_{(\tau,t)\in[\tau_{i,1}-1,\tau_{i,1}+1]\times[0,1]} |\nabla^k (w - w^{\text{flat}}_{\gamma,s_1})|(\tau,t) > \epsilon \tag{62.107}$$

for any Reeb chords γ or points $\Theta_1 \in S^{2n-1}$ and for s_1.

Applying Proposition 62.88 and the diagonal sequence argument, we may take a sequence $s_{2,i}$ and then subsequences thereof, still denoted by the same i's such that

$$(\tau,t) \mapsto \mathfrak{T}_{s_{2,i}} w_i(\tau + \tau_{i,1}, t)$$

converges to a J_0-holomorphic map

$$w_\infty : \mathbb{R} \times [0,1] \to \mathbb{R} \times S^{2n-1}$$

in compact C^∞ topology. As before, we derive

$$E_{d\lambda}(w_\infty) = 0, \qquad E(w_\infty) \leq E_0.$$

Then it follows either w_∞ is constant or

$$w_\infty = w^{\text{flat}}_{\gamma,s_1'}$$

for some Reeb chord γ and s_1'. This contradicts to (62.107) and hence the proof. $\square$

Lemma 62.108. *Under the same hypotheses as Lemma* 62.105*, assume* (62.86.3) *in addition. Then for any given* $\tau_1 \in [-R+3, R-3]$ *there exist* $s_1 \in \mathbb{R}$ *and* $a \in S^{n-1}$ *such that*

$$|\nabla^k (w - w^{\text{flat}}_{a,s_1})|(\tau, t) \le \epsilon \tag{62.109}$$

on $[\tau_1 - 1, \tau_1 + 1] \times [0,1]$ *for* $k \le k_0$*. Here we recall* $w^{\text{flat}}_{a,s_1} = w^{\text{flat}}_{\gamma_a,s_1}$ *in the notation of Lemma* 62.105*.*

Proof. It suffices to show that $\gamma = \gamma_a$ for some $a \in S^{n-1}$ in the proof of Lemma 62.105. This is immediate from (62.86.3) if $\tau_1 = 0$.

To consider general τ_1, we deform it from 0 to the given τ_1 in $[R-3, R+3]$. As we mentioned the Reeb strip γ may change accordingly. However by (62.106), it can 'jump' only up to ϵ. (Here we use the C^0 distance between two Reeb strips to measure the size of the jump.)

We now use the fact that the set of Reeb chords of the form γ_a, $a \in S^{n-1}$ is minimal nondegenerate in the Bott-Morse sense and so positive C^0-distance away from all other possible Reeb chords, including constant orbits. Therefore if we take ϵ small enough then the Reeb chord γ must be of the form γ_a for some a. This finishes the proof. $\square$

Once Lemma 62.108 is established, one can prove Theorem 62.85 by the general scheme of handling the Bott-Morse type Floer theory, which was developed in [Fuk96II] §7, or in the proof of Lemma 11.2 of §14 [FuOn99II]. Here we again take a short cut by exploiting our special circumstance.

Proof of Theorem 62.85. Let w be as in Theorem 62.85. Define

$$\overline{w} : [-R, R] \times [0,1] \to \mathbb{C}P^{n-1}$$

to be the composition of w with

$$\pi : \mathbb{R} \times S^{2n-1} \to S^{2n-1} \to \mathbb{C}P^{n-1}$$

as in the proof of Proposition 62.88. We note

$$\pi(\mathbb{R} \times (S^{n-1}_{\mathbb{R}^n} \cup S^{n-1}_{\Lambda})) \subseteq \mathbb{R}P^{n-1}$$

and so $\overline{w}$ satisfies the real boundary condition

$$\overline{w}([-R, R] \times \{0,1\}) \subset \mathbb{R}P^{n-1}. \tag{62.110}$$

Moreover, since $d\lambda$ is a pull back of the Kähler form $\omega_{\mathbb{C}P^{n-1}}$ of $\mathbb{C}P^{n-1}$, it follows that

$$\int_{[-R,R]\times[0,1]} \overline{w}^* \omega_{\mathbb{C}P^{n-1}} = \int_{[-R,R]\times[0,1]} w^* d\lambda \le e_0. \tag{62.111}$$

We use the boundary condition (62.110) to apply the reflection principle and obtain a holomorphic cylinder

$$\widehat{\overline{w}} : [-R, R] \times S^1 \to \mathbb{C}P^{n-1}$$

whose area has the bound

$$\int_{[-R,R]\times S^1} \widehat{\overline{w}}^* \omega_{\mathbb{C}P^{n-1}} \le 2e_0. \tag{62.112}$$

Moreover Lemma 62.108 implies

$$\sup_{s\in[-R,R]} \operatorname{Diam}(\widehat{\overline{w}}(\{s\} \times S^1)) \le o(e_0)$$

where $o(e_0)$ stands for any function of e_0 satisfying $\lim_{e_0\to 0} o(e_0) = 0$.

Therefore, by the monotonicity formula (see Lemma 4.2.1 [Mul94], for example), we obtain

$$\operatorname{Diam}(w([-R, R] \times S^1)) \le o(e_0) \tag{62.113}$$

Here we use the following well-known fact on the harmonic function whose proof we omit.

Lemma 62.114. *If $f : [-R, R] \times S^1 \to \mathbb{R}$ is a harmonic function satisfying*

$$\operatorname{Diam}(f([-R, R] \times S^1)) \le 1,$$

then there exists $c \in \mathbb{R}$ such that

$$|\nabla^k(f - c)|(\tau, t) \le C_k e^{-c_k(R-|\tau|)}. \tag{62.115}$$

Here C_k, c_k depends only on $k = 0, 1, 2, \cdots$.

We derive from (62.113) and Lemma 62.114 that there exists a point $\overline{a} \in \mathbb{C}P^{n-1}$ such that

$$|\nabla^k(\overline{w} - \overline{a})|(\tau, t) \le C_k e^{-c_k(R-|\tau|)}. \tag{62.116}$$

We may replace $\overline{a}$ with $\overline{w}(0, 0) \in \mathbb{R}P^{n-1}$ without destroying the inequality (62.116).

We remark that

$$\pi^{-1}(\overline{a}) = \mathbb{C}a \setminus \{0\} \subset \mathbb{C}^n \setminus \{0\} \cong \mathbb{R} \times S^{2n-1}$$

is a one dimensional complex vector space minus origin. Here $a \in S^{n-1} \subset \mathbb{R}^n$.

We define

$$w' : [-R, R] \times [0, 1] \to \mathbb{C} = \mathbb{C}a \subset \mathbb{C}^n$$

as the composition of w and the unitary projection $\Pi_a : \mathbb{C}^n \to \mathbb{C}^n$ with $\operatorname{Image}\Pi_a = \mathbb{C}a$. There exists R_3 depending only on C_0, c_0 in (62.116) such that

$$w'([-R + R_3, R - R_3] \times [0, 1]) \cap \{0\} = \emptyset.$$

Therefore (62.116), and (62.109) (combined with Lemma 62.114 again) imply

$$|\nabla^k(w - w')|(\tau, t) \le C_k e^{-c_k(R-|\tau|)}. \tag{62.117}$$

Here we use the *product metric* of the target space $\mathbb{C}^n \setminus \{0\} \cong \mathbb{R} \times S^{2n-1}$. Now the following lemma will complete the proof of Theorem 62.85.

Lemma 62.118. *There exists τ_0 such that*

$$|\nabla^k(w' - w'_{\tau_0})|(\tau,t) \le C_k e^{-c_k(R-|\tau|)}. \tag{62.119}$$

Here

$$w'_{\tau_0}(\tau,t) = \exp(\alpha(\tau+\tau_0+\sqrt{-1}t)).$$

We use the product metrics of $\mathbb{C}\setminus\{0\} \cong \mathbb{R}\times S^1$ *both on the target and* $[-R+R_3, R-R_3]\times[0,1]$ *and on the domain.*

Proof. We put

$$F(\tau,t) = \log w'(\tau,t) - \alpha(\tau+\sqrt{-1}t).$$

Note Lemma 62.108 implies that the image of w' is in the sector $\{z \mid -\epsilon < \operatorname{Arg} z < \alpha+\epsilon\}$. So log above is well defined. We have

$$\mathrm{Im}F(\tau,0) = \mathrm{Im}F(\tau,1) = 0.$$

Hence we apply reflection principle to obtain a holomorphic function

$$\widehat{F} : [-R+R_3, R-R_3]\times S^1 \to \mathbb{C}.$$

By Lemma 62.108 and (62.116), we have

$$|\mathrm{Im}\widehat{F}(\tau,t)| < \epsilon.$$

Therefore we can use Lemma 62.114 to derive

$$|\nabla^k \mathrm{Im}F|(\tau,t) \le C_k e^{-c_k(R-|\tau|)}.$$

(62.119) follows easily. This finishes the proof □

The proof of Theorem 62.85 is finally complete. □

Now we go back to the situation of Theorem 62.13. We will use Theorem 62.85 to prove (62.14), which establishes exponential convergence on the neck region.

Let

$$E_0 = \sup_i E(w_i) < \infty.$$

We take e_0 as in Theorem 62.85. By Corollary 62.78 there exists S' and I_0 such that if $i > I_0$ then

$$\int_{\Sigma_{i,\ge \frac{1}{2}\log\epsilon_{1,i}+S'}} w_i^* d\lambda < e_0. \tag{62.120}$$

We may assume $S' \ge \log S_0$ and $\frac{1}{2}\log\epsilon_{1,i} + S' \le \log\epsilon_0$ for large i.

We put

$$\mathcal{U}_{i,\mathrm{out}} = w_i^{-1}\left(\left[\frac{1}{2}\log\epsilon_{1,i} + S', \infty\right] \times S^{2n-1}\right).$$

The inclusion $\mathcal{U}_{i,\mathrm{out}} \supset \mathbb{H} \setminus \Sigma_i$ is obvious. The main step is to construct the parametrization

$$\psi_{i,\mathrm{neck}} : (-T_i + C_{1,i}, S_1 + C_{2,i}) \times [0,1] \to \mathcal{U}_{i,\mathrm{out}}$$

satisfying (62.14). Recall the definition

$$T_i = -\alpha^{-1}\left(\frac{1}{2}\log\epsilon_{1,i} + \log S_0\right)$$

and S_1 is the constant appearing in (62.10.3). We put

$$T_i' = -\alpha^{-1}\left(\frac{1}{2}\log\epsilon_{1,i} + S'\right).$$

One consequence of Proposition 62.79 is that the intersection $\mathcal{U}_{i,\mathrm{neck}}$ is diffeomorphic to a disc with 4 corners. Namely, the boundary $\partial(\mathcal{U}_{i,\mathrm{neck}})$ is decomposed into

$$\begin{aligned}\partial(\mathcal{U}_{i,\mathrm{neck}}) = \widehat{\gamma}_{i,-\alpha T_i'} &\cup [\widehat{\gamma}_{i,-\alpha T_i'}(0), \widehat{\gamma}_{i,\mathrm{out}}(0)] \\ &\cup \widehat{\gamma}_{i,\mathrm{out}} \cup [\widehat{\gamma}_{i,\mathrm{out}}(1), \widehat{\gamma}_{i,-\alpha T_i'}(1)].\end{aligned}$$

Here we note

$$\begin{aligned}\widehat{\gamma}_{i,-\alpha T_i'}(0) &= w_i^{-1}(\gamma_{i,-\alpha T_i'}(0)) \\ \widehat{\gamma}_{i,-\alpha T_i'}(1) &= w_i^{-1}(\gamma_{i,-\alpha T_i'}(1))\end{aligned}$$

lie in $\mathbb{R} = \partial\mathbb{H}$ and so regard them as real numbers. The intervals $[\widehat{\gamma}_{i,-\alpha T_i'}(0), \widehat{\gamma}_{i,\mathrm{out}}(0)]$ and $[\widehat{\gamma}_{i,\mathrm{out}}(1), \widehat{\gamma}_{i,-\alpha T_i'}(1)]$ lie in $\mathbb{R}_+$ and $\mathbb{R}_-$ respectively.

Figure 62.6.

By the convergence in Corollary 62.9, it follows that $\widehat{\gamma}_{i,\text{out}}$ is C^∞ approximates the half circle $\{z \in \mathbb{H} \mid |z| = \epsilon_0\}$. On the other hand, at this stage we are unable to exclude the possibility that the shape of arc $\widehat{\gamma}_{i,-\alpha T'_i}$ could be wild in $\mathbb{H}$ and hence the standard (cylindrical) coordinate might not be a good one to use to parameterize the map w_i. The role of $\psi_{i,\text{neck}}$ is to transform $\widehat{\gamma}_{i,-\alpha T'_i}$ to a more tame curve (See the argument of §62.6.)

Applying the Riemann mapping theorem, we obtain a sequence $R_{1,i} > 0$ and a conformal isomorphism

$$\psi_i : [-R_{1,i}, 0] \times [0,1] \to \mathcal{U}_{i,\text{neck}} \tag{62.121}$$

such that (See Figure 62.6.)

$$(w_i \circ \psi_i)(-R_{1,i}, 0) = \gamma_{i,-\alpha T'_i}(0), \tag{62.122.1}$$
$$(w_i \circ \psi_i)(-R_{1,i}, 1) = \gamma_{i,-\alpha T'_i}(1), \tag{62.122.2}$$
$$(w_i \circ \psi_i)(0, 0) = \gamma_{i,\text{out}}(0), \tag{62.122.3}$$
$$(w_i \circ \psi_i)(0, 1) = \gamma_{i,\text{out}}(1). \tag{62.122.4}$$

The following lemma intuitively looks obvious. For the completeness' sake, we give its proof based on the method of extremal length. (See [AhBe50] §4.)

Lemma 62.123.

$$\lim_{i\to\infty} R_{1,i} = +\infty.$$

Proof. We consider the submanifold

$$w_i(\mathcal{U}_{i,\text{neck}}) \subset \mathbb{R} \times S^{2n-1}$$

and denote by g_{ind} the Riemannian metric induced from the product metric on $\mathbb{R} \times S^{2n-1}$.

We also consider the product

$$\left[\frac{1}{2}\log \epsilon_{1,i} + S', \log \epsilon_0\right] \times [0, \alpha]$$

equipped with standard product metric g_0. Then it follows from Proposition 62.79 that there is a diffeomorphism

$$\Phi_i : w_i(\mathcal{U}_{i,\text{neck}}) \cong \left[\frac{1}{2}\log \epsilon_{1,i} + S', \log \epsilon_0\right] \times [0, \alpha]$$

that has the properties,

$$\Phi_i(\mathcal{U}_{i,\text{neck}} \cap (\{s\} \times S^{2n-1})) = \{s\} \times [0, \alpha]. \tag{62.124.1}$$

(62.124.2) For each ϵ there exists $C = R_0$ such that

$$|\Phi_{i*}(g_{\text{ind}}) - g_0|_{C^1} < \epsilon$$

on $[\frac{1}{2}\log 2\epsilon_{1,i} + S' + C, \log \epsilon_0 - C] \times [0, \alpha]$ for all sufficiently large i.

Next we denote $g_2 = \Phi_{i*}(g_{\text{ind}})$ and by g_1 to be the standard metric on $[-R_{1,i}, 0] \times [0, 1]$.

Figure 62.7.

Since there are several metrics that we consider, we enlist them here for the purpose of referencing (See Figure 62.7.) :

(1) g_{ind} = the induced metric on the image $w_i(\mathcal{U}_{i,\text{neck}}) \subset \mathbb{R} \times S^{2n-1}$,
(2) g_0 = the standard metric on the product $[\frac{1}{2}\log \epsilon_{1,i} + S', \log \epsilon_0] \times [0, \alpha]$,
(3) g_1 = the standard metric on $[-R_{1,i}, 0] \times [0, 1]$,
(4) $g_2 = \Phi_*(g_{\text{ind}})$.

Since $w_i \circ \psi_i$ is holomorphic and so g_1-g_{ind} conformal, we have $(w_i \circ \psi_i)_*(g_1) = f^2 g_{\text{ind}}$ and so

$$(\Phi_i \circ w_i \circ \psi_i)_*(g_1) = f^2 g_2 \tag{62.125}$$

for a positive function f on $[\frac{1}{2}\log \epsilon_{1,i} + S', \log \epsilon_0] \times [0, \alpha]$.

