Open-Closed Gromov-Witten theory
and
Floer homology

Chern 100th birthday conference
MSRI

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Does (Genus 0 one boundary component)
Open Gromov-Witten theory
determine
Closed Gromov-Witten theory (genus zero) ?

(X,ω) : symplectic manifold.

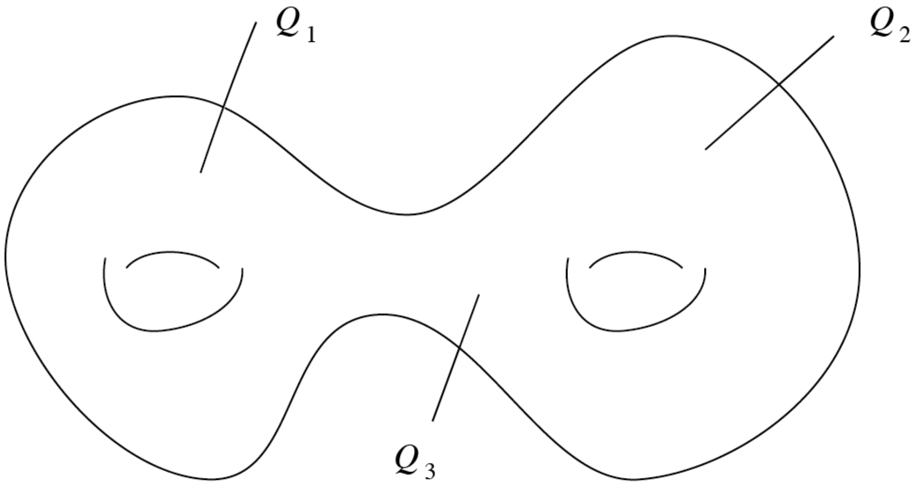
Take compatible almost complex structure J

Close (topological) string

= Gromov-Witten invariant

=Count the number of

$\varphi:\Sigma \to X;\quad \Sigma$: Riemann surface, $\quad \varphi$: holomorphic



Q_i : cycles on X

In genus zero it defines $Comm_\infty$ structure.

$$\mathcal{GW}_\ell \quad : \; H(X;\Lambda_0)^\ell \to H(X;\Lambda_0)$$

$$\Lambda_0 = \left\{ \sum a_i T^{\lambda_i} \,\middle|\, a_i \in \mathbb{C}, \quad \lambda_i \in \mathbb{R}_{\ge 0}, \quad \lambda_i \uparrow +\infty \right\}$$

$$\sum_{I+J=\{4,\ldots,\ell\}} \mathcal{GW}_{|J|+1}(\mathcal{GW}_{|I|+1}(y_1, y_2, y_I), y_3, y_J)$$

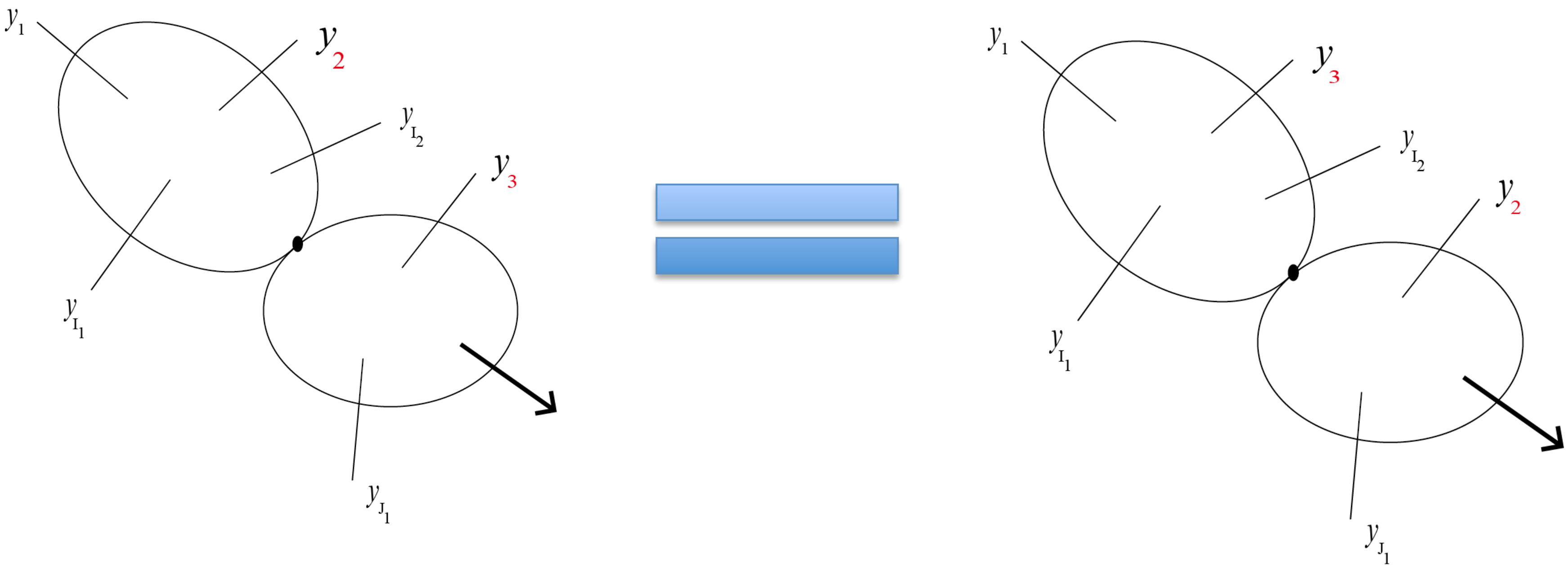
$$= \sum_{I+J=\{4,\ldots,\ell\}} \mathcal{GW}_{|J|+1}(\mathcal{GW}_{|I|+1}(y_1, y_3, y_I), y_2, y_J)$$

Special case: Associativity

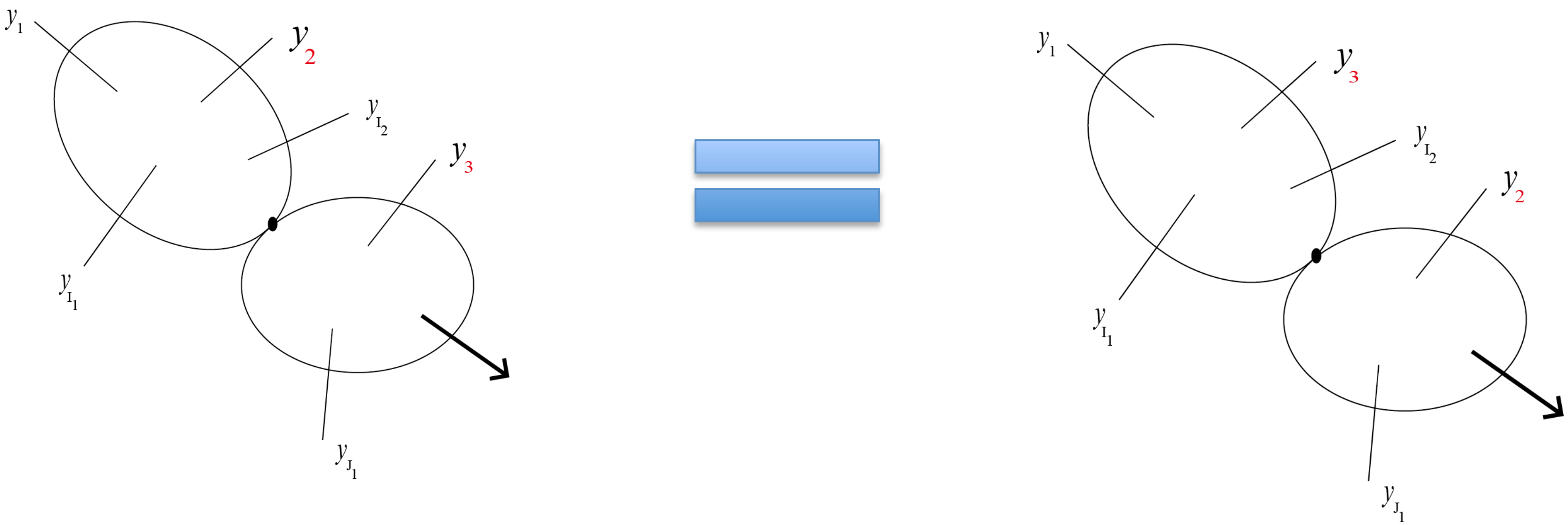
$$\mathcal{GW}_2(\mathcal{GW}_2(y_2, y_1), y_3) = \mathcal{GW}_2(y_2, \mathcal{GW}_2(y_1, y_3))$$

$$\sum_{I+J=\{4,\ldots,\ell\}} \mathcal{GW}_{|J|+1}(\mathcal{GW}_{|I|+1}(y_1, y_2, y_I), y_3, y_J)$$

$$= \sum_{I+J=\{4,\ldots,\ell\}} \mathcal{GW}_{|J|+1}(\mathcal{GW}_{|I|+1}(y_1, y_3, y_I), y_2, y_J)$$



$Comm_\infty$ relation

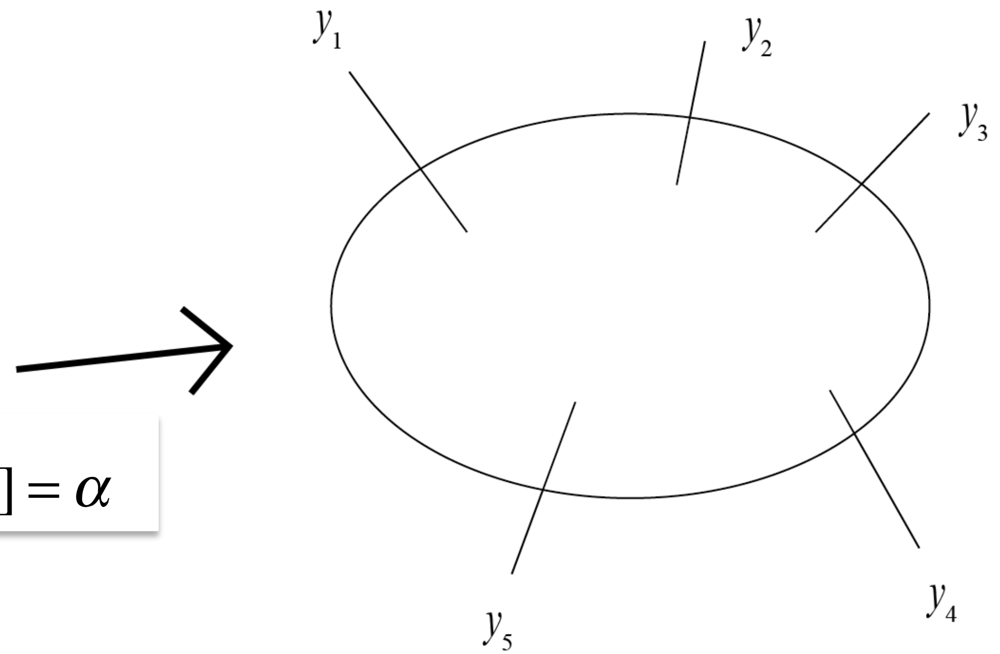


$$\sum_{I+J=\{4,\ldots,\ell\}} \mathcal{GW}_{|J|+1}(\mathcal{GW}_{|I|+1}(y_1,y_2,y_I),y_3,y_J)$$

$$= \sum_{I+J=\{4,\ldots,\ell\}} \mathcal{GW}_{|J|+1}(\mathcal{GW}_{|I|+1}(y_1,y_3,y_I),y_2,y_J)$$

$$
\left\langle \mathcal{GW}_\ell(y_1,\ldots,y_\ell),y_0\right\rangle_{\mathrm{PD}_X}
= \sum_{\alpha\in\pi_2(X)} T^{\alpha\cap\omega}\,\#\left\{(\varphi;z_1^+,\ldots,z_\ell^+,z_0^+)\;\middle|\;\begin{array}{l}\varphi : S^2 \to X \quad \text{holomorphic}\\ [\varphi]=\alpha\\ z_i^+\in S^2,\quad \varphi(z_i^+)\in Q_i\end{array}\right\}
$$

$$[Q_i]=y_i$$



$$[\varphi]=\alpha$$

Symmetry : (Frobenius algebra etc.)

