# Lagrangian surgery and Rigid analytic family of Floer homologies

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A part of this talk is based on joint work with
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# Why Family of Floer cohomology ?

It is expected that
if family Floer cohomology is built in an ideal way
then homological Mirror symmetry conjecture will be proved.

F-Oh-Ohta-Ono

(AMS/IP 46)

$(X, \omega)$ a sympectic manifold

$L \subset X$ (relatively spin) Lagrangian submanifold

$$m_k : H(L;\Lambda_0)^{\otimes k} \to H(L;\Lambda_0), \quad k = 0,1,2,\ldots$$

(Filtered) A infinity structure

$$\mathfrak{M}(L) = \left\{ b \in H^1(L;\Lambda_0) \,\middle|\, \sum_{k=0}^{\infty} m_k(b,\ldots,b) = 0 \right\}$$

Maurer-Cartan Scheme

$b_i \in \mathfrak{M}(L_i)$ ⟹ $HF((L_1,b_1),(L_2,b_2);\Lambda_0)$

Floer homology

Suppose Maslov index of $L \subset X$ is 0.

$$\Lambda_0 = \left\{ \sum_i a_i T^{\lambda_i} \,\middle|\, a_i \in \mathbb{C}, \quad \lambda_i \ge 0, \quad \lim_{i\to\infty} \lambda_i = \infty \right\} \qquad \Lambda = \Lambda_0[T^{-1}]$$

$HF((L_1,b_1),(L_2,b_2);\Lambda_0)$ is Z graded.

Homological Mirror symmetry conjecture (Kontsevitch)

$$(X,\omega) \Longrightarrow (X^{\vee}, J)$$ Mirror complex manifold.

$$L \subset X \Longrightarrow E(L) \to X^{\vee}$$ Object of derived category of coherent sheaves.

$$HF(L_1, L_2) \cong \mathrm{Ext}(E(L_1), E(L_2))$$

Guess (related to (a version of) Stronminger-Yau-Zaslow conjecture)

$$X^{\vee} = \bigcup_{u \in B} \mathfrak{M}(L(u))$$

$L(u)$ : a family of Lagrangian submanifold parametrized by $u \in B$

$$E(L)_{(L(u),b)} = HF(L,(L(u),b);\Lambda)$$

Mirror Object = family of Floer homologies

$$\begin{array}{ccc} \mathfrak{L} & \xrightarrow{\ i\ } & X \\ {\scriptstyle\pi}\downarrow & & \\ B & & \end{array}$$

$\pi$ is a locally trivial fiber bundle

$$L(u) = \pi^{-1}(u)$$

$$i\big|_{L(u)} : L(u) \to X$$

is a Lagrangian embedding.

$s : B \to \mathfrak{L}$ is a section.

Assume $T_u B \to H^1(L(u); \boldsymbol{R})$ is an isomorphism.

$B$ has flat affine structure.

Suppose Maslov index of $L(u) \subset X$ is 0.

# Theorem 1 (to be written up, a part is in arXiv:0908.0148)

(1) $$\mathfrak{M}(\mathfrak{L}) = \bigcup_{u \in B} \mathfrak{M}(L(u)) \big/ H^1(L(u); 2\pi\sqrt{-1}\mathbf{Z})$$

has a structure of rigid analytic space.

(2) If $L'$ is a another Lagrangian submanifold (relatively spin, Maslov = 0).

$$b' \in \mathfrak{M}(L')$$

then

$$(u,b) \mapsto HF((L',b'),(L(u),b);\Lambda)$$

defines an object of derived category of coherent sheaves on $\mathfrak{M}(\mathfrak{L})$

Kontsevich-Soibelman proposed to use Rigid analytic geometry to study homological Mirror symmetry around 2000.

Various operators etc. appears in Floer theory and Gromov-Witten theory is one over (universal) Novikov ring = a kind of formal power series ring, and its convergence is not know.

