Floer homology of Lagrangian Foliation and Noncommutative Mirror Symmetry I

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§ 0 Introduction

In this paper and Part II, we study mirror symmetry of symplectic and complex torus. It leads us the study of a generalization of a part of the theory of theta functions (line bundles on complex torus) to the case of (finite or infinite dimensional) vector bundles (or sheaves) and to multi theta function.

We will define noncommutative complex torus, holomorphic vector bundles on it, and noncommutative theta functions. We also will show a way to calculate coefficients of theta series expansion (or theta type integrals) of holomorphic sections of vector bundles on (commutative or noncommutative) complex torus in terms of counting problem of holomorphic polygons in $\mathbf{C}^n$ with affine boundary conditions. We will prove that this counting problem reduces to the Morse theory of quadratic functions in in the "semi classical limit".

In the case of (usual) complex torus, the author conjectures that special values of these multi theta functions give a coefficients of polynomials describing the moduli space of sheaves and of linear equations describing its cohomology.

Let (M,ω) be a $2n$-dimensional symplectic manifold.

Definition 0.1 A *Lagrangian foliation* on (M,ω) is a foliation $\mathcal{F}$ on M such that each leaf is a Lagrangian submanifold. (Namely each leaf F of $\mathcal{F}$ is an n-dimensional submanifold of M such that $\omega|_F = 0$.)

In this paper we are mainly concern with the following simple (but nontrivial) example. (One may find other examples in solve or nil manifolds.)

Example 0.2 Let us consider a torus $T^{2n} = \mathbf{C}^n/\Gamma$. (Here Γ is a lattice in $\mathbf{C}^n$). We put a homogeneous nondegenerate two form ω on T^{2n} and consider a symplectic manifold (T^{2n},ω). We consider affine Lagrangian submanifolds of it. Let $\tilde{L} \subset \mathbf{C}^n$ be a Lagrangian linear subspace. Namely $\tilde{L} \subset \mathbf{C}^n$ is an n- dimensional $\mathbf{R}$-linear subspace and $\omega|_{\tilde{L}} = 0$. We consider a foliation $\mathcal{F}_{\tilde{L}}$ induced by the linear action of $\tilde{L}$ on T^{2n}. In case when $\tilde{L} \cap \Gamma \cong \mathbf{Z}^n$, all leaves of $\mathcal{F}_{\tilde{L}}$ are compact. Otherwise they are noncompact. In particular if $\overline{\tilde{L} + \Gamma} \cong \mathbf{C}^n$, all leaves are dense.

Hereafter we assume that $\mathbf{R}^n \cap \Gamma$ is a lattice in $\mathbf{R}^n$, without loosing generality. Then, in case when $[\omega] \in H^{1,1}(T^{2n})$, there are Lagrangian linear subspaces $\tilde{L}$ such that $\tilde{L} \cap \Gamma \cong \mathbf{Z}^n$. In fact we can take $\tilde{L} = \mathbf{R}^n \subset \mathbf{C}^n$. However, in case when $[\omega] \notin H^{1,1}(T^{2n})$, there may not exist such $\tilde{L}$.

This fact is related to Mirror symmetry in the following way. Strominger, Yau, Zaslow [29] observed that a mirror of our symplectic manifold (T^{2n},ω) is a component of the moduli space of pairs $(L,\mathcal{L})$ of Lagrangian submanifold L and a flat line bundle $\mathcal{L}$ on it. (In general, we need to use complexified symplectic form $\Omega = \omega + \sqrt{-1}B$. In that case, the flatness condition of $\mathcal{L}$ should be replaced by $F_{\mathcal{L}} = 2\pi\sqrt{-1}B$.)

In the case when $[\omega+\sqrt{-1}B]\in H^{1,1}(T^{2n})$, (and $\boldsymbol{R}^n\cap\Gamma\cong\boldsymbol{Z}^n$), we can certainly find a complex manifold in this way. (See Part II.)

Let us denote by $(T^{2n},\Omega)^{\vee}$ the mirror of $(T^{2n},\Omega=\omega+\sqrt{-1}B)$. Deformation of the complex structure of $(T^{2n},\Omega)^{\vee}$ is parametrized by $H^1\left((T^{2n},\Omega)^{\vee},T(T^{2n},\Omega)^{\vee}\right)$ which is isomorphic to $H^{1,n-1}\left((T^{2n},\Omega)^{\vee}\right)$ since $\Lambda^n TT^{2n}$ is trivial. Here $\Lambda^k TT^{2n}$ is the k-th exterior power (over $\boldsymbol{C}$) of the tangent bundle of T^{2n}. Since $H^{1,n-1}\left((T^{2n},\Omega)^{\vee}\right)\cong H^{1,1}(T^{2n})$ by the definition of Mirror symmetry, the deformation of compexified symplectic form $\Omega=\omega+\sqrt{-1}B$ corresponds to the deformation of complex structure of the mirror.

[17], [25], [2] considered extended deformation space of complex structure of the Calabi-Yau manifold M. It is described by the larger vector space $\bigoplus_{p,q} H^p\left(M,\Lambda^q TM\right)\cong\bigoplus_{p,q} H^{p,n-q}(M)$. In [2], the Frobenius structure is constructed in this extended moduli space. However geometric meaning of this deformation (other than those corresponding to $H^1(M,TM)$) is mysterious. If $M^{\vee}$ is a mirror of M then we have $H^p\left(M^{\vee},\Lambda^q TM^{\vee}\right)\cong H^{p,q}(M)$. The deformation of symplectic structure of M is parametrized by $H^2(M)\cong\bigoplus_{p+q=2} H^{p,q}(M)$. This group is strictly bigger than $H^1\left(M^{\vee},TM^{\vee}\right)$. For example in case $M=T^{2n}$ deformation of the symplectic structure belonging to $H^2(T^{2n})-H^{1,1}(T^{2n})$ is a deformation which does *not* corresponds to the usual deformation of complex structure of $(T^{2n},\Omega)^{\vee}$. (It corresponds to $\bigoplus_{p+q=2} H^p\left((T^{2n},\Omega)^{\vee},\Lambda^q T(T^{2n},\Omega)^{\vee}\right)-H^1\left((T^{2n},\Omega)^{\vee},T(T^{2n},\Omega)^{\vee}\right)$.) The goal of this paper is to find a "geometric" objects which corresponds to such a deformation. Our proposal is :

Heorem[1] 0.3 *The deformation of complex torus $(T^{2n},\Omega=\omega+\sqrt{-1}B)^{\vee}$ to the direction in $\bigoplus_{p+q=2} H^p\left((T^{2n},\Omega)^{\vee},\Lambda^q T(T^{2n},\Omega)^{\vee}\right)-H^1\left((T^{2n},\Omega)^{\vee},T(T^{2n},\Omega)^{\vee}\right)$ is realized by a noncommutative complex torus corresponding to a complexication of the C^*-algebra of a Lagrangian foliation in a symplectic manifold $(T^{2n},\Omega'=\omega'+\sqrt{-1}B')$ where $\Omega'\notin H^{1,1}(T^{2n})$.*

Heorem 0.3 might be generalized to K3 surfaces and Calabi-Yau manifolds embedded in toric variety somehow if we include singular Lagrangian foliation.

C^*-algebra of a foliation is used by A.Connes extensively in his noncommutative geometry [3]. In § 1, we recall its definition in the case we need. We remark that the C^*-algebra of a foliation is regarded as a "noncommutative space" which is the space of leaves of the foliation. In many cases (for example in the case of the foliation $\mathcal{F}_{\tilde L}$ in Example 0.2 with $\overline{\tilde L+\Gamma}\cong\boldsymbol{C}^n$), the space of leaves is not a Hausdorff space. Connes' idea is to regard the noncommutative C^*-algebra $C(M,\mathcal{F})$ as the set of functions on this "space".

We remark that the space of leaves is the real part of the moduli space of $(L,\mathcal{L})$ we mentioned above.

The "imaginary part" is the moduli space of connections on $\mathcal{L}$ such that $F_{\nabla}=\sqrt{-1}B$. We find, by a simple dimension counting, that there is an n-dimensional family of Lagrangian vector spaces $\tilde L$ such that $\omega+\sqrt{-1}B\big|_{\tilde L}=0$. So we restrict ourselves to a Lagrangian foliation

[1] 0.3 is not a theorem in the sense of Mathematics. So I removed "T".

$\mathcal{F}_{\tilde{L}}$ such that $\omega + \sqrt{-1}B\big|_{\tilde{L}} = 0$. For simplicity, we suppose that $\mathcal{F}_{\tilde{L}}$ is ergodic. We consider the set $\mathcal{A}$ of all homomorphisms $\tilde{L} \to Lie(U(1)) = \sqrt{-1}\boldsymbol{R}$ and regard an element of it as a leafwise connections of a trivial line bundle on T^{2n}. We next consider the gauge transformations. The set of gauge transformations (of trivial bundle on T^{2n}) which preserves $\mathcal{A}$ is identified with $Hom(T^{2n}, U(1)) \cong \boldsymbol{Z}^{2n}$. Its action on $\mathcal{A}$ is obtained by logarithm. The key observation is that the action of $Hom(T^{2n}, U(1))$ on $\mathcal{A}$ is ergodic. Hence the "imaginary part" we need to consider is the quotient space $\mathcal{A}/Hom(T^{2n}, U(1))$ which is not Hausdorff. So again we need a similar construction using C^*-algebra. We will discuss "imaginary part" and "complex structure" of our "noncommutative space" in part II of this paper.

To see more explicitly the meaning of Heorem 0.3, we recall the following dictionary between symplectic geometry and complex geometry. This idea is initiated by M. Kontsevich [17], [18].

Symplectic manifold M	Complex manifold $M^\vee$
$Lag(M)$: Moduli space of the pair $(L,\mathcal{L})$ where L is a Lagrangian submanifold and $\mathcal{L}$ is a line bundle on it together with a connection ∇ with $F_\nabla = \sqrt{-1}B$. We identify $(L,\mathcal{L})$ and $(L',\mathcal{L}')$ if $L' = \varphi(L)$ and $\mathcal{L}' = \varphi^*(\mathcal{L})$ for a Hamiltonian diffeomorphism φ	$Hilb(M^\vee)$: The Hilbert scheme, that is the compactification of the moduli space of the complex subvarieties of $M^\vee$.
$Hom(Lag(M),ch)$: the set of all holomorphic A^∞functors from the A^∞category $Lag(M)$ to the category of chain complex. (See [10], [12], [9] for the definition of the terminology we used here.)	$Der(Sh(M^\vee))$: Derived category of the category of all coherent sheaves on $M^\vee$.
$HF((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2))$: Floer homology.	$Ext(i_*O(C_1),i_*O(C_2))$: where $C_i \in Hilb(M^\vee)$ and $O(C_i)$ is a structure sheaf and $i : C_i \to M^\vee$ is the inclusion.
$H_*(Hom(F_1,F_2))$: where $F_i \in Hom(Lag(M),ch)$ are A^∞ functors and $Hom(F_1,F_2)$ is a chain complex of all pre natural transformations. (See [12].)	$Ext(F_1,F_2)$: where $F_i \in Der(Sh(M^\vee))$.
$HF((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)) \otimes HF((L_2,\mathcal{L}_2),(L_3,\mathcal{L}_3)) \to HF((L_1,\mathcal{L}_1),(L_3,\mathcal{L}_3))$: Product strucure of Floer homology ([10], [13], [12]).	$Ext(i_*O(C_1),i_*O(C_2)) \otimes Ext(i_*O(C_2),i_*O(C_3)) \to Ext(i_*O(C_1),i_*O(C_3))$: Yoneda Product.
Higher multiplication of Floer homology and of A^∞ functors [12].	(Higher) Massey Yoneda Product.

In the symplectic side, Floer homology of Lagrangian submanifold ([8], [21], [14]) plays the key role in the dictionary. So the main part of this paper is devoted to the study of "Floer homology between leaves of Lagrangian foliation".

We recall that Floer homology theory [8], [21] associates a graded vector space $HF(L_1, L_2)$ to a pair of Lagrangian submanifolds L_1, L_2, (if they are spin and the obstruction class we defined in [14] vanishes.) It satisfies

(0.4) $$\sum_k (-1)^k \operatorname{rank} HF_k(L_1, L_2) = [L_1] \bullet [L_2],$$

where right hand side is the intersection number.

Let us consider the case of Example 0.2 with $\overline{\tilde{L}_1 + \Gamma} = \overline{\tilde{L}_2 + \Gamma} = \boldsymbol{C}^n$, $\tilde{L}_1 \cap \tilde{L}_2 = \{0\}$. Let L_i be leaves of $\mathcal{F}_{\tilde{L}_i}$. We find that $\#\left(L_1 \cap L_2\right) = \infty$. Hence if we want to find a Floer homology $HF(L_1, L_2)$ of leaves of our Lagrangian foliation satisfying (0.4) , then $HF(L_1, L_2)$ is necessary of infinite dimension. This is a consequence of the noncompactness of the leaves.

This trouble is similar to the index theory of noncompact manifolds. The idea by Atiyah [1] is to regard an infinite dimensional vector space (the space of L^2 solutions of an elliptic operator in Atiyah's case and Floer homology in our case) as a module of an appropriate C^*-algebra, then the infinite dimensional vector space becomes manageable.

Our approach is similar to this approach and we will construct Floer homology $HF(\mathcal{F}_{\tilde{L}_1}, \mathcal{F}_{\tilde{L}_2})$ as a bimodule over $C(M, \mathcal{F}_{\tilde{L}_1})$ and $C(M, \mathcal{F}_{\tilde{L}_2})$. Here $C(M, \mathcal{F}_{\tilde{L}})$ is the C^*-algebra of foliation. (See [3] and § 1.)

One important idea of noncommutative geometry is that a module of a C^*-algebra C is a "vector bundle" or a "sheaf" on the "space" corresponding to C. Hence $HF(\mathcal{F}_{\tilde{L}_1}, \mathcal{F}_{\tilde{L}_2})$ may be regarded as a "sheaf" on a direct product of the leaf spaces of $\mathcal{F}_{\tilde{L}_1}$ and $\mathcal{F}_{\tilde{L}_2}$. (But it is not coherent in any reasonable sense.)

There might be a generalization of (0.4) which is similar to Atiyah's Γ-index theorem [1] and Connes' index theorem of foliation [3].

We next generalize the product structure of Floer homology $HF((L_1, \mathcal{L}_1), (L_2, \mathcal{L}_2)) \otimes HF((L_2, \mathcal{L}_2), (L_3, \mathcal{L}_3)) \to HF((L_1, \mathcal{L}_1), (L_3, \mathcal{L}_3))$ introduced in [10], [12]. Let us fix a transversal measure τ_i of $\mathcal{F}_{\tilde{L}_i}$. Then we will construct :

(0.5) $$\begin{aligned} m_2(\tau_2) : HF^p(\mathcal{F}_{\tilde{L}_1}, \mathcal{F}_{\tilde{L}_2}; \tau_1 \otimes \tau_2) \otimes_{C(M, \mathcal{F}_{\tilde{L}_2})} HF^q(\mathcal{F}_{\tilde{L}_2}, \mathcal{F}_{\tilde{L}_3}; \tau_2 \otimes \tau_3) \\ \to HF^r(\mathcal{F}_{\tilde{L}_1}, \mathcal{F}_{\tilde{L}_3}; \tau_1 \otimes \tau_3). \end{aligned}$$

Here HF^p etc. is an appropriate L^p completion of $HF(\mathcal{F}_{\tilde{L}_1}, \mathcal{F}_{\tilde{L}_2})$ and $1/p + 1/q = 1/r$.

(0.5) is a $C(T^{2n}, \mathcal{F}_{\tilde{L}_1})$ $C(T^{2n}, \mathcal{F}_{\tilde{L}_3})$ bimodule homomorphism and satisfies associativity relation

(0.6) $$m_2(m_2(x \otimes y) \otimes z) = m_2(x \otimes m_2(y \otimes z)).$$

We can generalize A^∞ structure (see [12]) also to our foliation case. More precisely we are going to construct an A^∞ category whose object is a linear Lagrangian foliation $\mathcal{F}_{\tilde L}$ together with transversal measure and a morphism between them is an element of the Floer homology $HF(\mathcal{F}_{\tilde L_1}, \mathcal{F}_{\tilde L_2})$.

In case $\tilde L_1 \cap \Gamma \cong \tilde L_2 \cap \Gamma \cong \boldsymbol{Z}^n$ (namely in the case all leaves are compact) each leaf L_i of $\mathcal{F}_{\tilde L_i}$ (which is compact) determines a transversal measure τ_i. In that case, we have

$$HF^p(\mathcal{F}_{\tilde L_i},\mathcal{F}_{\tilde L_j};\tau_i \otimes \tau_j) = HF(L_i,L_j) \otimes L^p(L_i \times L_j) \tag{0.7}$$

and (0.5) reduces to the tensor product of the map $HF(L_1,L_2)\otimes HF(L_2,L_3) \to HF(L_1,L_3)$ and a trivial map :

$$\overline{m}_2 : L^p(L_1 \times L_2) \otimes L^q(L_2 \times L_3) \to L^r(L_1 \times L_3). \tag{0.8}$$

Namely

$$\overline{m}_2(f \otimes g)(x,z) = \int_{y \in L_2} f(x,y) g(y,z).$$

Thus, in this case, we can identify the map (0.5) with the usual multiplicative structure of Floer homology.

This map $HF(L_1,L_2)\otimes HF(L_2,L_3) \to HF(L_1,L_3)$, in the case of elliptic curve, is calculated by Kontsevich [18] and is a theta function. Polishchuk and Zaslow studied the case of elliptic curve in detail by explicit calculation, in [24], [23]. (See § 4, where we discuss the case of higher dimensional torus.) Thus (0.5) is regarded as a noncommutative theta function.

We will prove, in part II, that this map m_2 is a "holomorphic section" of a "holomorphic vector bundle" on a noncommutative complex torus.

These constructions may give something new also in the case when the mirror $(T,\Omega)^\vee$ is a complex torus. (Namely the case when $\Omega \in H^{1,1}$.) We recall that, in this case, $(T,\Omega)^\vee$ is a moduli space of the pairs $(L_0,\mathcal{L}_0)$ of Lagrangian torus L_0 and a flat line bundle $\mathcal{L}_0$ on it such that the universal cover of L_0 is parallel to $\boldsymbol{R}^n \subseteq \boldsymbol{C}^n$. We consider the set $\mathcal{W}$ of all Lagrangian linear subspaces $\tilde L$ such that $\Omega|_{\tilde L} = 0$. ($\mathcal{W}$ is n dimensional if Ω is generic.) For each $\tilde L \in \mathcal{W}$ we have Floer homology $HF(\mathcal{F}_{\boldsymbol{R}^n}, \mathcal{F}_{\tilde L})$ which is a $C(M,\mathcal{F}_{\boldsymbol{R}^n})$ $C(M,\mathcal{F}_{\tilde L})$ bimodule. We remark that the space corresponding to $C(M,\mathcal{F}_{\boldsymbol{R}^n})$ is the moduli space of affine Lagrangian submanifold L_0 such that its universal cover is parallel to $\boldsymbol{R}^n$. This is the "real part" of $(T,\Omega)^\vee$. We can include imaginary part in a way similar to [18] and [24], then $HF(\mathcal{F}_{\boldsymbol{R}^n}, \mathcal{F}_{\tilde L})$ as a module over $C(M,\mathcal{F}_{\boldsymbol{R}^n})$ will be a holomorphic vector bundle of infinite dimension over $(T,\Omega)^\vee$. (If $\boldsymbol{R}^n$ is not transversal to $\tilde L$ then we will obtain a complex of infinite dimensional holomorphic vector bundles.)

Suppose furthermore that $\tilde L \cap \Gamma$ is a lattice in $\tilde L$. (If Ω is of rational coefficient there are many such Γ.) Then the space corresponding to $C(M,\mathcal{F}_{\tilde L})$ is also a usual (commutative) space. If we include imaginary part then we obtain another torus, (which is also a mirror of

(T,Ω).) In this case, each leaf of $\mathcal{F}_{\tilde{L}}$ is compact and defines a transversal delta measure. Using this transversal measure, we take a completion of Floer homology $HF(\mathcal{F}_{\mathbf{R}^n}, \mathcal{F}_{\tilde{L}})$. What we get is then equivalent to a vector bundle on $(T,\Omega)^\vee$. (The rank is $L_0 \bullet L$ where L is a leaf of $\mathcal{F}_{\tilde{L}}$.) This construction, that is a family of Floer homologies, gives a systematic way to construct a vector bundle or a sheaf on $(T,\Omega)^\vee$ from a Lagrangian submanifold of (T,Ω). This construction is regarded as a map :

(0.9) Object of a category $Lag(M)$ $\mapsto$ The functor represented by it,

as Kontsevich explained to the author in summer 1997. We studied the homological algebra of the map (0.9) in [12]. Then we conjecture that our multiplicative structure m_k on Floer homology coincides with (higher) Massey Yoneda product. We will prove this conjecture in case $k = 2$ in Part II. In the case of Elliptic curve, this fact was verified by an explicit calculation in [24], [23].

Thus, (including imaginary part) , the C^* algebra $C(M, \mathcal{F}_{\tilde{L}})$ is regarded as a moduli space of vector bundles. It seems that, in the case when $L_0 \bullet L = 1$, this construction together with m_2 reproduce some part of the theory of theta functions. One might obtain something new if we consider the case when $L_0 \bullet L > 1$ (namely the case of vector bundle), or higher composition m_k (see § 5.)

What seems more novel is the case when $\mathcal{F}_{\tilde{L}}$ is ergodic. In this case, we recall that we regard $C(M, \mathcal{F}_{\tilde{L}})$ as a "moduli space" of a vector bundles on $(T,\Omega)^\vee$. It follows that a "point" of $C(M, \mathcal{F}_{\tilde{L}})$ is supposed to correspond to a "vector bundle" on $(T,\Omega)^\vee$. However, in our case, $C(M, \mathcal{F}_{\tilde{L}})$ is a "noncommutative space". As a consequence, it does not make sense to say a point on it. Therefore, in place of a family of finite dimensional vector bundles parametrized by a moduli space, we find one infinite dimensional vector bundle on which $C(M, \mathcal{F}_{\tilde{L}})$ acts.