$$\left\langle \mathcal{GW}_\ell(y_1,\dots,y_\ell), y_0 \right\rangle_{\mathrm{PD}_X}$$
$$= \pm \left\langle \mathcal{GW}_\ell(y_{\sigma(1)},\dots,y_{\sigma(\ell)}), y_{\sigma(0)} \right\rangle_{\mathrm{PD}_X}$$

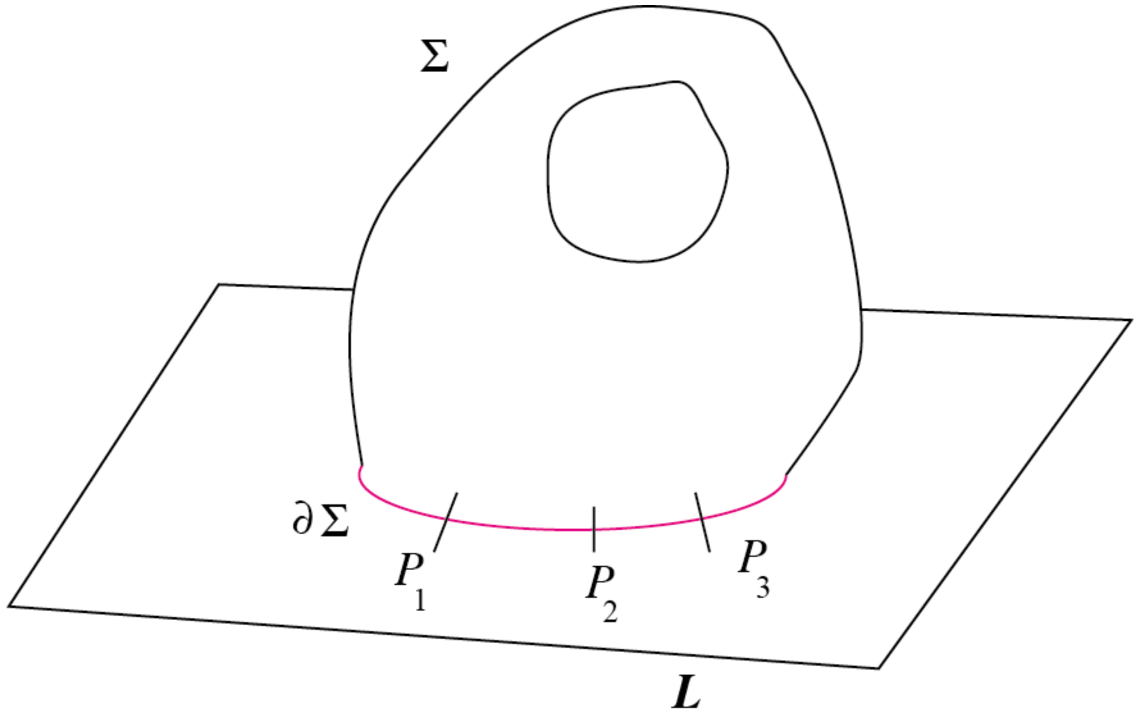
(X,ω) : symplectic manifold.

Take compatible almost complex structure J

$L \subset X$: Lagrangian submanifold

Open (topological) string

= Count of $\varphi : (\Sigma, \partial\Sigma) \to (X, L)$ φ holomorphic



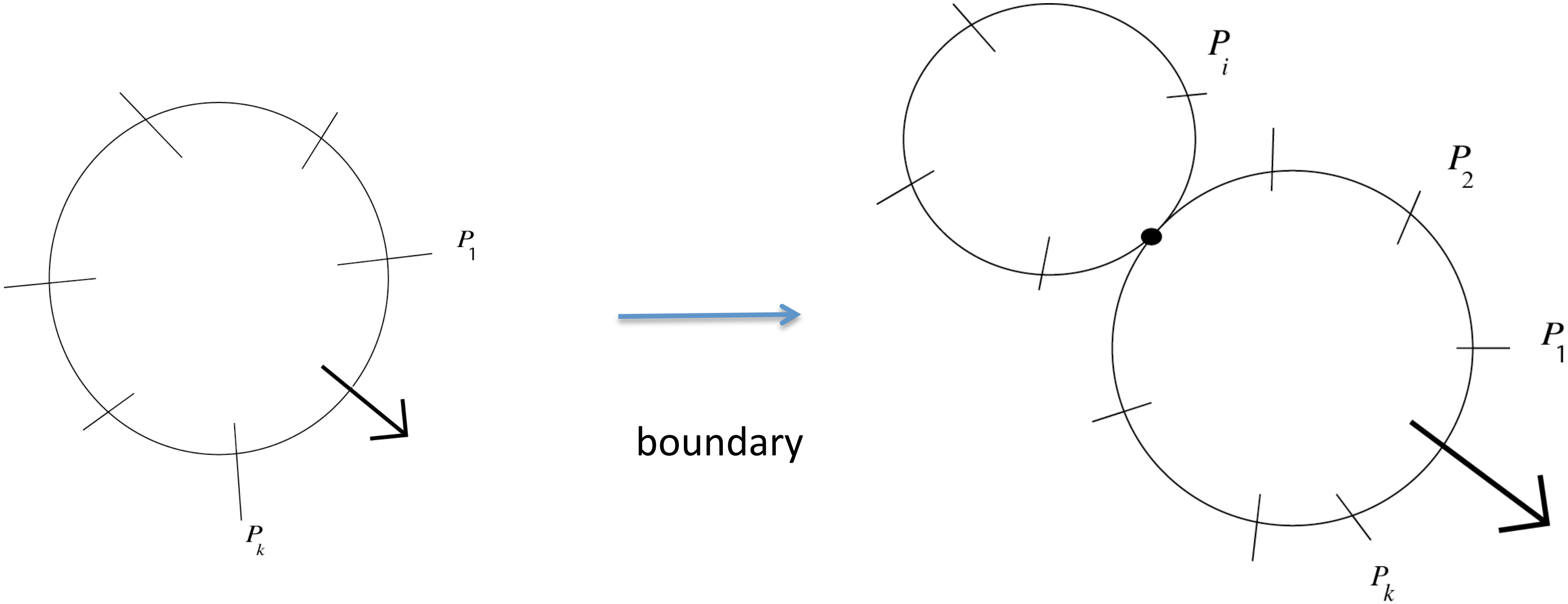
In case $\Sigma = D^2$ it defines A_∞ structure.

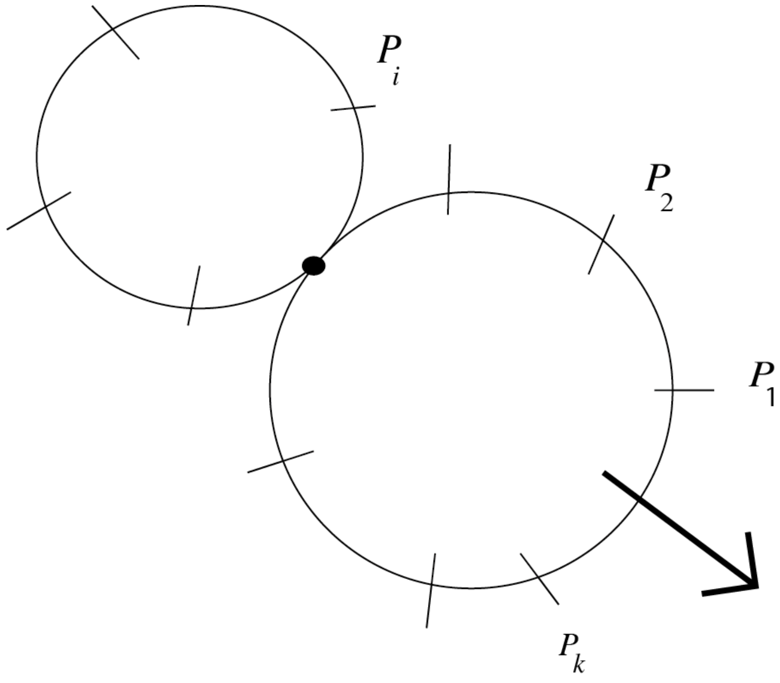
$$m_k : H(L;\Lambda_0)^k \to H(L;\Lambda_0)$$

$$k = 0,1,2,\ldots$$

A_∞ relation

$$0 = \sum_{k_1+k_2=k+1} \sum_{i=1}^{k_1} \pm m_{k_1}(x_1,\ldots,m_{k_2}(x_i,\ldots,x_{i+k_2-1}),\ldots,x_k)$$

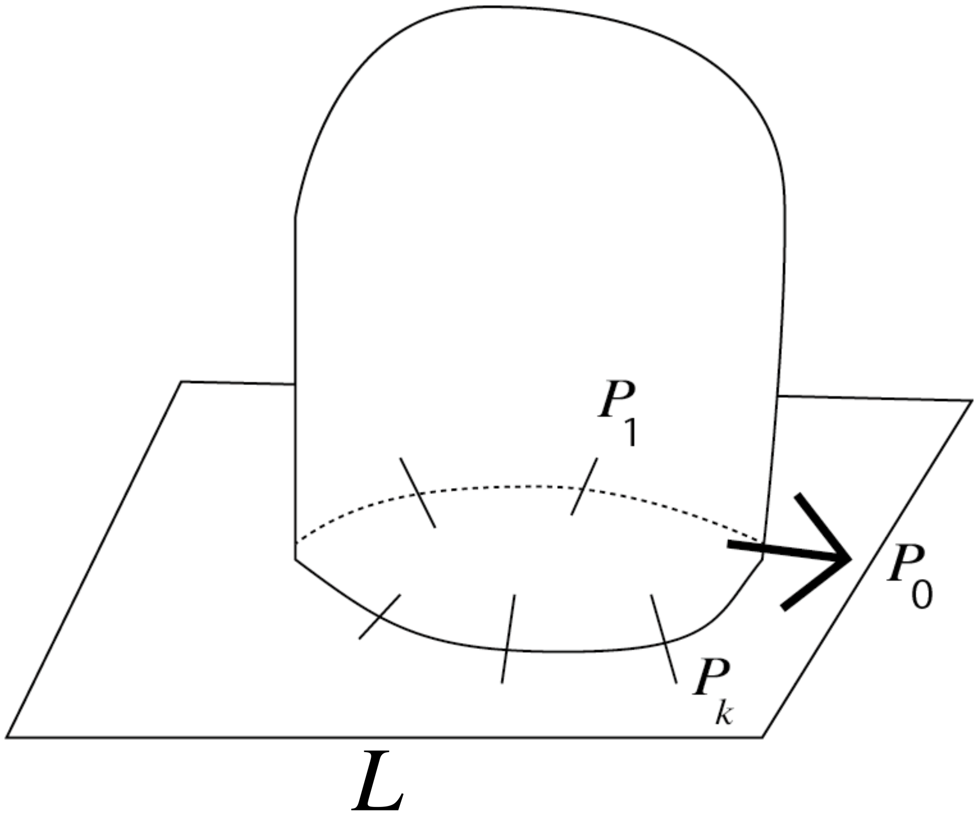

P_1
P_k
boundary
P_i
P_2
P_1
P_k



$$0 = \sum_{k_1+k_2=k+1} \sum_{i=1}^{k_1} \pm m_{k_1}(x_1,\ldots,m_{k_2}(x_i,\ldots,x_{i+k_2-1}),\ldots,x_k)$$

$$\langle m_k(x_1,\ldots,x_k),x_0\rangle$$

$$= \sum_{\beta\in\pi_2(X,L)} T^{\beta\cap\omega}\,\#\left\{(\varphi;z_1,\ldots,z_k,z_0)\;\middle|\;\begin{array}{l}\varphi:(D^2,\partial D^2)\to(X,L)\quad\text{holomorphic}\\ [\varphi]=\beta\\ z_i\in\partial D^2,\quad\text{respect cyclic order}\\ \varphi(z_i)\in P_i\end{array}\right\}$$



$$[P_i]=x_i$$

Cyclic symmetry

$$\langle m_k(x_1,\ldots,x_k),x_0\rangle = \pm\langle m_k(x_0,x_1,\ldots,x_{k-1}),x_k\rangle$$

$$L$$

$$\downarrow$$

Cyclic (= Calabi Yau) A_∞ algebra on $H(L;\Lambda_0)$

(Fukaya-Oh-Ohta-Ono,
hereafter abbreviated to FOOO)