An idea to use rigid analytic geometry is first to construct everything in the lebel of formal power series (Novikov ring) and prove a version of Mirror symmetry (formal power series version) and use GAGA of rigid analytic geometry to prove convergence later (in the complex side).

$$\mathfrak{M}(L) = \left\{ b \in H^1(L;\Lambda_0) \middle| \sum_{k=0}^{\infty} m_k(b,\ldots,b) = 0 \right\}$$

$$P_u(x) = \sum_{k=0}^{\infty} m_k(x,\ldots,x) \in H^2(L(u);\Lambda_0), \quad P_u(x) = (P_u^l(x))_{l=1\ldots,\mathrm{rank}\, H^2}$$

$$x = \sum x_i \mathbf{e}_i$$

$\mathbf{e}_i$ is a basis of $H^1(L(u);\boldsymbol{R})$ $\quad y_i = \exp(x_i)$

$$P_u^l(x) = \sum_{i=1}^{\infty} T^{\lambda_i} P_{i,u}^l(y_1,\ldots,y_m) \qquad P_{i,u}^l(y_1,\ldots,y_m) \in \boldsymbol{R}\left[y_1,\ldots,y_m,y_1^{-1},\ldots,y_m^{-1}\right]$$

$$P_{u'}(y'_1,\ldots,y'_m) = P_u(y_1,\ldots,y_m)$$

$u'_i, u_i$ are affine coordinate of $u', u$

$$y'_i = T^{u'_i - u_i} y_i + Q_i(T^{u'_1-u_1}y_1,\ldots,T^{u'_m-u_m}y_m)$$

$$Q_i(T^{u'_1-u_1}y_1,\ldots,T^{u'_m-u_m}y_m) = \sum_{j=1}^{\infty} T^{\lambda_{i,j}} Q_{i,j}(T^{u'_1-u_1}y_1,\ldots,T^{u'_m-u_m}y_m)$$

Theorem 1 is good enough to construct homological Mirror functor for torus.

To go beyond the case of torus we need to include singular fiber.

The main result of this talk says that we can do it in the case of simplest singular fiber.

$$\begin{array}{ccc} \mathfrak{L} & \xrightarrow{\ i\ } & X \\ {\scriptstyle \pi}\downarrow & & \\ B & & \end{array}$$

$\pi$ is a locally trivial fiber bundle over $B^o$

$u \in B^o \qquad L(u) = \pi^{-1}(u)$

$i|_{L(u)} : L(u) \to X$

is a Lagrangian embedding.

Assume $T_u B^o \to H^1(L(u); \boldsymbol{R})$ is an isomorphism.

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$\dim_{\boldsymbol{R}} X = 4 \qquad L(u) = \pi^{-1}(u) = T^2$

$B - B^o$ is a finitely many point

$u \in B - B^o \Longrightarrow L(u) = \pi^{-1}(u)$ is immersed $S^2$ with one self intersection point which is transversal.

# Theorem 2 (to be written up)

(1) $$\mathfrak{M}(\mathfrak{L}) = \bigcup_{u \in B} \mathfrak{M}(L(u)) \Big/ H^1(L(u); 2\pi\sqrt{-1}\mathbf{Z})$$

has a structure of rigid analytic space.

(2) If $L'$ is a another Lagrangian submanifold (relatively spin, Maslov = 0).

$$b' \in \mathfrak{M}(L')$$

then

$$(u,b) \mapsto HF((L',b'),(L(u),b);\Lambda)$$

defines an object of derived category of coherent sheaves on $\mathfrak{M}(\mathfrak{L})$

## Application

Construction of homological Mirror functor for K3 surface.

Note: Homological Mirrror symmetry was proved for quartic surface by P. Seidel.

Method of proof:

- Lagrangian surgery and behavior of Floer homologies via surgery.

  (F,Oh,Ohta,Ono; Chapter 10 of Lagrangian Floer thoery book.)