Denote $X = [\log \epsilon_{1,i} + S' + C, \log \epsilon_0 - C] \times [0, \alpha]$ and its area form by dA_{g_2} for the metric g_2. We compute

$$\begin{aligned}\left(\int_X f\, dA_{g_2}\right)^2 &\leq \left(\int_X f^2 dA_{g_2}\right)\left(\int_X dA_{g_2}\right) \\ &\leq \text{Vol}([-R_{1,i}, \log \epsilon_0] \times [0,1]; g_1) \times (1+\epsilon)\left(\int_X dA_{g_0}\right) \\ &\leq (1+\epsilon)\alpha(R_{1,i} + \log \epsilon_0)(-\frac{1}{2}\log \epsilon_{1,i} - S' + \log \epsilon_0 - 2C).\end{aligned} \tag{62.126}$$

On the other hand, we derive, from (62.124.2) and (62.125),

$$\int_X f\, dA_{g_2} \geq (1+\epsilon)^{-1} \int_{\frac{1}{2}\log \epsilon_{1,i}+S'+C}^{\log \epsilon_0 - C} \text{length}_{g_0}(w_i^{-1} \circ \gamma_{i,s})\, ds.$$

Since $w_i^{-1}\gamma_{i,s}(0) \in \mathbb{R} \times \{0\}$, $w_i^{-1}\gamma_{i,s}(\alpha) \in \mathbb{R} \times \{1\}$, it follows that $\text{length}_{g_0}(w_i^{-1} \circ \gamma_{i,s}) \geq 1$ and so

$$(1+\epsilon)\int_X f\, dA_{g_2} \geq -\frac{1}{2}\log \epsilon_{1,i} - S' - 2C + \log \epsilon_0. \tag{62.127}$$

Substituting this into (62.126), we have obtained

$$R_{1,i} + \log \epsilon_0 \geq \frac{1+\epsilon}{\alpha}\left(-\frac{1}{2}\log \epsilon_{1,i} - S' - 2C + \log \epsilon_0\right) \tag{62.128}$$

The lemma now follows from the convergence $\epsilon_{1,i} \to 0$. □

We define

$$w_i' : [-R_{1,i}, 0] \times [0,1] \to \mathbb{R} \times S^{2n-1}$$

by

$$w_i' = w_i \circ \psi_i.$$

By our choice of $\mathcal{U}_{i,\text{out}}$ and (62.120), we have

$$\int w_i'^* d\lambda < e_0$$

for $i > I_0$. Therefore, we can apply Theorem 62.85 to w_i' for sufficiently large i. It follows from (62.87) that $|R_{1,i} - T_i|$ is uniformly bounded. (Note $T_i - T_i'$ is independent of i.)

Now take $C_{3,i}$ such that

$$(s \circ w_i \circ \psi_i)\left(\frac{S_1 - T_i}{2} + C_{3,i}, \frac{1}{2}\right) = \alpha\left(\frac{S_1 - T_i}{2}\right)$$

and define

$$\psi_{i,\text{neck}}(\tau, t) = \psi_i(\tau + C_{3,i}, t).$$

Again from (62.87), $C_{3,i}$ are uniformly bounded. By Theorem 62.85 for $k = 1$ and integrating over $[0, |\tau|]$, we derive

$$\left| s \circ w_i \circ \psi_{i,\text{neck}}\left(\frac{S_1 - T_i}{2} + \tau, t\right) - \alpha\left(\frac{S_1 - T_i}{2} + \tau\right)\right| \leq \int_0^{|\tau|} C_1 e^{-c_1\left|\frac{R_{1,i} - C'}{2} - |x|\right|} dx.$$

We put $C_{1,i} = T_i - R_{1,i} - C_{3,i}$, $C_{2,i} = S_1 - C_{3,i}$. Then

$$\psi_{i,\mathrm{neck}} : [-T_i + C_{1,i}, S_1 + C_{2,i}] \times [0,1] \to \mathcal{U}_{i,\mathrm{neck}}$$

is a biholomorphic map which satisfies

$$\left|\nabla^k (w_i \circ \psi_{i,\mathrm{neck}} - w^{\mathrm{flat}}_{a_i,0})\right| \left(\frac{S_1 - T_i}{2} + \tau, t\right) \le C_k e^{-c_k|\tau|}$$

by Theorem 62.85 again. Then (62.14.2) follows. (62.14.3) is trivial. This finishes the proof of (62.14).

We also remark that the convergence statement $\lim_{i\to\infty} a_i = a_0$ in Theorem 62.13 follows from (62.14) and Lemma 62.8.

62.6. C^∞ convergence in a neighborhood of p_{12}.

In this subsection, we prove (62.15) of Theorem 62.13 and completes the proof thereof. We first prepare some notations. Let $w_i : \mathbb{H} \to M$ be as in Theorem 62.13. It restricts to a map

$$w_i : \Sigma_i \to B(p_{12}; \epsilon_0) (= I(B^{2n}(\epsilon_0)))$$

where $\Sigma_i \subset \mathbb{H}$ is as in §62.2. We identify $B(p_{12}; \epsilon_0)$ with $B^{2n}(\epsilon_0)$ via the Darboux chart I.

Lemma 62.129. *$w_i(\Sigma_i)$ does not contain $0 \in \mathbb{C}^n$.*

Proof. Using estimate (62.14) the proof is similar to the proof of Proposition 60.59 and hence is omitted. □

Applying an element of $\mathrm{Aut}(\mathbb{H}; \{\pm 1\})$ to the domain $\mathbb{H}$, may assume that w_i satisfies

$$|w_i(0)|_{\mathbb{C}^n} = \inf\{|w_i(z)|_{\mathbb{C}^n} \mid z \in \partial\mathbb{H}\}. \tag{62.130.1}$$

We put

$$\epsilon_{3,i} = \sup\{|z| \mid z \in \Sigma_i, |w_i(z)|_{\mathbb{C}^n} \le 2\sqrt{\epsilon_{1,i}} S_0\} > 0. \tag{62.130.2}$$

Conditions (62.7), Lemma 62.8 and (62.130.1) imply $\lim_{i\to\infty} \epsilon_{3,i} = 0$.

We define a rescaled map

$$\widetilde{w}_i : \epsilon_{3,i}^{-1} \Sigma_i \to \mathbb{C}^n$$

by

$$\widetilde{w}_i(z) = \epsilon_{1,i}^{-1/2} w_i(\epsilon_{3,i} z).$$

(62.10.3) shows that $\widehat{\gamma}_{i,\rm out} = \partial\Sigma_i \subset \mathbb{H}$ is close to the image of the curve $t \mapsto e^{\pi(S_1+\sqrt{-1}t)} \in \mathbb{H}$. This implies $\Sigma_i \supset \mathbb{H}_{|z|\le e^{\pi S_1}/2}$ and $\epsilon_{3,i}^{-1}\Sigma_i \supset \mathbb{H}_{|z|\le \epsilon_{3,i}^{-1}e^{\pi S_1}/2}$. We put

$$S_{1,i} = \epsilon_{3,i}^{-1}e^{\pi S_1}/2$$

and consider the restriction

$$\widetilde{w}_i : \mathbb{H}_{|z|\le S_{1,i}} \to \mathbb{C}^n \setminus \{0\} \cong \mathbb{R}\times S^{2n-1}$$

of $\widetilde{w}_i$. We remark $\lim_{i\to\infty} S_{1,i} = \infty$.

We have

$$\widetilde{w}_i(\partial\mathbb{H}_{|z|<S_{1,i}}) \subset \epsilon_{1,i}^{-1/2}(H^{\alpha}_{-\epsilon_{1,i}})' = (H^{\alpha}_{-1})'$$

where the right hand side is independent of i, and

$$\begin{aligned}(H^{\alpha}_{-1})' &\cap ([\log S_0, \log\epsilon_0 - \log\epsilon_{1,i}/2]\times S^{2n-1}) \\ &= [\log S_0, \log\epsilon_0 - \log\epsilon_{1,i}/2]\times (S^{n-1}_{\mathbb{R}^n}\cup S^{n-1}_{\Lambda}).\end{aligned} \tag{62.131}$$

We next consider the energies of $\widetilde{w}_i$ given by

$$E_{\rm int}(\widetilde{w}_i) = \int_{\{z\in\mathbb{H}_{|z|\le S_{1,i}} \mid |\widetilde{w}_i(z)|_{\mathbb{C}^n}\le 2S_0\}} \widetilde{w}_i^* d(e^{2s}\lambda) \tag{62.132.1}$$

and

$$E_{d\lambda}(\widetilde{w}_i;S) = \int_{\{z\in\mathbb{H}_{|z|\le S_{1,i}} \mid |\widetilde{w}_i(z)|_{\mathbb{C}^n}\ge e^{S})\}} \widetilde{w}_i^* d\lambda. \tag{62.132.2}$$

Next let $\mathcal{C}$ be the set of all nonnegative smooth function $\rho:\mathbb{R}\to\mathbb{R}$ whose support is compact and is contained in $[\log S_0,\infty)$ and such that $\int\rho(s)=1$. Then we define

$$E_{\lambda}(\widetilde{w}_i) = \sup_{\rho\in\mathcal{C}} \int \widetilde{w}_i^*(\rho\, ds\wedge\lambda), \tag{62.133.1}$$

$$E_{\rm neck}(\widetilde{w}_i) = E_{d\lambda}(\widetilde{w}_i;\log S_0) + E_{\lambda}(\widetilde{w}_i). \tag{62.133.2}$$

Lemma 62.134. *$E_{\rm neck}(\widetilde{w}_i)$ and $E_{\rm int}(\widetilde{w}_i)$ are uniformly bounded above over i.*

Proof. It is easy to see from the scaling property that

$$E_{\rm neck}(\widetilde{w}_i) \le E(w_i)$$

and hence it is uniformly bounded by Proposition 62.30.

On the other hand, we have

$$E_{\rm int}(\widetilde{w}_i) \le \epsilon_{1,i}^{-1}\int_{\Sigma_{i,\frac{1}{2}\log\epsilon_{1,i}+\log 2S_0},{\rm int}} w_i^*\omega_0$$

by definition which becomes uniformly bounded by Proposition 62.51. This finishes the proof. □

Lemma 62.135.

$$\lim_{S\to\infty} \limsup_{i\to\infty} E_{d\lambda}(\widetilde{w}_i; S) = 0.$$

Proof. This is a consequence of Corollary 62.78. □

We next describe the metrics on the domain $\mathbb{H}$ and the target $\mathbb{C}^n$, with which we evaluate the C^k norms of $\widetilde{w}_i$'s.

For the target, we required the metric, denoted by $g'_{\mathbb{C}^n}$, to satisfy the following properties :

(62.136.1) $g'_{\mathbb{C}^n}$ is a flat Euclidean metric on the Euclidean ball $B^{2n}(S_0)$.

(62.136.2) Outside the (Euclidean) ball $B^{2n}(2S_0)$ of radius $2S_0$, it is the standard product metric on $[\log 2S_0, \infty) \times S^{2n-1}(3S_0/2)$. (Here $S^{2n-1}(3S_0/2)$ is the round sphere of radius $3S_0/2$.

(62.136.3) $g'_{\mathbb{C}^n}$ is of nonnegative curvature.

Figure 62.8.

For the domain, we require the metric, denoted by $g'_{\mathbb{H}}$, to have totally geodesic boundary and to satisfy the following properties :

(62.137.1) $g'_{\mathbb{H}}$ is a flat Euclidean metric on the Euclidean ball $B_1(0, \mathbb{H})$.

(62.137.2) Outside the (Euclidean) ball $B_2(0, \mathbb{H})$ of radius 2, $g'_{\mathbb{H}}$ is the standard product metric $[0, \infty) \times [0, 3\pi/2]$.

(62.137.3) $\quad g'_{\mathbb{H}}$ is of nonnegative curvature.

Figure 62.9.

We divide our analysis into the following two cases :

Case A : For each R, $|\nabla \widetilde{w}_i|$ are uniformly bounded on $\mathbb{H}_{|z| \leq R}$.

Case B : There exists a bounded sequence of points $z_i \in \mathbb{H}$ such that $|\nabla \widetilde{w}_i|(z_i)$ goes to infinity.

We start with Case A. In this case, by the elliptic regularity, the C^k norm of $\widetilde{w}_i$ is uniformly bounded on any bounded subset of $\mathbb{H}$. (We here use the fact that $\widetilde{w}_i$ satisfy the same Lagrangian boundary condition, independent of i.)

Therefore, by Ascoli-Arzela's theorem, we can find a subsequence of $\widetilde{w}_i$ that converges to a holomorphic map

$$\widetilde{w}_\infty : (\mathbb{H}, \partial\mathbb{H}) \to (\mathbb{C}^n, (H^\alpha_{-1})')$$

in compact C^∞ topology. The following energy bound is an immediate consequences of Lemma 62.134.

Lemma 62.138. *$E_{\text{int}}(\widetilde{w}_\infty)$ and $E_{\text{neck}}(\widetilde{w}_\infty)$ are finite.*

Next we prove the following

Lemma 62.139. *$\widetilde{w}_\infty$ is unbounded.*

Proof. The proof is by contradiction. Suppose that $\sup |\widetilde{w}_\infty|_{\mathbb{C}^n} \leq C < \infty$. Then we can choose $\rho \in \mathcal{C}$ such that

$$\text{supp } \rho \subset \{(s, \Theta) \in \mathbb{R} \times S^{2n-1} \mid \log 2S_0 - 1 \leq s \leq \log C + 1\}$$

and

$$\rho(s) \equiv \frac{1}{\log C + 1 - \log 2S_0} \qquad \text{for } s \in [\log 2S_0, \log C].$$

Denote $c = \frac{1}{\log C - \log 2S_0}$. Then we have

$$\begin{aligned}\int_{\mathbb{H}} \widetilde{w}_\infty^* d(e^{2s}\lambda) &\leq E_{\text{int}}(\widetilde{w}_\infty) + C^2 \int_{\{z|\, s(\widetilde{w}_\infty(z)) \geq \log 2S_0\}} w_\infty^*(ds \wedge \lambda + d\lambda) \\ &\leq E_{\text{int}}(\widetilde{w}_\infty) + C^2 E_\lambda(\widetilde{w}_\infty)/c + C^2 E_{d\lambda}(\widetilde{w}_\infty) < \infty.\end{aligned}$$

Recalling $\omega_0 = d(e^{2s}\lambda)$ is the standard symplectic structure on $\mathbb{C}^n$, we have shown that $\widetilde{w}_\infty$ has finite area. Applying a conformal diffeomorphism $(\mathbb{H}, \partial\mathbb{H}) \cong (D^2 \setminus \{1\}, \partial D^2 \setminus \{1\})$ and the removable singularity theorem, we can extend $\widetilde{w}_\infty$ to a holomorphic map

$$\widetilde{w}_\infty^+ : (D^2, \partial D^2) \to (\mathbb{C}^n, (H_{-1}^\alpha)').$$

Since $(H_{-1}^\alpha)'$ is an *exact* Lagrangian submanifold, it follows that $\widetilde{w}_\infty^+$ and so $\widetilde{w}_\infty$ must be a constant map.

We will next prove that $\widetilde{w}_\infty$ can not be a constant, which will finish the proof. By (62.130.2) and by the definition of $\widetilde{w}_i$ there exists z_i with $z_i \in \mathbb{H}$ such that

$$|z_i| = 1, \qquad |\widetilde{w}_i(z_i)|_{\mathbb{C}^n} = 2S_0 \tag{62.140.1}$$

and

$$\inf\{|\widetilde{w}_i(z)|_{\mathbb{C}^n} \mid |z| \geq 1\} \geq 2S_0. \tag{62.140.2}$$

Recall $(H_{-1}^\alpha)' \cap ([\log S_0, \infty) \times S^{2n-1})$ has two connected components

$$[\log S_0, \infty) \times S_{\mathbb{R}^n}^{n-1}, \quad [\log S_0, \infty) \times S_\Lambda^{n-1}.$$

And (62.140.2) implies

$$\begin{aligned}\widetilde{w}_i([1,\infty)) &\subset [\log S_0, \infty) \times S_{\mathbb{R}^n}^{n-1} \\ \widetilde{w}_i((-\infty,-1]) &\subset [\log S_0, \infty) \times S_\Lambda^{n-1}\end{aligned} \tag{60.140.3}$$

where $[1,\infty), (-\infty,-1] \subset \mathbb{R} = \partial\mathbb{H}$. Therefore there must exist a sequence z_i' such that

$$|\widetilde{w}_i(z_i')|_{\mathbb{C}^n} \leq 2S_0 \quad \text{and} \quad |\widetilde{w}_i(z_i) - \widetilde{w}_i(z_i')|_{\mathbb{C}^n} \geq 2S_0 \tan\left(\frac{\alpha}{2}\right). \tag{62.140.4}$$

Figure 62.10.