Can be generalized by including several Lagrangian submanifolds

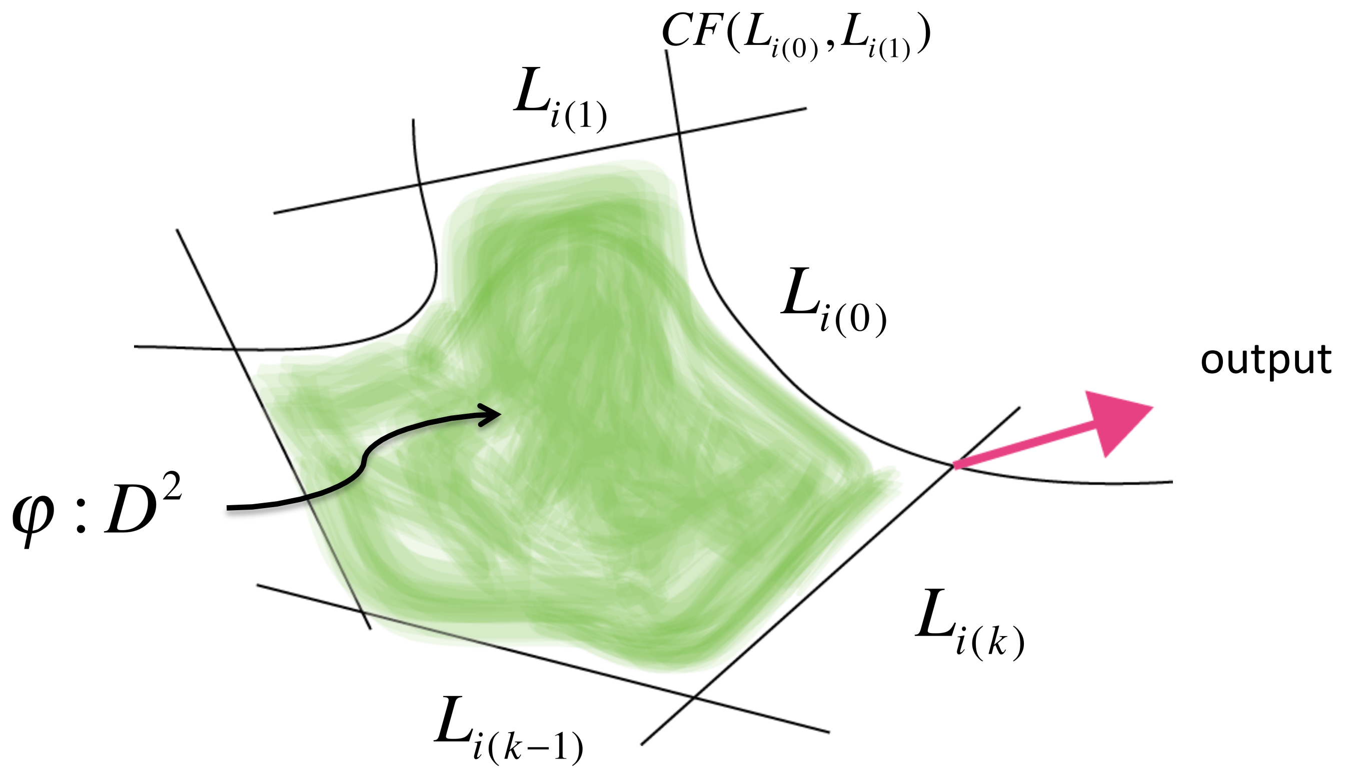
$$\mathcal{L} = \{L_1, \ldots, L_N\} \qquad L_i \text{ is transversal to } L_j \qquad i \neq j$$

$$CF(L_i, L_j) = \begin{cases} \bigoplus_{p \in L_i \cap L_j} \Lambda_0[p] & i \neq j \\ H(L; \Lambda_0) & L_i = L_j = L \end{cases}$$

Floer's chain complex

$$m_k : CF(L_{i(0)}, L_{i(1)}) \otimes CF(L_{i(1)}, L_{i(2)}) \otimes \cdots$$
$$\otimes CF(L_{i(k-2)}, L_{i(k-1)}) \otimes CF(L_{i(k-1)}, L_{i(k)}) \to CF(L_{i(0)}, L_{i(k)})$$

$$m_k : CF(L_{i(0)}, L_{i(1)}) \otimes CF(L_{i(1)}, L_{i(2)}) \otimes \cdots$$
$$\otimes CF(L_{i(k-2)}, L_{i(k-1)}) \otimes CF(L_{i(k-1)}, L_{i(k)}) \rightarrow CF(L_{i(0)}, L_{i(k)})$$



A_∞ relation

$$0 = \sum_{k_1+k_2=k+1} \sum_{i=1}^{k_1} \pm m_{k_1}(x_1,\ldots,m_{k_2}(x_i,\ldots,x_{i+k_2-1}),\ldots,x_k)$$

Symmetry Cyclic symmetry

$$\langle m_k(x_1,\ldots,x_k),x_0\rangle = \pm\langle m_k(x_0,x_1,\ldots,x_{k-1}),x_k\rangle$$

Cyclic (filtered) A_∞ category $\mathcal{A}(\mathcal{L})$

$$\mathcal{L} = \{L_1,\ldots,L_N\}$$

Question: (genus 0 only)

(1)

DOES Open topological string = Cyclic (filtered) A_∞ category $\mathcal{A}(\mathcal{L})$

determine closed topological string = $Comm_\infty$ algebra ?

$$\mathcal{GW}_\ell \quad : \; H(X;\Lambda_0)^\ell \to H(X;\Lambda_0)$$

(2) if yes how big does $\mathcal{L} = \{L_1,\ldots,L_N\}$ must be for (1) to hold.

Answer:

(1) NO

$$X = \mathbb{C}P^2$$

$\mathscr{L}$ The set of all Lagrangian submanifolds with nontrivial Floer homology

$\mathscr{A}(\mathscr{L})$ determine $\mathscr{GW}_2$ = quantum cup product

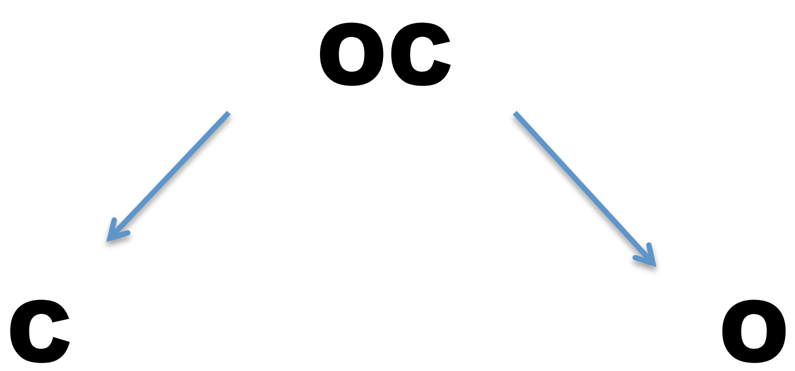
but not determine $\mathscr{GW}_3$ or higher.

Remark

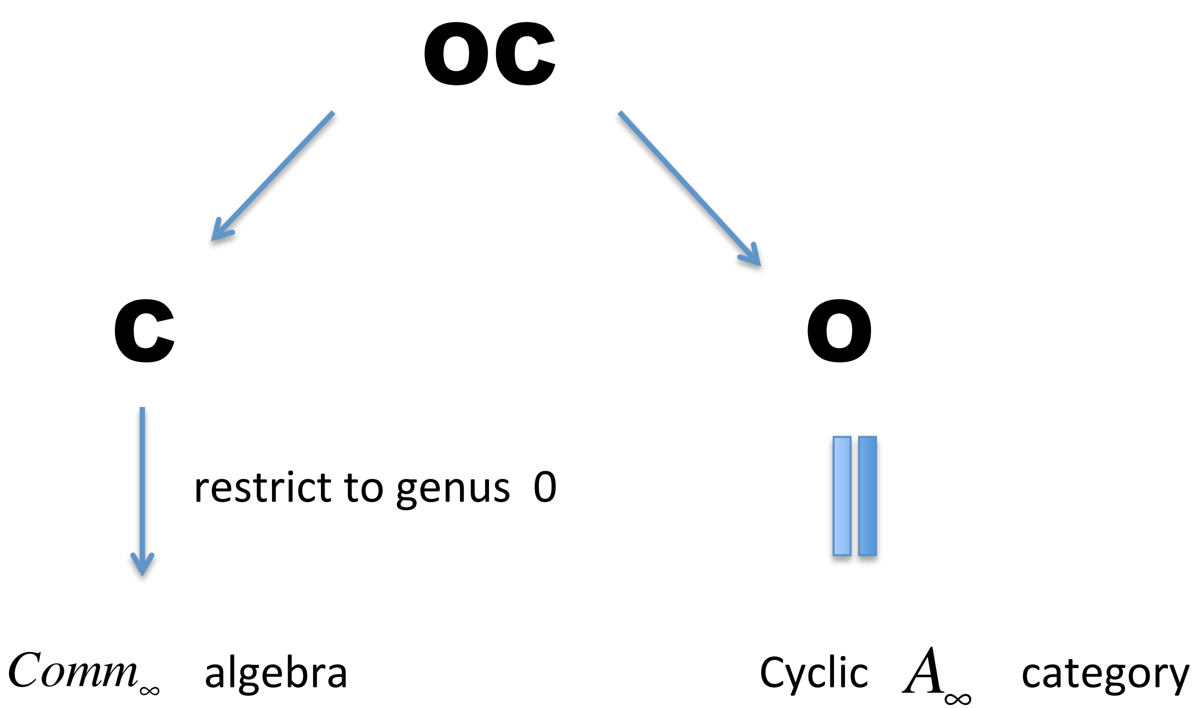
Open closed topological conformal field theory = **oc**