Thus what we find is a family of infinite dimensional vector bundles parametrized by $\mathcal{W}$. At special values (which is at most countable) this infinite dimensional vector bundle splits into a family of finite dimensional vector bundles. It seems that similar stories are known in representation theory.

There are various works [27], [19], [30] studying noncommutative torus and its relation to C^* algebra and to theta functions. It seems that they are closely related to this paper. We remark that the deformation constructed in [2] is closely related to the deformation quatization and [27], [30] are based on deformation quantization. From our point of view, a theorem of [26] , which gives a relation of C^* algebra of foliation to a C^* algebra obtained from deformation quatization of a torus, may be regarded as a mirror symmetry.

Recently, several authors (for example [4], [6]) discussed a relation of Matrix theory to noncommutative torus. They might be related also to this paper.

In this paper, we put several lemmata and theorems in quote. The argument we offer to justify them is not enough to prove them rigorously. The gaps left without proof are, for example, convergence of integral, justification of the change of variables, transversality etc.

Many of those statements in the quote will then be proved rigorously in the case of Example 0.2. The proofs of the results without “ ” are all rigorous.

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§ 1 C^* algebra of Foliation

In this section, we review the construction of $C(M,\mathcal{F})$ of the C^*-algebra of foliation $\mathcal{F}$. See [7], [31], [3] § II.8 for more detail. There is nothing new in this section. We include it here for the convenience of the reader and to fix a notation. The reader will find that our construction in § 2 is a natural generalization of the construction of this section.

Definition 1.1 The *holonomy groupoid* $G(M,\mathcal{F})$ of the foliation $\mathcal{F}$ is the set of all $(x,y;[\ell])$ where $x,y\in M$ and ℓ is a path joining x and y and is contained in a leaf of $\mathcal{F}$. (Hence x and y lie in the same leaf.) We identify $[\ell]$ and $[\ell']$ if they have the same holonomy $\in Diff(\boldsymbol{R}^n,\boldsymbol{R}^n)_0$. (Here $Diff(\boldsymbol{R}^n,\boldsymbol{R}^n)_0$ is the group germ of the local diffeomorphisms $(\boldsymbol{R}^n,0)\to(\boldsymbol{R}^n,0)$.) We define

$$(x,y;[\ell])\cdot(y,z;[\ell'])=(x,z;[\ell\circ\ell']). \tag{1.2}$$

(Here $\ell\circ\ell'$ is the path obtained by joining ℓ and ℓ' at y.) (1.2) defines a groupoid structure on $G(M,\mathcal{F})$. We say

$$\lim_{i\to\infty}(x_i,y_i;[\ell_i])=(x_\infty,y_\infty;[\ell_\infty])$$

if there exists a representatives ℓ_i such that ℓ_i converges to ℓ_∞ in C^0 topology (and $\lim_{i\to\infty}x_i=x_\infty$, $\lim_{i\to\infty}y_i=y_\infty$). This defines a topology on $G(M,\mathcal{F})$.

Let $\mathcal{F}$ be an n-dimensional foliation on a $2n$-dimensional manifold M. We can prove the following :

Lemma 1.3 *$G(M,\mathcal{F})$ is a $3n$-dimensional smooth manifold and $T_{(x,y,[\ell])}G(M,\mathcal{F})\cong T_xM\oplus T_y\mathcal{F}\cong T_x\mathcal{F}\oplus T_yM$. Here $T_xM\oplus T_y\mathcal{F}\cong T_x\mathcal{F}\oplus T_yM$ is obtained by the isomorphism $T_xM/T_x\mathcal{F}\cong T_yM/T_y\mathcal{F}$ which is induced from the holonomy group of foliation along ℓ.*

We omit the proof. See [31]. We remark that $G(M,\mathcal{F})$ is not Hausdorff in the general case. However it is so in the case of the foliation in Example 0.2. We define :

Definition 1.4 $C_{comp}(M,\mathcal{F})$ is the set of all compact support continuous sections of the line bundle $\left|\Lambda_x^{top}\mathcal{F}\otimes\Lambda_y^{top}\mathcal{F}\right|^{1/2}\otimes\boldsymbol{C}$ on $G(M,\mathcal{F})$. (Here and hereafter $|L|^{1/2}$ is the (real) line bundle whose transition function is $|g|_{ij}^2$ where g_{ij} is the transition function of a line bundle L.) For $f(x,y;[\ell]),g(x,y;[\ell])\in C_{comp}(M,\mathcal{F})$, we put

$$(f*g)(x,y;[\ell])=\int_{(x,z,[\ell'])}f(x,z;[\ell'])g(z,y;[\ell'^{-1}\circ\ell])\,d(x,z;[\ell']).$$

$$f^*(x,y;[\ell]) = \overline{f(y,x;[\ell^{-1}])}.$$

They satisfy axiom of *algebra. Namely

$$\begin{aligned}
(c_1 f + c_2 g) * h &= c_1 f * h + c_2 g * h \\
(c_1 f + c_2 g)^* &= \bar{c}_1 f^* + \bar{c}_2 g^* \\
(f * g) * h &= f * (g * h) \\
(f * g)^* &= g^* * f^*.
\end{aligned} \tag{1.5}$$

To obtain a C^* algebra we need a completion of $C_{comp}(M,\mathcal{F})$. We omit it and refer [3] section II.8 . (See however § 3.)

We remark that, in the case of Example 0.2, the holonomy group of foliation is always trivial. Hence we can simply write (x,y) in place of $(x,y;[\ell])$. We remark however that $\lim_{i\to\infty} x_i = x_\infty$, $\lim_{i\to\infty} y_i = y_\infty$ does *not* imply $\lim_{i\to\infty}(x_i,y_i) = (x_\infty,y_\infty)$.

§ 2 Floer homology of Lagrangian Foliation

Let $\mathcal{F}_1$, $\mathcal{F}_2$ be Lagrangian foliations on M. We fix orientations of them. We assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are of general position. It is rather delicate to define precisely what one means by two Lagrangian foliations to be of general position. We do not discuss this point in this paper, since, in our main example (Example 0.2), we find that all leaves of $\mathcal{F}_{\tilde{L}_1}$ is transversal to $\mathcal{F}_{\tilde{L}_2}$ if $\tilde{L}_1 \cap \tilde{L}_2 = \{0\}$. Our goal of this section is to define a $C_{comp}(M, \mathcal{F}_1)$, $C_{comp}(M, \mathcal{F}_2)$ $\boldsymbol{Z}_2$-graded differential bimodule $(CF_k^{comp}(\mathcal{F}_1, \mathcal{F}_2), \partial)$. (We discuss the case when $\tilde{L}_1 \cap \tilde{L}_2 \neq \{0\}$ at the end of this section.)

Definition 2.1 $X(M;\mathcal{F}_1,\mathcal{F}_2)$ is the set of all $(x,y,z;\ell_1,\ell_2)$ such that $(x,y;\ell_1) \in G(M,\mathcal{F}_1)$, $(y,z;\ell_2) \in G(M,\mathcal{F}_2)$ and that the leaf of $\mathcal{F}_1$ is transversal to the leaf of $\mathcal{F}_2$ at y. Topologies of $G(M,\mathcal{F}_i)$ induce a topology of $X(M;\mathcal{F}_1,\mathcal{F}_2)$.

We say $(x,y,z;\ell_1,\ell_2) \in X_0(M;\mathcal{F}_1,\mathcal{F}_2)$ if the isomorphism $T_y\mathcal{F}_1 \oplus T_y\mathcal{F}_2 \cong T_yM$ is orientation preserving. We say $(x,y,z;\ell_1,\ell_2) \in X_1(M;\mathcal{F}_1,\mathcal{F}_2)$ otherwise. We put $\deg(y) = d$ if $(x,y,z;\ell_1,\ell_2) \in X_d(M;\mathcal{F}_1,\mathcal{F}_2)$.

We write an element of $X(M;\mathcal{F}_1,\mathcal{F}_2)$ as (x,y,z) in case when holonomy is trivial, (as in the case of Example 0.2).

Lemma 2.2 *$X(M;\mathcal{F}_1,\mathcal{F}_2)$ is a $4n$-dimensional smooth manifold and* $T_{(x,y,z;[\ell_1],[\ell_2])}X(M;\mathcal{F}_1,\mathcal{F}_2) \cong T_x\mathcal{F}_1 \oplus T_yM \oplus T_z\mathcal{F}_2$.

The proof is similar to Lemma 1.3 and is omitted.

Example 2.3 If $\tilde{L}_1 \cap \tilde{L}_2 = \{0\}$, $\tilde{L}_1 \cap \Gamma = \tilde{L}_2 \cap \Gamma = \{0\}$, then $X(M;\mathcal{F}_{\tilde{L}_1},\mathcal{F}_{\tilde{L}_2})$ is diffeomorphic to $\left(\boldsymbol{C}^n \times \boldsymbol{C}^n\right)/\Gamma$. (Here the action is the diagonal action.) To see this, let $(x,y,z) \in X(M;\mathcal{F}_{\tilde{L}_1},\mathcal{F}_{\tilde{L}_2})$. We lift $y \in T^{2n} = \boldsymbol{C}^n/\Gamma$ to $\tilde{y} \in \boldsymbol{C}^n$. Then lifting x, z along curves on the leaves, we obtain $\tilde{x}, \tilde{z} \in \boldsymbol{C}^n$. We put $I(x,y,z) \equiv [\tilde{x},\tilde{z}] \in \left(\boldsymbol{C}^n \times \boldsymbol{C}^n\right)/\Gamma$. It is easy to see that $I : X(M;\mathcal{F}_{\tilde{L}_1},\mathcal{F}_{\tilde{L}_2}) \to \left(\boldsymbol{C}^n \times \boldsymbol{C}^n\right)/\Gamma$ is a diffeomorphism.

Let us define a map $\pi : X(M;\mathcal{F}_1,\mathcal{F}_2) \to M^2$ by $\pi(x,y,z;\ell_1,\ell_2) = (x,z)$. It is easy to see that π is a local diffeomorphism. In case $\mathcal{F}_1$ is everywhere transversal to $\mathcal{F}_2$, $\pi : X(M;\mathcal{F}_1,\mathcal{F}_2) \to M^2$ is a covering space.

We remark that $X(M;\mathcal{F}_1,\mathcal{F}_2)$, $X_k(M;\mathcal{F}_1,\mathcal{F}_2)$, have left actions of the groupoid $G(M,\mathcal{F}_1)$ and right actions of $G(M,\mathcal{F}_2)$. Namely we define

(2.4)
$$(x',x;\ell)(x,y,z;\ell_1,\ell_2) = (x',y,z;\ell \circ \ell_1,\ell_2)$$
$$(x,y,z;\ell_1,\ell_2)(z,z';\ell') = (x,y,z';\ell_1,\ell_2 \circ \ell').$$

Now we define $CF_k^{comp}(\mathcal{F}_1, \mathcal{F}_2)$. Using isomorphism $T_{(x,y,z;[\ell_1],[\ell_2])}X(M;\mathcal{F}_1,\mathcal{F}_2) \cong T_x\mathcal{F}_1 \oplus T_yM \oplus T_z\mathcal{F}_2$ we find that

(2.5) $$\Lambda^{top}_{(x,y,z;[\ell_1],[\ell_2])}X(M;\mathcal{F}_1,\mathcal{F}_2) \cong \Lambda^{top}_x\mathcal{F}_1 \otimes \Lambda^{top}_y M \otimes \Lambda^{top}_z\mathcal{F}_2 .$$

Definition 2.6 $CF_k^{comp}(\mathcal{F}_1, \mathcal{F}_2)$ is the set of all continuous sections F of compact support of the line bundle $\left|\Lambda^{top}_x\mathcal{F}_1 \otimes \Lambda^{top}_z\mathcal{F}_2\right|^{1/2} \otimes \mathbf{C}$ on $X_k(M;\mathcal{F}_1,\mathcal{F}_2)$.

The actions of $G(M,\mathcal{F}_1)$ and $G(M,\mathcal{F}_2)$ on $X_k(M;\mathcal{F}_1,\mathcal{F}_2)$ determine a $C_{comp}(M,\mathcal{F}_1)$, $C_{comp}(M,\mathcal{F}_2)$ bimodule structure on $CF_k^{comp}(\mathcal{F}_1, \mathcal{F}_2)$ by the following formula. Let $F \in CF_k^{comp}(\mathcal{F}_1, \mathcal{F}_2)$, $f \in C_{comp}(M,\mathcal{F}_1)$, $g \in C_{comp}(M,\mathcal{F}_2)$.

(2.7.1) $$(f * F)(x,y,z;\ell_1,\ell_2) = \int f(x,x';\ell)F(x',y,z;\ell^{-1}\circ\ell_1,\ell_2)\, dx'$$

(2.7.2) $$(F * g)(x,y,z;\ell_1,\ell_2) = \int F(x,y,z';\ell_1,\ell_2\circ\ell'^{-1})g(z',z;\ell')\, dz' .$$

Here the integration (2.7.1) is taken over the set of pairs (x',ℓ) such that $(x,x';\ell) \in G(M,\mathcal{F}_1)$. We remark that $f(x,x')F(x',y,z)$ is a density with respect to x'. Similarly the integration (2.7.2) is taken over the leaf of $\mathcal{F}_2$ containing z.

Now we have :

Lemma 2.8 *Products defined by* (2.7) *are complex bilinear. We have also :* $(f_1 * f_2) * F = f_1 * (f_2 * F)$. $(f * F) * g = f * (F * g)$. $F * (g_1 * g_2) = (F * g_1) * g_2$.

The proof is immediate from definitions and Fubini's theorem. We discuss relation to * product in the next section (Lemma 3.5).

There also exists a left action of $C_{comp}(M,\mathcal{F}_1)$ to $\Gamma\left(M;\left|\Lambda^{top}_x\mathcal{F}_1\right|^{1/2}\otimes\mathbf{C}\right)$ and right action of $C_{comp}(M,\mathcal{F}_2)$ to $\Gamma\left(M;\left|\Lambda^{top}_x\mathcal{F}_2\right|^{1/2}\otimes\mathbf{C}\right)$. They are defined by

(2.9.1) $$(f * F)(x) = \int f(x,x';\ell)F(x')\, dx' .$$

(2.9.2) $$(F * g)(y) = \int F(y')g(y',y;\ell)\, dy' .$$

In case $\mathcal{F}_1$ is everywhere transversal to $\mathcal{F}_2$, the map $\pi : X(M;\mathcal{F}_1,\mathcal{F}_2) \to M^2$ induces a map

$$\pi_!: CF_k^{comp}(\mathcal{F}_1,\mathcal{F}_2) \to \Gamma\left(M;\left|\Lambda^{top}_x\mathcal{F}_1\right|^{1/2}\otimes\mathbf{C}\right)\otimes\Gamma\left(M;\left|\Lambda^{top}_x\mathcal{F}_2\right|^{1/2}\otimes\mathbf{C}\right),$$

by "integration along fiber". Namely

$$(2.10)\qquad (\pi ! F)(x,z) = \sum_{(y;\ell_1,\ell_2):(x,y,z;\ell_1,\ell_2)\in X(M;\mathcal{F}_1,\mathcal{F}_2)} F(x,y,z;\ell_1,\ell_2).$$

The right hand side of (2.10) is a finite sum since F is of compact support. (We remark that in case when $\mathcal{F}_1$ is not transversal to $\mathcal{F}_2$ somewhere the right hand side of (2.10) may be discontinuous.) It is easy to see that $\pi!$ is $C_{comp}(M,\mathcal{F}_1)$ $C_{comp}(M,\mathcal{F}_2)$ equivariant.

We remark that so far we did not use symplectic structure of our manifold. In fact all constructions so far work for a pair of oriented n-dimensional foliations in a $2n$-dimensional manifold. This is natural since Floer's chain complex of Lagrangian intersection (if we put $\boldsymbol{Z}_2$ grading) is independent of symplectic structure, as an abelian group. Symplectic structure is used in the construction of boundary operator and of product structure.

We next "define" boundary operator ∂. Our discussing on it is not rigorous yet because of several technical problems. We remark that, in the case of our main example $\mathcal{F}_{\tilde{L}_i}$ with $\tilde{L}_1 \cap \tilde{L}_2 = \{0\}$, vector space $CF_k^{comp}(\mathcal{F}_{\tilde{L}_1},\mathcal{F}_{\tilde{L}_2})$ can be nonzero for only one of $k = 0,1$. Therefore $\partial = 0$. In case $\tilde{L}_1 \cap \tilde{L}_2 \neq \{0\}$ we can also check that the construction below works directly (after moving $\mathcal{F}_{\tilde{L}_1}$ by a Hamiltonian diffeomorphism). Thus the construction is rigorous in the case of our main example. Another way to construct $CF_k^{comp}(\mathcal{F}_{\tilde{L}_1},\mathcal{F}_{\tilde{L}_2})$ in the case when $\tilde{L}_1 \cap \tilde{L}_2 \neq \{0\}$ is to work out Bott-Morse Floer theory (see [11]) in this case.

In this paper our main concern is Example 0.2, where the role of boundary operator is rather minor. Symplectic structure is used mainly in the construction of the product structure. This is the reason why our discussion on the construction of the boundary operators is sketchy.

We fix an almost complex structure on M (a compact symplectic manifold) compatible with our symplectic structure. In the case of our main example, we can take an (integrable) complex structure compatible with ω. (That is the homogeneous tensor $J : TT^{2n} \to TT^{2n}$. We recall that ω is homogeneous.) We remark that this complex structure is different from the obvious complex structure $T^{2n} = \boldsymbol{C}^n/\Gamma$ which we start with. (ω is not compatible with the original complex structure of $T^{2n} = \boldsymbol{C}^n/\Gamma$ unless ω is of 1.1 type.)

Let L_1, L_2 be Lagrangian submanifolds which are not necessary compact. Let $a,b \in L_1 \cap L_2$. We put

$$(2.11)\qquad \mathcal{M}(M;L_1,L_2;a,b) = \left\{ \varphi : D^2 \to M \,\middle|\, \begin{array}{l} \varphi \text{ is holomorphic} \\ \varphi(-1) = a, \quad \varphi(1) = b \\ \varphi(z) \in L_1 \text{ if } z \in \partial D^2, \operatorname{Im} z > 0 \\ \varphi(z) \in L_2 \text{ if } z \in \partial D^2, \operatorname{Im} z < 0 \end{array} \right\}.$$

In a "generic" situation, $\mathcal{M}(M;L_1,L_2;a,b)$ is a union of (infinitely many) components, whose dimension is equal to $\deg(a) - \deg(b)$ modulo 2. (We do not try to make the assumption "generic" precise in this paper. See [14].)

There is an action of $\boldsymbol{R} \cong Iso(D^2, J, (-1,1))$ on $\mathcal{M}(M;L_1,L_2;a,b)$. Let $\overline{\mathcal{M}}(M;L_1,L_2;a,b)$

be the quotient space.

Now we "define" boundary operator

$$\partial : CF_k^{comp}(\mathcal{F}_1,\mathcal{F}_2) \to CF_{k-1}^{comp}(\mathcal{F}_1,\mathcal{F}_2)$$

as follows. Let $F \in \Gamma_{compact}\left(X_k(M;\mathcal{F}_1,\mathcal{F}_2), \left|\Lambda_x^{top}\mathcal{F}_1 \otimes \Lambda_z^{top}\mathcal{F}_2\right|^{1/2} \otimes \mathbf{C}\right)$ and $(x,b,z;\ell_1,\ell_2) \in X_{k-1}(M;\mathcal{F}_1,\mathcal{F}_2)$.

"Definition 2.12" $(\partial F)(x,a,z;\ell_1,\ell_2) = \sum \pm \exp\left(-\int_\varphi \omega\right) F\left(x,b,z;\ell_1 \circ \partial_1\varphi, \partial_2\varphi \circ \ell_2\right)$.

Here the sum is taken over all $(b,[\varphi])$ such that :

(2.13.1) $\left(x,b,z;\ell_1 \circ \partial_1\varphi, \partial_2\varphi \circ \ell_2\right) \in X_{k-1}(M;\mathcal{F}_1,\mathcal{F}_2)$.
(2.13.2) Let L_i be the leaf of $\mathcal{F}_i$ containing a. Then $b \in L_1 \cap L_2$.
(2.13.3) $[\varphi] \in \overline{\mathcal{M}}(M;L_1,L_2;a,b)$.
(2.13.4) The component of $\overline{\mathcal{M}}(M;L_1,L_2;a,b)$ which contains $[\varphi]$ is of zero dimensional.

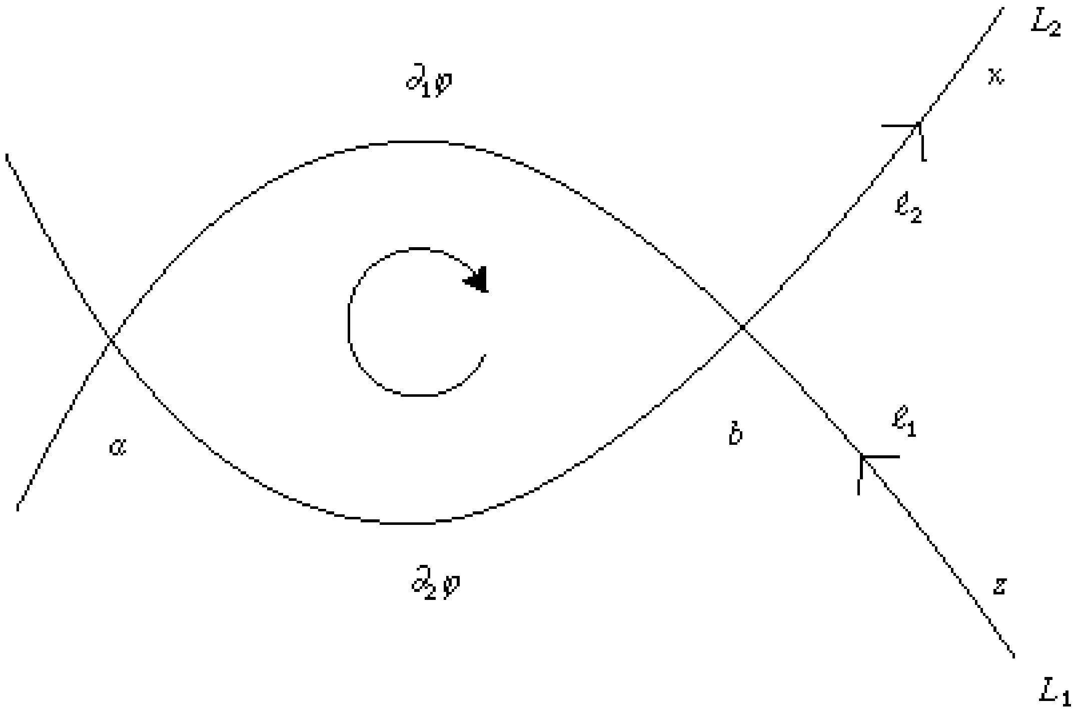


Figure 1

The set of all such $(b,[\varphi])$ is countable. $\pm$ in "Definition 2.12" is determined by the orientation of $\overline{\mathcal{M}}(M;L_1,L_2;a,b)$. $\partial_i\varphi = \varphi(\partial D^2) \cap L_i$ is an arc joining a and b. We remark that the leaves of Lagrangian foliation have canonical spin structure (since they have trivial tangent bundle). Hence $\overline{\mathcal{M}}(M;L_1,L_2;a,b)$ is oriented by [14] , [28].