Then (62.140.2) and (62.140.4) imply that we may assume both z_i' and z_i converge by taking subsequences. Denote

$$z_\infty' = \lim_{i\to\infty} z_i', \quad z_\infty = \lim_{i\to\infty} z_i.$$

Then $|\widetilde{w}_\infty(z_\infty) - \widetilde{w}_\infty(z_\infty')|_{\mathbb{C}^n} \geq 2S_0 \tan\left(\frac{\alpha}{2}\right)$ and so $\widetilde{w}_\infty$ cannot be constant. This finishes the proof. $\square$

Lemma 62.141. *We have*

$$\widetilde{w}_\infty \in \widetilde{\mathcal{M}}(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{-1})').$$

Proof. It remains to show that there exists a minimal Reeb chord γ_{a_∞} for some $a_\infty \in S^{n-1}_{\mathbb{R}^n}$ and a constant $s_1 \in \mathbb{R}$ such that $\widetilde{w}_\infty$ satisfies

$$|\widetilde{w}_\infty(z) - w^{\rm flat}_{a_\infty,s_1}(z)|_{C^k} \to 0$$

in exponential order as $|z| \to \infty$.

Let $E_0 = E_{\rm neck}(\widetilde{w}_\infty)$. We take e_0 as in Theorem 62.85. Since $E(\widetilde{w}_\infty) < \infty$, we can choose S such that

$$E(\widetilde{w}_\infty; S) \leq e_0.$$

Then, we can apply Theorem 62.85 to the restriction of $\widetilde{w}_\infty$ to $[S, S+2R] \times [0,1]$.

Note $\tilde{w}_\infty([S, S+2R] \times [0,1]) \subset [\log 2S_0, \infty) \times S^{2n-1}$ by (62.140.2) and $(H^\alpha_{-1})' \cap ([\log 2S_0, \infty) \times S^{2n-1}) = [\log 2S_0, \infty) \times (S^{n-1}_{\mathbb{R}^n} \cup S^{n-1}_\Lambda)$. Therefore, the boundary condition (62.83) follows from (62.140.3). Moreover (62.86.3) can be proved as follows : Put

$$\gamma(t) = \widetilde{w}_\infty(S+R,t), \quad \gamma_i(t) = \widetilde{w}_i(S+R,t).$$

Then, by Corollary 62.9 and (62.32.1), we have :

$$\int_{t=0}^1 \gamma^*\lambda = \lim_{i\to\infty}\int_{t=0}^1 \gamma_i^*\lambda \le \lim_{i\to\infty}\int_{t=0}^1 \gamma_{i,\rm out}^*\lambda \le \frac{3\alpha}{2},$$

as required.

Therefore we have constants $R_{2,j}$, a_j and $s_{1,j}$ such that $R_{2,j}\to\infty$ and

$$|\nabla^k(\widetilde{w}_\infty - w^{\rm flat}_{a_j,s_{1,j}})|_{g'_{\mathbb{H}},g'_{\mathbb{C}^n}}(\tau,t) \le C_k e^{-c_k|\tau - S - R_{2,j}|} \tag{62.142}$$

on $(\tau,t)\in[S+10, S-10+2R_{2,j}]\times[0,1]$.

We may assume $a_j\to a_\infty$ by compactness of S^{n-1}. And since the intervals $[S+10,S-10+2R_{2,j}]$ are nested as $R_{2,j}\nearrow\infty$, we should also have $s_{1,j}\to s_1$ as $j\to\infty$ for s_1 appearing in Theorem 62.85. Then (62.142) implies

$$|\nabla^k(\widetilde{w}_\infty - w^{\rm flat}_{a_\infty,s_1})|_{g'_{\mathbb{H}},g'_{\mathbb{C}^n}}(\tau,t) \le C'_k e^{-c'_k|\tau|},$$

on $(\tau,t)\in[S+10,\infty)\times[0,1]$. Lemma 62.141 follows. □

Now we are ready to complete the proof of (62.15) for Case A. We take an isomorphism $\psi:\mathbb{H}\to\mathbb{H}$ such that $\psi(\infty)=\infty$ and

$$\widetilde{w}_\infty\circ\psi \in \widetilde{\mathcal{M}}_0(\mathbb{H},\mathbb{C}^n;(H^\alpha_{-1})').$$

(See Definition 61.14.)

By definition (61.8) of $\widetilde{\mathcal{M}}(\mathbb{H},\mathbb{C}^n;(H^\alpha_{-1})')$ there exists a_∞ such that

$$\lim_{\tau\to\infty}|\nabla^k(\widetilde{w}_\infty - w^{\rm flat}_{a_\infty,S})|_{g'_{\mathbb{H}},g'_{\mathbb{C}^n}}(\tau,t) = 0 \tag{62.143}$$

for some S. Therefore recalling the definition of $w_{a,b}$ given right above Theorem 62.13, we have

$$\widetilde{w}_\infty\circ\psi = w_{a_\infty,b}$$

for some b after re-choosing ψ if necessary. We now define the map

$$\psi_{i,\rm int}:[-\infty,R_i)\times[0,1]\to\mathbb{H}.$$

by

$$\psi_{i,\rm int}(\tau,t) := \epsilon_{3,i}\psi(\tau,t). \tag{62.144}$$

(We will determine R_i later in the proof.) Since we have

$$\epsilon_{1,i}^{-1/2}(w_i \circ \psi_{i,\rm int}) = \widetilde{w}_i \circ \psi$$

by definition of $\widetilde{w}_i$, it follows that $\epsilon_{1,i}^{-1/2}(w_i \circ \psi_{i,\rm int})$ converges to $w_{a_\infty,b}$ on compact C^∞ topology.

By the diagonal sequence argument, we can choose a sequence $R_i \to \infty$ so that

$$\lim_{i\to\infty} \sup_{\tau \le 2R_i} |\nabla^k(\epsilon_{1,i}^{-1/2}(w_i \circ \psi_{i,\rm int}) - w_{a_\infty,b})|_{g'_{\mathbb{H}},g'_{\mathbb{C}^n}}(\tau,t) = 0. \tag{62.145}$$

Let e_0, R_0 be as in Theorem 62.85. It follows from (62.143) that there exist S_3, I_0 such that the following holds for $i \ge I_0$:

$$\int_{[S_3,2R_i)\times[0,1]} (\epsilon_{1,i}^{-1/2}(w_i \circ \psi_{i,\rm int}))^* d\lambda < e_0 \tag{62.146.1}$$

$$2R_i - S_3 \ge R_0. \tag{62.146.2}$$

We can apply Theorem 62.85 to obtain a'_i, s'_i such that

$$|\nabla^k(\epsilon_{1,i}^{-1/2}(w_i \circ \psi_{i,\rm int}) - w^{\rm flat}_{a'_i,s'_i})|(\tau,t) \le C_k e^{-c_k \min\{|2R_i-\tau|,|\tau-S_3|\}}. \tag{62.147}$$

Comparing (62.147) with (62.145) we have $s'_i \to 0$. Perturbing $\psi_{i,\rm int}$ slightly and re-choosing s_i, we may assume $s'_i = 0$.

(62.147) and (62.14) around $\tau = R_i$ imply

$$|a_i - a'_i| \le Ce^{-cR_i}.$$

Therefore we obtain

$$|\nabla^k(\epsilon_{1,i}^{-1/2}(w_i \circ \psi_{i,\rm int}) - w^{\rm flat}_{a_i,0})|(\tau,t) \le C_k e^{-c_k \min\{|2R_i-\tau|,|\tau-S_3|\}}. \tag{62.148}$$

Now (62.15.3.1) follows from (62.145) and (62.15.3.2) follows from (62.148) respectively. The proof of (62.15) in Case A is complete.

Now we turn to Case B. Using the boundedness of z_i, the following lemma can be proved by the same way as that of Lemma 62.89 and so its proof is omitted.

Lemma 62.149. *We can take a bounded sequence $z'_i \in \mathbb{H}$ with the following properties.*

(62.150.1) $\mathrm{dist}_{g'_{\mathbb{H}}}(z_i, z'_i) \le 1$ *for large i.*
(62.150.2) $C'_i = |\nabla \widetilde{w}_i(z'_i)|_{g'_{\mathbb{H}},g'_{\mathbb{C}^n}}$ *goes to infinity.*
(62.150.3) *If z' satisfies* $\mathrm{dist}_{g'_{\mathbb{H}}}(z', z'_i) \le C_i'^{-1/2}$, *then* $|\nabla \widetilde{w}_i(z')|_{g'_{\mathbb{H}},g'_{\mathbb{C}^n}} \le 2C'_i$.

We can now use a similar argument as the proof of Proposition 62.88 and prove the following C^0-bound.

Lemma 62.151. *The sequence $\widetilde{w}_i(z_i') \in \mathbb{C}^n$ is bounded.*

Proof. The proof is by contradiction.

Suppose to the contrary that $R_{3,i} = |\widetilde{w}_i(z_i')|_{\mathbb{C}^n} \to \infty$. We put

$$D_i = \{u \in \mathbb{C} \mid \operatorname{dist}_{g'_{\mathbb{H}}}(C_i'^{-1}u + z_i', z_i') < \min\{C_i'^{-1}\sqrt{R_{3,i}}/2, C_i'^{-1/2}\},\ \ C_i'^{-1}u + z_i' \in \mathbb{H}\}.$$

We note that D_i is a convex domain of its diameter with the order of

$$\min\{\sqrt{R_{3,i}}/2, C_i'^{1/2}\}$$

which goes to ∞ as $i \to \infty$.

We define $\widetilde{\widetilde{w}}_i : D_i \to \mathbb{C}^n$ by

$$\widetilde{\widetilde{w}}_i(u) = \widetilde{w}_i(C_i'^{-1}u + z_i').$$

We now prove

Sublemma 62.152.

$$\inf_{u \in D_i} |\widetilde{\widetilde{w}}_i(u)| \geq \sqrt{R_{3,i}}\left(\sqrt{R_{3,i}} - 1\right) > 2S_0$$

if i is sufficiently large.

Proof. We note

$$\begin{aligned} |\widetilde{\widetilde{w}}_i(u)| &\geq |\widetilde{\widetilde{w}}_i(0)| - |\widetilde{\widetilde{w}}_i(u) - \widetilde{\widetilde{w}}_i(0)| \\ &= |\widetilde{w}_i(z_i')| - |\widetilde{\widetilde{w}}_i(u) - \widetilde{\widetilde{w}}_i(0)|. \end{aligned} \tag{62.153}$$

We have $|\widetilde{w}_i(z_i')| = R_{3,i}$ and

$$\begin{aligned} |\widetilde{\widetilde{w}}_i(u) - \widetilde{\widetilde{w}}_i(0)| &\leq \int_0^1 |u \cdot \nabla \widetilde{\widetilde{w}}_i(su)|\, ds \\ &= \int_0^1 |u \cdot C_i'^{-1} \nabla \widetilde{w}_i(C_i'^{-1}(su) + z_i')|\, ds \\ &\leq \int_0^1 |C_i'^{-1}u||\nabla \widetilde{w}_i(C_i'^{-1}(su) + z_i')|\, ds. \end{aligned}$$

But since $su \in D_i$ for all $s \in [0,1]$, we have

$$\operatorname{dist}(C_i'^{-1}(su) + z_i', z_i') \leq C_i'^{-1/2}.$$

Then (62.150.3) implies

$$|\nabla \widetilde{w}_i(C_i'^{-1}(su) + z_i')| \le 2C_i'.$$

Therefore we have

$$|\widetilde{\widetilde{w}}_i(u) - \widetilde{\widetilde{w}}_i(0)| \le 2|u| \le \sqrt{R_{3,i}}.$$

Substituting these into (62.153), we derive

$$|\widetilde{\widetilde{w}}_i(u)| \ge R_{3,i} - \sqrt{R_{3,i}} = \sqrt{R_{3,i}}(\sqrt{R_{3,i}} - 1).$$

This finishes the proof of Sublemma 62.152. □

Since $(H_{-1}^{\alpha})' \cap (\mathbb{C}^n \setminus B^{2n}(2S_0)) \subset \mathbb{R}^n \cup \Lambda$, Sublemma 62.152 allows us to regard $\widetilde{\widetilde{w}}_i$ as a sequence of maps

$$\widetilde{\widetilde{w}}_i : (D_i, \partial D_i) \to (\mathbb{R} \times S^{2n-1}, \mathbb{R} \times (S^{n-1}_{\mathbb{R}^n} \cup S^{n-1}_{\Lambda})).$$

(Note ∂D_i could be empty.) We have

(62.154.1) $E(\widetilde{\widetilde{w}}_i) \le E_0$.
(62.154.2) $E_{d\lambda}(\widetilde{\widetilde{w}}_i) \to 0$.

Here (62.154.2) follows from Lemma 62.135. We can find $s_i' \to \infty$ and a subsequence such that $\mathfrak{T}_{s_i'} \circ \widetilde{\widetilde{w}}_i$ converges to a map

$$\widetilde{\widetilde{w}}_\infty : (D_\infty, \partial D_\infty) \to (\mathbb{R} \times S^{2n-1}, \mathbb{R} \times (S^{n-1}_{\mathbb{R}^n} \cup S^{n-1}_{\Lambda}))$$

in compact C^∞ topology. We can now deduce contradiction in the same way as the proof of Proposition 62.88 especially in the proof of (62.93). This finishes the proof of Lemma 62.151. □

We put

$$D_i' = \{u \in \mathbb{C} \mid \mathrm{dist}_{g'_{\mathbb{H}}}(C_i'^{-1}u + z_i', z_i') < C_i'^{-1/2},\ \ C_i'^{-1}u + z_i' \in \mathbb{H}\}$$

and define

$$\widehat{w}_i : D_i' \to \mathbb{C}^n$$

by

$$\widehat{w}_i(u) = \widetilde{w}_i(C_i'^{-1}u + z_i').$$

Then we derive the uniform bounds

$$|\nabla \widehat{w}_i(u)| \le C_i'^{-1}|\nabla \widetilde{w}_i(C_i'^{-1}u + z_i')| \le C_i'^{-1}(2C_i') = 2 \tag{62.155}$$

from (62.150.3). Then Lemma 62.151 and this derivative bound enable us to apply Ascoli-Arzela theorem to $\widehat{w}_i$ and derive that $\widehat{w}_i$ converges in C^∞ compact topology.

Now depending on whether $C_i' \operatorname{dist}(z_i, \partial D_i') \to \infty$ or $C_i' \operatorname{dist}(z_i, \partial D_i') < c$ for some $c > 0$, one of the following alternatives occurs : (Note D_i' is a convex domain of $\mathbb{C}$.)

(62.156.1) $\lim_{i\to\infty} D_i' = D_\infty' = \mathbb{C}$.

(62.156.2) There exists $c \geq 0$ such that

$$\lim_{i\to\infty} D_i' = D_\infty' = \{z \in \mathbb{C} \mid \mathrm{Im} z \geq -c\} = \mathbb{H} - c\sqrt{-1}.$$

Moreover (62.155) enables us to assume that $\widehat{w}_i$ converges to $\widehat{w}_\infty : D_\infty' \to \mathbb{C}^n$ in compact C^∞ topology by taking a subsequence if necessary.