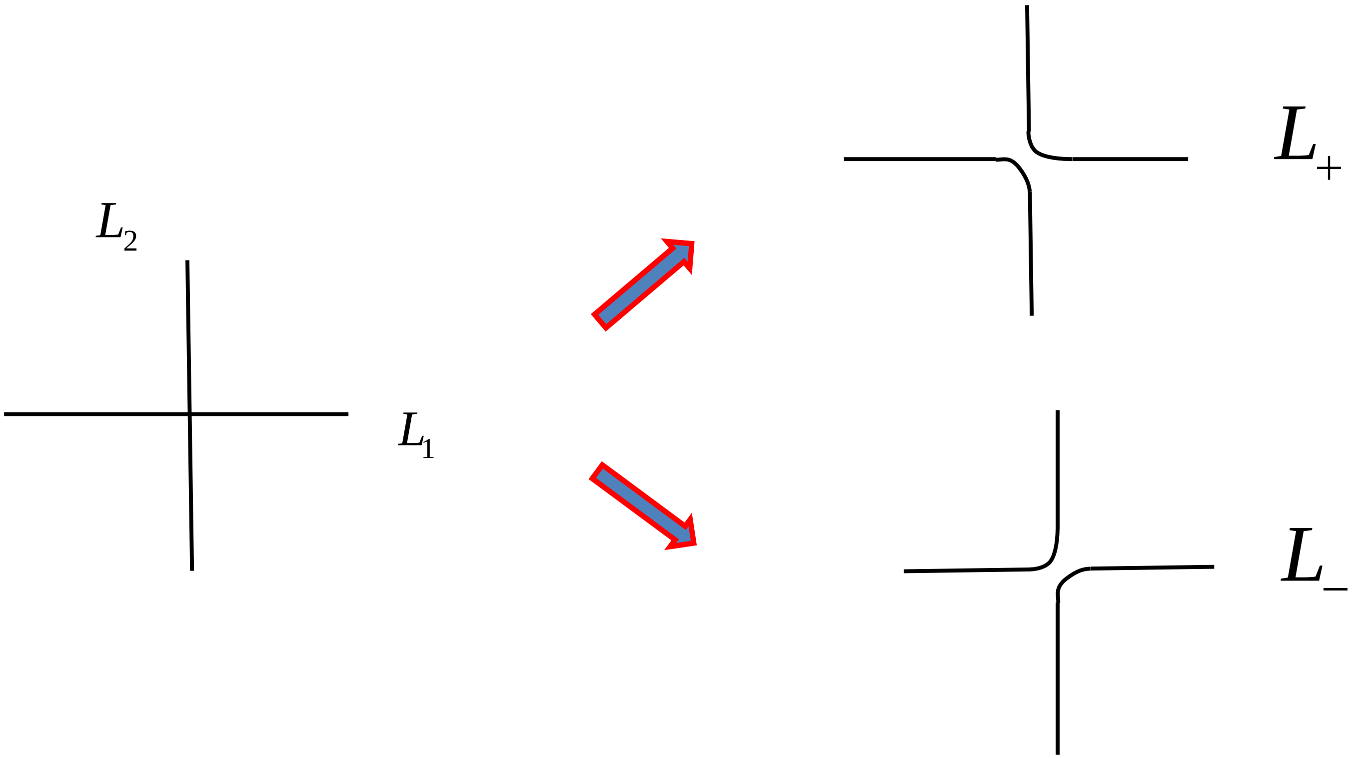
- Floer theory of Immersed Lagrangian submanifold.

  (M. Akaho – D. Joyce, arXiv:0803.0717)

# Lagrangian surgery

$X \supset L_1 \cup L_2$ Two Lagragian submanifolds which intersect at one point transversaly.

$L_{\pm} = L_1 \#_{\pm} L_2$ is the connected sum, embedded in $X$




(Lalonde-Sikovar, Polterovich)

# Theorem 3 (F-Oh-Ohta-Ono, will be in the revised version of Chapter 10)

Assueme Maslov index of $L_i$ are zero.