Conjecture 2.14 The right hand side of Definition 2.12 converges.

There are similar problems in the case of usual Floer homology (of Lagrangian intersection). In that case, we can go around it by introducing Novikov ring. (Roughly speaking this corresponds to defining the boundary operator as a *formal* power series on $t_\alpha = \exp\left(-\int_\alpha \omega\right)$ where $\alpha \in \pi_2(M, L_1 \cup L_2)$ runs over a generator.) It seems harder to do so in our case of Lagrangian foliations.

In fact we need to add various other terms to Definition 2.14 in the same way as [14]. Otherwise ∂F may not be continuous and the following Lemma 2.15 does not hold. We omit the detail. We do not need such a correction term in the case when $\pi_2(M, L) = 0$ hold for all leaves L of our foliation [8]. This condition is satisfied in the case of our main example.

"Lemma 2.15" *Definition* 2.12 *determines a* $C_{comp}(M, \mathcal{F}_1)$, $C_{comp}(M, \mathcal{F}_2)$ *bimodule homomorphism* $\partial : CF_k^{comp}(\mathcal{F}_1, \mathcal{F}_2) \to CF_{k-1}^{comp}(\mathcal{F}_1, \mathcal{F}_2)$. *If* $\pi_2(M, L) = 0$ *for all leaves we also have*

$$\partial\partial = 0.$$

The proof is similar to the proof of the same formula for Floer homology of (compact) Lagrangian intersection. ([8]). We put the lemma in the quote since ∂ is not defined in a rigorous way in general case. Namely we need to prove Conjecture 2.14 to define ∂. We put

$$HF_k^{comp}(\mathcal{F}_1, \mathcal{F}_2) = \frac{\mathrm{Ker}\left(\partial : CF_k^{comp}(\mathcal{F}_1, \mathcal{F}_2) \to CF_{k-1}^{comp}(\mathcal{F}_1, \mathcal{F}_2)\right)}{\mathrm{Im}\left(\partial : CF_{k+1}^{comp}(\mathcal{F}_1, \mathcal{F}_2) \to CF_k^{comp}(\mathcal{F}_1, \mathcal{F}_2)\right)}. \tag{2.16}$$

In fact it seems appropriate to take a completion of $CF_k^{comp}(\mathcal{F}_1, \mathcal{F}_2)$ (see § 3) before taking homology group. Also the author does not know under which condition the image of the boundary operator (after taking a completion) will have a closed range.

We consider the case of linear foliation in T^{2n}. Let $\tilde{L}_1, \tilde{L}_2 \subseteq \boldsymbol{C}^n$ be two Lagrangian linear subspaces such that $\tilde{L}_1 \cap \tilde{L}_2 \neq \{0\}$. Then generic leaf of $\mathcal{F}_{\tilde{L}_1}$ does not intersect with the generic leaf of $\mathcal{F}_{\tilde{L}_2}$. In case when a leaf L_1 of $\mathcal{F}_{\tilde{L}_1}$ intersects with a leaf L_2 of $\mathcal{F}_{\tilde{L}_2}$, we have $\dim(L_1 \cap L_2) \cong \dim(\tilde{L}_1 \cap \tilde{L}_2)$. To handle this case, we use a Hamiltonian perturbation in the following way. Let $\bar{h} : T^n \to \boldsymbol{R}$ be a Morse function. For example we take

$$\bar{h}(x_1, \cdots, x_n) = \sum \cos 2\pi x_i. \tag{2.17}$$

(Here we identify $T^n = \boldsymbol{R}^n / \boldsymbol{Z}^n$.) We use an affine diffeomorphism $T^{2n} = T^n \times T^n$ to define a projection $T^{2n} \to T^n$. Hence, composing $\bar{h}$, we obtain a map $h : T^{2n} \to \boldsymbol{R}$. We assume that the restriction of the differential of $T^{2n} \to T^n$ to $\tilde{L}_1 \cap \tilde{L}_2$ is injective. (This is possible by changing the affine projection $T^{2n} \to T^n$ if necessary.) Let H_h be the Hamiltonian vector field associate to $h : T^{2n} \to \boldsymbol{R}$. (We use our symplectic structure ω on T^{2n} to define H_h.) We put $\Phi = \exp \varepsilon H_h$ for small ε. Φ is a symplectic diffeomorphism and $\Phi(\mathcal{F}_{\tilde{L}_1})$ is of general position to $\mathcal{F}_{\tilde{L}_2}$.

In this case, we can prove Lemma 2.17 without changing the definition of the boundary operator ∂, since $\pi_2(M,L)=0$. Convergence of 2.16 can be checked also directly in this case. Hence we can define (rigorously) :

(2.18) $$HF_k^{comp}(\mathcal{F}_{\tilde{L}_1},\mathcal{F}_{\tilde{L}_2}) = HF_k^{comp}(\Phi(\mathcal{F}_{\tilde{L}_1}),\mathcal{F}_{\tilde{L}_2}).$$

However, in fact, we find that the group $HF_k^{comp}(\mathcal{F}_{\tilde{L}_1},\mathcal{F}_{\tilde{L}_2})$ is rather pathological. (Namely the image of the boundary operator is dense in the kernel.) The reason is that generic leaf of $\mathcal{F}_{\tilde{L}_1}$ does not intersect with the generic leaf of $\mathcal{F}_{\tilde{L}_2}$ in the case when $\tilde{L}_1 \cap \tilde{L}_2 \neq \{0\}$. If we take a completion as we will explain in the next section, we find the example that the Floer homology becomes nontrivial and $\tilde{L}_1 \cap \tilde{L}_2 \neq \{0\}$ (3.13).

§ 3 Transversal measure and completion.

In order to construct a von-Neuman algebra from our $*$ algebra $C_{comp}(M,\mathcal{F}_i)$ and to find a completion of our infinite dimensional vector space $CF_k^{comp}(\mathcal{F}_1,\mathcal{F}_2)$ to a Hilbert or Banach space, we need a transversal measure to our Lagrangian foliation $\mathcal{F}_i$.

Let τ_i be a transversal measure to $\mathcal{F}_i$. (See [22] for its definition.) It determines a (distribution valued) section of $\left|\Lambda^{top}\left(TM/T\mathcal{F}_i\right)\right|$. Let $(x,y,z)\in X(M;\mathcal{F}_1,\mathcal{F}_2)$. Then, since

$$T_y M = T_y\mathcal{F}_1 \oplus T_y\mathcal{F}_2,$$

τ_1 and τ_2 determine a distribution valued section $(\tau_1\otimes\tau_2)(y)$ on $\pi_2^*\left|\Lambda^{top}TM\right|$. Here $\pi_2 : X(M;\mathcal{F}_1,\mathcal{F}_2)\to M$ is defined by $(x,y,z;\ell_1,\ell_2)\mapsto y$. Now we define L^2 inner product $(\ )_{\tau_1\otimes\tau_2}$ on $CF_k^{comp}(\mathcal{F}_1,\mathcal{F}_2)$ as follows.

Definition 3.1

$$(F,G)_{\tau_1\otimes\tau_2} = \int_{X_k(M;\mathcal{F}_1,\mathcal{F}_2)} F(x,y,z;\ell_1,\ell_2)\overline{G(x,y,z;\ell_1,\ell_2)}(\tau_1\otimes\tau_2)(y) \ .$$

We remark that $F(x,y,z;\ell_1,\ell_2)\overline{G(x,y,z;\ell_1,\ell_2)}$ is a section of $\left|\Lambda_x^{top}\mathcal{F}_1\otimes\Lambda_z^{top}\mathcal{F}_2\right|\otimes\boldsymbol{C}$. On the other hand, we have $\Lambda^{top}_{(x,y,z;[\ell_1],[\ell_2])}X(M;\mathcal{F}_1,\mathcal{F}_2)\cong\Lambda_x^{top}\mathcal{F}_1\otimes\Lambda_y^{top}M\otimes\Lambda_z^{top}\mathcal{F}_2$. Therefore $F(x,y,z;\ell_1,\ell_2)\overline{G(x,y,z;\ell_1,\ell_2)}(\tau_1\otimes\tau_2)(y)$ is a top dimensional current (of compact support) on $X_k(M;\mathcal{F}_1,\mathcal{F}_2)$. Definition 3.1 therefore makes sense. The following lemma is easy to prove.

Lemma 3.2 $\left(CF_k^{comp}(\mathcal{F}_1,\mathcal{F}_2),(\ )_{\tau_1\otimes\tau_2}\right)$ *is a pre Hilbert space.*

Definition 3.3 $CF_k^2(\mathcal{F}_1,\mathcal{F}_2;\tau_1\otimes\tau_2)$ is the completion of $\left(CF_k^{comp}(\mathcal{F}_1,\mathcal{F}_2),(\ )_{\tau_1\otimes\tau_2}\right)$.

Hereafter we write $CF_k(\mathcal{F}_1,\mathcal{F}_2;\tau_1\otimes\tau_2)$ in place of $CF_k^2(\mathcal{F}_1,\mathcal{F}_2;\tau_1\otimes\tau_2)$ for simplicity.

Conjecture 3.4 ∂ is extended to a bounded operator $CF_k(\mathcal{F}_1,\mathcal{F}_2;\tau_1\otimes\tau_2)\to CF_{k-1}(\mathcal{F}_1,\mathcal{F}_2;\tau_1\otimes\tau_2)$.

Again, for our example $\Phi(\mathcal{F}_{\tilde L_1}),\mathcal{F}_{\tilde L_2}$ with $\tilde L_1\cap\tilde L_2\neq\{0\}$, we can prove Conjecture 3.4 by a direct calculation. (In the case when $\tilde L_1\cap\tilde L_2=\{0\}$, we have $\partial=0$ and hence there is nothing to show.)

Next we have :

Lemma 3.5 *Actions of* $C_{comp}(M,\mathcal{F}_i)$ *on* $CF_k^{comp}(\mathcal{F}_1,\mathcal{F}_2)$ *is extended to a continuous action on* $CF_k(\mathcal{F}_1,\mathcal{F}_2;\tau_1\otimes\tau_2)$. *We have*

$$(f*F,G)_{\tau_1\otimes\tau_2} = \left(F, f^* * G\right)_{\tau_1\otimes\tau_2}$$
$$(F*g,G)_{\tau_1\otimes\tau_2} = \left(F, G*g^*\right)_{\tau_1\otimes\tau_2}.$$

The proof is straightforward and is omitted. Lemma 3.5 means that we have a *-homomorphism

$$C_{comp}(M,\mathcal{F}_i) \to End\left(CF_k(\mathcal{F}_1,\mathcal{F}_2;\tau_1\otimes\tau_2)\right). \qquad (3.6)$$

Here $End\left(CF_k(\mathcal{F}_1,\mathcal{F}_2;\tau_1\otimes\tau_2)\right)$ is the algebra of all bounded operators.

Definition 3.7 $C(M,\mathcal{F}_i;\tau_1\otimes\tau_2)$ is the weak closure of the image of (3.6).

$C(M,\mathcal{F}_i;\tau_1\otimes\tau_2)$ is a von-Neumann algebra by definition.

Lemma 3.8 *If* $f\in C(M,\mathcal{F}_1;\tau_1\otimes\tau_2)$, $g\in C(M,\mathcal{F}_2;\tau_1\otimes\tau_2)$ *and* $F\in CF_k(\mathcal{F}_1,\mathcal{F}_2;\tau_1\otimes\tau_2)$, *then*

$$(f*F)*g = f*(F*g).$$

The lemma follows from von-Neumann's double commutation theorem.

We remark that we can find a completion of $\Gamma\left(M;\left|\Lambda_x^{top}\mathcal{F}_1\right|^{1/2}\otimes \boldsymbol{C}\right)$ and $\Gamma\left(M;\left|\Lambda_x^{top}\mathcal{F}_2\right|^{1/2}\otimes \boldsymbol{C}\right)$. We denote them by $L^2\left(M;\left|\Lambda_x^{top}\mathcal{F}_1\right|^{1/2}\otimes \boldsymbol{C};\tau_1\right)$, $L^2\left(M;\left|\Lambda_x^{top}\mathcal{F}_2\right|^{1/2}\otimes \boldsymbol{C};\tau_2\right)$.

Example 3.9 Let us consider the case of foliations in Example 0.2 such that $\tilde L_i\cap\Gamma\cong \boldsymbol{Z}^n$. This is the case when all leaves are compact. (Hence we do not need to use operator algebra to study Floer homology of leaves. We discuss this example to show that our construction is a natural generalization of the case when leaves are compact.) We first assume that $\tilde L_1\cap\tilde L_2=\{0\}$. We find that $\pi: X(T^{2n};\mathcal{F}_{\tilde L_1},\mathcal{F}_{\tilde L_2})\to\left(T^{2n}\right)^2$ is an $|L_1\bullet L_2|$ hold covering. (Here L_i is a leaf of $\mathcal{F}_{\tilde L_i}$.) Now let τ_i be the (transversal) delta measure supported on L_i. Then $L^2\left(T^{2n};\left|\Lambda_x^{top}\mathcal{F}_{\tilde L_i}\right|^{1/2}\otimes\boldsymbol{C};\tau_i\right)$ can be identified with $L^2(L_i)$. (Here we use usual Lebesgue measure on the leaf L_i to define $L^2(L_i)$.) We obtain a *-homomorphism

$$C_{comp}(M,\mathcal{F}_i)\to End\left(L^2\left(T^{2n};\left|\Lambda_x^{top}\mathcal{F}_{\tilde L_i}\right|^{1/2}\otimes\boldsymbol{C};\tau_i\right)\right)=End(L^2(L_i)). \qquad (3.10)$$

It is easy to see that the image of (3.10) is dense in weak topology.

Therefore, using the fact that $\pi: X(T^{2n};\mathcal{F}_{\tilde L_1},\mathcal{F}_{\tilde L_2})\to\left(T^{2n}\right)^2$ is a finite covering, we find that

$$(3.11) \qquad C(T^{2n}, \mathcal{F}_i; \tau_1 \otimes \tau_2) = End(L^2(L_i)).$$

Then again using the fact that $\pi : X(T^{2n}; \mathcal{F}_{\tilde{L}_1}, \mathcal{F}_{\tilde{L}_2}) \to \left(T^{2n}\right)^2$ is $\left|L_1 \bullet L_2\right|$ hold covering, we find

$$(3.12) \qquad CF\left(T^{2n}; \mathcal{F}_{\tilde{L}_1}, \mathcal{F}_{\tilde{L}_2}; \tau_1 \otimes \tau_2\right) = \bigoplus_{p \in L_1 \cap L_2} L^2(L_1) \hat{\otimes} L^2(L_2)[p],$$

where action of $C(T^{2n}, \mathcal{F}_i; \tau_1 \otimes \tau_2)$ is obtained from isomorphism (3.11). Hence by the isomorphism

$$K\left(End\left(L^2(L_1)\right) \otimes End\left(L^2(L_2)\right)\right) \cong \boldsymbol{Z},$$

our Floer homology corresponds to $L_1 \bullet L_2$, as expected.

Next let us consider the case when $\tilde{L}_i \cap \Gamma \cong \boldsymbol{Z}^n$ but $\tilde{L}_1 \cap \tilde{L}_2 \neq \{0\}$. Using, for example the explicit Morse function (2.15), we can prove the following. Let τ_i be the (transversal) delta measure supported on the leaf L_i. Then

$$(3.13) \qquad HF_*\left(T^{2n}; \mathcal{F}_{\tilde{L}_1}, \mathcal{F}_{\tilde{L}_2}; \tau_1 \otimes \tau_2\right) = H_{*+d}(L_1 \cap L_2; \boldsymbol{R}) \otimes_{\boldsymbol{R}} \left(L^2(L_1) \hat{\otimes} L^2(L_2)\right),$$

where $d = 0,1$. We remark that Floer homology of L_1 and L_2 is calculated as :

$$HF_*(L_1; L_2) = H_{*+d}(L_1 \cap L_2; \boldsymbol{R}),$$

in this case.

In case when $\pi : X(T^{2n}; \mathcal{F}_{\tilde{L}_1}, \mathcal{F}_{\tilde{L}_2}) \to \left(T^{2n}\right)^2$ is an infinite covering, the "rank" of Floer homology may be regarded as the "order" of the deck transformation group, (which is infinite). So to "count" it correctly, we need some kind of averaging process (similar to [1]). It them might be related to the "average intersection number" of the leaves. The author does not know the correct way to do it.

We need also L^p completion to study product structure. (We remark that the product structure m_2 is a nonlinear map hence it is unbounded if we use only L^2 norm.) We first take (any) Riemannian metric g_M on M. Let $\Omega_{T_x\mathcal{F}} \in \left|\Lambda_x^{top} \mathcal{F}\right|$ be the volume form induced by g_M. Let $F \in CF_k^{comp}(\mathcal{F}_1, \mathcal{F}_2)$. We may regard it as

$$(3.14) \qquad F(x,y,z;\ell_1,\ell_2) = F'(x,y,z;\ell_1,\ell_2)\left|\Omega_{T_x\mathcal{F}} \otimes \Omega_{T_z\mathcal{F}}\right|^{1/2}.$$

Here $F'(x,y,z;\ell_1,\ell_2)$ is a complex valued function. We put

$$(3.15) \qquad \|F\|^p_{L^p, g_M, \tau_1\otimes\tau_2} = \int_{X_k(M;\mathcal{F}_1,\mathcal{F}_2)} \left|F'(x,y,z;\ell_1,\ell_2)\right|^p \Omega_{T_x\mathcal{F}} \otimes \Omega_{T_z\mathcal{F}}(\tau_1 \otimes \tau_2)(y) \quad .$$

Here $(\tau_1 \otimes \tau_2)(y) \in |\Lambda_y^{top} M|$ is the distribution section induced by the measures τ_1, τ_2. It is easy to see that (3.15) defines a semi norm on $CF_k^{comp}(\mathcal{F}_1, \mathcal{F}_2)$. Let $CF_k^p(\mathcal{F}_1, \mathcal{F}_2)$ be the completion of it. Let g'_M be another metric on M. We remark that there exists a constant C such that

(3.15) $$\frac{1}{C}\|F\|^p_{L^p, g'_M, \tau_1 \otimes \tau_2} \le \|F\|^p_{L^p, g_M, \tau_1 \otimes \tau_2} \le C\|F\|^p_{L^p, g'_M, \tau_1 \otimes \tau_2}$$

for any F. Hence $CF_k^p(\mathcal{F}_1, \mathcal{F}_2)$ is independent of the choice of g_M. We also define a completion $C^p(M, \mathcal{F})$ as follows. Let $f \in C_{comp}(M, \mathcal{F})$. Then we put

(3.16) $$f(x, y; \ell) = f'(x, y; \ell)\left|\Omega_{T_x \mathcal{F}}\right|^{1/2} \otimes \left|\Omega_{T_y \mathcal{F}}\right|^{1/2},$$

We then put

(3.17) $$\|f\|^p_{L^p, g_M, \tau} = \int_{X_k(M; \mathcal{F}_1, \mathcal{F}_2)} |f'(x, y; \ell)|^p \Omega_{T_x \mathcal{F}} \otimes \Omega_{T_y \mathcal{F}} \otimes \tau(y) \quad .$$

We remark that $\Omega_{T_y \mathcal{F}} \otimes \tau$ is a distribution section of $|\Lambda_y^{top} M|$. Hence the integration in the right hand side of (3.17) makes sense. Let $C^p(M, \mathcal{F})$ be the completion of $C_{comp}(M, \mathcal{F})$ with respect to this norm. Using Hölder inequality, it is easy to see that if $1/r = 1/p + 1/q$ then

(3.18.1) $$\|f * g\|_{L^r, g_M, \tau} \le C\|f\|_{L^p, g_M, \tau}\|g\|_{L^q, g_M, \tau},$$

(3.18.2) $$\|f^*\|_{L^p, g_M, \tau} \le C\|f\|_{L^p, g_M, \tau},$$

(3.18.3) $$\|f * F\|_{L^r, g_M, \tau_1 \otimes \tau_2} \le C\|f\|_{L^p, g_M, \tau_1}\|F\|_{L^q, g_M, \tau_1 \otimes \tau_2},$$

(3.18.4) $$\|F * g\|_{L^r, g_M, \tau_1 \otimes \tau_2} \le C\|F\|_{L^q, g_M, \tau_1 \otimes \tau_2}\|g\|_{L^p, g_M, \tau_2}.$$

Therefore these maps are extended to the L^p completions. We remark that we do not use the metric g_M in the definition of the maps in (3.18).