Closed topological conformal field theory = **c**

Open topological conformal field theory = **o**



Remark



oc
c
o
restrict to genus 0
$Comm_\infty$ algebra
Cyclic A_∞ category

oc

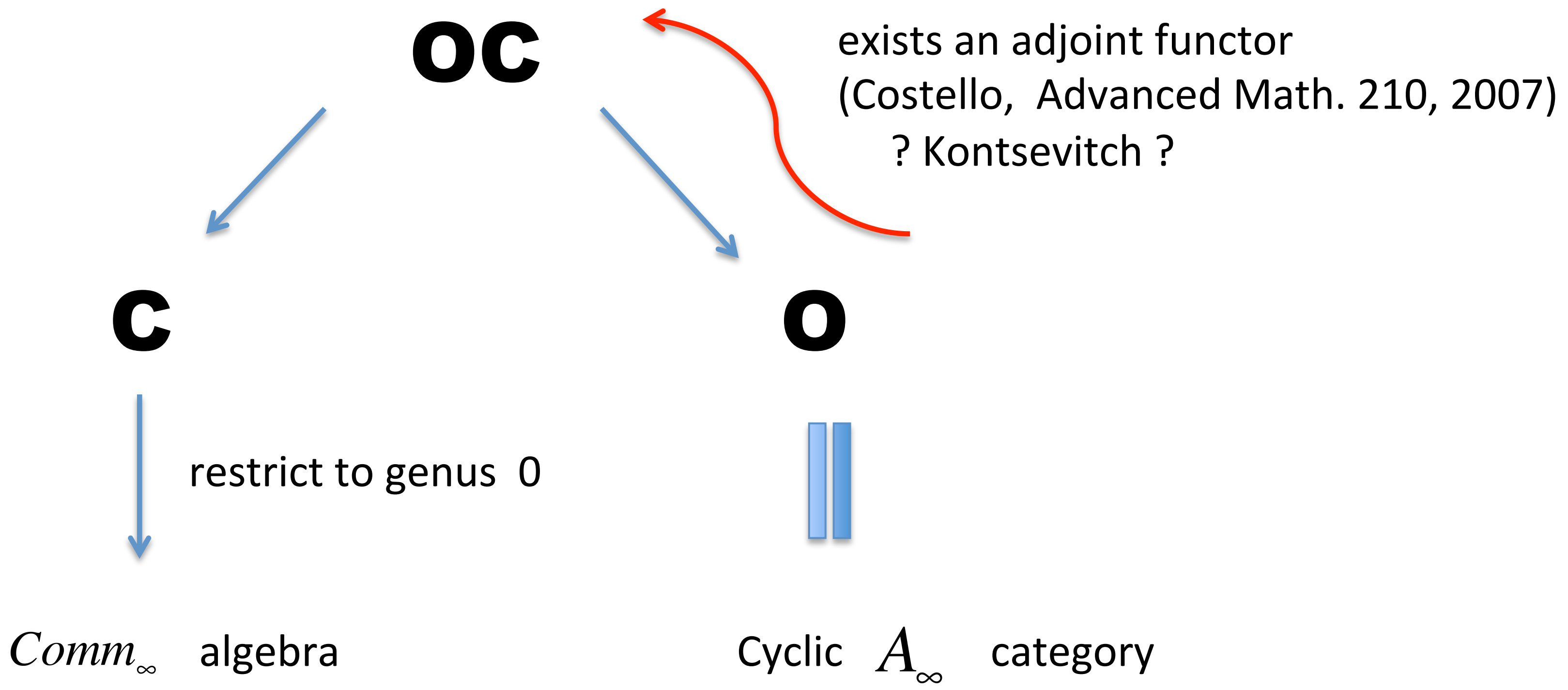
exists an adjoint functor
(Costello, Advanced Math. 210 2007)
? Kontsevitch ?

c **o**

restrict to genus 0

$Comm_{\infty}$ algebra

Cyclic A_{∞} category



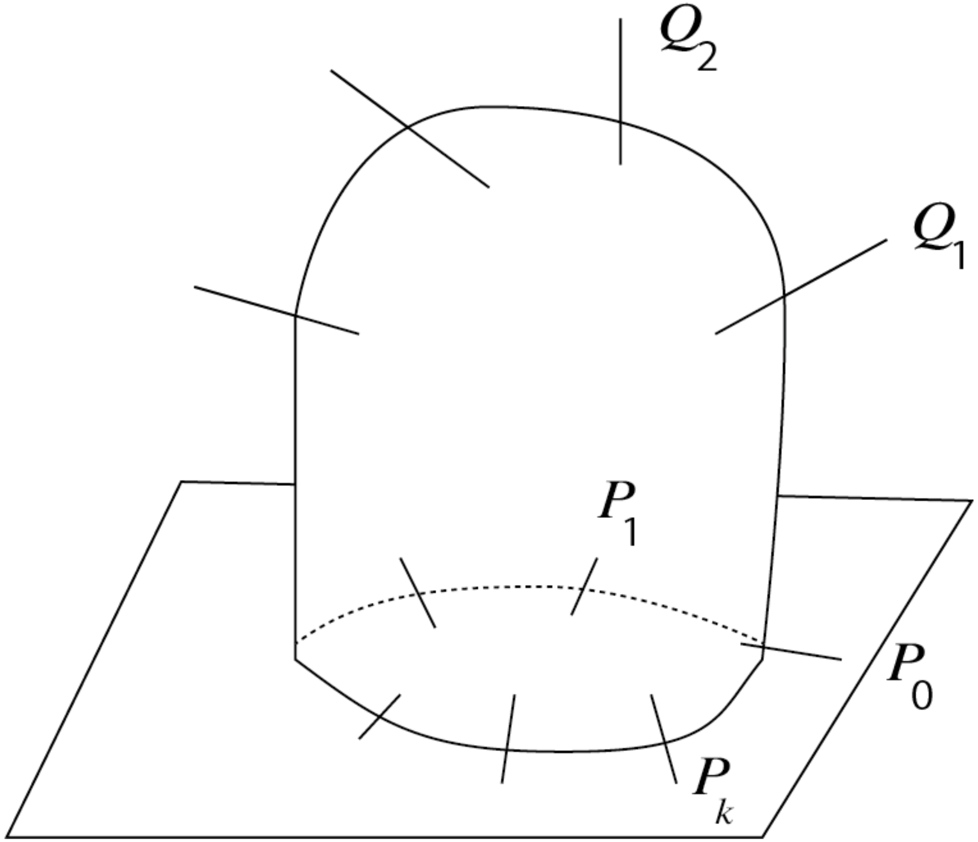
From cyclic A_∞ category $\mathcal{A}(\mathcal{L})$ we get a $Comm_\infty$ algebra structure on $H(X;\Lambda_0)$

However, in general it does not coicide with $Comm_\infty$ algebra structure defined by by Gromov-Witten invariant.

(There is another possibility that I will mention later.)

Open closed (topological) string (genus 0 and 0 loop)
= count of :

Use cycles of
both on L and on X



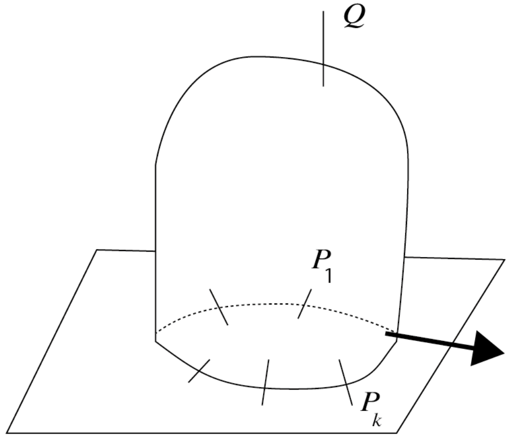
Open closed (topological) string determines two operators

Output is chain on *L*

Hochshild cohomology

$$\mathfrak{q} : H^*(X;\Lambda_0) \to HH^*(\mathcal{A}(\mathcal{L}),\mathcal{A}(\mathcal{L}))$$

(FOOO, (earlier work by Kontsevich, Seidel))



Output is chain on *X*

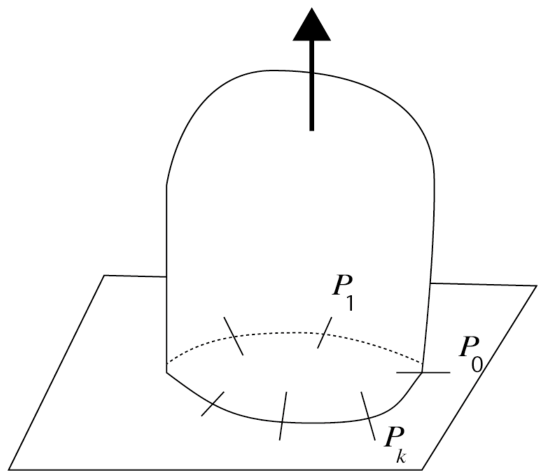
$$\mathfrak{p} : HH_*(\mathcal{A}(\mathcal{L})) \to H_*(X;\Lambda_0)$$

Hochshild homology

or

$$\mathfrak{p} : HC_*(\mathcal{A}(\mathcal{L})) \to H_*(X;\Lambda_0)$$

Cyclic homology



(FOOO, (monotone case by Albers independently))

(Hereafter abbraviated to AFOOO.)

Theorem (Kontsevich, Seidel, FOOO, Biran-Cornea, Abouzaid-FOOO etc.)

$$\mathfrak{q} : QH^*(X;\Lambda_0) \to HH^*(\mathcal{A}(\mathcal{L}),\mathcal{A}(\mathcal{L}))$$ is a ring homomorphism.

$\mathcal{GW}_2$ is detected by $\mathcal{A}(\mathcal{L})$ if $\mathfrak{q}$ is an isomorphism.

Theorem (FOOO(arXive1009.1648)) If X is compact toric then there exists a finite set $\mathcal{L}$ of torus oribit for which $\mathfrak{q}$ is an isomorphism.

Conjecture (FOOO, related to works by Mau-Wehrheim-Woodward)

$\mathfrak{q}: QH^*(X;\Lambda_0) \to HH^*(\mathcal{A}(\mathcal{L}),\mathcal{A}(\mathcal{L}))$ extends to an A_∞ homomorphism.

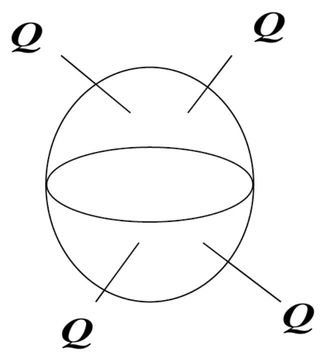
Remark

$QH^*(X;\Lambda_0)$ is an A_∞ algebra.

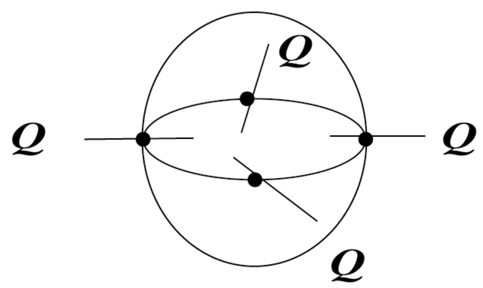
quantum Massey product (FOOO arXive 0912.2646)

$HH^*(\mathcal{A}(\mathcal{L}),\mathcal{A}(\mathcal{L}))$ is an A_∞ algebra. (general fact on homological algebra.)

But we want to know $Comm_\infty$ structure



$Comm_\infty$



A_∞

To see Poincare duality of X from the point of view of L and study $\mathcal{GW}_3$

or higher, we use $\mathfrak{p}$.

Theorem (AFOOO) $\langle \mathfrak{p}(\mathbf{x}), y \rangle_{\mathrm{PD}_X} = \langle \mathbf{x}, \mathfrak{q}(y) \rangle$

$$\mathbf{x} \in HH_*(\mathcal{A}(\mathcal{L}))$$
$$y \in H^*(X)$$

the right hand side is a duality between
Hochshild homology and Hochshild cohomology of
cyclic A_∞ category.

Theorem (AFOOO) If $[X] \in \mathrm{Im}(\mathfrak{p} : HH_*(\mathcal{A}(\mathcal{L}) \to H_*(X))$

then $\mathfrak{p} : HH_*(\mathcal{A}(\mathcal{L})) \to H_*(X)$

$\mathfrak{q} : QH^*(X;\Lambda_0) \to HH^*(\mathcal{A}(\mathcal{L}), \mathcal{A}(\mathcal{L}))$ are isomorphisms.

An answer to Problem (2).