Lemma 62.157. *$\widehat{w}_\infty$ is unbounded.*

Proof. Suppose contrary that $\widehat{w}_\infty(D_\infty')$ is bounded. Then in the same way as the proof of Lemma 62.139, we can prove that $\widehat{w}_\infty$ is constant. On the other hand, since $|\nabla \widehat{w}_i(0)| = 1$ it follows that $|\nabla \widehat{w}_\infty(0)| = 1$. This is a contradiction. □

Proposition 62.158. *$D_\infty' \neq \mathbb{C}$. Namely* (62.156.1) *does not occur.*

Proof. The proof will be by contradiction. Assume $D_\infty' = \mathbb{C}$ and identify $\mathbb{C} \setminus \{0\} \cong \mathbb{R} \times S^1$. By Lemma 62.157 and the derivative bound (62.155), we can find $S(k) \to \infty$, $\tau_k \to \infty$, such that

$$\widehat{w}_\infty([\tau_k - 1, \tau_k + 1] \times S^1) \subset (-10 + \log S(k), 10 + \log S(k)) \times S^{2n-1}.$$

Then by the same way as the proof of Lemma 62.105 we have the following :

Lemma 62.159. *By taking a subsequence if necessary we can find a closed Reeb orbit $\gamma : S^1 \to S^{2n-1}$ and $S_k' \to \infty$ such that*

$$\lim_{k\to\infty} \sup_{(\tau,t)\in[\tau_k-1,\tau_k+1]\times S^1} |\nabla^\ell(\widehat{w}_\infty - w^{\mathrm{flat}}_{\gamma,S_k'})|(\tau,t) = 0$$

for $\ell \leq \ell_0$. Here we define $w^{\mathrm{flat}}_{\gamma,S_k'}$ by $w^{\mathrm{flat}}_{\gamma,S_k'}(\tau,t) = (\alpha_\gamma \tau + S_k', \gamma(t))$, and $\alpha_\gamma = \int_{S^1} \gamma^*\lambda$.

We next prove :

Lemma 62.160. *γ in Lemma 62.159 is not a constant loop.*

Proof. (The argument below is a minor modification of one in [Hof93 p 538].) Suppose to the contrary that γ is a constant loop. If $\widehat{w}_\infty$ does not pass the origin, then Stokes' formula and Lemma 62.159 imply

$$\lim_{k\to\infty} \int_{[-\infty,\tau_k]\times S^1} \widehat{w}_\infty^* d\lambda = \int_{S^1} \gamma^*\lambda = 0.$$

Namely

$$\int_{\mathbb{C}} \widehat{w}_\infty^* d\lambda = 0.$$

This implies that the projectivization $[\widehat{w}_\infty] : \mathbb{C} \to \mathbb{C}P^{n-1}$ of $\widehat{w}_\infty$ has zero area and so must be constant, which implies that the image of $\widehat{w}_\infty$ must be contained in some complex line $\mathbb{C} \cdot a$ for some $a \in \mathbb{C}^n$.

We remark that the restriction of the one form λ to $\mathbb{C}a \setminus \{0\}$ is a closed form. It follows that

$$\int_{\hat{w}_\infty^{-1}([S'_k, S'_{k+1}] \times S^{2n-1})} \hat{w}_\infty^* d(e^{2s}\lambda) = 2e^{2S'_{k+1}} \int \gamma_{k+1}'^* \lambda - 2e^{2S'_k} \int \gamma_k'^* \lambda,$$

where γ'_k is the restriction of $\hat{w}_\infty$ to $\hat{w}_\infty^{-1}(\{S'_k\} \times S^{2n-1})$.

Since γ'_k converges to a constant loop, it is homologous to zero for large k. It follows from $d\lambda = 0$ that

$$\int \gamma_k'^* \lambda = 0$$

for large k. Therefore we have

$$\int_{\hat{w}_\infty^{-1}([S'_k, S'_{k+1}] \times S^{2n-1})} \hat{w}_\infty^* d(e^{2s}\lambda) = 0$$

for large k. Hence, by unique continuation and holomorphicity, $\hat{w}_\infty$ is a constant map. This is a contradiction.

Now consider the case $\widehat{w}_\infty^{-1}(0) \neq \emptyset$. It follows from the convergence in Lemma 62.159 as $\tau \to \infty$ that the set $\widehat{w}_\infty^{-1}(0)$ has finite order. We write the finite set $\widehat{w}_\infty^{-1}(0)$ as

$$\widehat{w}_\infty^{-1}(0) = \{z_1, \cdots, z_m\}$$

for some $m \in \mathbb{Z}_+$. We denote by n_i the order of vanishing as before in (60.61). Then we have

$$\lim_{\delta \to 0} \int_{\partial B_{z_i}(\delta)} \widehat{w}_\infty^* \lambda = 2\pi n_i \geq 2\pi$$

for all i.

On the other hand, we recall that

$$E_{d\lambda}(\widehat{w}_\infty) \leq \liminf_{i \to \infty} E_{d\lambda}(\widehat{w}_i)$$

the last of which is uniformly bounded by Lemma 62.134. In particular, the limit

$$\lim_{\delta \to 0} \int_{\mathbb{C} \setminus \cup_{i=1}^m B_{z_i}(\delta)} \widehat{w}_\infty^* d\lambda$$

exists and hence we obtain

$$\int_{\mathbb{C}\setminus\cup_{i=1}^m B_{z_i}(\delta)} \widehat{w}_\infty^* d\lambda = -2\pi \sum_{i=1}^m \int_{\partial B_{z_i}(\delta)} \widehat{w}_\infty^* \lambda \tag{62.161}$$

by Stokes' formula. We recall that $\widehat{w}_\infty^* d\lambda$ is a non-negative form and hence the left hand side of (62.161) is non-negative for all $\delta > 0$. But the right hand side thereof converges to $-2\pi \sum_{i=1}^m n_i \leq -2\pi m < 0$ as $\delta \to 0$.

This gives rise to a contradiction and hence γ cannot be constant. □

Lemmas 62.159 and 62.160 imply that for all sufficiently large k, the ϵ-neighborhood of

$$\widehat{w}_\infty([\tau_k - 1, \tau_k + 1] \times S^1)$$

contains $\{S'_k\} \times \gamma$ where γ is a nontrivial *closed* Reeb orbit.

Recall $\widehat{w}_i$ was defined as

$$\widehat{w}_i(u) = \epsilon_{1,i}^{-1/2} w_i(\epsilon_{3,i}(C_i'^{-1} u + z_i')).$$

Namely in cylindrical coordinates (s, Θ) of $\mathbb{C}^n \setminus \{0\} \cong \mathbb{R} \times S^{2n-1} \to \mathbb{R}$, we have

$$s(w_i(\epsilon_{3,i}(C_i'^{-1} u + z_i'))) = \frac{1}{2}\log \epsilon_{1,i} + s(\widehat{w}_i(u)).$$

Therefore, by the choice of τ_k we have :

$$w_i(\epsilon_{3,i}(C_i'^{-1} u + z_i')) \in \left[\frac{1}{2}\log \epsilon_{1,i} + \log S_0 + R_0, \infty\right) \times S^{2n-1} \tag{62.162}$$

if $u \in [\tau_k - 1, \tau_k + 1] \times S^1$, k is large and $i \geq C(k)$. Here R_0 is the constant as in Proposition 62.79.

It follows from Proposition 62.79 that the Θ-component of $\widehat{w}_\infty([\tau_k - 1, \tau_k + 1] \times S^1)$ must be contained in a small neighborhood of a *minimal Reeb chord* joining $S^{n-1}_{\mathbb{R}^n}$ to S^{n-1}_Λ.

This is impossible since it contains the whole closed Reeb orbit γ of S^{2n-1}. The proof of Proposition 62.158 is now complete. □

We now continue the proof of Theorem 62.13. We have proved $D'_\infty = \mathbb{H} - c\sqrt{-1}$. Replacing z'_i to a point $\in \partial\mathbb{H}$ closest to $z'_i + cC'_i\sqrt{-1}$, we may assume $z'_i \in \partial\mathbb{H}$ and $D'_\infty = \mathbb{H}$.

We identify $\mathbb{H} \setminus \{0\} \cong \mathbb{R} \times [0, 1]$ as before. It follows from Lemma 62.157 and the derivative bound (62.155), we can find a $S(k) \to \infty$, $\tau_k \to \infty$ such that

$$\widehat{w}_\infty([\tau_k - 1, \tau_k + 1] \times [0, 1]) \subset [-10 + \log S(k), 10 + \log S(k)] \times S^{2n-1}$$

and

$$\widehat{w}_\infty([\tau_k - 1, \tau_k + 1] \times \{0, 1\}) \subset [-10 + \log S(k), 10 + \log S(k)] \times (S^{n-1}_{\mathbb{R}^n} \cup S^{n-1}_\Lambda).$$

Then by the same way as the proof of Lemma 62.105 we prove the following :

Lemma 62.163. *By taking a subsequence if necessary we can find a Reeb chord* $\gamma : ([0,1], \partial[0,1]) \to (S^{2n-1}, S^{n-1}_{\mathbb{R}^n} \cup S^{n-1}_{\Lambda})$ *and* $S'_k \to \infty$ *such that*

$$\lim_{k\to\infty} \sup_{(\tau,t)\in[\tau_k-1,\tau_k+1]\times[0,1]} |\nabla^\ell(\widehat{w}_\infty - w^{\rm flat}_{\gamma,S'_k})|(\tau,t) = 0 \tag{62.164}$$

for $\ell \le \ell_0$. *Here we define* $w^{\rm flat}_{\gamma,S'_k}$ *by* $w^{\rm flat}_{\gamma,S'_k}(\tau,t) = (\alpha_\gamma\tau + S'_k, \gamma(t))$, *and* $\alpha_\gamma = \int_{[0,1]}\gamma^*\lambda$.

We next prove the following :

Lemma 62.165. γ *is equal to* γ_a, *a minimal Reeb chord joining* $S^{n-1}_{\mathbb{R}^n}$ *to* S^{n-1}_{Λ}.

Proof. We first prove by contradiction that γ is not a constant path. Let us assume If $\gamma \equiv p_0 \in S^{n-1}_{\mathbb{R}}$. We regard $(\tau_k, 0) = e^{\pi\tau_k}, (\tau_k, 1) = -e^{\pi\tau_k} \in \partial\mathbb{H} = \mathbb{R}$ and let

$$\widehat{\mu}_k = [-e^{\pi\tau_k}, e^{\pi\tau_k}] \subset \mathbb{R}.$$

By Stokes' formula

$$\lim_{k\to\infty}\int_{[-\infty,\tau_k]\times[0,1]} \widehat{w}^*_\infty d\lambda = \int_{[0,1]}\gamma^*\lambda - \lim_{k\to\infty}\int_{\widehat{\mu}_k}\widehat{w}^*_\infty\lambda - 2\pi m$$

where m is the sum of multiplicities of $\hat{w}^{-1}_\infty(0)$. We remark that the integral $\int_{\widehat{\mu}_k}\widehat{w}^*_\infty\lambda$ depends only on the relative homology class

$$\widehat{w}_{\infty *}([\widehat{\mu}_k;\partial\widehat{\mu}_k]) \in H_1(H^\alpha_{-\epsilon_{1,k}}; (\{a^0_k\}\times S^{n-1}_{\mathbb{R}^n}) \cup (\{a^1_k\}\times S^{n-1}_{\mathbb{R}^n})).$$

(Here $\widehat{\mu}_k(j) \in \{a^j_k\}\times S^{n-1}_{\mathbb{R}^n}$.) This fact can be proved in the same way as (60.63).

Therefore, we have $\int_{\widehat{\mu}_k}\widehat{w}^*_\infty\lambda = 0$ for large k. It follows from the similar argument as the proof of Lemma 62.160 that $\hat{w}_\infty$ is a constant map. This is a contradiction.

We can then prove that γ must coincide with γ_a for some a in the same way as the last step of the proof of Proposition 62.158. □

By Lemmata 62.163, 62.165, Proposition 62.79 and the convergence $\widehat{w}_i \to \widehat{w}_\infty$, we find that

Lemma 62.166.

$$\widehat{w}^{-1}_\infty(\{S'_k\}\times S^{2n-1}) \subset [\tau_k - 1, \tau_k + 1]\times[0,1] \tag{62.167}$$

for all sufficiently large k.

Now we prove the following lemma. Let R_0 be as in Proposition 62.79.

Lemma 62.168. *There exists $R_4 \geq \log S_0 + R_0$ with the following properties :*

(62.169.1) *$\widehat{w}_i$ is transversal to $\{R_4\}\times S^{2n-1}$ for large i. The preimage $\widehat{w}_i^{-1}(\{R_4\}\times S^{2n-1})$ is an arc, which we denote by $\widehat{\gamma}_i$.*

(62.169.2) *If we put*

$$\widehat{\gamma}'_i = \{\epsilon_{3,i}(C_i'^{-1}\widehat{\gamma}_i(t) + z'_i) \mid t \in [0,1]\},$$

$\Sigma_i \setminus \widehat{\gamma}'_i$ is a disjoint union of $D_i^{\rm int}$ and $D_i^{\rm ext}$ such that

$$w_i(D_i^{\rm ext}) \subset \left[\frac{1}{2}\log \epsilon_{1,i} + R_4, \epsilon_0\right) \times S^{2n-1}.$$

(62.169.3) *$\widehat{\gamma}_i \subset \mathbb{H}$ is uniformly bounded.*

Proof. We put $R_4 = S'_k$ for large k. (62.169.1) is a consequence of (62.162) and Lemmas 62.163, 62.166. (62.169.3) is a consequence of Lemma 62.166. Then recalling the definition of $\widehat{w}_i$

$$\widehat{w}_i(u) = \epsilon_{1,i}^{-1/2} w_i(\epsilon_{3,i}(C_i'^{-1}u + z'_i))$$

we derive (62.169.2) from (62.169.1) and Lemma 62.166. □

Lemma 62.168 and Proposition 62.79 imply that for large i

$$w_i^{-1}([-\infty, \log S_0 + R_0] \times S^{2n-1}) \subset D_i^{\rm int}. \tag{62.171}$$

(62.130.1) implies that

$$w_i(0) \in [-\infty, \frac{1}{2}\log \epsilon_{1,i} + \log 2S_0] \times S^{2n-1}.$$

Therefore

$$0 \in D_i^{\rm int}. \tag{62.172}$$

We can take $z''_i \in \mathbb{H}$ such that $|z''_i| = \epsilon_{3,i}$, $|w_i(z''_i)| = 2\sqrt{\epsilon_{1,i}}S_0$ by (62.130.2). Then we obtain

$$z''_i \in D_i^{\rm int} \tag{62.173}$$

from (62.171). And (62.172), (62.173) imply that both $-C'_i z'_i$ and $C'_i(\epsilon_{3,i}^{-1} z''_i - z'_i)$ lie in the convex hull of $\widehat{\gamma}_i$.

Therefore it follows from (62.169.3) that $|C'_i \epsilon_{3,i}^{-1} z''_i| = C'_i$ is bounded, which contradicts to the standing hypothesis $C'_i = |\nabla \widetilde{w}_i(z'_i)| \to \infty$ in Case B. This implies that Case B does not occur.

The proof of Theorem 62.13 is finally completed. □

62.7. Wrap up of the proof of Theorem 62.2.

In this subsection, we use Theorem 62.13 to complete the proof of Theorem 62.2.

Let w_i, $\epsilon_{1,i}$, $\epsilon_{2,i}$ be the sequences chosen in (62.7). Subsequently we obtain a_i, b, $\delta_{k,i}, \mathcal{U}_{i,\rm int}, \mathcal{U}_{i,\rm out}, \mathcal{U}_{i,\rm neck}$, $\psi_{i,\rm int}$, $\psi_{i,\rm neck}$, R_i, $C_{1,i}$, $C_{2,i}$ that appear in Theorem 62.13.

If i is enough large we can apply Theorem 61.46 (2) to $\epsilon_1 = -\epsilon_{1,i}$, $\epsilon_2 = \epsilon_{2,i}, b \in S^{n-2}$ and obtain another pseudo-holomorphic map

$$w'_i : \mathbb{H} \to M$$

satisfying the same Lagrangian boundary condition as w_i, which is

$$w'_i(z) \in L_{-\epsilon_{1,i}},\ z \in [-1,1] \subset \mathbb{R},$$
$$w'_i(z) \in L_0,\ z \in \mathbb{R} \setminus [-1,1].$$

We next apply Theorem 62.13 to this new sequence w'_i. We put primes on the objects corresponding to w'_i to tell them from those associated to w_i. Without loss of any generality, we may assume

$$\epsilon'_{2,k} = \epsilon_{2,k},\ \delta'_{2,k} = \delta_{2,k},\ R_i = R'_i,\ C_{1,i} = C'_{1,i},\ C_{2,i} = C'_{2,i}.$$

(In fact we may replace them by $\max(\epsilon_{2,k}, \epsilon'_{2,k})$, $\min(R'_i, R_i)$ and etc.)