We next remark that we can identify the dual space of $CF_k^p(\mathcal{F}_1, \mathcal{F}_2)$ to $CF_{n-k}^q(\mathcal{F}_2, \mathcal{F}_1)$ if $p > 1$, $1/p + 1/q = 1$. So we have a complex bilinear map

(3.19) $$\langle\ \rangle_{\tau_1 \otimes \tau_2} : CF_k^p(\mathcal{F}_1, \mathcal{F}_2) \otimes CF_{n-k}^q(\mathcal{F}_2, \mathcal{F}_1) \to \mathbf{C}.$$

More explicitly, for $F \in CF_k^p(\mathcal{F}_1, \mathcal{F}_2)$, $G \in CF_{n-k}^q(\mathcal{F}_2, \mathcal{F}_1)$, we put

(3.20) $$\langle F, G\rangle_{\tau_1 \otimes \tau_2} = \int_{X_k(M; \mathcal{F}_1, \mathcal{F}_2)} F(x, y, z; \ell_1, \ell_2) G(z, y, x; \ell_2, \ell_1)(\tau_1 \otimes \tau_2)(y) \quad .$$

We remark that $\langle\ \rangle_{\tau_1 \otimes \tau_2}$ is complex bilinear. We can verify

(3.21.1) $$\langle f * F, G\rangle_{\tau_1\otimes\tau_2} = \langle F, G * f\rangle_{\tau_1\otimes\tau_2}$$

(3.21.2) $$\langle F * g, G\rangle_{\tau_1\otimes\tau_2} = \langle F, g * G\rangle_{\tau_1\otimes\tau_2},$$

for $F \in CF_k^p(\mathcal{F}_1, \mathcal{F}_2)$, $G \in CF_{n-k}^q(\mathcal{F}_2, \mathcal{F}_1)$, $f \in C^r(M, \mathcal{F}_1)$, $g \in C^r(M, \mathcal{F}_2)$ with $1/p + 1/q + 1/r = 1$.

§ 4 Product Structure and Noncommutative Theta Function.

In this section we construct a map

$$m_2 : CF^{comp}_{k_1}(\mathcal{F}_1, \mathcal{F}_2) \otimes CF^{comp}_{k_2}(\mathcal{F}_2, \mathcal{F}_3) \to CF^{comp}_{k_1+k_2}(\mathcal{F}_1, \mathcal{F}_2).$$

We also construct its completion

$$m_2(\tau_2) : CF^{p}_{k_1}(\mathcal{F}_{\tilde L_1}, \mathcal{F}_{\tilde L_2};\tau_1 \otimes \tau_2) \otimes CF^{q}_{k_2}(\mathcal{F}_{\tilde L_2}, \mathcal{F}_{\tilde L_3};\tau_2 \otimes \tau_3) \to CF^{r}_{k_1+k_2}(\mathcal{F}_{\tilde L_1}, \mathcal{F}_{\tilde L_3};\tau_1 \otimes \tau_3),$$

in the case of our main example. (Here $1/p + 1/q = 1/r$.) We also calculate it.

Let L_1, L_2, L_3 be Lagrangian submanifolds of M which are not necessary compact. Let $p_i \in L_i \cap L_{i+1}$. ($L_{3+1} = L_1$ by convention.) We take $-1, -e^{2\pi/3}, -e^{-2\pi/3} \in \partial D^2$ and let $\partial_1 D^2$, $\partial_2 D^2$, $\partial_3 D^2$, be parts of $\partial D^2 = S^1$ between -1 and $-e^{-2\pi/3}$, $-e^{-2\pi/3}$ and $-e^{2\pi/3}$, $-e^{-2\pi/3}$ and -1, respectively. We define

$$\mathcal{M}(M; L_1, L_2, L_3; p_1, p_2, p_3) = \left\{ \varphi : D^2 \to M \;\middle|\; \begin{array}{l} \varphi \text{ is holomorphic} \\ \varphi(-1) = p_1, \varphi(-e^{-2\pi/3}) = p_2, \varphi(-e^{2\pi/3}) = p_3 \\ \varphi(\partial_i D^2) \subseteq L_i,\ i = 1,2,3 \end{array} \right\}.$$

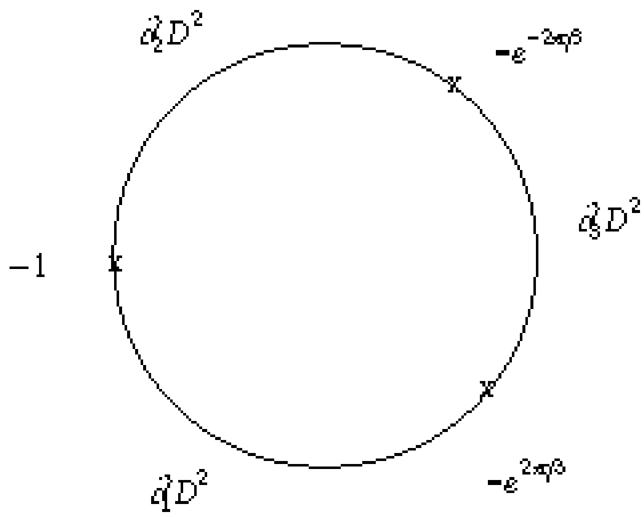


Figure 2

In the "generic situation" $\mathcal{M}(M; L_1, L_2, L_3; p_1, p_2, p_3)$ is a union of (infinitely many) components, which are oriented manifolds whose dimension is equal to $\deg(p_1) + \deg(p_2) + \deg(p_3)$ modulo 2. (See [14].) Here $\deg(p_i)$ is as in § 2. We denote by $\mathcal{M}_k(M; L_1, L_2, L_3; p_1, p_2, p_3)$ the union of components of dimension k. For $\varphi \in \mathcal{M}(M; L_1, L_2, L_3; p_1, p_2, p_3)$ we put $\partial_i \varphi = \varphi|_{\partial_i D^2}$. $\partial_i \varphi$ is a path joining p_i to p_{i+1}. We define

"Definition 4.1" Let $F \in CF^{comp}_{k_1}(\mathcal{F}_1, \mathcal{F}_2)$, $G \in CF^{comp}_{k_2}(\mathcal{F}_2, \mathcal{F}_3)$, and

$(x,c,z;\ell_1,\ell_3)\in X_{k_1+k_2}(M;\mathcal{F}_1,\mathcal{F}_3)$. Let τ_2 be transversal measure of $\mathcal{F}_2$. We put :

$$\begin{aligned}(4.2)\qquad &\big(m_2(\tau_2)(F\otimes G)\big)(x,c,z;\ell_1,\ell_3)\\ &=\int_{a,y,b}\sum_{\varphi\in\mathcal{M}_0(M;L_1,L_2,L_3;a,b,c)}\pm\exp\Big(-\int\varphi^*\omega\Big)\,F(x,a,y;\ell_1\circ\partial_1\varphi,\ell_2)\\ &\qquad G(y,b,z;\ell_2^{-1}\circ\partial_2\varphi,\partial_3\varphi^{-1}\circ\varphi_3)\,d\tau_2(a,b).\end{aligned}$$

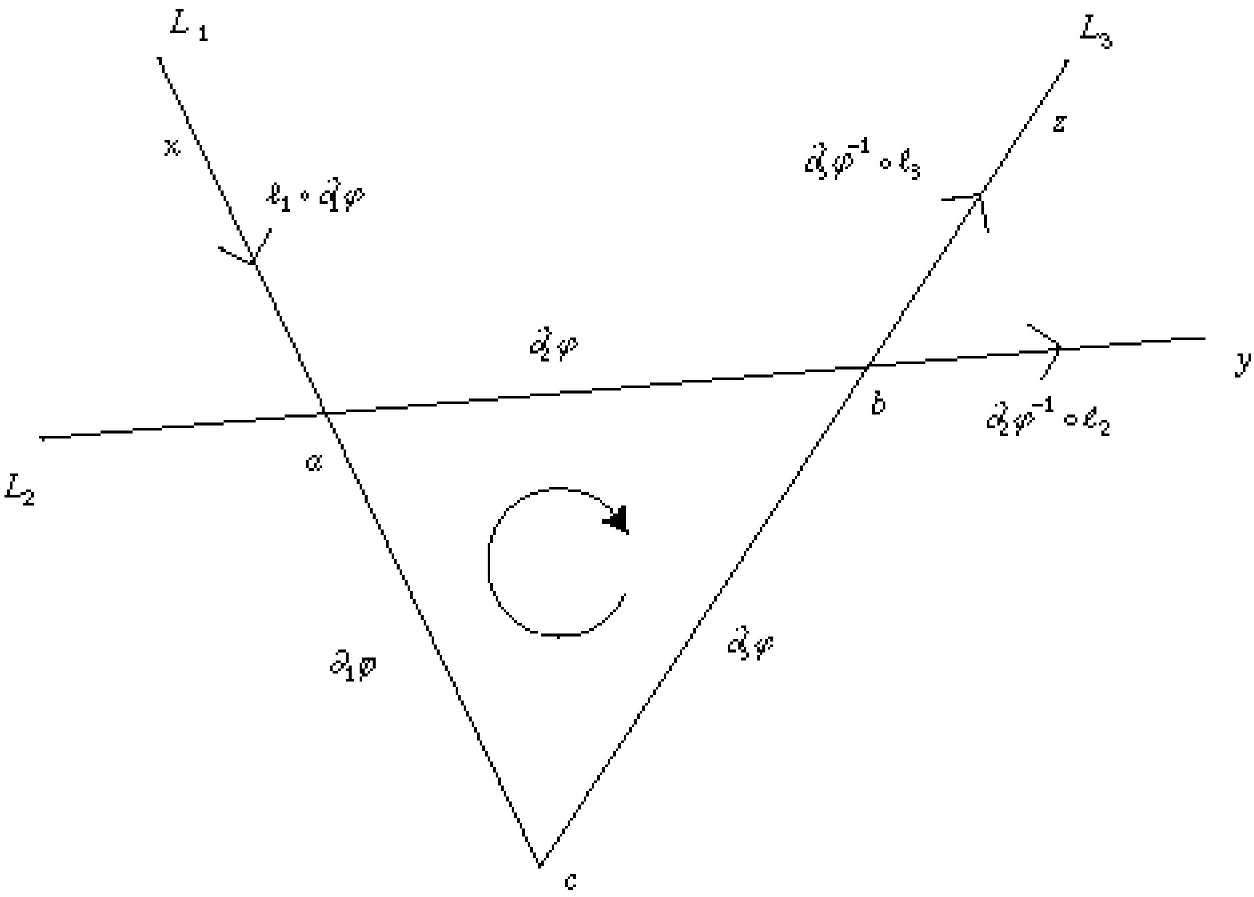


Figure 3

We write m_2 instead of $m_2(\tau_2)$ in case no confusion occur.

Let us explain the notations in (4.2). The domain of integration is the set of all triples $(a,y,b;\ell_2)$ such that $a\in L_1$, $b\in L_3$ and $(a,y;\ell_2),(b,y;\partial_2\varphi^{-1}\circ\ell_2)\in G(M,\mathcal{F}_2)$. ($L_1$ is the leaf of $\mathcal{F}_1$ containing x,c and L_3 is the leaf of $\mathcal{F}_3$ containing z,c.) The space of such triples is a $2n$-dimensional smooth manifold. Let us denote it by $X(L_1,L_3;\mathcal{F}_2)$. We find that

$$(4.3)\qquad T_{(a,y,b;\ell_1,\ell_3)}X(L_1,L_3;\mathcal{F}_2)=T_a\mathcal{F}_1\oplus T_y\mathcal{F}_2\cong T_b\mathcal{F}_3\oplus T_y\mathcal{F}_2.$$

Here the isomorphism $T_a\mathcal{F}_1\cong T_b\mathcal{F}_3$ is obtained by the holonomy of the foliation $\mathcal{F}_2$. We remark that

$$F(x,a,y) \in \left|\Lambda_x^{top}(L_1) \otimes \Lambda_y^{top}(\mathcal{F}_2)\right|^{1/2} \otimes \boldsymbol{C}$$
$$G(y,b,z) \in \left|\Lambda_y^{top}(\mathcal{F}_2) \otimes \Lambda_z^{top}(L_3)\right|^{1/2} \otimes \boldsymbol{C}.$$

(Here and hereafter we omit to write the path ℓ etc. to simplify the formula, in case no confusion can occur.) Hence

$$F(x,a,y)G(y,b,z) \in \left|\Lambda_x^{top}(L_1) \otimes \Lambda_z^{top}(L_3)\right|^{1/2} \otimes \left|\Lambda_y^{top}\mathcal{F}_2\right| \otimes \boldsymbol{C},$$

by (4.3). We remark that τ_2 determines a distribution section on $\left|\Lambda_a^{top}\mathcal{F}_1\right|$ and $\left|\Lambda_b^{top}\mathcal{F}_3\right|$. These two distribution sections are identified to each other by $T_a\mathcal{F}_1 \cong T_c\mathcal{F}_3$ since transversal measure is holonomy invariant. Therefore by (4.3), we can integrate $F(x,a,y)G(y,b,z)$ over $X(L_1,L_3;\mathcal{F}_2)$ and obtain an element of $\left|\Lambda_x^{top}(L_1) \otimes \Lambda_z^{top}(L_3)\right|^{1/2} \otimes \boldsymbol{C}$. We remark that $F(x,a,y;\ell_1 \circ \partial_1\varphi,\ell_2)G(y,b,z;\ell_2^{-1} \circ \partial_2\varphi,\partial_3\varphi^{-1} \circ \varphi_3)$ is zero outside a compact subset of $X(L_1,L_3;\mathcal{F}_{\tilde{L}_2})$ for given x,z,c,ℓ_1,ℓ_3.

The sign in Formula (4.2) is determined by the orientation of the moduli space $\mathcal{M}_0(M;L_1,L_2,L_3;a,b,c)$. (See [14].)

Unfortunately we can not prove the convergence of (4.2) in the general case since we do not have a control of the order of the set $\mathcal{M}_0(M;L_1,L_2,L_3;a,b,c)$.

Conjecture 4.4 For any $F \in CF_{k_1}^{comp}(\mathcal{F}_1,\mathcal{F}_2)$, $G \in CF_{k_2}^{comp}(\mathcal{F}_2,\mathcal{F}_3)$, (4.2) converges. It defines a bounded map :

$$m_2(\tau_2) : CF_{k_1}^p(\mathcal{F}_{\tilde{L}_1},\mathcal{F}_{\tilde{L}_2};\tau_1 \otimes \tau_2) \otimes CF_{k_2}^q(\mathcal{F}_{\tilde{L}_2},\mathcal{F}_{\tilde{L}_3};\tau_2 \otimes \tau_3) \to CF_{k_1+k_2}^r(\mathcal{F}_{\tilde{L}_1},\mathcal{F}_{\tilde{L}_3};\tau_1 \otimes \tau_3)$$

for $1/p + 1/q = 1/r$.

We can “prove”

(4.5) $$m_2(\partial F \otimes G) = (-1)^{k_1} m_2(F \otimes \partial G) = \partial m_2(F \otimes G)$$

formally, (that is modulo convergence). The “proof” of (4.5) is similar to the proof of the associativity formula we give in the next section. We can prove the following also formally.

“Theorem 4.6” *For* $F \in CF_{k_1}^p(\mathcal{F}_1,\mathcal{F}_2)$, $G \in CF_{k_2}^q(\mathcal{F}_2,\mathcal{F}_3)$, $f \in C^r(M,\mathcal{F}_1)$, $g \in C^r(M,\mathcal{F}_2)$, $h \in C^r(M,\mathcal{F}_3)$ *with* $1/p + 1/q + 1/r \le 1$. *We have*

(4.7.1) $$m_2((f * F) \otimes G) = f * m_2(F \otimes G),$$
(4.7.2) $$m_2((F * g) \otimes G) = m_2(F \otimes (g * G)),$$
(4.7.3) $$m_2(F \otimes (G * h)) = m_2(F \otimes G) * h.$$

"Proof" (4.7.2) is proved by the calculation below using definition :

$$\begin{aligned}
&\big(m_2((F * g)\otimes G)\big)(x,c,z)\\
&= \int_{a,y,b} \sum_{\varphi\in\mathcal{M}_0(M;L_1,L_2,L_3;a,b,c)} \pm\exp\Big(-\int\varphi^*\omega\Big)\,(F*g)(x,a,y)G(y,b,z)\,d\tau_2(a,b)\\
&= \int_{a,y,y',b} \sum_{\varphi\in\mathcal{M}_0(M;L_1,L_2,L_3;a,b,c)} \pm\exp\Big(-\int\varphi^*\omega\Big)\,F(x,a,y)g(y,y')G(y',b,z)\,d\tau_2(a,b)\\
&= \int_{a,y',b} \sum_{\varphi\in\mathcal{M}_0(M;L_1,L_2,L_3;a,b,c)} \pm\exp\Big(-\int\varphi^*\omega\Big)\,F(x,a,y')(g*G)(y',b,z)\,d\tau_2(a,b).
\end{aligned}$$

The "proofs" of (4.7.1), (4.7.3) are easier.

(We remark that we use Fubini's theorem in the "proof" above. Since we do not know how to find an appropriate estimate to justify it , in general case, we put " " .)

"Theorem 4.8" *If* $F\in CF^p_{k_1}(\mathcal{F}_1,\mathcal{F}_2)$, $G\in CF^q_{k_2}(\mathcal{F}_2,\mathcal{F}_3)$, $H\in CF^r_{k_3}(\mathcal{F}_1,\mathcal{F}_3)$ *with* $k_1+k_2+k_3\equiv n$ *mod 2*, $1/p+1/q+1/r=1$, *then we have*

$$\big\langle m_2(\tau_2)(F\otimes G),H\big\rangle_{\tau_1\otimes\tau_3} = \big\langle m_2(\tau_3)(G\otimes H),F\big\rangle_{\tau_2\otimes\tau_1} = \big\langle m_2(\tau_1)(H\otimes F),G\big\rangle_{\tau_3\otimes\tau_2}.$$

"Proof" We calculate

$$\begin{aligned}
&\big\langle m_2(\tau_2)(F\otimes G),H\big\rangle_{\tau_1\otimes\tau_3}\\
&= \int m_2(\tau_2)(F\otimes G)(x,c,z)H(z,c,x)(\tau_1\otimes\tau_3)(c)\\
&= \int_{a,b,c,x,y,z} \sum_{\varphi\in\mathcal{M}_0(M;L_1,L_2,L_3;a,b,c)} \pm\exp\Big(-\int\varphi^*\omega\Big)\,F(x,a,y)\\
&\qquad\qquad G(y,b,z)H(z,c,x)(\tau_1\otimes\tau_2\otimes\tau_3)(a,b,c).
\end{aligned} \tag{4.9}$$

We explain the notation in the last formula of (4.9). The domain of the integration is the set of all a,b,c,x,y,z such that

(4.10.1) a,x,c lies in the same leaf of $\mathcal{F}_1$.
(4.10.2) a,y,b lies in the same leaf of $\mathcal{F}_2$.
(4.10.3) c,z,b lines in the same leaf of $\mathcal{F}_3$.

(See Figure 3.) (In fact we need to include the path in the definition of $X(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)$. We omit it for simplicity.) Let $X(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)$ be the set of all (a,b,c,x,y,z) satisfying (4.10). We have

$$(4.11)\qquad \begin{aligned} T_{(a,b,c,x,y,z)}X(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3) &\cong T_aM\oplus T_c\mathcal{F}_1\oplus T_x\mathcal{F}_1\oplus T_y\mathcal{F}_2\oplus T_z\mathcal{F}_3\\ &\cong T_aM\oplus T_b\mathcal{F}_2\oplus T_x\mathcal{F}_1\oplus T_y\mathcal{F}_2\oplus T_z\mathcal{F}_3\\ &\cong T_bM\oplus T_a\mathcal{F}_2\oplus T_x\mathcal{F}_1\oplus T_y\mathcal{F}_2\oplus T_z\mathcal{F}_3\\ &\cong T_bM\oplus T_c\mathcal{F}_3\oplus T_x\mathcal{F}_1\oplus T_y\mathcal{F}_2\oplus T_z\mathcal{F}_3\\ &\cong T_cM\oplus T_b\mathcal{F}_3\oplus T_x\mathcal{F}_1\oplus T_y\mathcal{F}_2\oplus T_z\mathcal{F}_3\\ &\cong T_cM\oplus T_a\mathcal{F}_1\oplus T_x\mathcal{F}_1\oplus T_y\mathcal{F}_2\oplus T_z\mathcal{F}_3. \end{aligned}$$

Let us prove (4.11). We first choose a without restriction. Then (4.10.1) implies that the set of the directions to which c can move is $T_c\mathcal{F}_1$. If we fix a,c then (4.10.2) and (4.10.3) determine b locally. Once (a,b,c) is determined, the set of directions to which x,y,z move is $T_x\mathcal{F}_1$, $T_y\mathcal{F}_2$, $T_z\mathcal{F}_3$, (because of (4.10.1), (4.10.2), (4.10.3)) respectively. Thus we have

$$T_{(a,b,c,x,y,z)}X(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)\cong T_aM\oplus T_c\mathcal{F}_1\oplus T_x\mathcal{F}_1\oplus T_y\mathcal{F}_2\oplus T_z\mathcal{F}_3.$$

The proof of other equalities of (4.11) is similar.

We remark that the isomorpisms among right hand sides of (4.11) is obtained by holonomy of the foliations. For example $T_aM\oplus T_c\mathcal{F}_1\cong T_aM\oplus T_b\mathcal{F}_2$ is obtained by the holonomy of $\mathcal{F}_3$. On the other hand, $T_aM\oplus T_b\mathcal{F}_2\cong T_a\mathcal{F}_2\oplus T_a\mathcal{F}_1\oplus T_b\mathcal{F}_2\cong T_a\mathcal{F}_2\oplus T_b\mathcal{F}_3\oplus T_c\mathcal{F}_2\cong T_bM\oplus T_a\mathcal{F}_2$ is obtained by the isomorphism $T_a\mathcal{F}_1\cong T_b\mathcal{F}_3$ induced by holonomy of $\mathcal{F}_2$.

We next recall

$$F(x,a,y)G(y,b,z)H(z,c,x)\in\left|\Lambda_x^{top}\mathcal{F}_1\otimes\Lambda_y^{top}\mathcal{F}_2\otimes\Lambda_z^{top}\mathcal{F}_3\right|.$$

Hence the integration of x,y,z parameter makes sense. On the other hand, we have a distribution valued section $(\tau_2\otimes\tau_3)(a)\otimes(\tau_1)(b)$ of $\left|\Lambda^{top}T_aM\oplus\Lambda^{top}T_b\mathcal{F}_1\right|$. We write it $(\tau_1\otimes\tau_2\otimes\tau_3)(a,b,c)$. Holonomy invariance of transversal measure implies that we can use other isomorphism (4.11) and obtain the same result. Thus the last formula of (4.9) makes sense (modulo convergence).