Problem: Let us consider

$$\mathfrak{p} : HH_*(\mathcal{A}(\mathcal{L})) \to H_*(X) \qquad \mathfrak{p} : HC_*(\mathcal{A}(\mathcal{L})) \to H_*(X)$$

Can we calculate

$$Z_2(\mathbf{x}_1, \mathbf{x}_2) = \left\langle \mathfrak{p}(\mathbf{x}_1), \mathfrak{p}(\mathbf{x}_2) \right\rangle_{\mathrm{PD}_X}$$

$$Z_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \left\langle \mathcal{GW}_2(\mathfrak{p}(\mathbf{x}_1), \mathfrak{p}(\mathbf{x}_2)), \mathfrak{p}(\mathbf{x}_3) \right\rangle_{\mathrm{PD}_X}$$

$$\bullet \bullet \bullet$$

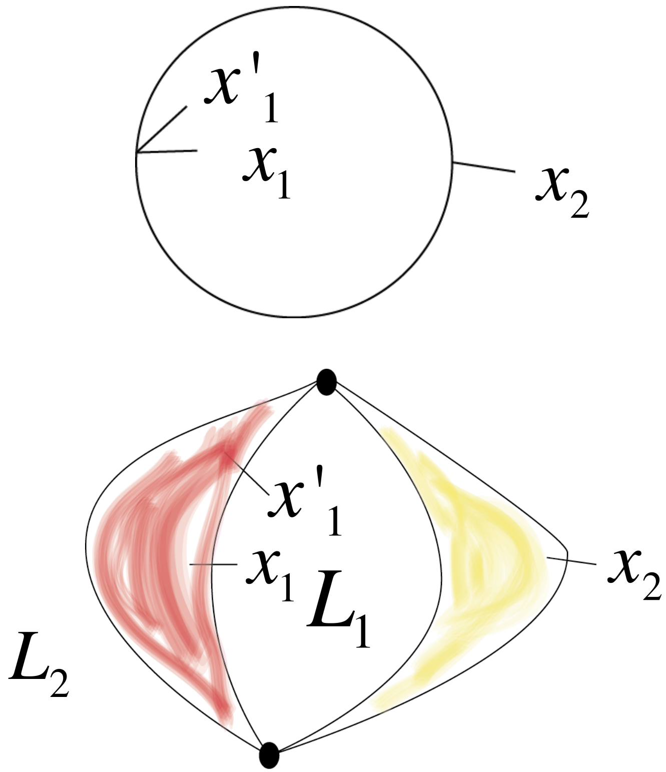
$$Z_k(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k) = \left\langle \mathcal{GW}_{k-1}(\mathfrak{p}(\mathbf{x}_1), \dots, \mathfrak{p}(\mathbf{x}_{k-1})), \mathfrak{p}(\mathbf{x}_k) \right\rangle_{\mathrm{PD}_X}$$

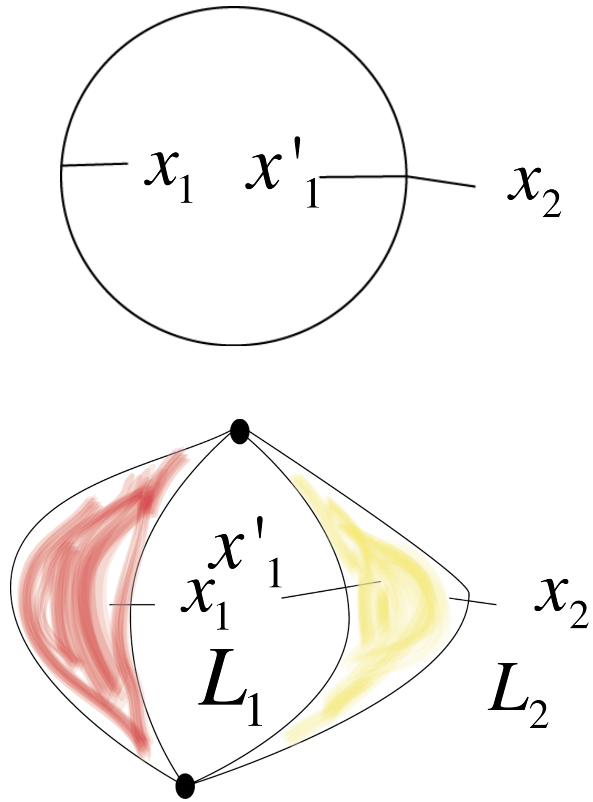
in terms of the cyclic filtered A_∞ category $\mathcal{A}(\mathcal{L})$?

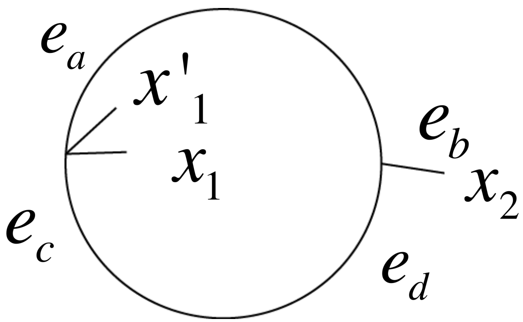
Yes for Z_2 and Z_3 No for Z_4 or higher.

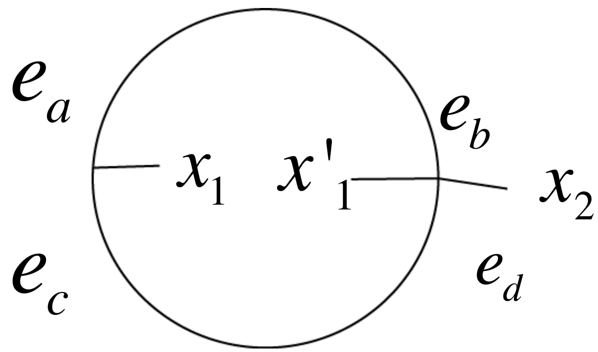
Formula for $Z_2(\mathbf{x}_1,\mathbf{x}_2)=\left\langle \wp(\mathbf{x}_1),\wp(\mathbf{x}_2)\right\rangle_{\mathrm{PD}_X}$

$x_1 \otimes x'_1 = \mathbf{x}_1$ $\quad$ $x_2 = \mathbf{x}_2$ $\quad$ $x_1, x'_1 \in H(L_1)$ $\quad$ $x_2 \in H(L_2)$









$$\sum_{a,b,c,d} g^{ab} g^{cd} \langle m_3(x_1, x'_1, e_a), e_c \rangle \langle m_2(x_2, e_b), e_d \rangle \qquad + \sum_{a,b,c,d} g^{ab} g^{cd} \langle m_2(x_1, e_a), e_c \rangle \langle m_3(x_2, e_b, x'_1), e_d \rangle$$

Theorem (AFOOO, FOOO ....)

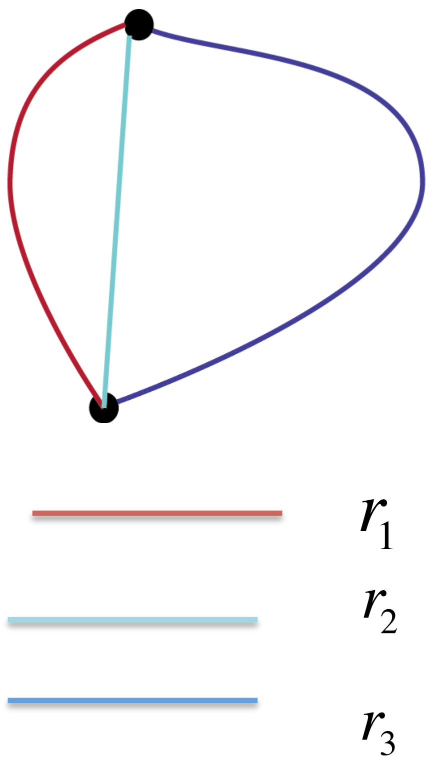
$$Z_2(\mathbf{x}_1, \mathbf{x}_2) = \langle \not{p}(\mathbf{x}_1), \not{p}(\mathbf{x}_2) \rangle_{\mathrm{PD}_X}$$

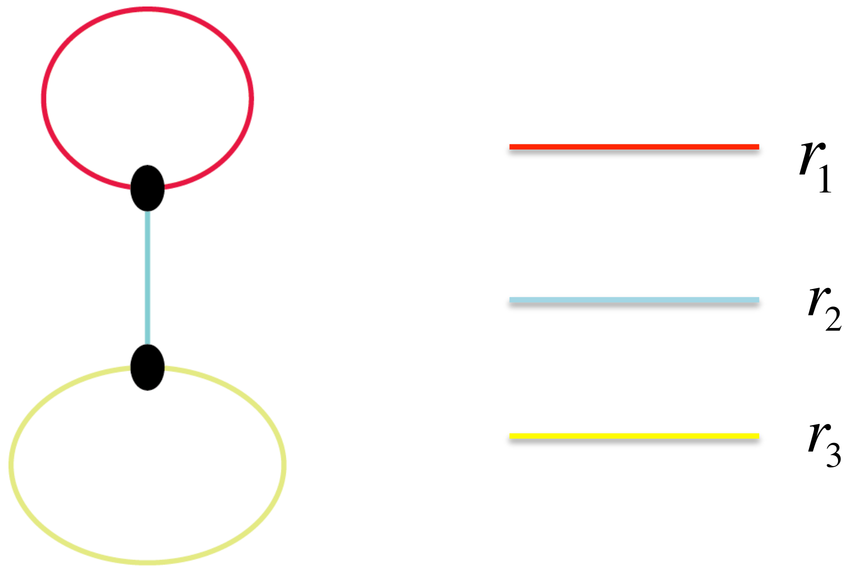
e_a a basis of $HF(L_1, L_2)$ $g_{ab} = \langle e_a, e_b \rangle$