Lemma 62.174. *We have $b = b'$.*

Proof. It follows from the gluing construction that there exist $g_i \in {\rm Aut}(\mathbb{H}\cup\infty; \pm 1)$ such that $\epsilon_{1,i}^{-1}(w'_i \circ g_i)$ converges to $w_{a_0,b}$ in compact C^∞ topology around $0 \in \mathbb{H}$. Here $a_0 = (1, 0, \cdots, 0)$.

On the other hand, (62.15.3.1) and $\lim_{i\to\infty} a_i = a_0$ (which is proved at the end of §62.5) say that the sequence of rescaled maps $\epsilon_{1,i}^{-1}(w'_i \circ \widetilde\psi_{i,\rm int})$ converges to $w_{a_0,b'}$ in compact C^∞ topology. The lemma follows easily. □

Our next task is to find a path $r \mapsto w_i^r$ joining w_i to w'_i. This is Step (62.5.2). We start with constructing a coordinated chart of the domain by interpolating the ones for w_i and w'_i.

Let R_i be the sequence that appears in Theorem 62.13. We will fix the constants $R_{\rm out}$ and $R_{\rm int}$ later, which are sufficiently large and independent of i. We take i's so large that $R_i > 10R_{\rm int}$. We define the coordinate transformations

(62.175.1) $$\Phi_{i;{\rm neck},{\rm int}} : (R_{\rm int}, R_{\rm int} + 1) \times [0, 1] \to (-T_i + C_{1,i}, S_1 + C_{2,i}) \times [0, 1]$$

(62.175.2) $$\Phi'_{i;{\rm neck},{\rm int}} : (R_{\rm int}, R_{\rm int} + 1) \times [0, 1] \to (-T_i + C_{1,i}, S_1 + C_{2,i}) \times [0, 1]$$

by the formula

(62.176.1) $$w_i \circ \psi_{i,\rm neck} \circ \Phi_{i;{\rm neck},{\rm int}} = w_i \circ \psi_{i,\rm int},$$

(62.176.2) $$w'_i \circ \psi'_{i,\rm neck} \circ \Phi'_{i;{\rm neck},{\rm int}} = w'_i \circ \psi'_{i,\rm int}.$$

Hereafter we denote by $o(n \mid a, b, c, \cdots)$ a sequence of constants depending only on $n, a, b, c, \cdots$ and satisfying $\lim_{n\to\infty} o(n \mid a, b, c, \cdots) = 0$ for each fixed $a, b, c, \cdots$. (In particular $o(n)$ is a sequence of constants such that $\lim_{n\to\infty} o(n) = 0$.) We may replace them several times in the proof with the same symbols.

Lemma 62.177. (1) *If i is sufficiently large, $\Phi_{i;\text{neck,int}}$, $\Phi'_{i;\text{neck,int}}$ are uniquely determined.*
(2) *Both $\Phi_{i;\text{neck,int}}$ and $\Phi'_{i;\text{neck,int}}$ are biholomorphic onto their images respectively.*
(3)

$$|\nabla^k(\Phi_{i;\text{neck,int}} - \Phi'_{i;\text{neck,int}})|(\tau,t) \le o(i \mid k, R_{\text{int}}) + C_k e^{-c_k R_{\text{int}}}. \tag{62.178.1}$$

(4)

$$|\Phi_{i;\text{neck,int}}(\tau,t) - (\tau + (2\alpha)^{-1}\log\epsilon_{1,i}, t)| \le o(i \mid R_{\text{int}}) + o(R_{\text{int}}), \tag{62.178.2}$$
$$|\Phi'_{i;\text{neck,int}}(\tau,t) - (\tau + (2\alpha)^{-1}\log\epsilon_{1,i}, t)| \le o(i \mid R_{\text{int}}) + o(R_{\text{int}}). \tag{62.178.3}$$

Proof. (1) is obvious from (62.14.2) and (62.15.3.2). (2) follows from (62.176).
Lemma 62.174 and $a_i, a'_i \to a_0$ imply

$$|\nabla^k(w_i \circ \psi_{i,\text{neck}} - w'_i \circ \psi'_{i,\text{neck}})| \le o(i \mid k) \tag{62.179.1}$$

on

$$[(2\alpha)^{-1}\log\epsilon_{1,i} + R_{\text{int}} - 10, (2\alpha)^{-1}\log\epsilon_{1,i} + R_{\text{int}} + 10] \times [0,1],$$

and

$$|\nabla^k(w_i \circ \psi_{i,\text{int}} - w'_i \circ \psi'_{i,\text{int}})| \le o(i \mid k) \tag{62.179.2}$$

on

$$[R_{\text{int}}, R_{\text{int}} + 1] \times [0,1].$$

We remark that the images of $\Phi_{i;\text{neck,int}}$, $\Phi'_{i;\text{neck,int}}$ is in $[(2\alpha)^{-1}\log\epsilon_{1,i} + R_{\text{int}} - 10, (2\alpha)^{-1}\log\epsilon_{1,i} + R_{\text{int}} + 10] \times [0,1]$ by (62.15.3.2) and (62.14.2). Therefore (16.179), (62.15.3.2) and (62.14.2) imply (62.178.1).
(62.178.2)-(62.178.3) also follow from (62.14.2), (62.15.3.2). □

We next define coordinate transformations

$$\Phi_{i;\text{neck,out}} : (S_1 - R_{\text{out}} - 1, S_1 - R_{\text{out}}) \times [0,1] \to \mathbb{H} \setminus \{0\} \tag{62.180.1}$$
$$\Phi'_{i;\text{neck,out}} : (S_1 - R_{\text{out}} - 1, S_1 - R_{\text{out}}) \times [0,1] \to \mathbb{H} \setminus \{0\} \tag{62.180.2}$$

to be the restrictions of $(\psi_{i,\text{neck}})^{-1}$, $(\psi'_{i,\text{neck}})^{-1}$, respectively.

Lemma 62.181. (1) $\Phi_{i;\mathrm{neck,out}}$, $\Phi'_{i;\mathrm{neck,out}}$ *are biholomorphic onto its images.*
(2) *They satisfy*

$$|\nabla^k(\Phi_{i;\mathrm{neck,out}} - \Phi'_{i;\mathrm{neck,out}})| \le o(i \mid k, R_{\mathrm{out}}) + C_k e^{-c_k R_{\mathrm{out}}}. \tag{62.182}$$

(3) *They also satisfy*

$$|\Phi_{i;\mathrm{neck,out}}(\tau,t) - (\tau,t)| \le o(i \mid R_{\mathrm{out}}) + o(R_{\mathrm{out}}), \tag{62.183.1}$$

$$|\Phi'_{i;\mathrm{neck,out}}(\tau,t) - (\tau,t)| \le o(i \mid R_{\mathrm{out}}) + o(R_{\mathrm{out}}). \tag{62.183.2}$$

Proof. By construction and from Lemma 62.8, we obtain

$$|\nabla^k(w_i \circ \Phi_{i;\mathrm{neck,out}} - w_{\mathrm{tri}} \circ \Phi_{i;\mathrm{neck,out}})|(\tau,t) \le o(i \mid k, R_{\mathrm{out}}), \tag{62.184.1}$$

$$|\nabla^k(w'_i \circ \Phi'_{i;\mathrm{neck,out}} - w_{\mathrm{tri}} \circ \Phi'_{i;\mathrm{neck,out}})|(\tau,t) \le o(i \mid k, R_{\mathrm{out}}). \tag{62.184.2}$$

By Theorem 54.17 and $w_{\mathrm{tri}} \in \widetilde{\mathcal{M}}_0(\mathbb{H},\mathbb{C}^n;(H^{\alpha}_{-\epsilon_{1,i}})')$, we have

$$|\nabla^k(w_{\mathrm{tri}} - w^{\mathrm{flat}}_{0,a_0})|(\tau,t) \le C_k e^{-c_k R_{\mathrm{out}}}, \tag{62.185}$$

on $(S_1 - R_{\mathrm{out}} - 1, S_1 - R_{\mathrm{out}}) \times [0,1]$.
From (62.14), we also obtain

$$|\nabla^k(w_i \circ \Phi_{i;\mathrm{neck,out}} - w^{\mathrm{flat}}_{0,a_i})|(\tau,t) \le C_k e^{-c_k R_{\mathrm{out}}}, \tag{62.186.1}$$

$$|\nabla^k(w'_i \circ \Phi'_{i;\mathrm{neck,out}} - w^{\mathrm{flat}}_{0,a'_i})|(\tau,t) \le C_k e^{-c_k R_{\mathrm{out}}} \tag{62.186.2}$$

on $(S_1 - R_{\mathrm{out}} - 1, S_1 - R_{\mathrm{out}}) \times [0,1]$.
Combining (62.184), (62.186) and the convergence $a_i, a'_i \to a_0$, we obtain

$$|\nabla^k(w_{\mathrm{tri}} \circ \Phi_{i;\mathrm{neck,out}} - w_{\mathrm{tri}} \circ \Phi'_{i;\mathrm{neck,out}})| \le o(i \mid k, R_{\mathrm{out}}) + C_k e^{-c_k R_{\mathrm{out}}}.$$

Therefore (62.185) implies (62.182) and (62.183) follows from (62.186) and Lemma 62.8. □

Now fix an identification $\mathbb{H} \setminus \{0\}$ with $\mathbb{R} \times [0,1]$ by the unique conformal isomorphism satisfying

$$0 \leftrightarrow -\infty,\ \infty \leftrightarrow +\infty,\ 0 + \sqrt{-1} \leftrightarrow (0, 1/2)$$

as before. Then for each $r \in [0,1]$ we define the maps $\Phi^r_{i;\mathrm{neck,int}} : [R_{\mathrm{int}}, R_{\mathrm{int}} + 1] \times [0,1] \to \mathbb{R} \times [0,1]$, and $\Phi^r_{i;\mathrm{neck,out}} : [S_1 - R_{\mathrm{out}} - 1, S_1 - R_{\mathrm{out}}] \times [0,1] \to \mathbb{R} \times [0,1]$ by

$$\Phi^r_{i;\mathrm{neck,int}} = (1-r)\Phi_{i;\mathrm{neck,int}} + r\Phi'_{i;\mathrm{neck,int}}, \tag{62.187.1}$$

$$\Phi^r_{i;\mathrm{neck,out}} = (1-r)\Phi_{i;\mathrm{neck,out}} + r\Phi'_{i;\mathrm{neck,out}}. \tag{62.187.2}$$

If we choose sufficiently large constants $R_{\rm out}$, $R_{\rm int}$ and then i's with $i \geq I(R_{\rm out}, R_{\rm int})$ for the constant $I(R_{\rm out}, R_{\rm int})$ depending only on $R_{\rm out}$, $R_{\rm int}$, it follows from (62.183) that these are well-defined and become diffeomorphisms onto their images respectively.

Next we put

$$\mathcal{U}^r_{i,\rm int} = [-\infty, R_{\rm int} + 1) \times [0,1], \tag{62.188.1}$$

$$\mathcal{U}^r_{i,\rm out} = (S_1 - R_{\rm out} - 1, \infty] \times [0,1]. \tag{62.188.2}$$

We define $\mathcal{U}^r_{i,\rm neck}$ to be the smallest connected open subset of $\mathbb{R} \times [0,1]$ which contains both the images of $\Phi^r_{i;\rm neck,int}$ and $\Phi^r_{i;\rm neck,out}$. It again follows from (62.183) that there exists $o(R_{\rm int})$, $o(R_{\rm out})$ such that

$$\begin{aligned} &[(2\alpha)^{-1}\log\epsilon_{1,i} + R_{\rm int} + o(R_{\rm int}), S_1 - R_{\rm out} - o(R_{\rm out})] \\ &\subseteq \mathcal{U}^r_{i,\rm neck} \subseteq [(2\alpha)^{-1}\log\epsilon_{1,i} + R_{\rm int} - o(R_{\rm int}), S_1 - R_{\rm out} + o(R_{\rm out})]. \end{aligned} \tag{62.188.3}$$

Figure 62.11.

Gluing $\mathcal{U}^r_{i,\rm int}$, $\mathcal{U}^r_{i,\rm neck}$, $\mathcal{U}^r_{i,\rm out}$ by the transition maps $\Phi^r_{i;\rm neck,int}$, $\Phi^r_{i;\rm neck,out}$ between the nearby domains, we obtain a real 2 dimensional compact manifold Σ_r with boundary : Here we regard $\mathcal{U}^r_{i,\rm int}$ as an open neighborhood of $0 \in \mathbb{H}$.

From now on, we equip Σ_r with a metric which we describe on each of the three domains separately.

First we decompose $\mathcal{U}^r_{i,\rm int}$ into

$$\mathcal{U}^r_{i,\rm int} = \mathbb{H}_{|z|\leq 1} \cup ([0, R_{\rm int}+1] \times [0,1]).$$

On $\mathbb{H}_{|z|\leq 1}$ we use standard Euclidean metric and on $[0, R_{\rm int}+1] \times [0,1]$ we use the product metric.

We use the product metric on

$$\mathcal{U}^r_{i,\text{neck}} \subset \mathbb{R} \times [0,1].$$

Finally we decompose

$$\mathcal{U}^r_{i,\text{out}} = ([S_1 - R_{\text{out}} - 1, S_1 - R_{\text{out}}] \times [0,1]) \cup \mathbb{H}_{|z| \geq e^{S_1 - R_{\text{out}}}}.$$

We use the product metric on $[S_1 - R_{\text{out}} - 1, S_1 - R_{\text{out}}] \times [0,1]$. On the other hand on the outside region $\mathbb{H}_{|z| \geq e^{S_1 - R_{\text{out}}}}$, we take the isomorphism

$$\mathbb{H}_{|z| \geq e^{S_1 - R_{\text{out}}}} \cong \mathbb{H}_{|z| \leq 1} \,;\; z \mapsto e^{S_1 - R_{\text{out}}}/z$$

and pushforward the standard Euclidean metric on $\mathbb{H}_{|z| \leq 1}$ onto $\mathbb{H}_{|z| \geq e^{S_1 - R_{\text{out}}}}$. See Figure 62.12.

Figure 62.12.

Note the metrics above do not match on the overlapped parts. So we need to modify them appropriately to get a smooth metric g'_{Σ_r} on Σ_r. We do this so that the ratio g'_{Σ_r}/g is uniformly bounded, where g is one of the above metrics. This process is not essential since we use the metric only to define C^k or Sobolev norms for the tensors or the maps defined on Σ_r. The norms obtained for different choices are all equivalent independent of the choice of smoothing as long as the ratio g'_{Σ_r}/g is uniformly bounded.

We next define a complex structure on Σ_r. Since $\Phi^r_{i;\text{neck,out}}$, $\Phi^r_{i;\text{neck,int}}$ for $r = 0$, 1 are biholomorphic onto their images, it follows that Σ_0, Σ_1 has the canonical glued complex structures denoted by $j^{(0)}$, $j^{(1)}$. Clearly $(\Sigma_0, j^{(0)}) \cong (\mathbb{H}, j_0) \cong (\Sigma_1, j^{(1)})$.

For the cases $r \neq 0, 1$, we remark that since the transition maps $\Phi^r_{i;\text{neck,out}}$, $\Phi^r_{i;\text{neck,int}}$ are not j_0-holomorphic in general, the standard complex structure j_0 on the three coordinate domains are not compatible with the transition maps and so do not glue globally. Therefore we manually put an almost complex structure interpolating them on the transition regions. By the two dimensionality of Σ_r, the constructed almost complex structure will be indeed integrable.