The equality (4.9) is then immediate from definition. Now, by the fact that we can use any of the right hand side of (4.11) to define the measure $(\tau_1\otimes\tau_2\otimes\tau_3)(a,b,c)$, we can check that the last term of (4.9) is invariant of the change $L_1\to L_2$ $L_2\to L_3$, $L_3\to L_1$, $(a,b,c)\to(b,c,a)$, $(x,y,z)\to(y,z,x)$. The "proof" of "Theorem 4.8" is complete.

Now we prove the following

Theorem 4.12 *Conjecture* 4.4 *and "Theorems"* 4.6,4.8 *hold in the case of our example* $\mathcal{F}_{\tilde{L}_i}$, *where* $\tilde{L}_i$ *are of general position.*

In fact we can prove more. Namely we calculate the map m_2 explicitly. The key result we need is the following Theorem 4.18 which follows from the main theorem in [15]. To state it we need notations.

Let us identify the universal cover $\tilde{T}^{2n}$ with $\boldsymbol{C}^n$ using a complex structure compatible with ω. (We remark again that this complex structure is different from the one with start with.) Let $\hat{L}_1, \hat{L}_2$ be affine Lagrangian subspaces of $\boldsymbol{C}^n$. By perturbing them a bit, we may assume that $J(\tilde{L}_1) \cap \tilde{L}_2 = \{0\}$. We regards $\boldsymbol{C}^n = T^* \hat{L}_1$. Then it is easy to see that $\hat{L}_2$ is identified with a graph of an exact one form $dV(\hat{L}_1, \hat{L}_2)$, where $V(\hat{L}_1, \hat{L}_2)$ is a quadratic function on $\hat{L}_1$. We identify $\hat{L}_1$ with the 0 section of $\boldsymbol{C}^n = T^* \hat{L}_1$. We assume that $V(\hat{L}_1, \hat{L}_2)$ is a Morse function. (In other words we assume that $\hat{L}_1$ is transversal to $\hat{L}_2$.)

Let $\hat{L}_3$ be another Lagrangian submanifold satisfying $\tilde{L}_1 \cap \tilde{L}_3 = \{0\}$, $\tilde{L}_2 \cap \tilde{L}_3 = \{0\}$, $J\tilde{L}_1 \cap \tilde{L}_3 = \{0\}$. Then $\hat{L}_3$ is a graph of $dV(\hat{L}_1, \hat{L}_3)$ for a quadratic function $V(\hat{L}_1, \hat{L}_3)$. We put

$$V(\hat{L}_3, \hat{L}_1) = -V(\hat{L}_1, \hat{L}_3). \tag{4.13.1}$$

$$V(\hat{L}_2, \hat{L}_3) = V(\hat{L}_1, \hat{L}_3) - V(\hat{L}_1, \hat{L}_2). \tag{4.13.2}$$

Let $\eta(\hat{L}_i, \hat{L}_j)$ be the Morse index of $V(\hat{L}_i, \hat{L}_j)$ at its unique critical point. (We remark that $V(\hat{L}_i, \hat{L}_j)$ is a Morse function since $\tilde{L}_i \cap \tilde{L}_j = \{0\}$.)

Definition 4.14 The *Maslov index* $\eta(\hat{L}_1, \hat{L}_2, \hat{L}_3)$ is defined by

$$\eta(\hat{L}_1, \hat{L}_2, \hat{L}_3) = n - \left(\eta(\hat{L}_1, \hat{L}_2) + \eta(\hat{L}_2, \hat{L}_3) + \eta(\hat{L}_3, \hat{L}_1)\right).$$

We put

$$\mathcal{MLG}(n,3) = \left\{(\hat{L}_1, \hat{L}_2, \hat{L}_3) \,\middle|\, \tilde{L}_i \cap \tilde{L}_j = \{0\}\right\}. \tag{4.15}$$

We can extend the function η to $\mathcal{MLG}(n,3)$. (Namely we can define $\eta(\hat{L}_1, \hat{L}_2, \hat{L}_3)$ in case when $J(\tilde{L}_1) \cap \tilde{L}_2 \neq \{0\}$ etc.) Namely we have :

Lemma 4.16 *η is extended continuously to $\eta : \mathcal{MLG}(n,3) \to \boldsymbol{Z}$. It satisfies $\eta(\hat{L}_1, \hat{L}_2, \hat{L}_3) = \eta(\hat{L}_2, \hat{L}_3, \hat{L}_1)$.*

We omit the proof, which is not difficult. We find easily that $\eta(\hat{L}_1, \hat{L}_2, \hat{L}_3)$ depends only on linear part $\tilde{L}_i$ of $\hat{L}_i$. Hence we write $\eta(\tilde{L}_1, \tilde{L}_2, \tilde{L}_3)$.

We remark that the definition of $\eta(\hat{L}_i, \hat{L}_j)$ is rather artificial since it depends on the choice of $J\tilde{L}_1$ and hence on complex structure. However $\eta(\tilde{L}_1, \tilde{L}_2, \tilde{L}_3)$ depends only on symplectic structure.

Lemma 4.17 *The formal dimension of $\mathcal{M}_0(\boldsymbol{C}^n; \hat{L}_1, \hat{L}_2, \hat{L}_3; \hat{p}_1, \hat{p}_2, \hat{p}_3)$ is $\eta(\tilde{L}_1, \tilde{L}_2, \tilde{L}_3)$.*

Here the formal dimension is the index of the linearized operator. This lemma is not new. We prove it later using [15].

Now, let $\{\hat{p}_i\} = \hat{L}_i \cap \hat{L}_{i+1}$. We have the following :

Theorem 4.18 *The order counted with sign of* $\mathcal{M}_0(\boldsymbol{C}^n;\hat{L}_1,\hat{L}_2,\hat{L}_3;\hat{p}_1,\hat{p}_2,\hat{p}_3)$ *is* ± 1 *if* $\eta(\tilde{L}_1,\tilde{L}_2,\tilde{L}_3)=0$. *Otherwise it is 0.*

We explain the sign at the end of this section.

We next define, for $a,b,c\in\boldsymbol{C}^n$

(4.19) $$Q(a,b,c;\omega)=\int_{\Delta_{ab,c}}\omega\ .$$

Here $\Delta_{a,b,c}$ is the (geodesic) triangle whose vertices are a,b,c.

Using Theorem 4.18 we calculate m_2 in the case of Theorem 4.12. Let $\hat{L}_1$, $\hat{L}_2$, $\hat{L}_3$ be n-dimensional affine subspaces of $\boldsymbol{C}^n$ parallel to $\tilde{L}_1$, $\tilde{L}_2$, $\tilde{L}_3$. We put $\{a\}=\hat{L}_1\cap\hat{L}_2$, $\{b\}=\hat{L}_2\cap\hat{L}_3$, $\{c\}=\hat{L}_3\cap\hat{L}_1$ and $x\in\hat{L}_1$, $y\in\hat{L}_2$, $z\in\hat{L}_3$. We recall that the universal cover of $X(T^{2n};\mathcal{F}_{\tilde{L}_1},\mathcal{F}_{\tilde{L}_2})$ is $\boldsymbol{C}^{2n}$. It is regarded as the set of all $(x,y,z)\in\boldsymbol{C}^{3n}$ such that $x-y\in\tilde{L}_1$, $z-y\in\tilde{L}_2$. In the next formula we regard F etc. as an $\pi_1\left(X(T^{2n};\mathcal{F}_{\tilde{L}_1},\mathcal{F}_{\tilde{L}_2})\right)$ etc. invariant functions on $\boldsymbol{C}^{2n}$.

Corollary 4.20 *If* $\eta(\tilde{L}_1,\tilde{L}_2,\tilde{L}_3)=0$, *then*

$$\big(m_2(\tau_2)(F\otimes G)\big)(x,c,z)=\int_{\hat{L}_2}\int_{y\in L_2}e^{-Q(a,b,c;\omega)}\,F(x,a,y)G(y,b,z)\,d\tau_2(\hat{L}_2).$$

Otherwise $m_2(\tau_2)(F\otimes G)=0$.

Here we take integration over the set of pairs $(\hat{L}_2,y)$ where $\hat{L}_2\subseteq\boldsymbol{C}^n$ is parallel to $\tilde{L}_2$ and $y\in\hat{L}_2$. The transversal measure τ_2 determines a measure on the set of $\hat{L}_2$.

To show the corollary we consider :

(4.21) $$\sum_{\varphi\in\mathcal{M}_0(M;L_1,L_2,L_3;\bar{a},\bar{b},\bar{c})}\pm\exp\left(-\int\varphi^*\omega\right)F(\bar{x},\bar{a},\bar{y}).$$

where $\pi(\hat{L}_i)=L_i$ and $\bar{a}\in T^{2n}$ is a mod Γ etc. We first remark that

(4.22) $$\int\varphi^*\omega=Q(a,b,c;\omega)$$

by Stokes' theorem.

On the other hand, by Theorem 4.18, we find that there exists unique $\varphi\in\mathcal{M}_0(M;L_1,L_2,L_3;\bar{a},\bar{b},\bar{c})$ for each lifts a,b,c of $\bar{a},\bar{b},\bar{c}$. Therefore the integration of (4.22) over the set of all triples (a,y,b) such that $a\in\hat{L}_1$, $b\in\hat{L}_3$ and $(a,y),(y,b)\in G(M,\mathcal{F}_2)$ is equal to the right hand sides of Corollary 4.20. The proof of Corollary 4.20 is now complete.

Corollary 4.20 looks similar to Weinstein's formula (2) in [30] p 329. However there is $\sqrt{-1}$ in the exponential in Weinstein's formula. This might be related to the fact that the deformation constructed by [2] is a deformation quantization with respect to an odd symplectic form.

We next consider $Q(a,b,c;\omega)$. We fix $\hat{L}_1$, $\hat{L}_3$ and $\{c\} = \hat{L}_3 \cap \hat{L}_1$. For each $v \in \boldsymbol{C}^n / \tilde{L}_2$. There exists unique $\hat{L}_2$ corresponding to it. We write it $\hat{L}_2(v)$. We put $\{a(v)\} = \hat{L}_1 \cap \hat{L}_2(v)$, $\{b(v)\} = \hat{L}_2(v) \cap \hat{L}_3$, and $Q(v;\hat{L}_1,\hat{L}_3;\omega) = Q(a(v),b(v),c;\omega)$.

We remark $\alpha(v) = a(v) - c$, $\beta(v) = b(v) - c$ define linear isomorphisms $\boldsymbol{C}^n / \tilde{L}_2 \to \tilde{L}_1$, $\boldsymbol{C}^n / \tilde{L}_2 \to \tilde{L}_3$. We regard ω as an anti symmetric $\boldsymbol{R}$ bilinear map $\boldsymbol{C}^n \otimes_{\boldsymbol{R}} \boldsymbol{C}^n \to \boldsymbol{R}$. (We recall that ω is of constant coefficient.) We then find

$$Q(v;\hat{L}_1,\hat{L}_3;\omega) = \frac{1}{2}\omega(\alpha(v),\beta(v)). \tag{4.23}$$

(4.23) implies that $Q(v;\hat{L}_1,\hat{L}_3;\omega)$ is a quadratic function. We have

Lemma 4.24 $Q(v;\hat{L}_1,\hat{L}_3;\omega) \ge 0$. *Equality holds only for* v *with* $\alpha(v) = \beta(v) = 0$.

Proof: Theorem 4.18 implies that there exists a holomorphic map φ such that

$$\int \varphi^* \omega = Q(v;\hat{L}_1,\hat{L}_3;\omega).$$

Hence $Q(v;\hat{L}_1,\hat{L}_3;\omega) \ge 0$. If $Q(v;\hat{L}_1,\hat{L}_3;\omega) = 0$ then φ must be a constant map. But then the boundary condition implies that $\hat{L}_1(v) \cap \hat{L}_2(v) \cap \hat{L}_3 \ne \emptyset$. Hence $a(v) = b(v) = c$. The proof of Lemma 4.24 is complete.

Let $\pi_{\tilde{L}_2} : \boldsymbol{C}^n \to \boldsymbol{C}^n / \tilde{L}_2$ be the projection. We have

$$\big(m_2(\tau_2)(F \otimes G)\big)(x,c,z) = \int_{\pi_{\tilde{L}_2}(y)=v} \int_{v \in \boldsymbol{C}^n/\tilde{L}_2} e^{-Q(v;\hat{L}_1,\hat{L}_3;\omega)} F(x,a,y)G(y,b,z)\, d\tau_2(v). \tag{4.25}$$

We fix a flat Riemannian metric on T^{2n}. (It induces one on $\boldsymbol{C}^n = \tilde{T}^n$.) Furthermore by Lemma 4.24 we have

$$Q(v;\hat{L}_1,\hat{L}_3;\omega) \ge \delta \; dist(v,\pi_{\tilde{L}_2}(c))^2. \tag{4.26}$$

Here δ is a positive constant depending only on $\tilde{L}_1,\tilde{L}_2,\tilde{L}_3$. Using (4.25), (4.26) and Hölder inequality it is easy to show Conjecture 4.4 in our case. Also estimate (4.26) gives enough control to justify the "proofs" of "Theorems 4.6 and 4.8". The proof of Theorem 4.12 modulo Theorem 4.18 is now complete.

We next consider the case when the foliations $\mathcal{F}_{\tilde{L}_i}$ have compact leaves. Let L_i be a compact leaf and τ_i be the (transversal) delta measure supported at L_i. We assume

$\eta(\tilde{L}_1, \tilde{L}_2, \tilde{L}_3) = n$ and put

(4.27) $$\{\bar{a}_1, \cdots, \bar{a}_I\} = L_1 \cap L_2, \ \{\bar{b}_1, \cdots, \bar{b}_J\} = L_2 \cap L_3, \ \{\bar{c}_1, \cdots, \bar{c}_K\} = L_3 \cap L_1 .$$

As we proved in the last section

$$\begin{aligned} HF^p(\mathcal{F}_1, \mathcal{F}_2; \tau_1 \otimes \tau_2) &= \bigoplus_i L^p(L_1 \times L_2)[\bar{a}_i], \\ HF^q(\mathcal{F}_2, \mathcal{F}_3; \tau_2 \otimes \tau_3) &= \bigoplus_j L^q(L_2 \times L_3)[\bar{b}_j], \\ HF^r(\mathcal{F}_3, \mathcal{F}_1; \tau_3 \otimes \tau_1) &= \bigoplus_k L^r(L_3 \times L_1)[\bar{c}_k]. \end{aligned}$$

Then we find from Corollary 4.24 that m_2 is the tensor product of the map $\overline{m}_2$ in the introduction and the map

$$\left(\bigoplus_i \boldsymbol{R}[\bar{a}_i]\right) \otimes \left(\bigoplus_j \boldsymbol{R}[\bar{b}_j]\right) \to \bigoplus_k \boldsymbol{R}[\bar{c}_k],$$

whose i, j, k component $Z_{ijk}(L_1, L_2, L_3)$ is given by as follows. Let $\pi : \boldsymbol{C}^n \to T^{2n}$ be the projection. We fix a lift c_k of $\bar{c}_k$. Let $\hat{L}_1, \hat{L}_3$ be the orbit of $\tilde{L}_1$, $\tilde{L}_3$ containing c_k. Let $\hat{L}_2(\gamma)$ $\gamma \in \boldsymbol{Z}^n$ be the components of $\pi^{-1}(L_2)$. (Here $\boldsymbol{Z}^n \cong \Gamma / \Gamma \cap \tilde{L}_2$.) We define a map

$$\mu : \boldsymbol{Z}^n \to \{1, \cdots, I\} \times \{1, \cdots, J\}$$

by

$$\pi\left(\hat{L}_2(\gamma) \cap \hat{L}_1\right) = \{a_i\}, \ \pi\left(\hat{L}_2(\gamma) \cap \hat{L}_3\right) = \{b_j\} \quad \Leftrightarrow \quad \mu(\gamma) = (i, j).$$

We put also

$$\{a(\gamma)\} = \hat{L}_1 \cap \hat{L}_2(\gamma), \ \{b(\gamma)\} = \hat{L}_3 \cap \hat{L}_2(\gamma).$$

Now Corollary 4.20 implies

Theorem 4.28 $$Z_{ijk}(L_1, L_2, L_3) = \sum_{\gamma : \mu(\gamma) = \{i, j\}} \exp\left(-Q(a(\gamma), b(\gamma), c_k; \omega)\right).$$

Moving L_i and also including flat line bundles on L_i we obtain a holomophic section of a vector bundle of the products of three complex tori which are mirrors of the torus (with complexified symplectic structure) we start with. This function is a Theta function as we can see from Theorem 4.28. This fact is due to Kontsevich [18] in the case of elliptic curve. [24] [23] studied the case of elliptic curve in more detail. (We remark that Theorem 4.18 is trivial in case $n = 1$.)

Finally we prove Theorem 4.18. The basic tool we use is Morse homotopy [13], [15]. We recall that we identified $\tilde{T}^{2n} = T^*\hat{L}_1$. And $\hat{L}_2$, $\hat{L}_3$ are identified with the graphs of $dV(\hat{L}_1,\hat{L}_2)$ and $dV(\hat{L}_1,\hat{L}_3)$ respectively. ($\hat{L}_1$ is identified with zero section.) Let $\hat{L}_2(\varepsilon)$ and $\hat{L}_3(\varepsilon)$ be the graphs of $\varepsilon dV(\hat{L}_1,\hat{L}_2)$ and $\varepsilon dV(\hat{L}_1,\hat{L}_3)$.

Let

$$\begin{gathered}
\{\hat{p}_1\} = \hat{L}_1 \cap \hat{L}_2,\ \{\hat{p}_1(\varepsilon)\} = \hat{L}_1 \cap \hat{L}_2(\varepsilon),\ \ p_1 = \Pi(\hat{p}_1) = \Pi(\hat{p}_1(\varepsilon)),\\
\{\hat{p}_2\} = \hat{L}_2 \cap \hat{L}_3,\ \{\hat{p}_2(\varepsilon)\} = \hat{L}_2 \cap \hat{L}_3(\varepsilon),\ \ p_2 = \Pi(\hat{p}_2) = \Pi(\hat{p}_2(\varepsilon)),\\
\{\hat{p}_3\} = \hat{L}_3 \cap \hat{L}_1,\ \{\hat{p}_3(\varepsilon)\} = \hat{L}_3 \cap \hat{L}_1(\varepsilon),\ \ p_3 = \Pi(\hat{p}_3) = \Pi(\hat{p}_3(\varepsilon)),
\end{gathered}$$

where $\Pi : \tilde{T}^{2n} = T^*\tilde{L}_1 \to \tilde{L}_1$ is the projection. We put

$$V(\hat{L}_2,\hat{L}_3) = V(\hat{L}_1,\hat{L}_3) - V(\hat{L}_1,\hat{L}_2).$$

We remark that

$$dV(\hat{L}_1,\hat{L}_2)(p_1) = dV(\hat{L}_2,\hat{L}_3)(p_2) = dV(\hat{L}_3,\hat{L}_1)(p_3) = 0.$$

We recall that we fix a complex structure on $\tilde{T}^{2n} = T^*\hat{L}_1$ compatible with symplectic structure ω. (The symplectic structure ω coincides with the canonical symplectic structure of the cotangent bundle $\tilde{T}^{2n} = T^*\hat{L}_1$.) Hence we obtain a Riemannian metric (Euclidean metric in fact) on $\hat{L}_1$. Using it we consider gradient vector fields

$$\operatorname{grad} V(\hat{L}_1,\hat{L}_2),\ \operatorname{grad} V(\hat{L}_2,\hat{L}_3),\ \operatorname{grad} V(\hat{L}_3,\hat{L}_1).$$

Let $U(p_i)$ be the unstable manifold of the vector field $\operatorname{grad} V(\hat{L}_i,\hat{L}_{i+1})$. By definition we have $\eta(p_i) = n - \dim U(p_i)$. The main theorem proved in [15] applied in this situation is

$$U(p_1) \cap U(p_2) \cap U(p_3) \cong \mathcal{M}(\boldsymbol{C}^n; \hat{L}_1(\varepsilon), \hat{L}_2(\varepsilon), \hat{L}_3(\varepsilon); \hat{p}_1(\varepsilon), \hat{p}_2(\varepsilon), \hat{p}_3(\varepsilon)) \tag{4.29}$$

for sufficiently small ε. (We remark that, in [15], we studied the case of cotangent bundle of compact manifold. However the proof there can be applied in our situation also.)

Since $V(\hat{L}_i,\hat{L}_{i+1})$ is a quadratic function it follows that $U(p_i)$ is an affine subspace. Therefore if $\hat{L}_i$ are of general position then $U(p_1) \cap U(p_2) \cap U(p_3)$ consists of one point. (In case when $\eta(\tilde{L}_1,\tilde{L}_2,\tilde{L}_3) = 0$.)

Lemma 4.17 also follows from (4.29) and independence of index under continuous deformation of Fredholm operators. (We proved in [15] that the index of the linearized operators of right and left sides coincide also.)

We next find that the order counted with sign of $\mathcal{M}(\boldsymbol{C}^n; \hat{L}_1(\varepsilon), \hat{L}_2(\varepsilon), \hat{L}_3(\varepsilon); \hat{p}_1(\varepsilon), \hat{p}_2(\varepsilon), \hat{p}_3(\varepsilon))$ is independent of ε. This follows from a well established cobordism argument using Lemma 4.30 below. Theorem 4.18 is proved.

Lemma 4.30 $\bigcup_{t\in[\varepsilon,1]} \mathcal{M}(\boldsymbol{C}^n;\hat{L}_1(t),\hat{L}_2(t),\hat{L}_3(t);\hat{p}_1(t),\hat{p}_2(t),\hat{p}_3(t))$ *is compact for generic* $\tilde{L}_i$.