(1) $$\mathfrak{M}(L_\pm) = \mathfrak{M}(L_1) \times \mathfrak{M}(L_2)$$

(2) If $b_i \in \mathfrak{M}(L_i)$ $b' \in \mathfrak{M}(L')$

There exist long exact sequences

$$\to HF((L',b'),(L_2,b_2)) \to HF((L',b'),(L_-,(b_1,b_2)) \to HF((L',b'),(L_1,b_1)) \to$$

and

$$\to HF((L',b'),(L_1,b_1)) \to HF((L',b'),(L_+,(b_1,b_2)) \to HF((L',b'),(L_2,b_2)) \to$$

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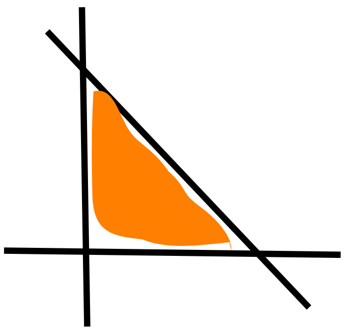
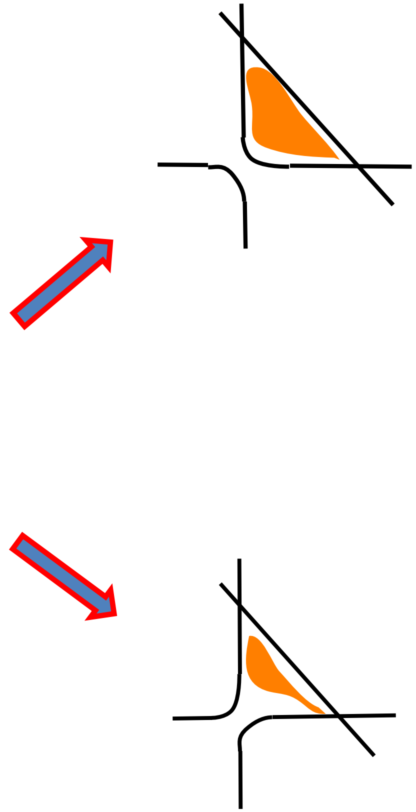
Note: Proved before by P. Seidel
in case $L_1$ or $L_2$ is a sphere and exact case
(= the case of Dehn twist).

Idea of the proof



Holomorphic triangle



Becomes
holomorphic 2 gon

Becomes $S^{n-2}$ parametrized
family of holomorphic 2 gons

Floer theory of Immersed Lagrangian submanifold.

(M. Akaho – D. Joyce, arXiv:0803.0717)

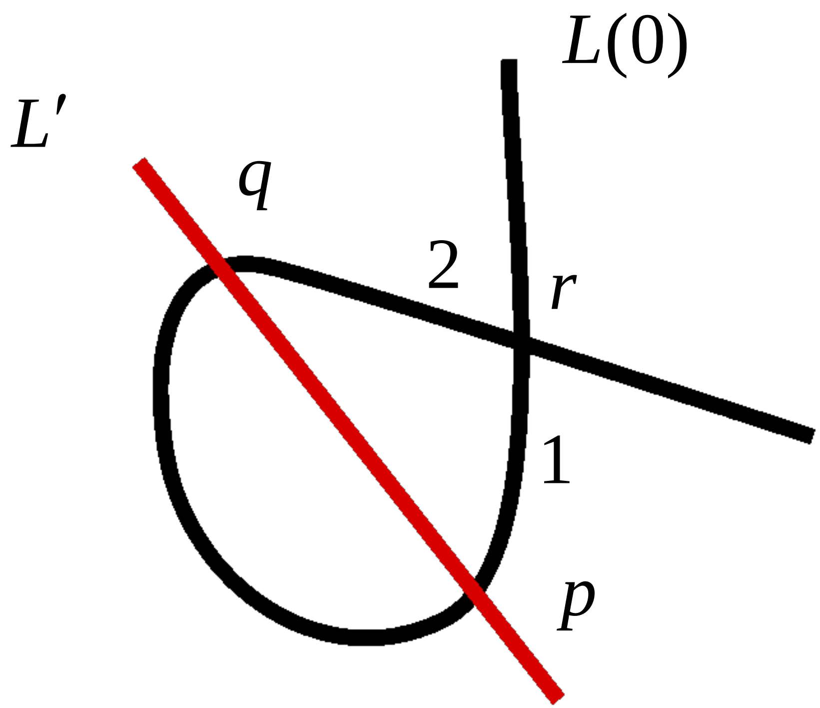
Special case: $L(0)$ is immersed 2-sphere with one self intersection point which is transversal.