Consider the set

$$\mathfrak{J}(\mathbb{R}^2) = \{j : \mathbb{R}^2 \to \mathbb{R}^2 \mid j^2 = -1\} \subset GL(2; \mathbb{R}).$$

We take its neighborhood $U(\mathfrak{J}(\mathbb{R}^2))$ in $GL(2; \mathbb{R})$ and a smooth retraction

$$\Pi : U(\mathfrak{J}(\mathbb{R}^2)) \to \mathfrak{J}(\mathbb{R}^2).$$

Consider a smooth cut-off function $\chi_A : \mathbb{R} \to [0, 1]$ such that

$$\chi_A(\tau) = \begin{cases} 0 & \tau < A + \frac{1}{10}, \\ 1 & \tau > A + \frac{9}{10}. \end{cases}$$

It follows from (62.178) and its C^1-analog that

$$|(\Phi^r_{i;\text{neck,int}})_* j_0 - j_0| \le o(i \mid R_{\text{int}}) + Ce^{-cR_{\text{int}}}. \tag{62.189}$$

Similar inequality also holds for $(\Phi^r_{i;\text{neck,out}})_* j_0$ by (62.184) and its C^1-version. Therefore we can define complex structures on $\mathcal{U}^r_{i,\text{int}}$ and $\mathcal{U}^r_{i,\text{out}}$ by

$$\begin{aligned} j^r_{i,\text{int}} &= \Pi((1 - \chi_{R_{\text{int}}}(\tau)) j_0 + \chi_{R_{\text{int}}}(\tau)(\Phi^r_{i;\text{neck,int}})^{-1}_* j_0), \\ j^r_{i,\text{out}} &= \Pi(\chi_{S_1 - R_{\text{out}} - 1}(\tau) j_0 + (1 - \chi_{S_1 - R_{\text{out}} - 1}(\tau))(\Phi^r_{i;\text{neck,out}})^{-1}_* j_0), \end{aligned}$$

respectively. Here j_0 is the standard complex structure of $\mathbb{R} \times [0, 1] \cong \mathbb{H} \setminus \{0\}$ and the summation is just the matrix sum in $M^{2\times 2}(\mathbb{R})$. (62.189) makes the sums lie in the neighborhood $U(\mathfrak{J}(\mathbb{R}^2))$ if R_{int} is sufficiently large and $i \ge I(R_{\text{int}})$. Therefore the almost complex structures $j^r_{i,\text{int}}$ are well defined. Similarly for R_{out} large and $i \ge I(R_{\text{out}})$, the almost complex structures $j^r_{i,\text{out}}$ are well defined.

We define an almost complex structure $j^r_{\text{i,neck}}$ on $\mathcal{U}^r_{i,\text{neck}}$ by the formula

$$j^r_{i,\text{neck}} = \begin{cases} (\Phi^r_{i;\text{neck,int}})_*(j^r_{i,\text{int}}) & \text{on the image of } \Phi^r_{i;\text{neck,int}}, \\ (\Phi^r_{i;\text{neck,out}})_*(j^r_{i,\text{out}}) & \text{on the image of } \Phi^r_{i;\text{neck,out}}, \\ j_0 & \text{elsewhere.} \end{cases}$$

It is easy to check that $j^r_{i,\text{neck}}$ is well defined when $j^r_{i,\text{int}}$, $j^r_{i,\text{out}}$ are well defined. They are glued to give an almost complex structure $j^r_{(i)}$ on Σ_r. This indeed defines a complex structure on Σ_r since Σ_r is two dimensional for which every almost complex structure is a complex structure.

Lemma 62.190. (1) $j^r_{(i)} = j_0$ *on* $\mathcal{U}^r_{i,\mathrm{int}} \setminus [R_{\mathrm{int}}, R_{\mathrm{int}} + 1] \times [0,1]$, *on* $\mathcal{U}^r_{i,\mathrm{out}} \setminus [S_1 - R_{\mathrm{out}} - 1, S_1 - R_{\mathrm{out}}] \times [0,1]$ *and on* $\mathcal{U}^r_{i,\mathrm{neck}} \setminus (\mathrm{Im}(\Phi^r_{i,\mathrm{neck,int}}) \cup \mathrm{Im}(\Phi^r_{i,\mathrm{neck,out}}))$.
(2) $|\nabla^k(j^r_{(i)} - j_0)|(\tau,t) \le o(i \mid k, R_{\mathrm{int}}) + Ce^{-cR_{\mathrm{int}}}$ *on* $[R_{\mathrm{int}}, R_{\mathrm{int}}+1] \times [0,1] \subset \mathcal{U}^r_{i,\mathrm{int}}$.
(3) $|\nabla^k(j^r_{(i)} - j_0)| \le o(i \mid k, R_{\mathrm{out}}) + Ce^{-cR_{\mathrm{out}}}$ *on* $[S_1 - R_{\mathrm{out}} - 1, S_1 - R_{\mathrm{out}}] \times [0,1] \subset \mathcal{U}^r_{i,\mathrm{out}}$.

We remark that j_0 in (1) and (2) means the standard complex structure j_0 on any open subset of $\mathbb{R} \times [0,1] \subset \mathbb{C}$.

Proof. (1) is immediate from definition. (2) follows from (62.189) and its C^k analogues. The proof of (3) is similar. □

Now we are ready to interpolate w_i and w'_i to define a family of approximate holomorphic maps

$$w^r_i : \Sigma_r \to M.$$

We recall the decomposition

$$M = B(p_{12};\epsilon_0) \cup (M \setminus B(p_{12};\epsilon_0))$$

and the identification or cylindrical coordinates

$$(s,\Theta) : B(p_{12};\epsilon_0) \setminus \{p_{12}\} \cong B^{2n}(\epsilon_0) \setminus \{0\} \to \mathbb{R} \times S^{2n-1}.$$

Denote by exp the exponential map of the product metric on $\mathbb{R} \times S^{2n-1}$ and

$$E(q,q') := (\exp_q)^{-1}(q') \in T_q(\mathbb{R} \times S^{2n-1}).$$

Note that E is well-defined as long as $\Theta' \neq -\Theta$ on S^{2n-1}, where $q = (s,\Theta)$ and $q' = (s',\Theta')$.

For the notational convenience, we make the following abuse of notations.

Definition 62.191. Let $q = (s,\Theta), q' = (s',\Theta') \in \mathbb{R} \times S^{2n-1}$, with $\Theta' \neq -\Theta$, we define

$$r(s,\Theta) + (1-r)(s',\Theta') =: \exp_q(rE(q,q'))$$

Definition 62.192. (1) For $(\tau,t) \in \mathcal{U}^r_{i,\mathrm{int}}$, we put $\Phi_{i;\mathrm{neck,int}}(\tau,t) =: (\tau',t')$ and define the maps

$$\begin{aligned} w^r_i(\tau,t) = &(1-\chi_{R_{\mathrm{int}}}(\tau))\left((1-r)w_i(\psi_{i,\mathrm{int}}(\tau,t)) + rw'_i(\psi'_{i,\mathrm{int}}(\tau,t))\right) \\ &+ \chi_{R_{\mathrm{int}}}(\tau)\left(((1-r)w_i(\psi_{i,\mathrm{neck}}(\tau',t')) + rw'_i(\psi'_{i,\mathrm{neck}}(\tau',t'))\right). \end{aligned} \tag{62.193.1}$$

In case (τ,t) is not in the domain of $\Phi_{i;\mathrm{neck,int}}$, we have $\chi_{R_{\mathrm{int}}}(\tau) = 0$. Hence the above formula makes sense. We remark that in (62.193.1) we use the notation in Definition 62.191 three times.

(2) For $(\tau, t) \in \mathcal{U}^r_{i,\mathrm{out}}$, we put $\Phi_{i;\mathrm{neck},\mathrm{out}}(\tau,t) = (\tau', t')$ and define :

(62.193.2)
$$\begin{aligned} w_i^r(\tau,t) =& \chi_{S_1 - R_{\mathrm{out}}-1}(\tau) \left((1-r)w_i(\psi_{i,\mathrm{out}}(\tau,t)) + r w_i'(\psi'_{i,\mathrm{out}}(\tau,t))\right) \\ &+ (1-\chi_{S_1 - R_{\mathrm{out}}-1}(\tau)) \left(((1-r)w_i(\psi_{i,\mathrm{neck}}(\tau',t')) + r w_i'(\psi'_{i,\mathrm{neck}}(\tau',t'))\right). \end{aligned}$$

(3) We define w_i^r on $\mathcal{U}^r_{i,\mathrm{neck}}$ as follows.

(62.193.3)
$$w_i^r(\tau,t) = \begin{cases} (w_i^r \circ \Phi^{-1}_{i;\mathrm{neck},\mathrm{int}})(\tau,t) & \text{on } \mathrm{Im}(\Phi_{i;\mathrm{neck},\mathrm{int}}), \\ (w_i^r \circ \Phi^{-1}_{i;\mathrm{neck},\mathrm{out}})(\tau,t) & \text{on } \mathrm{Im}(\Phi_{i;\mathrm{neck},\mathrm{out}}), \\ (1-r)w_i(\psi_{i,\mathrm{neck}}(\tau,t)) + r w_i'(\psi'_{i,\mathrm{neck}}(\tau,t)) & \text{eleswhere.} \end{cases}$$

In the next lemma and thereafter we equip a metric g'_M adapted to the above decomposition of M,
$$M = B_{p_{12}}(\epsilon_0) \cup (M \setminus B_{p_{12}}(\epsilon_0)).$$
On $M \setminus B_{p_{12}}(\epsilon_0)$ the metric g'_M is $\epsilon_0^{-1} g_M$. We further divide $B_{p_{12}}(\epsilon_0)$ into
$$B_{p_{12}}(\epsilon_0) = B_0(S_0\sqrt{\epsilon_{1,i}}, \mathbb{C}^n) \cup ([\frac{1}{2}\log \epsilon_{1,i} + \log S_0, \log \epsilon_0] \times S^{2n-1}).$$
Then we define the metric g'_M by
$$g'_M = \begin{cases} (\epsilon_0)^{-2} g_M & \text{on } M \setminus B(p_{12};\epsilon_0) \\ \varphi_* g_{\mathbb{R}\times S^{2n-1}} & \text{on } (B(p_{12};\epsilon_0) \setminus B(p_{12}; S_0\sqrt{\epsilon_{1,i}})) \\ S_0^{-2}\epsilon_{1,i}^{-1} I^* g_{\mathbb{C}^n} & \text{on } B(p_{12}; S_0\sqrt{\epsilon_{1,i}}) \end{cases}$$
with a suitable smoothing along the gluing hypersurfaces. See Figure 62.13.

Figure 62.13.

Lemma 62.194. (1) *w_i^r is well defined and smooth.*
(2) $w_i^0 = w_i$, $w_i^1 = w_i'$.
(3) $\lim_{i\to\infty}\sup|\nabla^k(w_i^r - w_i)| = 0$. *Here we identify $\mathcal{U}^r_{i,\rm int}$, $\mathcal{U}^r_{i,\rm out}$, $\mathcal{U}^r_{i,\rm neck}$ with $\mathcal{U}_{i,\rm int}$, $\mathcal{U}_{i,\rm out}$, $\mathcal{U}_{i,\rm neck}$ and then the difference $w_i^r - w_i$ makes sense on each of those charts.*
(4) $\lim_{i\to\infty}\sup|\nabla^k(\overline\partial_{j_i^r} w_i^r)| = 0$.

In (5),(6),(7),(8),(9) below, C_k, c_k are positive numbers independent of i and $R_{\rm int}$, $R_{\rm out}$.
(5) *For $(\tau,t) \in [0,R_{\rm int}]\times[0,1] \subset \mathcal{U}^r_{i,\rm int}$, we have*

$$|\nabla^k(\overline\partial_{j_i^r} w_i^r)|(\tau,t) \le C_k e^{-c_k|\tau|}.$$

(6) *For $(\tau,t) \in [S_1 - R_{\rm out},S_1]\times[0,1] \subset \mathcal{U}^r_{i,\rm out}$, we have*

$$|\nabla^k(\overline\partial_{j_i^r} w_i^r)|(\tau,t) \le C_k e^{-c_k|\tau|}.$$

(7) *For $(\tau,t) \in \mathcal{U}^r_{i,\rm neck} \setminus ({\rm Im}(\Phi_{i;{\rm neck,out}}) \cup {\rm Im}(\Phi_{i;{\rm neck,int}}))$, we have*

$$|\nabla^k(\overline\partial_{j_i^r} w_i^r)|(\tau,t) \le C_k e^{-c_k \min\{|\tau|,|\tau+T_i|\}}.$$

(8) *For $(\tau,t) \in [R_{\rm int},R_{\rm int}+1]\times[0,1] \subset \mathcal{U}^r_{i,\rm int}$, we have*

$$|\nabla^k(\overline\partial_{j_i^r} w_i^r)|(\tau,t) \le o(i \mid k,R_{\rm int}) + C_k e^{-c_k R_{\rm int}}.$$

(9) *For $(\tau,t) \in [S_1 - R_{\rm out}-1,S_1 - R_{\rm out}]\times[0,1] \subset \mathcal{U}^r_{i,\rm out}$, we have*

$$|\nabla^k(\overline\partial_{j_i^r} w_i^r)|(\tau,t) \le o(i \mid k,R_{\rm out}) + C_k e^{-c_k R_{\rm out}}.$$

Proof. (1),(2) and (3) are easy to see from the definition of w_i^r. (4) then follows.
Next we prove (7). We recall

$$|\nabla^k(w_i \circ \psi_{i,\rm neck} - w^{\rm flat}_{a_i,0})|(\tau,t) \le C_k e^{-c_k \min\{|\tau|,|\tau+T_i|\}},$$
$$|\nabla^k(w_i' \circ \psi'_{i,\rm neck} - w^{\rm flat}_{a_i',0})|(\tau,t) \le C_k e^{-c_k \min\{|\tau|,|\tau+T_i|\}}.$$

Note $a_i \ne a_i'$ in general. This is the reason we do not have exponential decay estimate for $w_i^r - w_i$. Nevertheless we have

$$|\nabla^k(w_i^r \circ \psi^r_{i,\rm neck} - w^{\rm flat}_{a_i^r,0})|(\tau,t) \le C_k e^{-c_k \min\{|\tau|,|\tau+T_i|\}}, \tag{62.195}$$

where

$$a_i^r = r a_i' + (1-r)a_i.$$

(Here we use a similar notation as Definition 62.191.)

(7) follows immediately from (62.195). The proofs of (5) and (6) are similar.

The proof of (8) and (9) is also similar to the proof of (7). The term $o(i \mid k, R_{\rm int})$ and $o(i \mid k, R_{\rm out})$ appear on the right hand side, since they appeared in Lemma 62.190 (2)(3) and (62.178),(62.182). □

We have thus constructed approximate solutions w_i^r and established their basic estimates. This finishes Step (62.5.2).

We next proceed to Step (62.5.3). Namely we deform w_i^r to a family of pseudo-holomorphic maps. We use the implicit function theorem for this purpose. For this we need the weighted Sobolev space similar to the one used in §61.6.

We begin with defining a weight function $\rho : \Sigma_r \to \mathbb{R}$. Let $\delta > 0$.

Definition 62.196. (1) If $(\tau, t) \in \mathcal{U}_{i,\rm int}$ we put

$$\rho_{\delta,i,\rm int}(\tau,t) = \begin{cases} 1 & \tau \le 0, \\ e^{\delta|\tau|} & 0 \le \tau \le R_{\rm int} + 1. \end{cases} \tag{62.197.1}$$

(2) $(\tau, t) \in \mathcal{U}_{i,\rm neck}$ we put

$$\rho_{\delta,i,\rm neck}(\tau,t) = \begin{cases} \exp(\delta|\tau - (2\alpha)^{-1}\log\epsilon_{1,i}|) & \tau \le \frac{(2\alpha)^{-1}\log\epsilon_{1,i}+S_1}{2} - 1, \\ \exp(\delta|\tau - S_1|) & \frac{(2\alpha)^{-1}\log\epsilon_{1,i}+S_1}{2} + 1 \le \tau, \end{cases} \tag{62.197.2}$$

and

$$\rho_{\delta,i,\rm neck}(\tau,t) = \exp\left(\delta\frac{((2\alpha)^{-1}\log\epsilon_{1,i} + S_1)}{2} - \delta\right) \tag{62.197.3}$$

if $\frac{(2\alpha)^{-1}\log\epsilon_{1,i}+S_1}{2} - 1 \le \tau \le \frac{(2\alpha)^{-1}\log\epsilon_{1,i}+S_1}{2} + 1$. (See Figure 62.14.)

(3) $(\tau, t) \in \mathcal{U}_{i,\rm out}$ we put

$$\rho_{\delta,i,\rm neck}(\tau,t) = \begin{cases} e^{\delta|\tau - S_1|} & S_1 - R_{\rm out} \le \tau \le S_1, \\ 1 & S_1 \le \tau, \end{cases} \tag{62.197.4}$$

Figure 62.14.