Proof: Suppose that $\varphi_i \in \mathcal{M}(\boldsymbol{C}^n;\hat{L}_1(t),\hat{L}_2(t),\hat{L}_3(t);\hat{p}_1(t),\hat{p}_2(t),\hat{p}_3(t))$ is a divergent sequence. Then there exists $w_i \in D^2$ such that $\varphi_i(w_i)$ diverges. Let R be a sufficiently large number determined later. Then for large i we have

(4.31) $$\#\left\{i \mid B(\varphi_i(w_i),R)\cap \hat{L}_i(t_i)\neq\emptyset\right\}\leq 1.$$

Here $B(\varphi_i(z_i),R)$ is the metric ball of radius R centered at $\varphi_i(w_i)$. Then, by the reflection principle, there exists a holomorphic map $\tilde{\varphi}_i : D^2 \to \boldsymbol{C}^n$ such that

(4.32.1) $$\tilde{\varphi}_i(\partial D^2)\subseteq \boldsymbol{C}^n - B(\varphi_i(w_i),R/2),$$
(4.32.2) $$\tilde{\varphi}_i(p)=\varphi_i(w_i),$$
(4.32.3) $$\int_{D^2}\tilde{\varphi}_i^*\omega < 2\int_{D^2}\varphi_i^*\omega .$$

Using (4.32.1) and (4.32.2) we have the following estimate :

$$\int_{D^2}\tilde{\varphi}_i^*\omega > CR^2 .$$

Hence (4.32.3) implies that

(4.33) $$\int_{D^2}\varphi_i^*\omega > CR^2 .$$

However by Stokes' theorem we have

(4.34) $$\int_{D^2}\varphi_i^*\omega = Q(\hat{p}_1(t_i),\hat{p}_2(t_i),\hat{p}_3(t_i);\omega).$$

We obtain a contradiction from (4.33) and (4.34) by choosing R sufficiently large. The proof of Lemma 4.30 is complete.

We finally determine the sign in Theorem 4.18. In fact, we need the following data to determine the orientation of the moduli space $\mathcal{M}(\boldsymbol{C}^n;\hat{L}_1,\hat{L}_2,\hat{L}_3;\hat{p}_1,\hat{p}_2,\hat{p}_3)$

(4.35.1) The orientation and the spin structure of L_i.
(4.35.2) The path joining $T_{p_i}L_i$ with $T_{p_i}L_{i+1}$ in the Lagrangian Grassmannian of $T_{p_i}M$. More precisely we need this date modulo two times H_1 of Lagrangian Grassmannian.

We refer [14] for the proof. We fixed orientation of our Lagrangian submanifolds. Since they are torus, we take their canonical spin structure (that is one corresponding to the trivialization of the tangent bundle), if we take an orientation of the torus itself. The data (4.35.2), in our case, is equivalent to fix an orientation of the unstable manifold $U(p_i)$ for each p_i.

Thus the orientation of $\mathcal{M}_0(\boldsymbol{C}^n;\hat{L}_1,\hat{L}_2,\hat{L}_3;\hat{p}_1,\hat{p}_2,\hat{p}_3)$ is determined by the choice of orientations of $\hat{L}_i$ and $U(p_i)$. We remark that, if we make this choice then the orientation of $U(p_1) \cap U(p_2) \cap U(p_3)$ is determined in an obvious way. Now by the proof in [15] we find (4.26) preserves orientation. Namely the order counted with sign of $\mathcal{M}_0(\boldsymbol{C}^n; \hat{L}_1, \hat{L}_2, \hat{L}_3; \hat{p}_1, \hat{p}_2, \hat{p}_3)$ is the intersection number $U(p_1) \bullet U(p_2) \bullet U(p_3)$.

§ 5 Associativity relation and A^∞ structure.

The "proof" of the following "theorem" is not rigorous because we do not know an estimate to justify the change of the order of the integral in the proof. (There is another problem to be clarified to make the "proof" rigorous. See Remark 5.5.) Later we will prove it rigorously in the case of Example 0.2. In this section, we consider only the case when $\partial = 0$ and $\pi_2(M,L)=0$, for simplicity.

"Theorem 5.1"

$$m_2(m_2(F\otimes G)\otimes H) = m_2(F\otimes m_2(G\otimes H)) \tag{5.2}$$

for any $\mathcal{F}_i,\tau_i$ $i=1,2,3,4$ *and* $F\in HF^p(\mathcal{F}_1,\mathcal{F}_2;\tau_1\otimes\tau_2)$, $G\in HF^q(\mathcal{F}_2,\mathcal{F}_3;\tau_2\otimes\tau_3)$, $H\in HF^r(\mathcal{F}_3,\mathcal{F}_4;\tau_3\otimes\tau_4)$ *with* $1/p+1/q+1/r\le 1$.

"Proof": Let L_i be a leaf of $\mathcal{F}_i$, $w\in L_4$, $e\in L_1\cap L_4$. Let x,y,z,a,b,c be as in Figure 3 and $f\in L_2\cap L_4$, $g\in L_3\cap L_4$. (See Figure 4.) We then find :

$$\begin{aligned}
&m_2\big(m_2(F\otimes G)\otimes H\big)(x,e,w)\\
&=\int_{c,z,g}\sum_{\psi\in\mathcal{M}_0(M;L_1,L_3,L_4;c,g,e)}\pm\exp\Big(-\int\psi^*\omega\Big)m_2(F\otimes G)(x,c,z)H(z,g,w)\,d\tau_3(c,g)\\
&=\int_{a,y,b,c,z,g}\sum_{\varphi\in\mathcal{M}_0(M;L_1,L_2,L_3;a,b,c)}\sum_{\psi\in\mathcal{M}_0(M;L_1,L_3,L_4;c,g,e)}\pm\exp\Big(-\int\varphi^*\omega\Big)\exp\Big(-\int\psi^*\omega\Big)\\
&\qquad F(x,a,y)G(y,b,z)H(z,g,w)\,d\tau_2(a,b)\,d\tau_3(c,g)
\end{aligned}\tag{5.3.1}$$

and

$$\begin{aligned}
&m_2\big(F\otimes m_2(G\otimes H)\big)(x,e,w)\\
&=\int_{a,y,f}\sum_{\varphi'\in\mathcal{M}_0(M;L_1,L_2,L_4;a,f,e)}\pm\exp\Big(-\int\varphi'^*\omega\Big)F(x,a,y)m_2(G\otimes H)(y,f,w)\,d\tau_2(a,f)\\
&=\int_{a,y,f,b,z,g}\sum_{\varphi'\in\mathcal{M}_0(M;L_1,L_2,L_4;a,f,e)}\sum_{\psi'\in\mathcal{M}_0(M;L_2,L_3,L_4;b,g,f)}\pm\exp\Big(-\int\varphi'^*\omega\Big)\exp\Big(-\int\psi'^*\omega\Big)\\
&\qquad F(x,a,y)G(y,b,z)H(z,g,w)\,d\tau_2(a,f)\,d\tau_3(b,g).
\end{aligned}\tag{5.3.2}$$

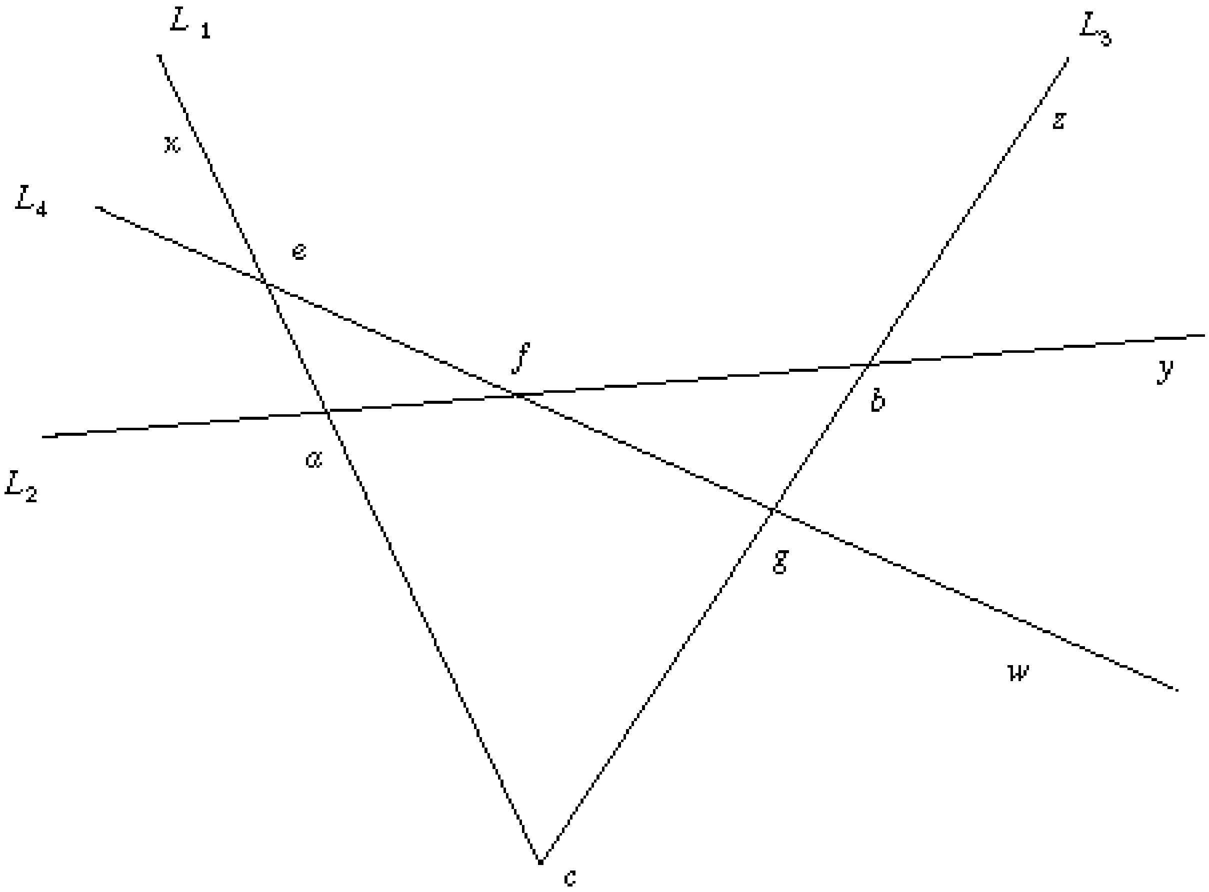


Figure 4

$d\tau_2(a,b)\, d\tau_3(c,g) = d\tau_2(a,f)\, d\tau_3(b,g)$ follows from holonomy invariance of the transversal measure. Therefore we are only to show the following :

"Lemma 5.4" *For generic* L_i *with* $\pi_2(M,L_i) = 0$ *and* a,b,c,e,f,g *we have*

$$\sum_{\varphi \in \mathcal{M}_0(M;L_1,L_2,L_3;a,b,c)} \sum_{\psi \in \mathcal{M}_0(M;L_1,L_3,L_4;c,g,e)} \pm \exp\left(-\int \varphi^*\omega\right) \exp\left(-\int \psi^*\omega\right)$$
$$= \sum_{\varphi' \in \mathcal{M}_0(M;L_1,L_2,L_4;a,f,e)} \sum_{\psi' \in \mathcal{M}_0(M;L_2,L_3,L_4;b,g,f)} \pm \exp\left(-\int \varphi'^*\omega\right) \exp\left(-\int \psi'^*\omega\right).$$

The idea of the "proof" of "Lemma 5.4" is in [10]. So we do not repeat it. The formula can be "proved" also in the case when $\pi_2(M,L_i) \neq 0$ if we add correction terms similar to [14].

Remark 5.5 For the reader who is familiar with the technique of pseudoholomorphic curve in symplectic geometry, we mention another reason we put " " to Lemma 5.4. The trouble is the transversality. The "proof" in [10] is based on the compactification of the moduli space of holomorphic rectangle which bounds $L_1 \cup L_2 \cup L_3 \cup L_4$. For fixed $L_1, \cdots, L_4$, it is possible to find an appropriate perturbation so that the moduli space of such pseudoholomorphic curves especially under our assumption $\pi_2(M, L_i) = 0$. However we are considering a family of such Lagrangian submanifolds. So we need to show that the equality in Lemma 5.3 holds for L_1, L_2, L_3, L_4 outside a measure 0 subset with respect to the transversal measure. This requires some additional arguments. We explain this point more later. See Conjecture 5.30 for example.

Theorem 5.6 (5.2) *holds for the foliations* $\mathcal{F}_{\tilde L_i}$ *in Example 0.2 such that* $\tilde L_i \cap \tilde L_j = \{0\}$.

Proof: Using Theorem 4.18 and (4.26) we can justify the calculation in the "proof" . So we only need to establish "Lemma 5.4" rigorously in our case. We prove it by using a series of lemmata and Theorem 4.18.

We first generalize Definition 4.14. Let $\tilde L_i$ be such that $J\tilde L_1 \cap \tilde L_i = \{0\}$, $\tilde L_i \cap \tilde L_j = \{0\}$. Then $\hat L_i$ is a graph of $V(\hat L_1, \hat L_i)$. We put

$$V(\hat L_i, \hat L_j) = V(\hat L_1, \hat L_j) - V(\hat L_1, \hat L_i)$$

and define

(5.7) $$\eta(\tilde L_1, \cdots, \tilde L_k) = n - \left(\eta(\tilde L_1, \tilde L_2) + \cdots + \eta(\tilde L_k, \tilde L_1)\right).$$

We also put

(5.8) $$\mathcal{MLG}(n,k) = \left\{(\hat L_1, \cdots, \hat L_k) \,\middle|\, \tilde L_i \cap \tilde L_j = \{0\}, \quad i \ne j\right\}.$$

Then η is extended continuously to $\mathcal{MLG}(n,k)$. It satisfies

(5.9.1) $$\eta(\tilde L_1, \cdots, \tilde L_k) = \eta(\tilde L_2, \cdots, \tilde L_k, \tilde L_1)$$

(5.9.2) $$\eta(\tilde L_1, \cdots, \tilde L_k) = \eta(\tilde L_1, \cdots, \tilde L_{\ell+1}) + \eta(\tilde L_\ell, \cdots, \tilde L_k).$$

Lemma 5.10 *Assume* $\eta(\tilde L_1, \cdots, \tilde L_4) = 0$. *Then the following four conditions are equivalent to each other.*

(5.11.1) $$\eta(\tilde L_1, \tilde L_2, \tilde L_3) = 0.$$

(5.11.2) $$\eta(\tilde L_1, \tilde L_3, \tilde L_4) = 0.$$

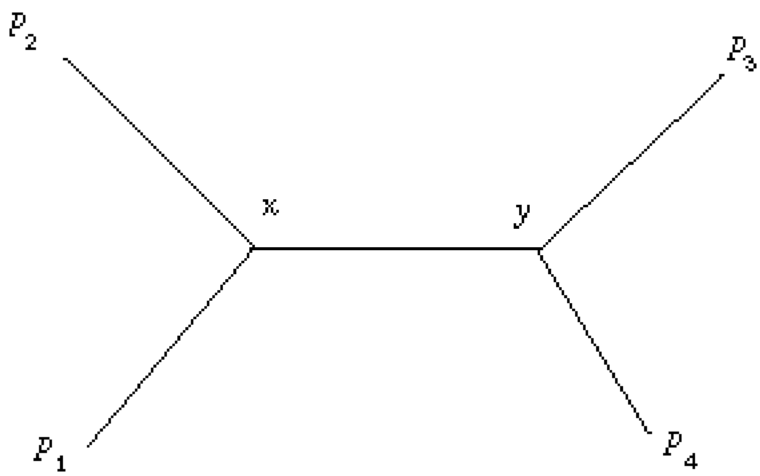
(5.11.3) $$\eta(L_2, L_3, L_4) = 0.$$

(5.11.4) $$\eta(L_1, L_2, L_4) = 0.$$

Proof: (5.9.1), (5.9.2) and the assumption imply that (5.11.1) $\Leftrightarrow$ (5.11.2) and (5.11.3) $\Leftrightarrow$ (5.11.4). Let $\hat L_i$ be a connected component of the inverse image of L_i in $\tilde T^{2n} = \boldsymbol{C}^n$. Let us identify $\boldsymbol{C}^n = T^* \hat L_1$ as in § 4. Then there exist quadratic functions $V(\hat L_1, \hat L_i)$ such that $\hat L_i$ is a graph of $dV(\hat L_1, \hat L_i)$. We put $f_i = V(\hat L_1, \hat L_i)$ and $f_1 = 0$. $\eta(L_i, L_j)$ is the number of negative eigenvalues of the quadratic function $f_j - f_i$. Thus Lemma 5.10 is an elementary assertion about quadratic forms. It is possible to give purely algebraic proof of it. But we prove it by using Morse homotopy. We use the notation in [13], [15]. Let p_i is the unique critical point of $f_{i+1} - f_i$. We consider the Morse moduli space $\mathcal{M}_{g_{\boldsymbol{R}^n}}\left(\boldsymbol{R}^n : (f_1, f_2, f_3, f_4),(p_1, p_2, p_3, p_4)\right)$ defined in [15] page 101. ($\mu(p_i)$ there is related to $\eta(L_i, L_{i+1})$ by $\mu(p_i) = n - \eta(L_i, L_{i+1})$.) Namely $\mathcal{M}_{g_{\boldsymbol{R}^n}}\left(\boldsymbol{R}^n : (f_1, f_2, f_3, f_4),(p_1, p_2, p_3, p_4)\right)$ is the union of the following three spaces.

$$\left\{(x, y, t) \mid x \in U(p_1) \cap U(p_2),\ y \in U(p_3) \cap U(p_4),\ t > 0,\ y = \exp\left(t\left(\mathrm{grad} f_3 - \mathrm{grad} f_1\right)\right)x\right\}$$
$$\left\{(u, v, t) \mid u \in U(p_1) \cap U(p_4),\ v \in U(p_2) \cap U(p_3),\ t > 0,\ v = \exp\left(t\left(\mathrm{grad} f_4 - \mathrm{grad} f_2\right)\right)u\right\}$$
$$U(p_1) \cap U(p_2) \cap U(p_3) \cap U(p_4).$$

(See Figure 5.) Here $U(p_i)$ is the unstable manifold of $\mathrm{grad}(f_{i+1} - f_i)$. The curvature x, y, u, v are gradient lines of $f_4 - f_2$ and $f_3 - f_1$, respectively.



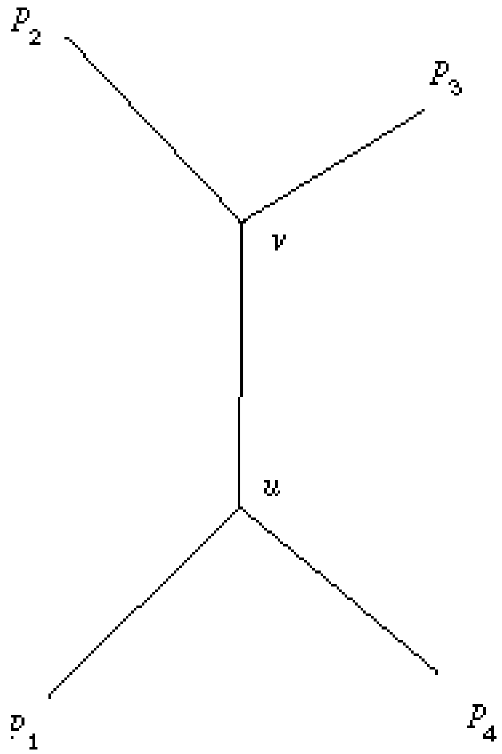


Figure 5

By [15] § 12, we can perturb f_i without changing it in a neighborhoods of $f_j - f_k$, so that $\mathcal{M}_{g_{\boldsymbol{R}^n}}\big(\boldsymbol{R}^n : (f_1, f_2, f_3, f_4), (p_1, p_2, p_3, p_4)\big)$ is a one dimensional manifold with boundary. (We use the assumption $\eta(\tilde{L}_1,\cdots,\tilde{L}_4) = 0$ here.) Its boundary is the union of

(5.12.1) $$\mathcal{M}_{g_{\boldsymbol{R}^n}}\big(\boldsymbol{R}^n : (f_1, f_2, f_3), (p_1, p_2, q)\big) \times \mathcal{M}_{g_{\boldsymbol{R}^n}}\big(\boldsymbol{R}^n : (f_1, f_3, f_4), (q, p_3, p_4)\big)$$

and

(5.12.2) $$\mathcal{M}_{g_{\boldsymbol{R}^n}}\big(\boldsymbol{R}^n : (f_1, f_2, f_4), (p_1, r, p_4)\big) \times \mathcal{M}_{g_{\boldsymbol{R}^n}}\big(\boldsymbol{R}^n : (f_2, f_3, f_4), (p_2, p_3, r)\big),$$

here q, r are unique critical point of $f_3 - f_1$ and $f_4 - f_2$ respectively. (See [10], [15].)

Sublemma 5.13 *$\mathcal{M}_{g_{\boldsymbol{R}^n}}\big(\boldsymbol{R}^n : (f_1, f_2, f_3, f_4), (p_1, p_2, p_3, p_4)\big)$ is compact.*

Before proving Sublemma 5.13, we complete the proof of Lemma 5.10. We assume (5.11.1) and (5.11.2). Then (5.12.1) consists of one point. Hence, by cobordism argument, (5.12.2) is nonempty. It then implies (5.11.3) and (5.114). (Otherwise one of the factors of (5.11.2) has negative dimension and is empty in the generic case.) The proof of Lemma 5.10 is complete.