$$\mathfrak{M}(L(0)) = \Lambda_0 \oplus \Lambda_0$$

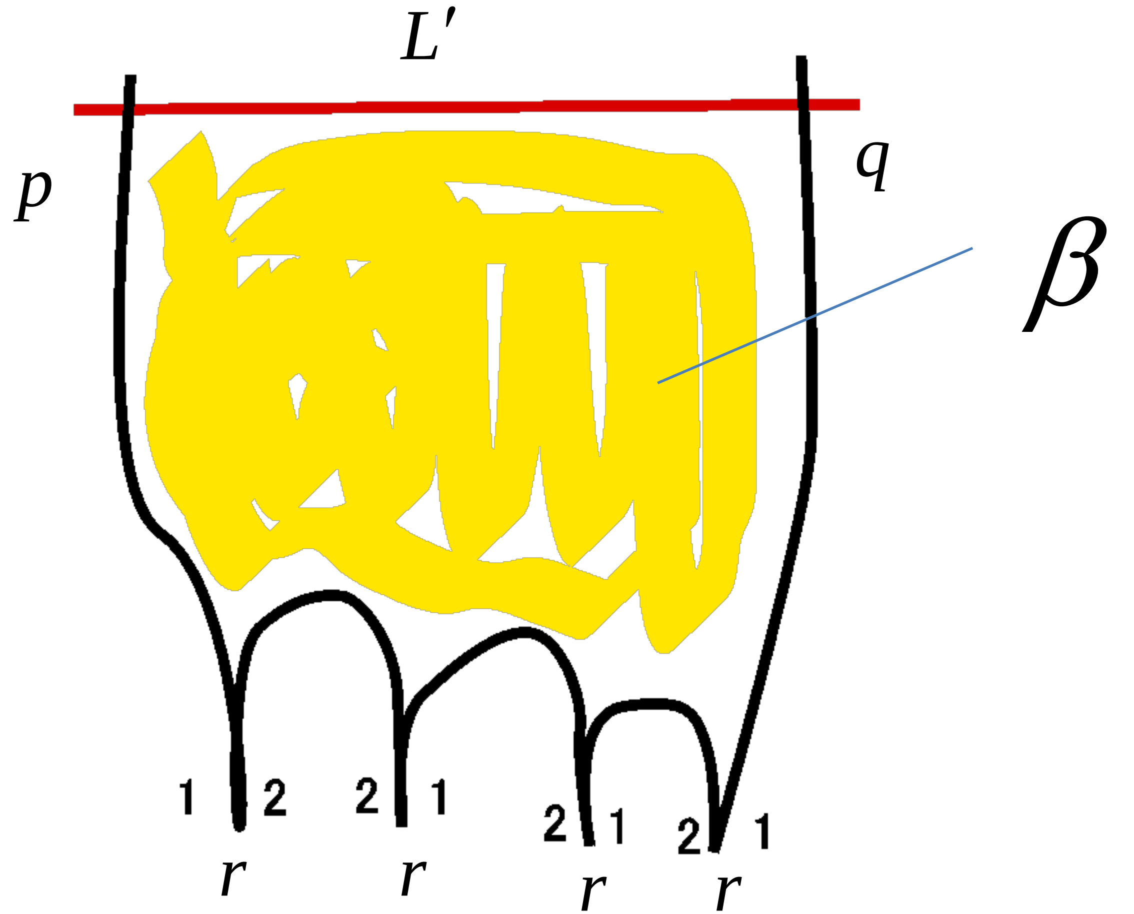
$HF((L'b'),(L(0),b))$ is parametrized by