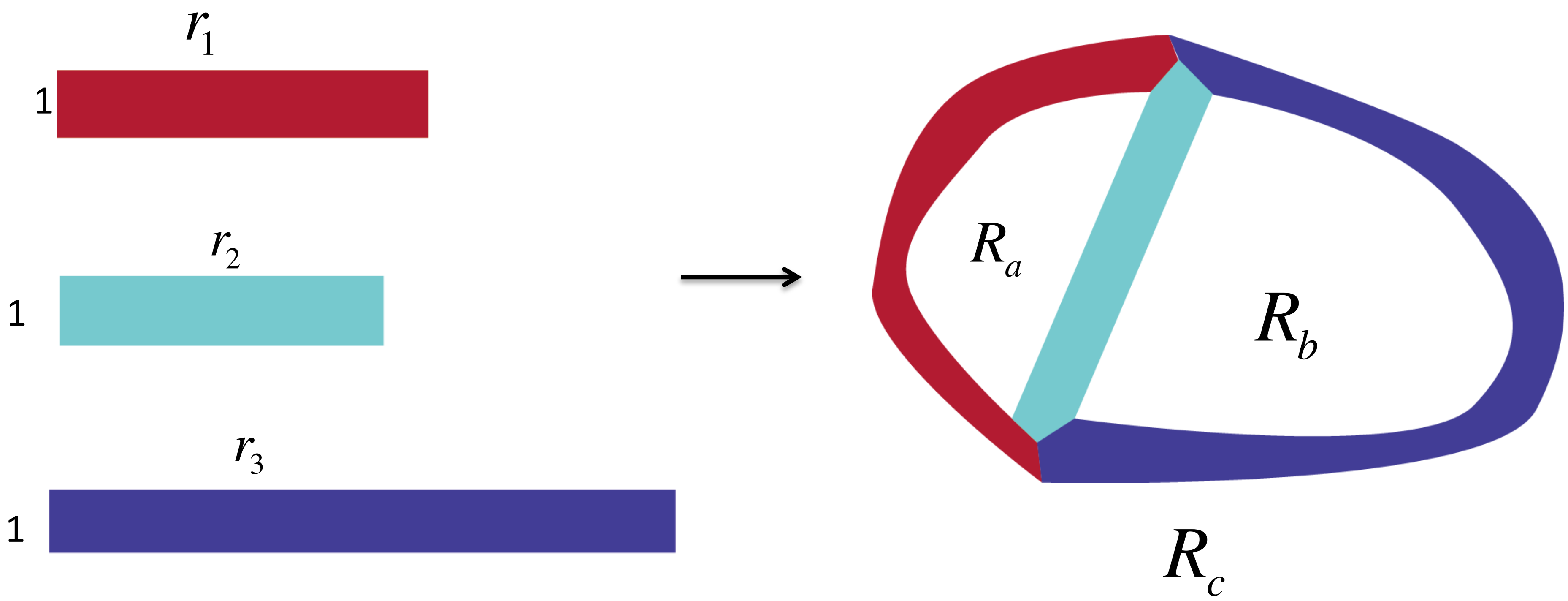
How to obtain a formula for Z_3 or Z_2

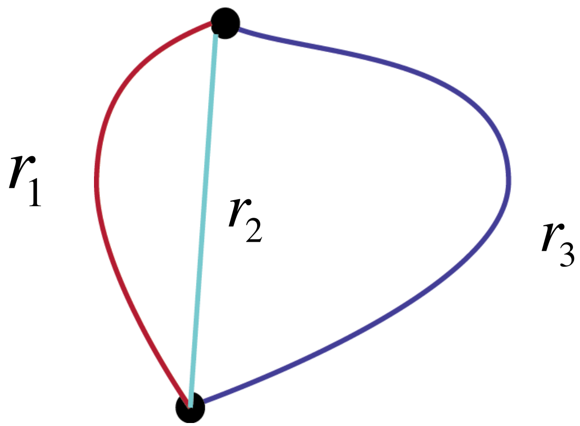
Metric Ribbon tree → Bordered Riemann surface

Example: 2 loop

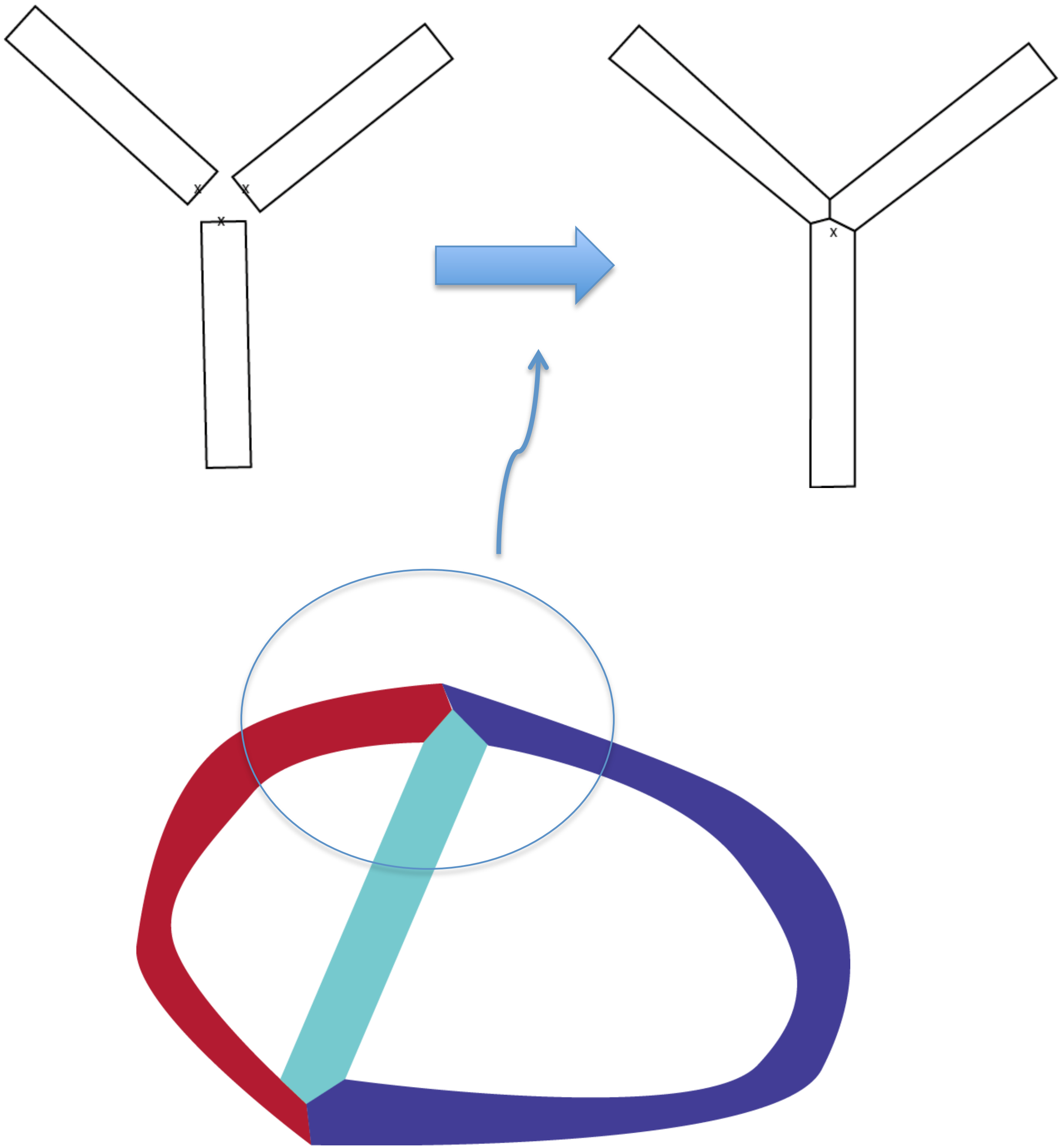


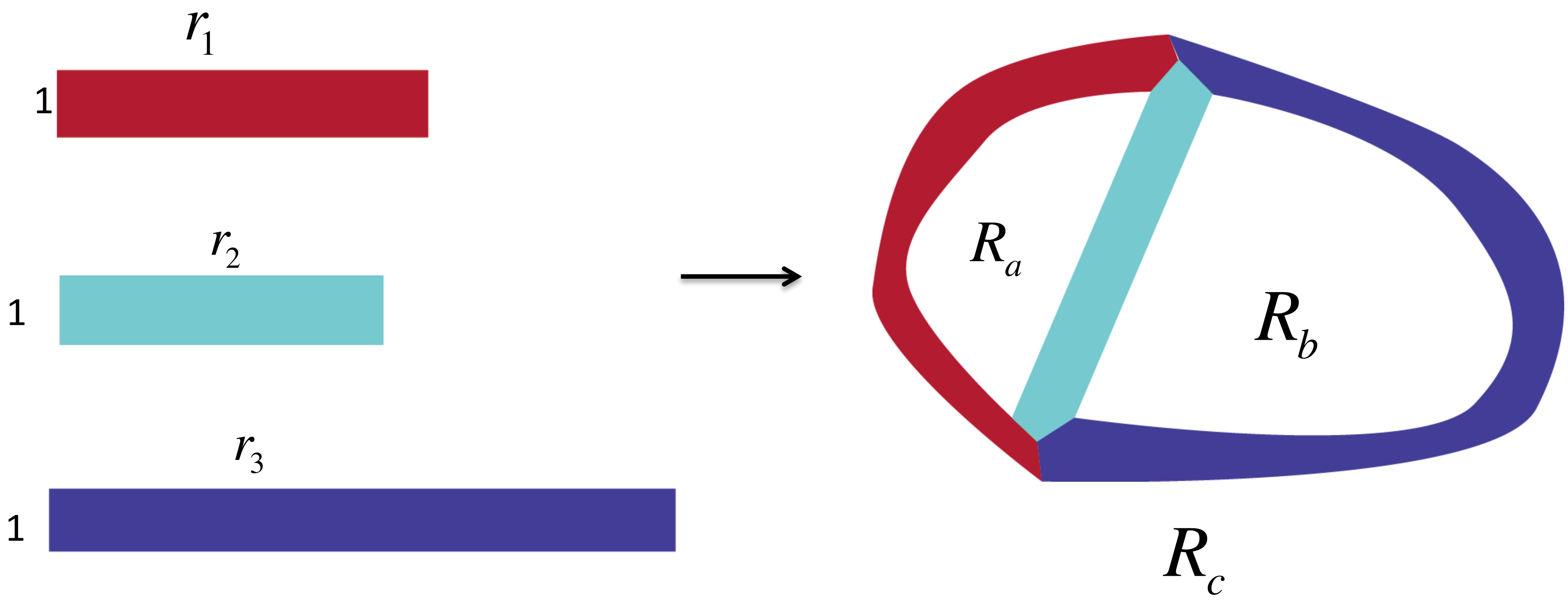


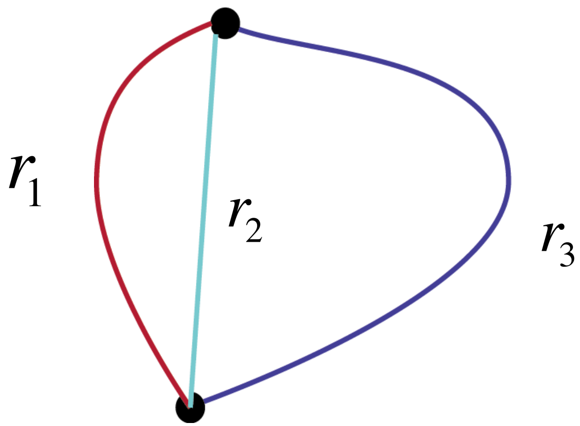




$$R_a = r_1 + r_2$$
$$R_b = r_2 + r_3$$
$$R_c = r_1 + r_3$$


x
x
x
x





$$R_a = r_1 + r_2$$

$$R_b = r_2 + r_3$$

$$R_c = r_1 + r_3$$

Moduli of metric ribbon graph $=$ $\mathcal{M}_{l,g} \times \mathbb{R}_+^l$

R_a

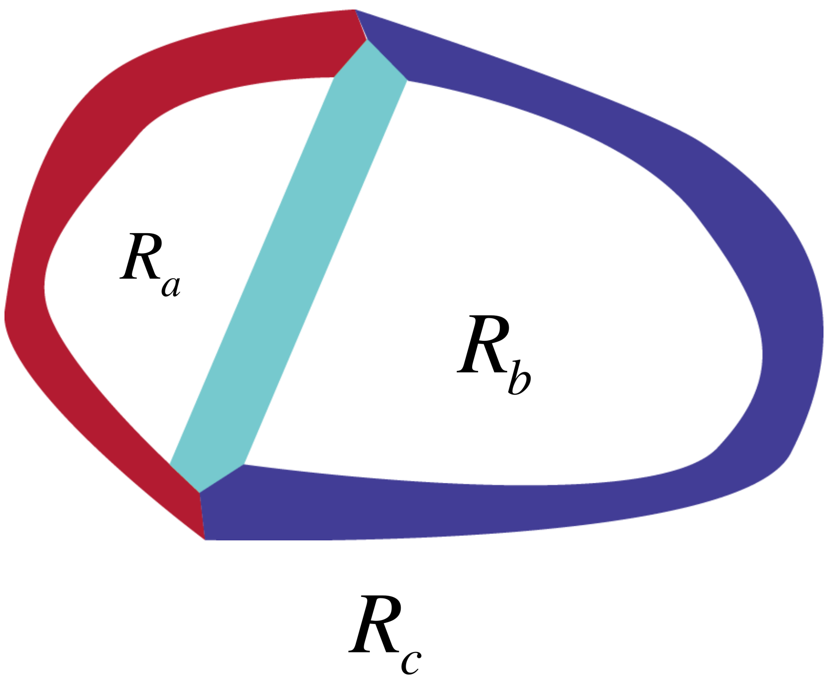
R_b

etc.