Proof of Sublemma 5.13: We consider a divergent sequence in $\mathcal{M}_{g_{\boldsymbol{R}^n}}\big(\boldsymbol{R}^n : (f_1, f_2, f_3, f_4), (p_1, p_2, p_3, p_4)\big)$. Without loss of generality we may assume that we have $x_i \in U(p_1) \cap U(p_2)$, $y_i \in U(p_3) \cap U(p_4)$ and $t_i \geq 0$ such that $y_i = \exp\big(t_i(\mathrm{grad} f_3 - \mathrm{grad} f_1)\big)$. If there exists $s_i \in [0,1]$ such that

$$\lim_{i \to \infty} \exp\big(s_i t_i(\mathrm{grad} f_3 - \mathrm{grad} f_1)\big) = q$$

then the limit of such a sequence is in (5.12.2). Hence we may assume that

$$\|\mathrm{grad} f_3 - \mathrm{grad} f_1\|_{\exp(t_i(\mathrm{grad} f_3 - \mathrm{grad} f_1))} \geq c > 0. \tag{5.14}$$

Here $s \in [0,1]$ and c is independent of i. We have

$$\begin{gathered}
(f_2 - f_1)(x_i) - (f_2 - f_1)(p_1) \geq 0 \\
(f_3 - f_2)(x_i) - (f_3 - f_2)(p_2) \geq 0 \\
(f_4 - f_3)(y_i) - (f_4 - f_3)(p_3) \geq 0 \\
(f_1 - f_4)(y_i) - (f_1 - f_4)(p_4) \geq 0 \\
(f_3 - f_1)(y_i) \geq (f_3 - f_1)(x_i)
\end{gathered} \tag{5.15}$$

Hence

$$\begin{aligned}
(f_4 - f_1)(p_4) + (f_3 - f_4)(p_3) &\geq (f_3 - f_1)(y_i) \\
&\geq (f_3 - f_1)(x_i) \geq (f_2 - f_1)(p_1) + (f_3 - f_2)(p_2).
\end{aligned} \tag{5.16}$$

(5.14) and (5.16) imply that t_i is uniformly bounded. Moreover (5.15) and (5.16) imply that $(f_2 - f_1)(x_i) - (f_2 - f_1)(p_1)$ and $(f_4 - f_3)(y_i) - (f_4 - f_3)(p_3)$ are uniformly bounded. Therefore, since $x_i \in U(p_1) \cap U(p_2)$, $y_i \in U(p_3) \cap U(p_4)$, x_i and y_i are bounded.

This contradicts to the fact that (x_i, y_i, t_i) gives a divergent series of $\mathcal{M}_{g_{\boldsymbol{R}^n}}\big(\boldsymbol{R}^n : (f_1, f_2, f_3, f_4), (p_1, p_2, p_3, p_4)\big)$. The proof of Sublemma 5.13 is complete.

Lemma 5.17 *Let* a, b, c, e, f, g *be as in Figure* 4. *Then we have :*

$$Q(a,b,c;\omega) + Q(c,g,e;\omega) = Q(a,f,e;\omega) + Q(b,g,f;\omega).$$

Proof: We use Stokes' theorem and the fact that L_i are Lagrangian submanifolds to show

$$Q(a,b,c;\omega) = Q(f,c,a;\omega) + Q(f,g,c;\omega) + Q(f,b,g;\omega)$$

and

$$Q(c,g,e;\omega) = Q(a,c,f;\omega) + Q(f,c,g;\omega) + Q(a,f,e;\omega).$$

Since $Q(x,y,z;\omega) = Q(y,z,x;\omega) = -Q(x,z,y;\omega)$, we obtain Lemma 5.17.

Now we are in the position to prove Lemma 5.4 in our case. Let $\eta(\tilde{L}_1,\tilde{L}_2,\tilde{L}_3,\tilde{L}_4)=0$. (Otherwise both sides are zero.) Theorem 4.15 implies that

$$\sum_{\varphi\in\mathcal{M}_0(M;L_1,L_2,L_3;a,b,c)}\ \sum_{\psi\in\mathcal{M}_0(M;L_1,L_3,L_4;c,g,e)} \pm\exp\left(-\int\varphi^*\omega\right)\exp\left(-\int\psi^*\omega\right) \tag{5.18}$$

is

$$\exp\left(-Q(a,b,c;\omega)-Q(c,g,e;\omega)\right) \tag{5.19}$$

if (5.11.1) and (5.11.2) hold . (5.18) is zero otherwise. On the other hand

$$\sum_{\varphi'\in\mathcal{M}_0(M;L_1,L_2,L_4;a,f,e)}\ \sum_{\psi'\in\mathcal{M}_0(M;L_2,L_3,L_4;b,g,f)} \pm\exp\left(-\int\varphi'^*\omega\right)\exp\left(-\int\psi'^*\omega\right) \tag{5.20}$$

is

$$\exp\left(-Q(a,f,e;\omega)-Q(b,g,f;\omega)\right) \tag{5.21}$$

if (5.11.3) and (5.11.4) hold and is zero otherwise. Therefore Lemma 5.4 follows from Lemmata 5.10 and 5.17.

The proof of Lemma 5.4 in our case (and hence the proof of Theorem 5.6) is complete.

We remark that we can go around Theorem 4.18 by regarding Corollary 4.20 as a definition. (We need to prove Lemma 4.21 by another way. We can prove it also by Morse homotopy.) The proof of Theorem 5.6 above works also. If we take that way, we do not need to study holomorphic disks in order to define m_2. The story then becomes more elementary and easier to establish.

We next follow the way taken by [10] and “define” higher multiplication m_k :

$$\begin{aligned} m_k(\tau_2,\cdots,\tau_k): HF^{p_1}(\mathcal{F}_1,\mathcal{F}_2;\tau_1,\tau_2)\otimes\cdots\otimes \\ \otimes HF^{p_k}(\mathcal{F}_k,\mathcal{F}_{k+1};\tau_k,\tau_{k+1})\to HF^{p_{k+1}}(\mathcal{F}_1,\mathcal{F}_{k+1};\tau_1,\tau_{k+1}) \end{aligned} \tag{5.22}$$

where $\sum_{i=1}^{k} 1/p_i = 1/p_{k+1}$.

For simplicity we concentrate to our Example 0.2 such that $\tilde{L}_i\cap\tilde{L}_j=\{0\}$. Unfortunately, our discussion is not rigorous even in this case. (But if we assume furthermore that $\tilde{L}_i$ and $\tilde{L}_j$ are almost parallel, then our result is rigorous.)

Let $\hat{L}_i$ be affine Lagrangian submanifolds of $\boldsymbol{C}^n$ and $\{\hat{p}_{ij}\}=\hat{L}_i\cap\hat{L}_j$. We define :

$$\tilde{\mathcal{M}}(\hat{L}_1,\cdots,\hat{L}_{k+1}) = \left\{ (\varphi; z_1,\cdots,z_{k+1}) \left| \begin{array}{l} \varphi : D^2 \to \mathbf{C}^n \text{ is holomorhpic} \\ z_i \in \partial D^2, (z_1,\cdots,z_{k+1}) \text{ respects the cyclic order of } \partial D^2 \\ \varphi(z_i) = p_{i,i+1}, \varphi(\partial_i D^2) \subseteq \hat{L}_i \end{array} \right. \right\}.$$

Here $\partial_i D^2$ is a part of ∂D^2 between z_i and z_{i+1}. $PSL(2,\mathbf{R})$ act on it. Let $\mathcal{M}(\hat{L}_1,\cdots,\hat{L}_{k+1})$ be the quotient space.

"Lemma 5.23" *After appropriate perturbation,* $\mathcal{M}(\hat{L}_1,\cdots,\hat{L}_{k+1})$ *is a smooth manifold of dimension* $\eta(\tilde{L}_1,\cdots,\tilde{L}_{k+1}) + k - 2$.

Modulo transversality problem mentioned in Remark 5.5 we can prove "Lemma 5.23". The "proof" is in [10]. We remark that $k-2$ is the dimension of the moduli space of disks with $k+1$ marked points on the boundary.

"Definition 5.24" Let $\eta(\tilde{L}_1,\cdots,\tilde{L}_{k+1}) + k - 2 = 0$. We let $m(\hat{L}_1,\cdots,\hat{L}_{k+1})$ be the number of elements of $\mathcal{M}(\hat{L}_1,\cdots,\hat{L}_{k+1})$ counted with sign.

In fact the following Lemma 5.25 is necessary for the definition.

"Lemma 5.25" *If* $\eta(\tilde{L}_1,\cdots,\tilde{L}_{k+1}) + k - 2 = 0$ *then* $\mathcal{M}(\hat{L}_1,\cdots,\hat{L}_{k+1})$ *is compact for generic* $\mathcal{M}(\hat{L}_1,\cdots,\hat{L}_{k+1})$.

We explain the argument to "prove" it later. We potpone the discussion on the sign (orientation) until the end of this section.

"Definition 5.26" $m(\hat{L}_1,\cdots,\hat{L}_{k+1})$ is the order counted with sign of $\mathcal{M}(\hat{L}_1,\cdots,\hat{L}_{k+1})$.

"Definition 5.27" Let $F_i \in CF^{comp}_{k_i}(\mathcal{F}_{\tilde{L}_i},\mathcal{F}_{\tilde{L}_{i+1}})$, $x_i \in L_i$. Then we put

$$\begin{aligned} & m_k(\tau_2,\cdots,\tau_k)(F_1\otimes\cdots\otimes F_k)(x_1,p_{k+1,1},x_{k+1}) \\ (5.28)\qquad & = \int_{\hat{L}_2,\cdots,\hat{L}_k} \int_{x_2,\cdots,x_k} m(\hat{L}_1,\cdots,\hat{L}_{k+1}) \exp\left(-Q(\hat{L}_1,\cdots,\hat{L}_{k+1};\omega)\right) F_1(x_1,p_{1,2},x_2) \\ & \qquad \cdots F_k(x_k,p_{k,k+1},x_{k+1})\, d\tau_2(\hat{L}_2)\cdots d\tau_k(\hat{L}_k) \end{aligned}$$

Here we regards F_i, $m_k(\tau_2,\cdots,\tau_{k-1})(F_1\otimes\cdots\otimes F_k)$ as Γ invariant functions on $\mathbf{C}^n \times \mathbf{C}^n$. We take integration over all $\hat{L}_i$ parallel to $\tilde{L}_i$ using transversal measure, and we put $\{p_{i,j}\} = \hat{L}_i \cap \hat{L}_j$.

We put "" since we do not know the convergence of the right hand side. Also the transversality problem and the sign convention in "Definition 5.27" must be clarified to make "Definition 5.27" rigorous. We will clarify these points in some special cases later.

Definition 5.29 For n and k we consider the space

$$\mathcal{MLG}(n,k,\ell) = \left\{ (\tilde{L}_1,\cdots,\tilde{L}_{k+1}) \,\middle|\, \begin{array}{l} \tilde{L}_i \subseteq \boldsymbol{C}^n \text{ are Lagrangian linear subspace} \\ \tilde{L}_i \cap \tilde{L}_j = \{0\}, \text{for } i \neq j, \\ \eta(\tilde{L}_1,\cdots,\tilde{L}_{k+1}) + k - 2 + \ell = 0 \end{array} \right\}.$$

$$\Delta(n,k) = \pi_0\big(\mathcal{MLG}(n,k,0)\big)\ , \ \Delta(n,k,\ell) = \pi_0\big(\mathcal{MLG}(n,k,\ell)\big).$$

For $\Delta \in \Delta(n,k;\ell)$ let $\mathcal{MLG}(\Delta)$ be the connected component of $\mathcal{MLG}(n,k,\ell)$ corresponding to it. Let $(\tilde{L}_1,\cdots,\tilde{L}_{k+1}) \in \mathcal{MLG}(\Delta)$ and $1 \le i < j \le k+1$. The space $\boldsymbol{C}^n/\tilde{L}_1 \times \cdots \times \boldsymbol{C}^n/\tilde{L}_{i-1} \times \boldsymbol{C}^n/\tilde{L}_{i+1} \times \cdots \times \boldsymbol{C}^n/\tilde{L}_{j-1} \times \boldsymbol{C}^n/\tilde{L}_{j+1} \times \cdots \times \boldsymbol{C}^n/\tilde{L}_{k+1}$ is identified with the set of all configuration of $(\hat{L}_1,\cdots,\hat{L}_{k+1})$ modulo translation. Hence we can identify

$$\begin{aligned} &\boldsymbol{C}^n/\tilde{L}_1 \times \cdots \times \boldsymbol{C}^n/\tilde{L}_{i-1} \times \boldsymbol{C}^n/\tilde{L}_{i+1} \times \cdots \times \boldsymbol{C}^n/\tilde{L}_{j-1} \times \boldsymbol{C}^n/\tilde{L}_{j+1} \times \cdots \times \boldsymbol{C}^n/\tilde{L}_{k+1} \\ &\qquad \cong \boldsymbol{C}^n/\tilde{L}_1 \times \cdots \times \boldsymbol{C}^n/\tilde{L}_{i'-1} \times \boldsymbol{C}^n/\tilde{L}_{i'+1} \times \cdots \times \boldsymbol{C}^n/\tilde{L}_{j'-1} \times \boldsymbol{C}^n/\tilde{L}_{j'+1} \times \cdots \times \boldsymbol{C}^n/\tilde{L}_{k+1} \end{aligned}.$$

We denote this affine space by $\mathcal{V}(\tilde{L}_1,\cdots,\tilde{L}_{k+1})$ and put

$$\mathcal{V}(\Delta) = \bigcup_{(\tilde{L}_1,\cdots,\tilde{L}_{k+1}) \in \mathcal{MLG}(\Delta)} \mathcal{V}(\tilde{L}_1,\cdots,\tilde{L}_{k+1}).$$

Conjecture 5.30 For $\Delta \in \Delta(n,k)$, there exists a codimension one real analytic subset $W(\Delta)$ of $\mathcal{V}(\Delta)$ such that $m(\hat{L}_1,\cdots,\hat{L}_{k+1})$ is well-defined and locally constant on $\mathcal{V}(\Delta) - W(\Delta)$.

We can verify conjecture easily in the case when $n = 1$. In fact, if $n = 1$ $W(\Delta)$ is a finite union of codimension one affine subspaces. The fact that $m(\hat{L}_1,\cdots,\hat{L}_{k+1})$ can jump was observed in [23] in case $n = 1$.

Let us explain an argument to "prove" Conjecture 5.30 modulo transversality. We consider the moduli space

$$\mathcal{M}(\Delta) = \bigcup_{(\hat{L}_1,\cdots,\hat{L}_{k+1}) \in \mathcal{V}(\Delta)} \mathcal{M}(\hat{L}_1,\cdots,\hat{L}_{k+1}). \tag{5.31}$$

"Lemma 5.23" implies that for each $(\tilde{L}_1,\cdots,\tilde{L}_{k+1}) \in \Delta \in \Delta(n,k;\ell)$, the space $\mathcal{M}(\hat{L}_1,\cdots,\hat{L}_{k+1})$ is of $-\ell$ dimensional. We use its family version and find that

$$\operatorname{vir}\dim \mathcal{M}(\Delta) = \dim \mathcal{V}(\Delta) - \ell. \tag{5.32}$$

We remark that $\operatorname{vir}\dim$ in (5.32) is the virtual dimension, that is the index of the linearized operator. Now we consider the case when $\ell = 0$. Then if the transversality is satisfied the space $\mathcal{M}(\Delta)$ is a smooth manifold and

$$\dim \mathcal{M}(\Delta) = \dim \mathcal{V}(\Delta).$$

We consider the projection $\pi : \mathcal{M}(\Delta) \to \mathcal{V}(\Delta)$. Then $m(\hat{L}_1, \cdots, \hat{L}_{k+1})$ should be the order counted with sign of $\pi^{-1}(\hat{L}_1, \cdots, \hat{L}_{k+1})$. One would be able to prove easily that this number be independent of $(\hat{L}_1, \cdots, \hat{L}_{k+1})$ if $\pi : \mathcal{M}(\Delta) \to \mathcal{V}(\Delta)$ were proper. (Note that we used Lemma 4.30 to show that $m(\hat{L}_1, \hat{L}_2, \hat{L}_3)$ is independent of the deformation of $(\hat{L}_1, \hat{L}_2, \hat{L}_3)$ in § 4.) However in case when $k+1 \geq 4$ the map $\pi : \mathcal{M}(\Delta) \to \mathcal{V}(\Delta)$ is not proper. We have the following :

Lemma 5.33 *Let $\Delta \in \Delta(n,k)$ and $I \subseteq \mathcal{V}(\Delta)$ be a compact subset. Then we can compactify $\mathcal{M}(\Delta)(I) = \pi^{-1}(I)$ to $C\mathcal{M}(\Delta)(I)$ such that*

$$C\mathcal{M}(\Delta)(I) - \mathcal{M}(\Delta)(I) \subseteq \bigcup_{\substack{(\hat{L}_1, \cdots, \hat{L}_{k+1}) \in I \\ 2 \leq j-i \leq k-3}} \mathcal{M}(\hat{L}_1, \cdots \hat{L}_i, \hat{L}_j, , \hat{L}_{k+1}) \times \mathcal{M}(\hat{L}_i, \cdots, \hat{L}_j).$$

Lemma 5.33 is proved in a similar way as the proof of Lemma 4.30.

Now we assume that the dimension of I is one. Then by "Lemma 5.23"

$$\text{virdim} \bigcup_{(\hat{L}_1, \cdots, \hat{L}_{k+1}) \in I} \mathcal{M}(\hat{L}_1, \cdots \hat{L}_i, \hat{L}_j, \cdots, \hat{L}_{k+1}) = \eta(\tilde{L}_1, \cdots \tilde{L}_i, \tilde{L}_j, \cdots, \tilde{L}_{k+1}) + k - (j-i), \tag{5.34}$$

$$\text{virdim} \bigcup_{(\hat{L}_1, \cdots, \hat{L}_{k+1}) \in I} \mathcal{M}(\hat{L}_i, \cdots, \hat{L}_j) = \eta(\tilde{L}_i, \cdots, \tilde{L}_j) + (j-i) - 1. \tag{5.35}$$

We recall

$$\eta(\tilde{L}_1, \cdots, \tilde{L}_{k+1}) + k - 2 = 0. \tag{5.36}$$

Therefore, using (5.9), we find :

$$\text{virdim} \bigcup_{(\hat{L}_1, \cdots, \hat{L}_{k+1}) \in I} \mathcal{M}(\hat{L}_1, \cdots \hat{L}_i, \hat{L}_j, \cdots, \hat{L}_{k+1}) \times \mathcal{M}(\hat{L}_i, \cdots, \hat{L}_j) = 0. \tag{5.37}$$

Namely the boundary $C\mathcal{M}(\Delta)(I) - \mathcal{M}(\Delta)(I)$ consists of finitely many points if virtual dimension is equal to the actual dimension. In that case (5.37) is nonempty only if

$$\eta(\tilde{L}_i, \cdots, \tilde{L}_j) + (j-i) = 1 \text{ or } 2. \tag{5.38}$$

If we are allowed to apply various perturbation methods established in the theory of pseudoholomorphic curve then certainly the transversality is achieved. However in our situation it is not clear what kind of perturbation is allowed because the wall seems to move if we change the perturbation.

If the transversality holds then the wall will be described by the union of

$$(5.39.1)\qquad \left\{ (\tilde L_1,\cdots,\tilde L_{k+1}) \in \mathcal{MLG}(\Delta) \;\middle|\; \begin{array}{l} \eta(\tilde L_i,\cdots,\tilde L_j)+(j-i)=0, \\ \mathcal{M}(\hat L_i,\cdots,\hat L_j)\neq\emptyset \text{ for some } (\hat L_i,\cdots,\hat L_j) \\ \quad \text{where } \hat L_a \text{ is paralell to } \tilde L_a \end{array} \right\}$$

and

$$(5.39.2)\left\{ (\tilde L_1,\cdots,\tilde L_{k+1}) \in \mathcal{MLG}(\Delta) \;\middle|\; \begin{array}{l} \eta(\tilde L_1,\cdots \tilde L_i,\tilde L_j,\cdots,\tilde L_{k+1})+k-(j-i)=0, \\ \mathcal{M}(\hat L_1,\cdots \hat L_i,\hat L_j,\cdots,\hat L_{k+1})\neq\emptyset \text{ for some} (\hat L_1,\cdots \hat L_i,\hat L_j,\cdots,\hat L_{k+1}) \\ \quad \text{where } \hat L_a \text{ is paralell to } \tilde L_a \end{array} \right\}.$$

This “proves” Conjecture 5.30. We remark that “Lemma 5.25” is “proved” also by the same argument.

Remark 5.40 As we mentioned, our number $m(\hat L_1,\cdots,\hat L_{k+1})$ jumps at the walls $W(\Delta)$. There is a similar phenomenon in Gauge theory, that is Donaldson invariant in the case when $b_2^+=1$, and is called wall crossing formula [5]. We remark that certain remarkable relations between wall crossing formula to automorphic forms are discovered recently. ([16], [20]) .

At the end of this section, we will prove that Conjecture 5.30 holds in the subdomain of $\mathcal{V}(\Delta)$ where $\tilde L_i$ are almost parallel to each other.

We put

Definition 5.41 $Q(\hat L_1,\cdots,\hat L_{k+1};\omega)=\sum_{i=2}^{k-1} Q(p_{k+1,1},p_{i,i+1},p_{i+1,i+2};\omega)$.

In a way similar to Lemma 5.17, we can prove the following two lemmata :

Lemma 5.42 $Q(\hat L_1,\cdots,\hat L_{k+1};\omega)=Q(\hat L_2,\cdots,\hat L_{k+1},\hat L_1;\omega)$.

Lemma 5.43 $Q(\hat L_1,\cdots,\hat L_{k+1};\omega)=Q(\hat L_1,\cdots,\hat L_{\ell+1};\omega)+Q(\hat L_\ell,\cdots,\hat L_{k+1};\omega)$.

The following lemma is a generalization of Lemma 4.24. We remark that $Q(\hat L_1,\cdots,\hat L_{k+1};\omega)$ is regarded as a function on $\boldsymbol{C}^n/\tilde L_2\times\cdots\times\boldsymbol{C}^n/\tilde L_k$.

Lemma 5.44 *If* $(\tilde L_1,\cdots,\tilde L_k)\in\Delta\in\Delta(n,k)$ *and* $m(\hat L_1,\cdots,\hat L_{k+1})\neq 0$ *then* $Q(\hat L_1,\cdots,\hat L_{k+1};\omega)\geq 0$.

The proof is the same as the proof of Lemma 4.24. We remark that Lemma 5.44 itself is rigorous (if we replace $m(\hat L_1,\cdots,\hat L_{k+1})\neq 0$ by $\mathcal{M}(\hat L_1,\cdots,\hat L_{k+1})\neq\emptyset$.)