$$b = (x_1, x_2) \in \mathfrak{M}(L(0)) = \Lambda_0 \oplus \Lambda_0$$

$$\left\langle \partial_{(x_1,x_2)} p, q \right\rangle = \sum_{\substack{(\beta,k_1,k_2) \\ k_1,k_2=0,1,2,\dots}} T^{\beta\cap\omega} x_1^{k_1} x_2^{k_2} \# M((\beta,k_1,k_2); p,q)$$



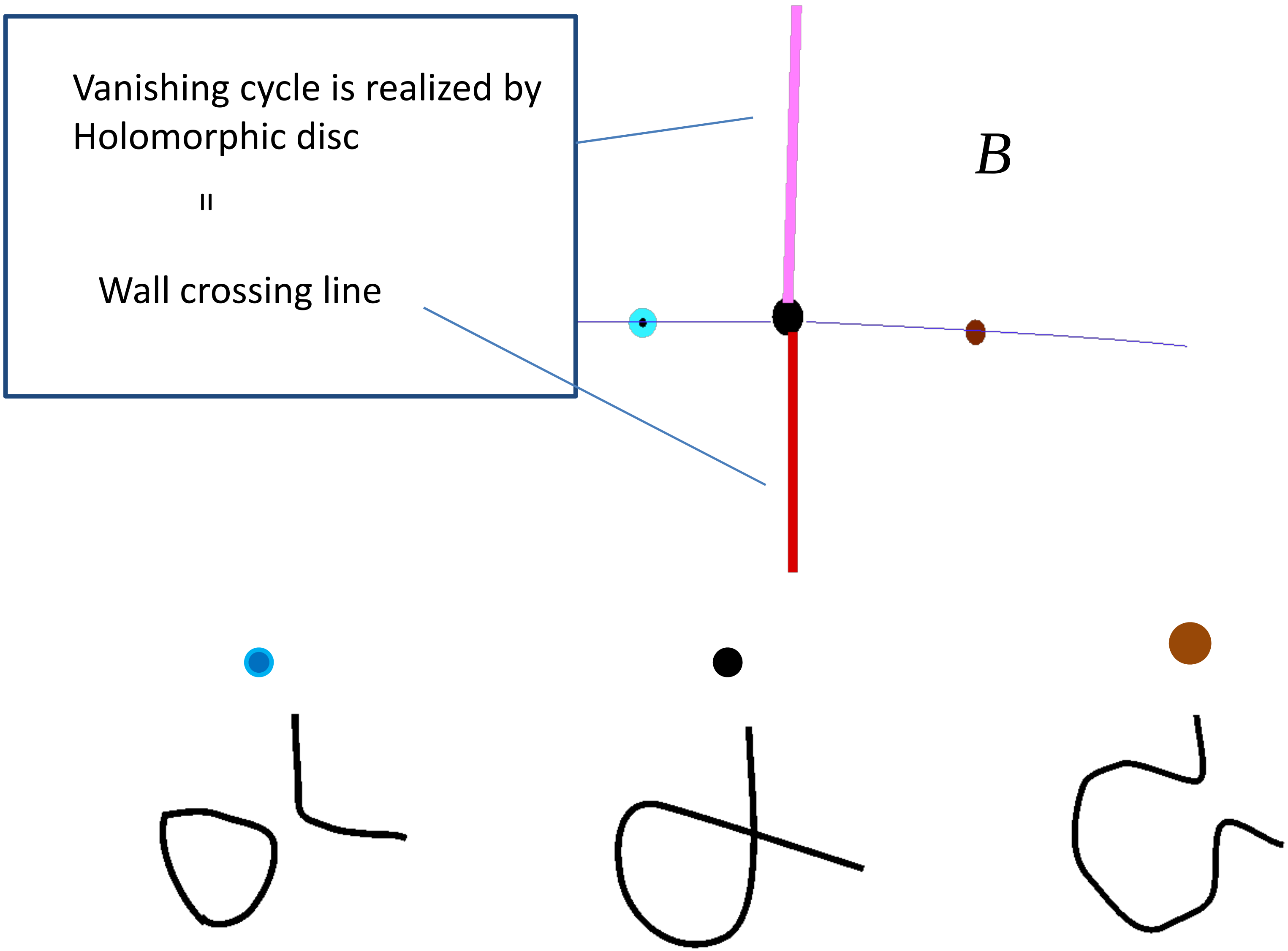