Lemma 62.198. *There exists $\rho_{\delta,i,r}$ such that the ratios $\rho_{\delta,i,r}/\rho_{\delta,i,\mathrm{neck}}$, $\rho_{\delta,i,r}/\rho_{\delta,i,\mathrm{int}}$, $\rho_{\delta,i,r}/\rho_{\delta,i,\mathrm{out}}$, are all bounded from above and from below by positive constants independent of i and R_{int}, R_{out}.*

Proof. Lemma 62.177 (4), Lemma 62.181 (3) and (62.188.3) imply that the ratio $\rho_{\delta,i,\mathrm{int}}/\rho_{\delta,i,\mathrm{neck}}$ and $\rho_{\delta,i,\mathrm{out}}/\rho_{\delta,i,\mathrm{neck}}$ are uniformly bounded on the overlapped parts. The lemma then can be proved by using partitions of unity in an obvious way. □

Definition 62.199. (1) Let V be a smooth section of $w_i^{r*}TM$ over Σ_r such that $V(z) \in w_i^{r*}T(L_{-\epsilon_{1,i}})$ or $V(z) \in w_i^{r*}T(L_0)$ for $z \in \partial\Sigma_r$.

We take $V(\frac{(2\alpha)^{-1}\log\epsilon_{1,i}+S_1}{2}, 1/2)$ where $(\frac{(2\alpha)^{-1}\log\epsilon_{1,i}+S_1}{2}, 1/2) \in \mathcal{U}_{i,\mathrm{neck}}$. We extend V to a section V_0 defined on the union $\mathcal{U}^+_{i,\mathrm{neck}}$ of

$$\begin{aligned}
&\mathcal{U}_{i,\mathrm{neck}}\\
&[0,R_{\mathrm{int}}]\times[0,1]\subset\mathcal{U}_{i,\mathrm{int}}\\
&[S_1-R_{\mathrm{out}},S_1]\times[0,1]\subset\mathcal{U}_{i,\mathrm{out}}
\end{aligned}$$

in the same way as (61.39). Now we put

$$\begin{aligned}
&\|V\|^p_{1,p,\rho_{\delta,i,r}}\\
&= \left|V\left(\frac{(2\alpha)^{-1}\log\epsilon_{1,i}+S_1}{2}, 1/2\right)\right|^p + \int_{\Sigma_r\setminus\mathcal{U}^+_{i,\mathrm{neck}}} \left(|\nabla V|^p_g + |V|^p_g\right)\Omega_{g'_{\Sigma_r}} \qquad (62.200)\\
&\quad + \int_{\mathcal{U}^+_{i,\mathrm{neck}}} \rho_{\delta,i,r}\left(|\nabla(V-V_0)|^p_{g'_M} + |(V-V_0)|^p_{g'_M}\right) dA_{g'_{\Sigma_r}}.
\end{aligned}$$

We denote by $W^{1,p}_{\rho_{\delta,i,r}}(\Sigma_r : w_i^{r*}TM; w_i^{r*}T(L_{-\epsilon_{1,i}}))$ the completion of the set of such V's with respect to the norm $\|\cdot\|_{1,p,\rho_{\delta,i,r}}$.

(2) Let V be a section of $w_i^{r*}T(L_0)\otimes\Lambda^{0,1}(\Sigma_r,j_r)$. We define

$$\|V\|^p_{p,\rho_{\delta,i,r}} = \int_{\Sigma_r}\rho_{\delta,i,r}|V|^p_{g'_M}\ dA_{g'_{\Sigma_r}}. \tag{62.201}$$

We denote by $L^p_{\rho_{\delta,i,r}}(\Sigma_r : w_i^{r*}TM; w_i^{r*}T(L_{-\epsilon_{1,i}})\otimes\Lambda^{0,1}(\Sigma_r,j_r))$ the completion of the set of all such V by the norm $\|\cdot\|_{p,\rho_{\delta,i,r}}$.

Now we have

Lemma 62.202. *If $R_{\rm out}$, $R_{\rm int}$ are sufficiently large and if $i > I(R_{\rm out},R_{\rm int})$, then the following holds.*
(1) *The operator :*

$$\begin{aligned}D_{w_i^r}\overline{\partial}_{j_r} :& W^{1,p}_{\rho_{\delta,i,r}}(\Sigma_r : w_i^{r*}TM; w_i^{r*}T(L_{-\epsilon_{1,i}}))\\ &\to L^p_{\rho_{\delta,i,r}}(\Sigma_r : w_i^{r*}TM; w_i^{r*}T(L_{-\epsilon_{1,i}})\otimes\Lambda^{0,1}(\Sigma_r,j_r))\end{aligned}$$

is a Fredholm operator.
(2) *There exists*

$$\begin{aligned}Q_{i,r} : L^p_{\rho_{\delta,i,r}}&(\Sigma_r : w_i^{r*}TM; w_i^{r*}T(L_{-\epsilon_{1,i}})\otimes\Lambda^{0,1}(\Sigma_r,j_r))\\ &\to W^{1,p}_{\rho_{\delta,i,r}}(\Sigma_r : w_i^{r*}TM; w_i^{r*}T(L_{-\epsilon_{1,i}}))\end{aligned}$$

such that

$$D_{w_i^r}\overline{\partial}_{j_r}\circ Q_{i,r} = \textit{identity}.$$

The operator norm of $Q_{i,r}$ is bounded by a number independent of i, r, $R_{\rm out}$, $R_{\rm int}$ as far as $i > I(R_{\rm out},R_{\rm int})$.

The same holds if we replace w_i^r by a map which is sufficiently close to w_i^r with respect to the $W^{1,p}_{\rho_{\delta,i,r}}$ norm.

The proof is the same as the proof of Lemma 61.44 and hence is omitted.

Lemma 62.203. *If $\delta > 0$ is sufficiently small then we have*

$$\|\overline{\partial}_{j_i^r}w_i^r\|_{p,\rho_{\delta,i,r}} \le o(i\mid R_{\rm int},R_{\rm out}) + o(R_{\rm int}) + o(R_{\rm out}).$$

Proof. We may choose δ smaller than the constant c_0 in (5),(6),(7),(8),(9) of Lemma 62.194. For $S\le\min(R_{\rm int},R_{\rm out})$, we consider the union $\mathcal{U}_S^r$ of the following three sets :

$$\begin{aligned}&\mathcal{U}^r_{i,\rm neck}\\ &[S,R_{\rm int}+1]\times[0,1]\subset\mathcal{U}^r_{i,\rm int}\\ &[S_1-R_{\rm out}-1,S_1-S]\times[0,1]\subset\mathcal{U}^r_{i,\rm out}.\end{aligned}$$

We put

$$\mathcal{U}^r_{\rm glue,int} = [R_{\rm int}, R_{\rm int}+1] \times [0,1] \subset \mathcal{U}^r_{i,\rm int},$$
$$\mathcal{U}^r_{\rm glue,out} = [S_1 - R_{\rm out} - 1, S_1 - R_{\rm out}] \times [0,1] \subset \mathcal{U}^r_{i,\rm int}.$$

Then by (5),(6),(7) of Lemma 62.194 we have

$$(\rho_{\delta,i,r}|\overline{\partial}_{j_i^r} w_i^r|^p_{g'_M})(\tau,t) \le Ce^{-(c_0-\delta)S}e^{-(c_0-\delta)\,\mathrm{dist}((\tau,t),\partial\mathcal{U}^r_S)} \tag{62.204.1}$$

on

$$\mathcal{U}_S \setminus (\mathcal{U}^r_{\rm glue,int} \cup \mathcal{U}^r_{\rm glue,out}).$$

Using (8),(9) of Lemma 62.194 we have

$$(\rho_{\delta,i,r}|\overline{\partial}_{j_i^r} w_i^r|^p_{g'_M})(\tau,t) \le e^{\delta R_{\rm int}} o(i \mid R_{\rm int}) + Ce^{-(c_0-\delta)R_{\rm int}} \tag{62.204.2}$$

on $\mathcal{U}^r_{\rm glue,int}$ and

$$(\rho_{\delta,i,r}|\overline{\partial}_{j_i^r} w_i^r|^p_{g'_M})(\tau,t) \le e^{\delta R_{\rm out}} o(i \mid R_{\rm out}) + Ce^{-(c_0-\delta)R_{\rm out}} \tag{62.204.3}$$

on $\mathcal{U}^r_{\rm glue,out}$.

Moreover, by (4) of Lemma 62.194, we have

$$\int_{\mathbb{H}\setminus\mathcal{U}^r_S} \rho_{\delta,i,r}|\overline{\partial}_{j_i^r} w_i^r|^p_{g'_M} dA_{g'_{\Sigma_r}} \le o(i \mid S).$$

Therefore we have

$$\int_{\mathcal{U}_S} \rho_{\delta,i,r}|\overline{\partial}_{j_i^r} w_i^r|^p_{g'_M}\; dA_{g'_{\Sigma_r}} \le Ce^{-(c_0-\delta)S} + o(i \mid R_{\rm int}, R_{\rm out}) + o(R_{\rm int}) + o(R_{\rm out}).$$

The lemma follows. □

Using Lemmas 62.202 and 62.203 we can apply the implicit function theorem in a standard way and obtain the following.

Proposition 62.205. *For a sufficiently large i, there exists a continuous family of $w_i^{r\prime}$ such that*

(62.206.1) $w_i^{0\prime} = w_i$, $w_i^{1\prime} = w_i'$.
(62.206.1) *$w_i^{r\prime}$ is j_r-J_M holomorphic.*

We have thus worked out Step (62.5.3).

Now we are to wrap up the proof of Theorem 62.2. We remark that (Σ_r, j_r) is biholomorphic to $\mathbb{H}$. By our choice $w_i^{1\prime} = w_i'$ is in the family constructed in Theorem 61.46. So using Proposition 62.205, an index calculation and a continuity argument, we prove that w_i^r lies in the family constructed in Theorem 61.46. In particular so is $w_i^{0\prime} = w_i$. The proof of Theorem 62.2 is now complete. □

62.8. Proof of Theorem 60.50.

In this subsection, we prove Theorem 60.50. We first prove the following result which is slightly easier to prove than Theorem 60.50.

Proposition 62.207. *For each fixed ϵ, The map*

$$\bigcup_{\alpha\in(0,\pi)} \mathcal{M}(\mathbb{H},\mathbb{C}^n;(H_\epsilon^\alpha)') \to (0,\pi) \tag{62.208}$$

is proper. Here (62.208) *maps elements in $\mathcal{M}(\mathbb{H},\mathbb{C}^n;(H_\epsilon^\alpha)')$ to α.*

Since we have a natural diffeomorphism between $\mathcal{M}(\mathbb{H},\mathbb{C}^n;(H_\epsilon^\alpha)')$ and $\mathcal{M}(\mathbb{H},\mathbb{C}^n;(H_{\text{sign}\,\epsilon}^\alpha)')$, we may assume $\epsilon = \pm 1$ without loss of generality. We will consider the case $\epsilon = -1$ only since the case $\epsilon = 1$ is easier.

Before proving Proposition 62.207, we define a topology of the total space of the projection

$$\bigcup_{\alpha\in(0,\pi)} W_\delta^{1,p}(\mathbb{H},\mathbb{C}^n;(H_\epsilon^\alpha)') \to (0,\pi)$$

which forms a locally trivial fibration. This topology is used in the statement of Proposition 62.207. It will suffice to prove local triviality of this projection which will in turn induce a topology in an obvious way from the model fiber of the above fibration.

For this purpose, we will construct a trivialization of $\bigcup_{\alpha_1<\alpha<\alpha_2} W_\delta^{1,p}(\mathbb{H},\mathbb{C}^n;(H_\epsilon^\alpha)')$ over the interval (α_1,α_2) explicitly, when the difference $\alpha_2-\alpha_1$ is sufficiently small.

We first construct a smooth family of diffeomorphisms

$$\widetilde{\Phi}_\alpha : (\mathbb{C}^n,(H_\epsilon^{\alpha_1})') \to (\mathbb{C}^n,(H_\epsilon^\alpha)')$$

parameterized by α's with $\alpha_1 < \alpha < \alpha_2$ in several steps.

We start with defining a family of diffeomorphisms

$$\Phi_\alpha : S^{2n-1} \to S^{2n-1}.$$

Let $p \in S^{2n-1}$ be a point in a neighborhood of $\bigcup_{c\in[0,\alpha]} S_c^{n-1}$. ($S_c^{n-1}$ is defined in (60.55).) We take a minimal geodesic

$$\gamma_p : [0,\text{dist}(S_0^{n-1},p)] \to S^{2n-1}$$

parameterized by arc length and with $\gamma_p(0) \in S_0^{n-1}$, $\gamma_p(\text{dist}(S_0^{n-1},p)) = p$. We extend it to a geodesic (parameterized by the arc length) up to the cut locus and denote it by the same symbol. We put

$$\Phi_\alpha(p) = \gamma_p(\text{dist}(S_0^{n-1},p)\alpha/\alpha_1).$$

In this way we define an diffeomorphism Φ_α on a neighborhood of $\bigcup_{c\in[0,\alpha]} S_c^{n-1}$ such that

$$\Phi_\alpha(S_c^{n-1}) = S_{\alpha c/\alpha_1}^{n-1}$$

for $c \in [0, \alpha_1 + \epsilon]$. We extend it to a diffeomorphism $\Phi_\alpha : S^{2n-1} \to S^{2n-1}$ so that it is close to identity. We next identify $\mathbb{C}^n \setminus \{0\} \cong \mathbb{R} \times S^{2n-1}$ and define

$$\widetilde{\Phi}_\alpha : \mathbb{R} \times S^{2n-1} \to \mathbb{R} \times S^{2n-1}$$

such that

$$\widetilde{\Phi}_\alpha(s, x) = (s, \Phi_\alpha(x))$$

if s is large,

$$\widetilde{\Phi}_\alpha(s, x) = (s, x)$$

if s is small, and

$$\widetilde{\Phi}_\alpha(H_{-1}^{\alpha_1})' = (H_{-1}^{\alpha})'.$$

We thus obtain :

$$\widetilde{\Phi}_\alpha : (\mathbb{C}^n, (H_\epsilon^{\alpha_1})') \to (\mathbb{C}^n, (H_\epsilon^{\alpha})').$$

We use it to identify

$$\bigcup_{\alpha_1<\alpha<\alpha_2} W_\delta^{1,p}(\mathbb{H}, \mathbb{C}^n; (H_\epsilon^\alpha)') \cong (\alpha_1, \alpha_2) \times W_\delta^{1,p}(\mathbb{H}, \mathbb{C}^n; (H_\epsilon^{\alpha_1})').$$

The right hand side has a direct product topology. So we define a topology on the left hand side by this identification.

The topology on $\bigcup_\alpha W_\delta^{1,p}(\mathbb{H}, \mathbb{C}^n; (H_\epsilon^\alpha))$ which is used in Theorem 60.50 can be defined in the same way.

Proof of Proposition 62.207. Let $\alpha_i \in (0, \pi)$ be a sequence converging to $\alpha \in (0, \pi)$, and $a_i \in S^{n-1}$ converging to a_∞. Consider any sequence $w_i \in \widetilde{\mathcal{M}}(\mathbb{H}, \mathbb{C}^n; (H_{-1}^{\alpha_i})')$ and denote the corresponding class by

$$[w_i] \in \mathcal{M}(\mathbb{H}, \mathbb{C}^n; (H_{-1}^{\alpha_i})', a_i) = \widetilde{\mathcal{M}}(\mathbb{H}, \mathbb{C}^n; (H_{-1}^{\alpha_i})')/\operatorname{Aut}(\mathbb{H}).$$

We will find a sequence $g_i \in \operatorname{Aut}(\mathbb{H})$ such that a subsequence of $w_i \circ g_i$ converges in $\bigcup_\alpha W_\delta^{1,p}(\mathbb{H}, \mathbb{C}^n; (H_{-1}^\alpha)')$.

We first prove the following.

Lemma 62.209. *There exists a sequence of holomorphic diffeomorphisms*

$$\psi_i : [0, \infty) \times [0, 1] \to \mathbb{H}$$

onto its image and constants c_k, C_k, S_2 independent of i such that

$$|\nabla^k((w_i \circ \psi_i) - w^{\rm flat}_{a_i,0}))|(\tau,t) < C_k e^{-c_k\tau} \tag{62.210}$$

and

$$\psi_i([0,\infty)\times[0,1]) \supset w_i^{-1}(\mathbb{C}^n \setminus B^{2n}(S_2)), \tag{62.211}$$

for large i.