Let $\pi : \mathcal{V}(\Delta) \to \mathcal{MLG}(\Delta)$ be the projection. If we assume Conjecture 5.30 then we can construct our operator m_k rigorously on the dense subset of $\mathcal{MLG}(\Delta)$ with $(\tilde{L}_1, \cdots, \tilde{L}_k) \in \Delta \in \Delta(n,k)$. Namely we have :

Theorem 5.45 *Let $(\tilde{L}_1, \cdots, \tilde{L}_{k+1}) \in \Delta \in \Delta(n,k)$. Assume that Conjecture 5.30 holds in a neighborhood of $(\tilde{L}_1, \cdots, \tilde{L}_{k+1})$, and $\pi^{-1}(\tilde{L}_1, \cdots, \tilde{L}_{k+1})$ intersects transversely with the wall $W(\Delta)$. We assume also that the measure $\tau_i(\hat{L}_i)$ is either equivalent to the standard Euclidean measure or a delta measure whose support is disjoint from the wall. Then the integral* (5.28) *converges.*

Proof: We consider the vector space $\mathbf{C}^n / \tilde{L}_2 \times \cdots \times \mathbf{C}^n / \tilde{L}_k$ and regard $m(\hat{L}_1, \cdots, \hat{L}_{k+1})$ as a function on it. Then it is easy to see that $m(\hat{L}_1, \cdots, \hat{L}_{k+1})$ is invariant of $\mathbf{R}^+$ action $(\hat{L}_1, \hat{L}_2(v_2), \cdots, \hat{L}_k(v_k), \hat{L}_{k+1}) \mapsto (\hat{L}_1, \hat{L}_2(cv_2), \cdots, \hat{L}_k(cv_k), \hat{L}_{k+1})$. Therefore Conjecture 5.30 implies that m_k is uniformly bounded.

Let D be a connected component of the domain

$$\left\{ (v_2, \cdots, v_k) \,\middle|\, \sum |v|_i^2 = 1, \quad (\hat{L}_1, \hat{L}_2(v_2), \cdots, \hat{L}_k(v_k), \hat{L}_{k+1}) \notin W(\Delta) \right\}. \tag{5.46}$$

In variance of the wall of $\mathbf{R}^+$ action implies that there are only finitely many connected components of (5.46) if Conjecture 5.30 hold. By Lemma 5.44 and its proof we find that

$$Q((\hat{L}_1, \hat{L}_2(v_2), \cdots, \hat{L}_k(v_k), \hat{L}_{k+1}); \omega) > 0 \tag{5.47}$$

if $(v_2, \cdots, v_k)$ is in the closure of D. In fact, in the case when $(v_2, \cdots, v_k)$ is in the closure of D, there exists a union of holomorphic disks such that the sum of the symplectic area of them is $Q(\hat{L}_1, \cdots, \hat{L}_{k+1}; \omega)$. (Lemma 5.33). Since $\hat{L}_1 \cap \hat{L}_2(v_2) \cap \cdots \cap \hat{L}_k(v_k) \cap \hat{L}_{k+1} = \emptyset$, one of such holomorphic disks is necessary non constant. (5.47) follows.

(5.47) implies that there exists $\delta > 0$ such that

$$Q((\hat{L}_1, \hat{L}_2(v_2), \cdots, \hat{L}_k(v_k), \hat{L}_{k+1}); \omega) > \delta \sum |v_i|^2 \tag{5.48}$$

for each $(v_2, \cdots, v_k)$ with $m(\hat{L}_1, \hat{L}_2(v_2), \cdots, \hat{L}_k(v_k), \hat{L}_{k+1}) \neq 0$. It is then easy to prove the convergence of (5.28). The proof of Theorem 5.45 is now complete.

We remark that in the case when all the leaves of $\mathcal{F}_{\tilde{L}_i}$ are compact and τ_i are delta measure supported on a leaf L_i, the integral (5.28) will be a tensor product of matrix $m_k(L_1, \cdots, L_k)$ and the trivial map

$$L^{p_1}(L_1 \times L_2) \times \cdots \times L^{p_k}(L_k \times L_{k+1}) \to L^q(L_1 \times L_{k+1}).$$

Let us write the formula of matrix $m_k(L_1, \cdots, L_k)$ we obtain. Let $L_i \cap L_{i+1} = \left\{ p_{i,1}, \cdots, p_{i,N_i} \right\}$. Then

$$HF(L_i, L_{i+1}) = \bigoplus_j \boldsymbol{C}[p_{i,j}].$$

We fix $\hat{L}_1,\cdots,\hat{L}_k$. So that $\pi\left(\hat{L}_{k+1} \cap \hat{L}_1\right) = p_{k+1,j_{k+1}}$ where $\pi : \boldsymbol{C}^n \to \boldsymbol{C}^n/\Gamma$. We define

$$\pi : \Gamma^{k-1} \to \prod \{1,\cdots,N_i\}$$

by

$$\mu(\gamma)_i = j \ \text{ if } \ \pi\left(\gamma_i(\hat{L}_i) \cap \gamma_{i+1}(\hat{L}_{i+1})\right) = p_{i,j} \ \text{ where } \ \pi : \boldsymbol{C}^n \to \boldsymbol{C}^n/\Gamma.$$

Now $m_k(L_1,\cdots,L_k) : \boldsymbol{C}^{N_1} \otimes \cdots \otimes \boldsymbol{C}^{N_k} \to \boldsymbol{C}^{N_{k+1}}$ is the higher multiplication of Floer homology whose coefficient is are described by :

$$\tag{5.49} \begin{aligned} &Z_k(L_1,\cdots,L_k)_{j_1,\cdots,j_{k+1}} \\ &= \sum_{\substack{\gamma = (\gamma_i) \in \Gamma^{k-1} \\ \mu(\gamma) = (j_1,\cdots,j_{k+1})}} m(\hat{L}_1,\gamma_2(\hat{L}_2)\cdots,\gamma_k(\hat{L}_k),\hat{L}_{k+1}) \exp\left(-Q(\hat{L}_1,\gamma_2(\hat{L}_2)\cdots\gamma_k(\hat{L}_k),\hat{L}_{k+1};\omega)\right). \end{aligned}$$

Moving L_i we may regards (5.49) as a function. If we include imaginary part in the same way as [18], [24] then we obtain a holomophic function. (See Part II.) However as was observed in [23] in case $n = 1$, this function is discontinuous at the point where $(\hat{L}_1,\gamma_2(\hat{L}_2)\cdots,\gamma_k(\hat{L}_k),\hat{L}_{k+1})$ meets the wall $W(\Delta)$.

We go back to the case when the foliations $\mathcal{F}_{\tilde{L}_i}$ can be ergodic and will discuss the properties of m_k. The following is an analogy of Theorems 4.6 and 4.8.

"Theorem 5.50" *Let* $F_i \in HF(\mathcal{F}_{L_i}, \mathcal{F}_{L_{i+1}})$, $f_i \in C(T^{2n}, \mathcal{F}_{\tilde{L}_i})$. *Then we have*

$$\tag{5.51.1} m_k(\tau_2,\cdots,\tau_k)\left(\left(f_1 * F_1\right) \otimes \cdots \otimes F_k\right) = f_1 * m_k(\tau_2,\cdots,\tau_k)\left(F_1 \otimes \cdots \otimes F_k\right),$$

$$\tag{5.51.2} \begin{aligned} &m_k(\tau_2,\cdots,\tau_k)\left(F_1 \otimes \cdots \otimes F_i \otimes \left(f_{i+1} * F_{i+1}\right) \otimes \cdots \otimes F_{k-1}\right) \\ &\qquad = m_k(\tau_2,\cdots,\tau_{k-1})\left(F_1 \otimes \cdots \otimes \left(F_i * f_{i+1}\right) \otimes F_{i+1} \otimes \cdots \otimes F_{k-1}\right), \end{aligned}$$

$$\tag{5.51.3} m_k(\tau_2,\cdots,\tau_k)\left(F_1 \otimes \cdots \otimes \left(F_k * f_{k+1}\right)\right) = m_k(\tau_2,\cdots,\tau_k)\left(F_1 \otimes \cdots \otimes F_k\right) * f_{k+1}$$

$$\tag{5.51.4} \begin{aligned} &\left\langle m_k(\tau_2,\cdots,\tau_k)\left(F_1 \otimes \cdots \otimes F_k\right), F_{k+1}\right\rangle_{\tau_k \otimes \tau_1} \\ &\qquad = \pm\left\langle m_k(\tau_3,\cdots,\tau_{k+1})\left(F_2 \otimes \cdots \otimes F_{k+1}\right), F_1\right\rangle_{\tau_1 \otimes \tau_2}. \end{aligned}$$

We put this theorem in the quote since we assume Conjecture 5.30 to "prove" it. The sign is also to be clarified to make it rigorous.

As we pointed out in [15], Formula (5.51.4) seems to be closely related to the theory of cyclic homology (see [3]). It may therefore suggests a relation of this paper to [32].

The "proof" of "Theorem 5.50" is similar to Theorems 4.6 and 4.8 and is omitted.

We next state the higher associativity relation.

"Theorem 5.52" *If* $(\tilde{L}_1,\cdots,\tilde{L}_{k+1}) \in \Delta \in \Delta(m,k,1)$ *we have*

(5.53)
$$\sum_{n,\ell} \pm m_{k-\ell+1}(F_1 \otimes \cdots \otimes F_n \otimes m_\ell(F_{n+1},\cdots F_{n+\ell}) \otimes F_{n+\ell+1} \otimes \cdots \otimes F_k) = 0 .$$

"Theorem 5.52" will follow from Lemma 5.43 and the following "Lemma 5.54" , in a way similar to the "proof" of "Theorem 5.1".

"Lemma 5.54" *If* $(\tilde{L}_1,\cdots,\tilde{L}_{k+1}) \in \Delta \in \Delta(n,k,1)$ *then*

(5.55)
$$\sum_{n,\ell} \pm m_{k-\ell+1}(\hat{L}_1,\cdots,\hat{L}_n, m_\ell(\hat{L}_{n+1},\cdots \hat{L}_{n+\ell}), \hat{L}_{n+\ell+1},\cdots \hat{L}_{k+1}) = 0 .$$

The idea of the proof of "Lemma 5.54" is in [10]. Namely we consider $\mathcal{M}(\hat{L}_1,\cdots,\hat{L}_{k+1})$. It is a one dimensional manifold. Its boundary gives the right hand side of (5.55).

In the case when $\hat{L}_i$ are almost parallel to each other, we can reduce the calculation of $m(\hat{L}_1,\cdots,\hat{L}_{k+1})$ to a problem on quadratic Morse function as follows.

We regards $T^*\hat{L}_1 = \tilde{T}^{2n}$ and let $\hat{L}_i$ be the graph of the exact form df_i on $\hat{L}_1$. Here f_i is a quadratic function on $\hat{L}_1$. We put $0 = f_1$. Let $L_i(\varepsilon)$ be the graph of εdf_i. Let $\hat{L}_i(\varepsilon) \cap \hat{L}_{i+1}(\varepsilon) = \{\hat{p}_i(\varepsilon)\}$ and $\pi(\hat{p}_i(\varepsilon)) = p_i$. Then we proved in [15] the following equality in the case when $\eta(\tilde{L}_1,\cdots,\tilde{L}_{k+1}) + k - 2 = 0$.

(5.56)
$$\mathcal{M}(\hat{L}_1(\varepsilon),\cdots,\hat{L}_{k+1}(\varepsilon)) = \mathcal{M}_{g_{\boldsymbol{R}^n}}\left(\boldsymbol{R}^n : (f_1,\cdots,f_{k+1}),(p_1,\cdots,p_{k+1})\right).$$

Here the right hand side is the Morse moduli space defined in [15] and is similar to $\mathcal{M}_{g_{\boldsymbol{R}^n}}\left(\boldsymbol{R}^n : (f_1,f_2,f_3,f_4),(p_1,p_2,p_3,p_4)\right)$ which we explained during the proof of Lemma 5.7.

We remark that (5.56) is not enough to calculate the number $m(\hat{L}_1,\cdots,\hat{L}_{k+1})$ in general since we do not have an analogy of Lemma 4.30 and hence the order of $\mathcal{M}(\hat{L}_1(\varepsilon),\cdots,\hat{L}_{k+1}(\varepsilon))$ may depend on ε.

Now we are in the position to clarify two points we postponed in the case when $\hat{L}_i$ are almost parallel to each other. One is the orientation of the moduli space $\tilde{\mathcal{M}}(\hat{L}_1,\hat{L}_2(\varepsilon),\cdots,\hat{L}_k(\varepsilon),\hat{L}_{k+1})$ and the other is the proof of Conjecture 5.30.

To prove Conjecture 5.30 in the case when $\hat{L}_i$ are almost parallel to each other, we only need to show the same statement for the Morse moduli space $\mathcal{M}_{g_{\boldsymbol{R}^n}}\left(\boldsymbol{R}^n : (f_1,\cdots,f_{k+1}),(p_1,\cdots,p_{k+1})\right)$. But this is almost obvious in the case of Morse homotopy of quadratic functions. Instead of giving the detail of the proof, we will describe the "Morse homotopy limit" of the wall $W(\Delta)$ in the case when $k=4$ later. (Proposition 5.59).

Next we consider the orientation. If we find an orientation on Morse moduli space $\mathcal{M}_{g_{\boldsymbol{R}^n}}\left(\boldsymbol{R}^n : (f_1,\cdots,f_{k+1}),(p_1,\cdots,p_{k+1})\right)$, then we can use the number of $\mathcal{M}_{g_{\boldsymbol{R}^n}}\left(\boldsymbol{R}^n : (f_1,\cdots,f_{k+1}),(p_1,\cdots,p_{k+1})\right)$ counted with sign to define $m(\hat{L}_1(\varepsilon),\cdots,\hat{L}_{k+1}(\varepsilon))$. This is justified by (5.56). To define an orientation on Morse moduli space, we use the map EXP defined in [15] p 160. We recall that we fixed orientation of the unstable manifolds $U(p_i)$ and orientation of $\tilde{L}_1$ (end of § 4.) The map EXP in our situation is

$$EXP(\mathfrak{t}) : \tilde{L}_1 \times Gr(\mathfrak{t}) \times \prod_{h=1}^{m} \prod_{j=i_h}^{i_{h+1}-1} U(p_i) \to \prod_{h=1}^{m} \left(\tilde{L}_1 \times \prod_{j=i_h}^{i_{h+1}-1} \tilde{L}_1 \right). \tag{5.57}$$

Here $Gr(\mathfrak{t})$ is the moduli space of metric Ribbon tree introduced in [15]. See [15] for other notations.

A dense subset of $\mathcal{M}_{g_{\boldsymbol{R}^n}}\left(\boldsymbol{R}^n : (f_1,\cdots,f_{k+1}),(p_1,\cdots,p_{k+1})\right)$ is the union of

$$EXP(\mathfrak{t})^{-1}(\text{Diagonal}), \tag{5.58}$$

where $\mathfrak{t}$ runs over trivalent graphs. (See [15] p 160 the definition of Diagonal.)

We proved in [15] § 14 that $Gr(\mathfrak{t})$ is diffeomorphic to an open subset of the moduli space of $[z_1,\cdots,z_{k+1}]$ where $z_i \in \partial D^2$ and z_i respects cyclic order. We identify $[z_1,\cdots,z_{k+1}]$ with $[z'_1,\cdots,z'_{k+1}]$ if there exits $\varphi \in PSL(2,\boldsymbol{R})$ such that $z'_i = \varphi(z_i)$.

Using this diffeomorphism we find an orientation on $Gr(\mathfrak{t})$. The spaces in (5.57) then are all oriented. Therefore (5.58) is oriented. Thus we obtain an orientation of $\mathcal{M}_{g_{\boldsymbol{R}^n}}\left(\boldsymbol{R}^n : (f_1,\cdots,f_{k+1}),(p_1,\cdots,p_{k+1})\right)$.

Finally, we show how the Morse moduli space $\mathcal{M}_{g_{\boldsymbol{R}^n}}\left(\boldsymbol{R}^n : (f_1,\cdots,f_{k+1}),(p_1,\cdots,p_{k+1})\right)$ jumps in the case when $k=3$. Let us consider the following figure where $k=3$, $n=2$.

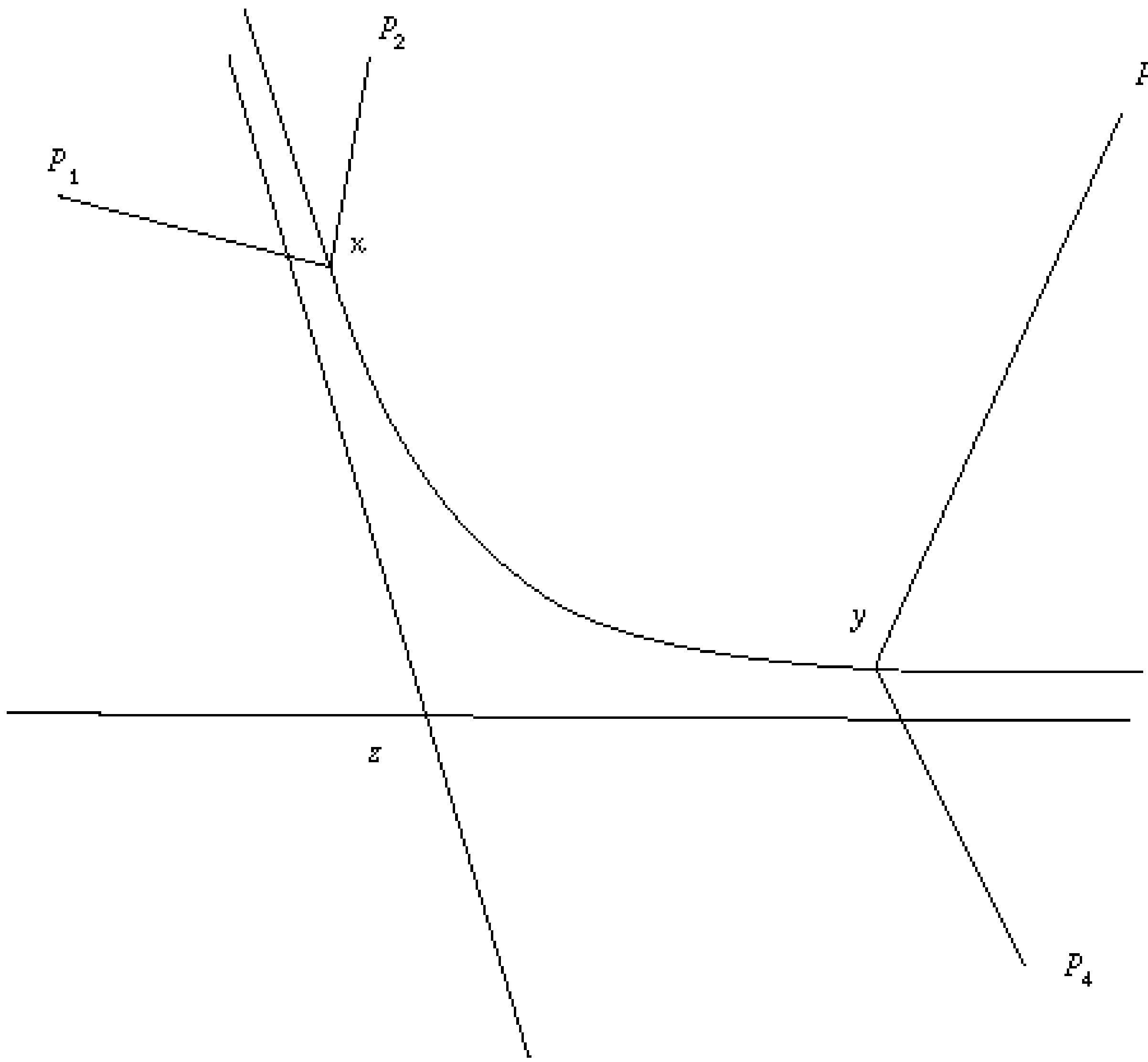


Figure 6

Here $\eta(p_1)=\eta(p_2)=\eta(p_3)=1$ and $\eta(p_4)=0$. The lines p_1x p_2x p_3y are unstable manifolds $U(p_1)$, $U(p_2)$, $U(p_3)$. $(U(p_4)=\boldsymbol{R}^2.)$ The curve passing x,y is the gradient line of f_3-f_1. The figure shows that $\mathcal{M}_{g_{\boldsymbol{R}^n}}\left(\boldsymbol{R}^n:(f_1,\cdots,f_4),(p_1,\cdots,p_4)\right)$ contains one in this case.

If we move p_2 to the left, then x also moves to the left. Let z be the critical point of f_3-f_1. (We assume that the index of f_3-f_1 is 1.) Then, at some moment x will meet the stable manifold of z. After that the gradient line of f_3-f_1 containing x will not meet the unstable manifold p_3y. Namely the moduli space $\mathcal{M}_{g_{\boldsymbol{R}^n}}\left(\boldsymbol{R}^2:(f_1,\cdots,f_4),(p_1,\cdots,p_4)\right)$ jumps.

We recall that all stable and unstable manifolds are affine in the case of quadratic Morse function. On the other hand, the jump of the moduli space $\mathcal{M}_{g_{\boldsymbol{R}^n}}\left(\boldsymbol{R}^2:(f_1,\cdots,f_4),(p_1,\cdots,p_4)\right)$ occurs when

$$\mathcal{M}_{g_{\boldsymbol{R}^n}}\left(\boldsymbol{R}^2:(f_1,f_2,f_3),(p_1,p_2,z)\right)\times\mathcal{M}_{g_{\boldsymbol{R}^n}}\left(\boldsymbol{R}^2:(f_3,f_4,f_1),(p_3,p_4,z)\right)$$

or

$$\mathcal{M}_{g_{\mathbf{R}^n}}\left(\mathbf{R}^2 : (f_4, f_1, f_2), (p_4, p_1, w)\right) \times \mathcal{M}_{g_{\mathbf{R}^n}}\left(\mathbf{R}^2 : (f_2, f_3, f_4), (p_2, p_3, w)\right)$$

becomes nonempty. Here z is the critical point of $f_3 - f_1$ and w is a critical point of $f_4 - f_2$. Therefore the following proposition holds.

Proposition 5.59 *If $k = 3$ (and any n) the Morse homotopy analogue of the Wall (in Conjecture 5.30) is a union of codimension one affine subspaces in* $\mathbf{R}^{2n} = \mathbf{C}^n / \tilde{L}_2 \times \mathbf{C}^n / \tilde{L}_3$.

The author has no idea how to describe the wall in the general case.

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