Moduli of genus g Riemann surface
with l marked points

This was appeared in Kontsevich's proof of Witten conjecture

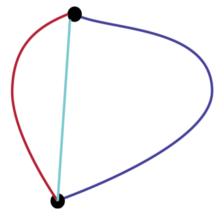
$\mathcal{M}_{l,g} \times \mathbb{R}_+^l$ is identified with moduli space of bordered Riemann surface.



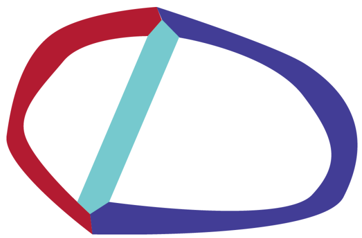
In our case

Moduli of metric ribbon graph of 2 loop genus 0 = $\mathcal{M}_{3,0} \times \mathbb{R}_+^3$

r_1, r_2, r_3



One point



R_a

R_b

R_c

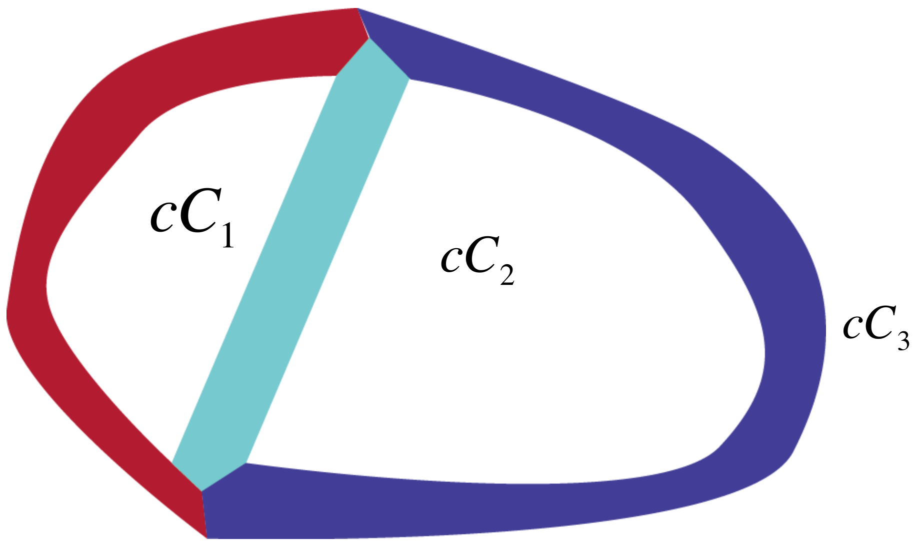
$$R_a = r_1 + r_2$$

$$R_b = r_2 + r_3$$

$$R_c = r_1 + r_3$$

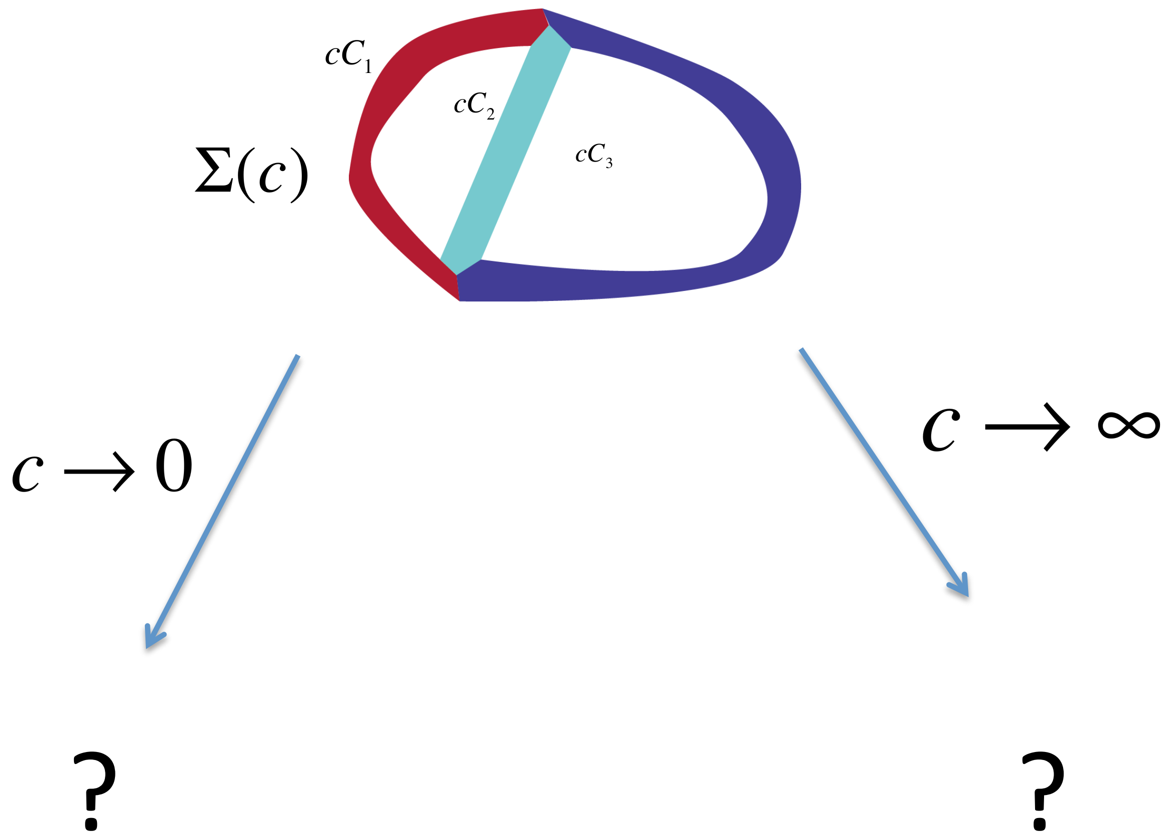
Fix C_1, C_2, C_3 and consider the family $\vec{R}(c) = (cC_1, cC_2, cC_3)$

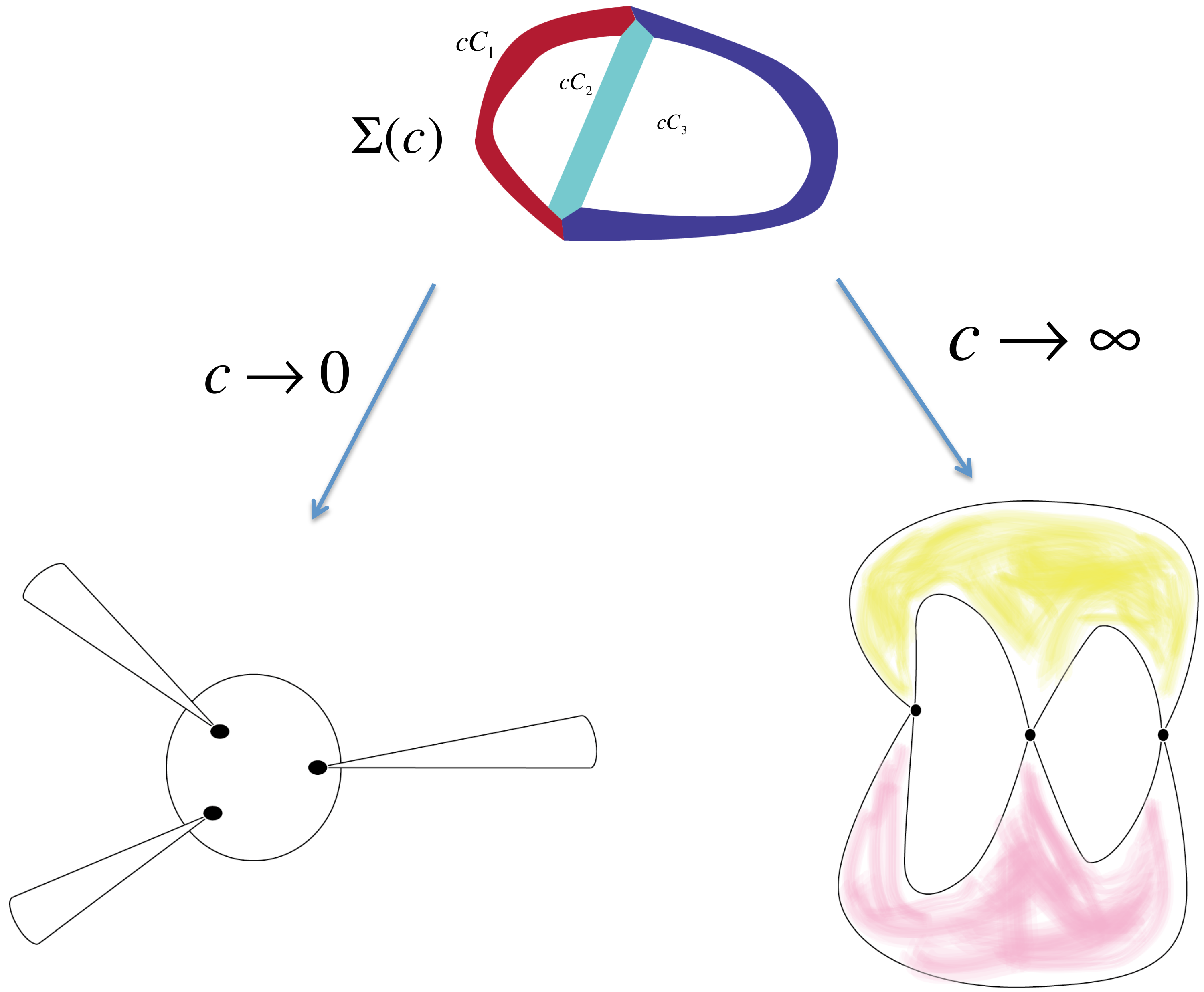
We obtain a family of bordered Riemann surfaces $\Sigma(c)$ parametrized by $c \in \mathbb{R}_{>0}$



$\Sigma(c)$

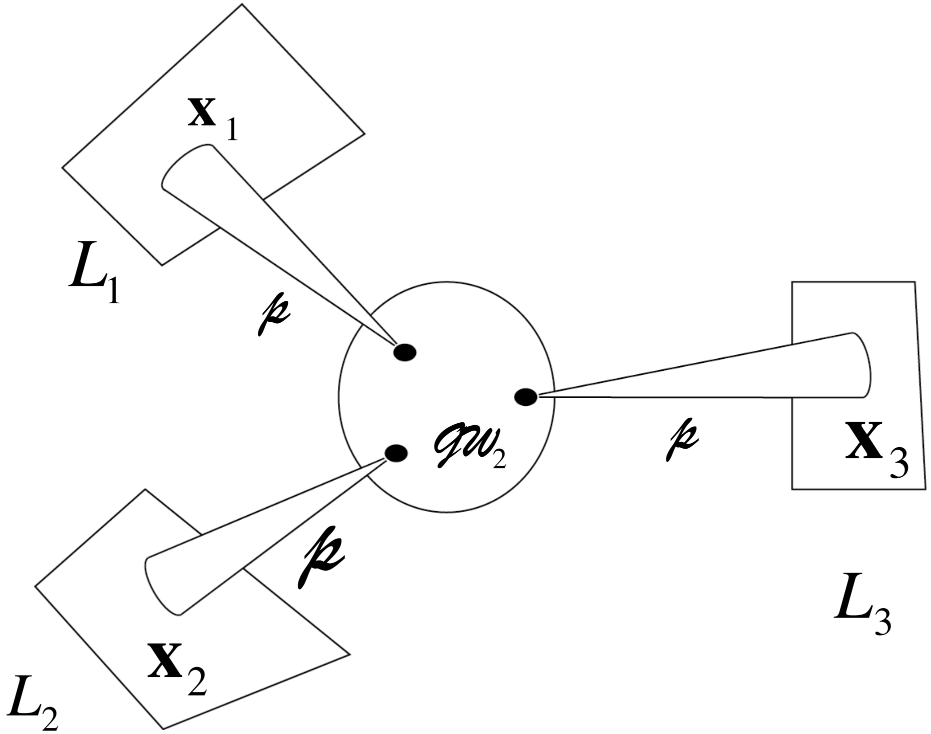
Study the limit when $c \to 0$ $c \to \infty$