Proof. The proof is similar to that of Theorem 62.13. So the discussion below is rather brief.

We identify

$$\mathbb{C}^n \setminus \{0\} \cong \mathbb{R} \times S^{2n-1}$$

as before, and denote

$$\Sigma_{i,\geq s} = w_i^{-1}([s,\infty)\times S^{2n-1})) \subset \mathbb{H}.$$

We consider

$$\Sigma_{i,\geq \log 2S_0} = w_i^{-1}([\log 2S_0,\infty)\times S^{2n-1})) \subset \mathbb{H}$$

and define energies as follows :

$$\begin{aligned} E_{d\lambda}(w_i) &= \int_{\Sigma_{i,\geq \log 2S_0}} w_i^* d\lambda, \\ E_{\lambda,\rm neck}(w_i) &= \sup_{\rho\in\mathcal{C}} \int_{\Sigma_{i,\geq \log 2S_0}} w_i^*(\rho ds \wedge \lambda), \\ E_{\rm neck}(w_i) &= E_{d\lambda}(w_i) + E_{\lambda,\rm neck}(w_i), \end{aligned}$$

where $\mathcal{C}$ is the set of all smooth nonnegative functions on $[\log 2S_0,\infty)$ with compact support such that $\int \rho ds = 1$.

The following sublemma can be proved in the same way as Proposition 62.30 in §62.4 whose proof is omitted.

Sublemma 62.212. *$E_{\rm neck}(w_i)$ is uniformly bounded.*

Moreover by the argument of §62.4 (especially Proposition 62.79) we can prove that there exist $S_3 > 0$ and I_2 such the following holds for $i > I_3$ and $s_3 > S_3$.

(62.213.1) s_3 is a regular level. The curve $w_i(\Sigma_i)\cap(\{s_3\}\times S^{2n-1})$ is parameterized by an arc $\gamma_{i,s_3} : [0,1] \to \{s_3\}\times S^{2n-1}$ for which there exists $a \in S^{n-1}$ such that

$$|\nabla^k(\gamma_a - \gamma_{i,s})| < o(i, S_3 \mid k). \tag{62.214.1}$$

(62.213.2) Moreover, the set

$$\Sigma_{i,s_3-1\le s\le s_3+1} = w_i(\mathbb{H}) \cap ([s_3-1, s_3+1] \times S^{2n-1})$$

has a parametrization

$$w_{i,s_3-1\le s\le s_3+1} : [-1/\alpha, 1/\alpha] \times [0,1] \to \Sigma_{i,s_3-1\le s\le s_3+1}$$

for which we have

$$|\nabla^k(w_{i,s_3-1\le s\le s_3+1} - w^{\text{flat}}_{a,s_3})| < o(i, S_3 \mid k). \tag{62.214.2}$$

Here we put $w^{\text{flat}}_{a,s_3}(\tau,t) = (\alpha\tau + s_3, \gamma_a(t))$.

Denote $E_0 = \sup E_{\text{neck}}(w_i)$ and let e_0 be as in Theorem 62.85. We remark that we can take e_0 independent of α_i since the set $\{\alpha_i \mid i = 1, 2, \cdots\}$ is relatively compact in $(0,\pi)$ by the choice.

We may also choose S_3 and I_3 so that if $s > S_3$ and $i > I_3$, then we have

$$\int_{\Sigma_{i,\ge s}} w_i^* d\lambda < e_0. \tag{62.215}$$

Using (62.213)-(62.215) we can apply Theorem 62.85. The rest of the proof is similar to the argument presented in §62.5 and omitted. □

Composing w_i with an element $v \in \mathbb{R} \subset \text{Aut}(\mathbb{H})$ (the group consisting of translations $z \mapsto z + v$), we may assume

$$|w_i(0)|_{\mathbb{C}^n} = \inf\{|w_i(z)|_{\mathbb{C}^n} \mid z \in \partial\mathbb{H}\}. \tag{62.216}$$

We recall that $w_i(\mathbb{H})$ does not contain $0 \in \mathbb{C}^n$ by Proposition 60.54. Therefore composing w_i with an element $\lambda \in \mathbb{R}_+ \subset \text{Aut}(\mathbb{H})$ (the group consisting of homotheties $z \mapsto \lambda z$), we may also assume

$$\sup\{|z| \mid z \in \mathbb{H}, |w_i(z)|_{\mathbb{C}^n} = 2S_2\} = 1. \tag{62.217}$$

Lemma 62.218. *w_i has a convergent subsequence in compact C^∞ topology.*

Proof. The proof is similar to that of (62.15) given in §62.6.

By elliptic regularity and Ascoli-Arzela's theorem, it suffices to show that

$$\sup\{|\nabla w_i|(z) \mid |z| < R\}$$

is bounded for each R. (We use the standard metric for $\mathbb{H}$ and the product metric on $\mathbb{C}^n \setminus \{0\} \cong \mathbb{R} \times S^{2n-1}$.) We will prove this by contradiction. Supposing to the contrary and taking a subsequence if necessary, we may assume the following :

There exists a bounded sequence $z_i \in \mathbb{H}$ such that

$$\lim_{i\to\infty} |\nabla w_i|(z_i) = \infty. \tag{62.219}$$

We apply Lemma 62.89 to find z_i' such that

(62.220.1) $|\nabla w_i|(z_i') := C_i \to \infty$ as $i \to \infty$.
(62.220.2) If $|z - z_i'| \le C_i^{-1/2}$ then $|\nabla w_i|(z) \le 2C_i$.
(62.220.3) $|z_i'|$ is bounded.

Using Lemma 62.209 and the proof of Lemma 62.151, we can show that $|w_i(z_i')|$ is bounded.

We put

$$D_i = \{u \in \mathbb{C} \mid |C_i u| \le C_i^{-1/2}, \quad C_i^{-1}u + z_i' \in \mathbb{H}\} \tag{62.221}$$

and define $\widetilde{w}_i : D_i \to \mathbb{C}^n$ by

$$\widetilde{w}_i(u) = w_i(C_i^{-1}u + z_i'). \tag{62.222}$$

Let $D_\infty = \lim D_i$. By taking a subsequence we may assume that one of the following occurs.

(62.223.1) $D_\infty = \mathbb{C}$.
(62.223.2) $D_\infty = \mathbb{H} - c\sqrt{-1}$.

We use boundedness of $|w_i(z_i')|$ and (62.220.2) to show that $\widetilde{w}_i$ has a subsequence (still denoted by $\widetilde{w}_i$) which converges to

$$\widetilde{w}_\infty : D_\infty \to \mathbb{C}^n \tag{62.224}$$

in compact C^∞ topology. Since $|\nabla \widetilde{w}_i|(0) = 1$ and hence $|\nabla \widetilde{w}_\infty|(0) = 1$ it follows that $\widetilde{w}_\infty$ is nontrivial. We can then prove that $\widetilde{w}_\infty$ must be unbounded in the same way as Lemma 62.157.

Using Lemma 62.209 and the proof of Propositoin 62.158, we prove that (62.223.1) can not occur.

Now assume (62.223.2). Slightly perturbing z_i', we may assume that $z_i' \in \partial\mathbb{H}$, $D_\infty = \mathbb{H}$. We use Lemma 62.209 and the fact that $\widetilde{w}_\infty$ is unbounded to show that there exists $R > 10S_1$ with the following properties (Compare this with Lemma 62.168.) :

(62.225.1) $\widetilde{w}_i$ is transversal to $\{R\} \times S^{2n-1}$ for large i. The preimage $\widehat{w}_i^{-1}(\{R\} \times S^{2n-1})$ is an arc, which we denote by $\widehat{\gamma}_i$.
(62.225.2) If we put

$$\widehat{\gamma}_i' = \{C_i^{-1}\widehat{\gamma}_i(t) + z_i' \mid t \in [0,1]\},$$

then $\Sigma_i \setminus \widehat{\gamma}'_i$ is a disjoint union of $D_i^{\rm int}$ and $D_i^{\rm ext}$ such that

$$w_i(D_i^{\rm ext}) \subset [R,\infty) \times S^{2n-1}.$$

(62.225.3) $\quad \widehat{\gamma}_i \subset D_i \subset \mathbb{H}$ is uniformly bounded.

It follows that

$$\lim_{i\to\infty} {\rm Diam}\{z \in \mathbb{H} \mid |w_i(z)| \le e^R\} = 0.$$

On the other hand (62.216) and (62.217) imply

$$\text{(62.226)} \qquad {\rm Diam}\{z \in \mathbb{H} \mid |w_i(z)| \le 2S_2\} \ge 1.$$

This is a contradiction. The proof of Lemma 62.218 is complete. □

Taking a subsequence if necessary, we may assume that

$$\lim_{i\to\infty} w_i = w_\infty$$

in compact C^∞ topology. Then (62.216) and (62.217) imply w_∞ must be nonconstant. We also derive from the conformal invariance of the energy and from the convergence that its energies are all finite. Then in the same way as the proof of Lemma 62.141, we use Lemma 62.209 to prove that w_∞ satisfies the correct asymptotic condition and hence

$$\text{(62.227)} \qquad w_\infty \in \widetilde{\mathcal{M}}(\mathbb{H},\mathbb{C}^n;(H^\alpha_{-1})').$$

This implies there exists a constant S such that

$$\text{(62.228)} \qquad w_\infty^{-1}((-\infty,\log 2S_2] \times S^{2n-1}) \subset B^{2n}(S) \cap \mathbb{H}.$$

Since $w_i^{-1}(\{\log 2S_2\}\times S^{2n-1})$ is connected by Lemma 62.209, it follows from (62.228) that

$$\text{(62.229)} \qquad w_i^{-1}((-\infty,\log 2S_2] \times S^{2n-1}) \subset B^{2n}(S+1) \cap \mathbb{H},$$

for all sufficiently large i's. (We remark that the convergence in Lemma 62.218 is compact C^∞ convergence. Therefore (62.228) does *not* directly imply (62.229) but we can use Lemma 62.209 to obtain (62.229).)

It follows from (62.229) and (62.215) that

$$\text{(62.230)} \qquad \int_{[\log S+10,\infty)\times[0,1]} w_i^*(d\lambda) < e_0.$$

Here we regard $[\log S+10,\infty)\times[0,1]$ as a subset of $\mathbb{H}$ as before. Note S is independent of i.

By (62.230) we can apply Theorem 62.85 to show that

$$\text{(62.231)} \qquad |\nabla^k(w_i - w^{\rm flat}_{a_i,s_i})|(\tau,t) \le C_k e^{-c_k(\tau-S)}$$

here c_k, C_k is independent of i. Using compact C^∞ convergence we derive that s_i converges to s_∞.

Now using compact C^∞ convergence and exponential decay (62.231) we can prove that w_i converges to w_∞ in $\bigcup_\alpha W^{1,p}_\delta(\mathbb{H},\mathbb{C}^n;(H^\alpha_{-1})')$. The proof of Proposition 62.207 is now complete □

Remark 62.232. The proof of the last step of Proposition 62.207 is simpler than that of the proof of Theorem 62.2 given in §62.7. This is because we are proving compactness here which is easier to prove than the surjectivity in general.

Now we are ready to wrap up the proof of Theorem 60.50. Let $0 < \alpha_1 < \alpha_2 < \pi$. We use Proposition 62.207 and Theorem 60.26 (Remark 60.28) in the same way as in §60.4 and show that there exists a constant $S_{0,1}$ such that for any $S_0 > S_{0,1}$ we have a diffeomorphism

$$\bigcup_{\alpha\in[\alpha_1,\alpha_2]} \mathcal{M}(\mathbb{H},\mathbb{C}^n;(H^{\alpha}_{-1})',a) \cong [\alpha_1,\alpha_2]\times S^{n-2}. \tag{62.233}$$

Here the map (62.208) is the projection to the $[\alpha_1,\alpha_2]$ factor. (Note S_0 appears in the definition of $(H^{\alpha}_{-1})'$.) Now we use (the proof of) Proposition 61.9 to obtain

$$[\alpha_1,\alpha_2]\times S^{n-2} \subseteq \bigcup_{\alpha\in[\alpha_1,\alpha_2]} \mathcal{M}(\mathbb{H},\mathbb{C}^n;(H^{\alpha}_{-1}),a). \tag{62.234}$$

If (62.234) were not an equality, there would exist an element $[w]$ in

$$\bigcup_{\alpha\in[\alpha_1,\alpha_2]} \mathcal{M}(\mathbb{H},\mathbb{C}^n;(H^{\alpha}_{-1}),a) \setminus ([\alpha_1,\alpha_2]\times S^{n-2}).$$

Then we could take S_0 large enough (which may depend on w) so that $[w]$ produce an element of $\mathcal{M}(\mathbb{H},\mathbb{C}^n;(H^{\alpha}_{-1})')$ by Proposition 61.9. This would contradict to (62.233). This implies Theorem 60.50 in case $\epsilon = -1$. The proof for the other ϵ is similar. The proof of Theorem 60.50 is now complete. □

Remark 62.235. We here remark one rather delicate point in the above proof. Namely to deduce Theorem 62.2 from Proposition 62.207 we use (the proof of) Proposition 62.207 which claims that the moduli space $\mathcal{M}(\mathbb{H},\mathbb{C}^n;(H^{\alpha}_{-1})',a)$ is diffeomorphic to $\mathcal{M}(\mathbb{H},\mathbb{C}^n;(H^{\alpha}_{-1}),a)$ if S_0 (which appears in the definition of $(H^{\alpha}_{-1})'$) is sufficiently large. Actually we use the compactness of these moduli spaces to obtain a global diffeomorpism betwee them.

The argument however is not circular as we explain below.

First by the argument of §59.3, the moduli space $\mathcal{M}(\mathbb{H},\mathbb{C}^n;(H^{\pi/2}_{-1})',a)$ is diffeomorphic to S^{n-2} and is in particular compact.

On the other hand, by Propositio 62.207, the moduli space $\mathcal{M}(\mathbb{H},\mathbb{C}^n;(H^{\alpha}_{-1})',a)$ is compact for any α.

So we can use (the proof of) Proposition 61.9 to show that $\mathcal{M}(\mathbb{H},\mathbb{C}^n;(H^{\pi/2}_{-1})',a)$ is diffeomorphic to S^{n-2} for sufficiently large S_0. We next use the argument of §60.4 to show that $\mathcal{M}(\mathbb{H},\mathbb{C}^n;(H^{\alpha}_{-1})',a)$ is diffeomorphic to S^{n-2} for any $\alpha\in(0,\pi)$.

We next use (the proof of) Proposition 61.9 to find an open embedding :

$$S^{n-2} \cong \mathcal{M}(\mathbb{H},\mathbb{C}^n;(H^{\alpha}_{-1})',a) \to \mathcal{M}(\mathbb{H},\mathbb{C}^n;(H^{\alpha}_{-1}),a).$$

We suppose that this is not surjective. We take an element w in $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{-1}), a) \setminus S^{n-2}$. We remark that $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{-1}), a)$ is independent of S_0, since S_0 is used only to define $(H^\alpha_{-1})'$.

Therefore using the compactness of $S^{n-2} \cup \{w\}$ and (the proof of) Proposition 61.9 we can find S_0' (which may depend on w) such that there exists an injective map

$$S^{n-2} \cup \{w\} \to \mathcal{M}(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{-1})', a).$$

(Here we use S_0' in place of S_0 to define $(H^\alpha_{-1})'$.) This is a contradiction since $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; (H^\alpha_{-1})', a)$ is diffeomorphic to S^{n-2} for any sufficiently large S_0.

Remark 62.236. The proof of Proposition 62.207 given in this subsection itself does not work if we replace $(H^\alpha_{-1})'$ by H^α_{-1}. This is because H^α_{-1} is only asymptotically flat and we took several short cut in this section using the fact that $(H^\alpha_{-1})'$ is flat outside a compact set.

Indeed, we can avoid using Proposition 62.207 and directly prove Theorem 60.50, if we develop analysis of pseudo-holomorphic curves with *asymptotically cylindrical* ends. This is certainly possible but will be carried out elsewhere since we do not need such general analysis in this book.

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