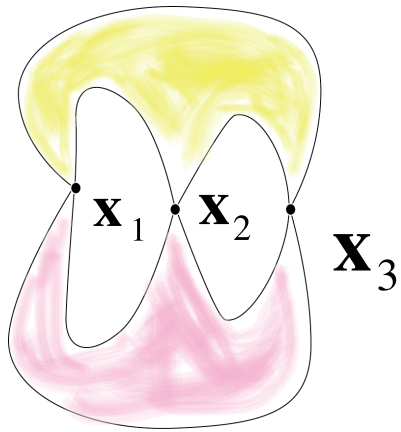
cC_1
cC_2
cC_3
$\Sigma(c)$
$c \to 0$
$c \to \infty$

Counting holomorphic maps:

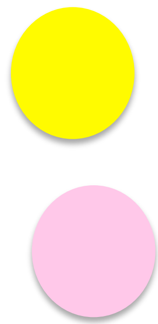


$$=$$

$$\left\langle gw_2\big(p(\mathbf{x}_1), p(\mathbf{x}_2)\big), p(\mathbf{x}_3) \right\rangle$$

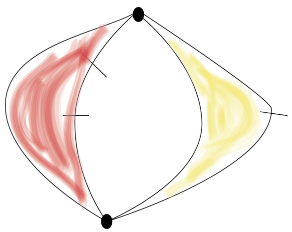
Counting holomorphic maps from 

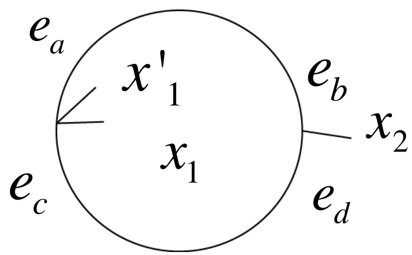
gives a formula of Z_3



It becomes a similar formula as Z_2

are both disks





$$\sum_{a,b,c,d} g^{ab} g^{cd} \langle m_3(x_1, x'_1, e_a), e_c \rangle \langle m_2(x_2, e_b), e_d \rangle$$

In particular it is written explicitly by m_k and $\langle \quad \rangle$

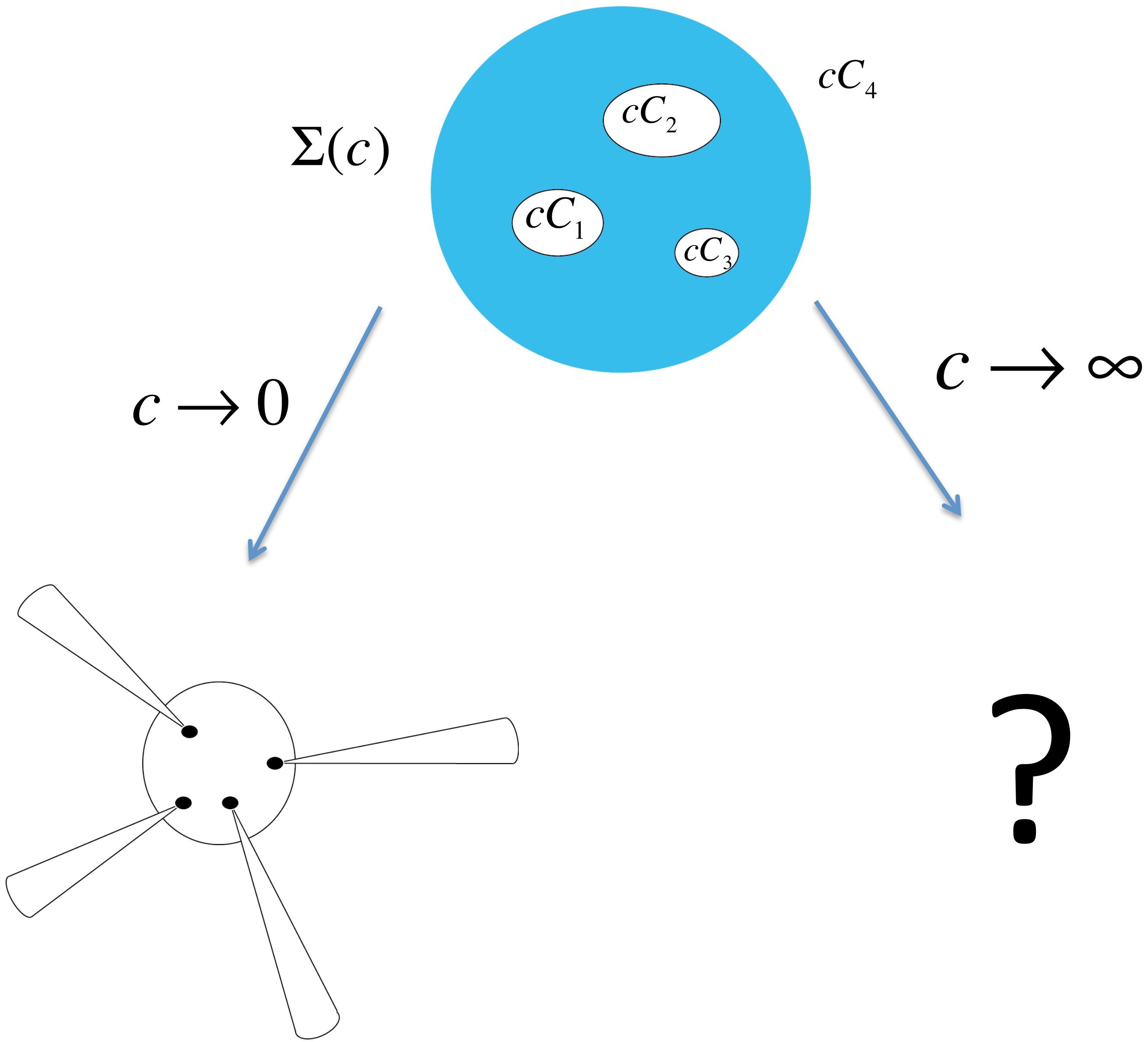
Thus

$$\left\langle \mathscr{GW}_2(\mathscr{p}(\mathbf{x}_1),\mathscr{p}(\mathbf{x}_2)),\mathscr{p}(\mathbf{x}_3)\right\rangle$$

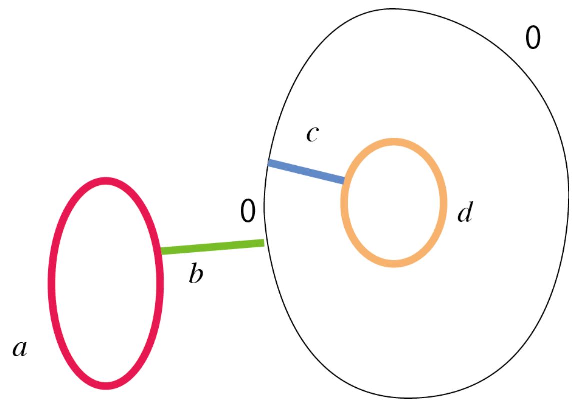
= something written explicitly by m_k and $\langle \quad \rangle$

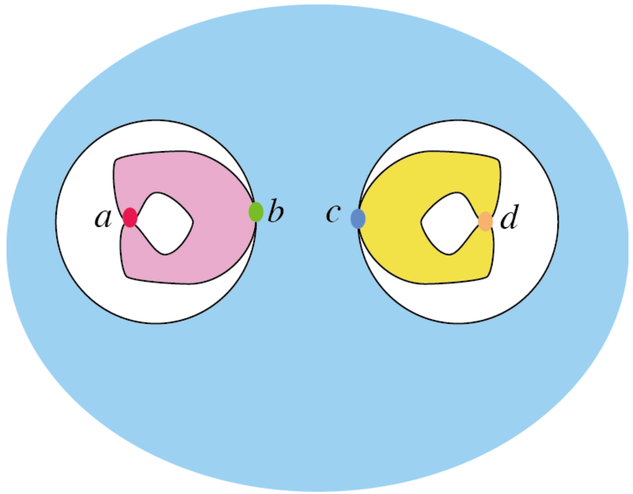
What's wrong when we try to do the same for

$$\left\langle \mathscr{GW}_3(\mathscr{p}(\mathbf{x}_1),\mathscr{p}(\mathbf{x}_2),\mathscr{p}(\mathbf{x}_3)),\mathscr{p}(\mathbf{x}_4)\right\rangle \qquad ?$$

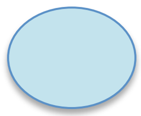

$\Sigma(c)$
cC_1
cC_2
cC_3
cC_4
$c \to 0$
$c \to \infty$
?

It appears maps from





That can not be calculated by operators m_k

because  is not a disk but is an annulus

(Note the point where length becomes 0 is related to the Deligne-Mumford compactification.
So the relation with Costello's work is related to this.
(Costello may not take compactification.)

Proof of the fact

$$X = \mathbb{C}P^2$$

$\mathcal{L}$ The set of all Lagrangian submanifolds with nontrivial Floer homology

$\mathcal{A}(\mathcal{L})$ do not determine $\mathcal{GW}_3$ or higher.

Need to recall some more facts (FOOO)

Object of $\mathcal{A}(\mathcal{L})$ is a pair (L,b)

$$b \in H^{odd}(L;\Lambda_0) \qquad \sum_{k=0}^{\infty} m_k(b,\ldots,b) = W(b)\,1_L$$

$$W(b) \in \Lambda_0$$

$$\mathcal{A}(\mathcal{L}) = \coprod_{\lambda} \mathcal{A}(\mathcal{L};\lambda)$$

$$(L,b) \in \mathcal{A}(\mathcal{L};\lambda) \Longleftrightarrow W(b) = \lambda$$

Theorem (AFOOO)

$$\mathbf{x}_1 \in HH(\mathcal{A}(\mathcal{L};\lambda_1))$$
$$\mathbf{x}_2 \in HH(\mathcal{A}(\mathcal{L};\lambda_2))$$

$$\lambda_1 \neq \lambda_2$$

$$\Rightarrow \quad \left\langle p(\mathbf{x}_1), p(\mathbf{x}_2) \right\rangle_{\mathrm{PD}_X} = 0$$

$$\mathcal{GW}_2\left(p(\mathbf{x}_1), p(\mathbf{x}_2)\right) = 0$$

$$QH(\mathbb{C}P^2) = \Lambda \times \Lambda \times \Lambda$$

$$\mathcal{A}(\mathcal{L}) = \mathcal{A}(\mathcal{L};\lambda_1) \cup \mathcal{A}(\mathcal{L};\lambda_2) \cup \mathcal{A}(\mathcal{L};\lambda_3)$$

$HH(\mathcal{A}(\mathcal{L};\lambda_i)) = \Lambda$ Corresponds to each of the factors

$\mathcal{GW}_2$ between two different factors are zero.

$$QH(\mathbb{C}P^2) = \Lambda \times \Lambda \times \Lambda$$

$$\mathcal{A}(\mathcal{L}) = \mathcal{A}(\mathcal{L};\lambda_1) \cup \mathcal{A}(\mathcal{L};\lambda_2) \cup \mathcal{A}(\mathcal{L};\lambda_3)$$

$HH(\mathcal{A}(\mathcal{L};\lambda_i)) = \Lambda$ Corresponds to each of the factors

$\mathcal{GW}_2$ between two different factors are zero.

However

$\mathcal{GW}_3$ between two different factors are NOT zero.

So they do not come from m_k

QED.