$$M((\beta, k_1, k_2); p, q)$$

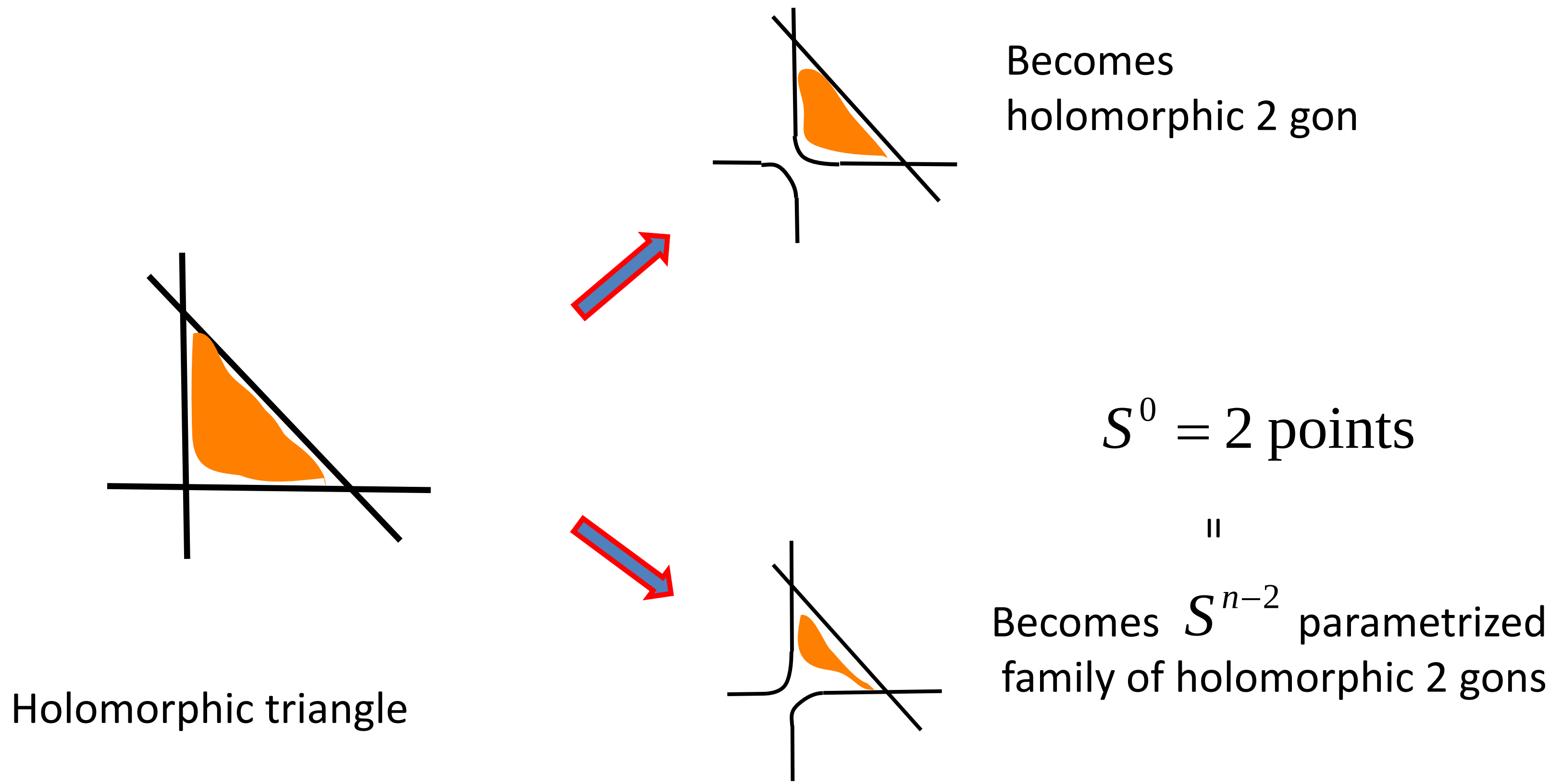


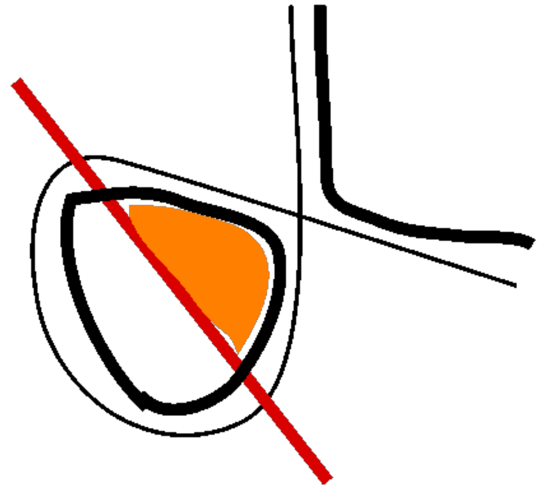

$$k_1 = 1, \quad k_2 = 3$$

Resolve singularity by surgery.

There are 2 parameter family of smooth Lagrangian $T^2$ obtained.

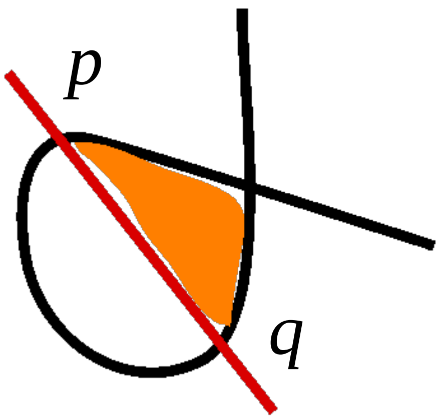

Vanishing cycle is realized by
Holomorphic disc
||
Wall crossing line
B


Becomes
holomorphic 2 gon
$S^0 = 2$ points
=
Becomes $S^{n-2}$ parametrized
family of holomorphic 2 gons
Holomorphic triangle

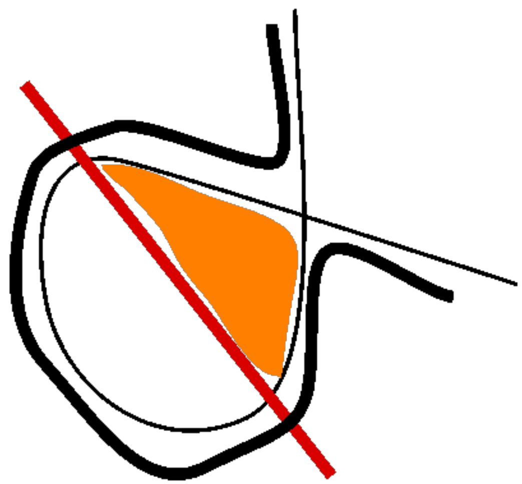


$T^{\beta\cap\omega} y_1$

One disc



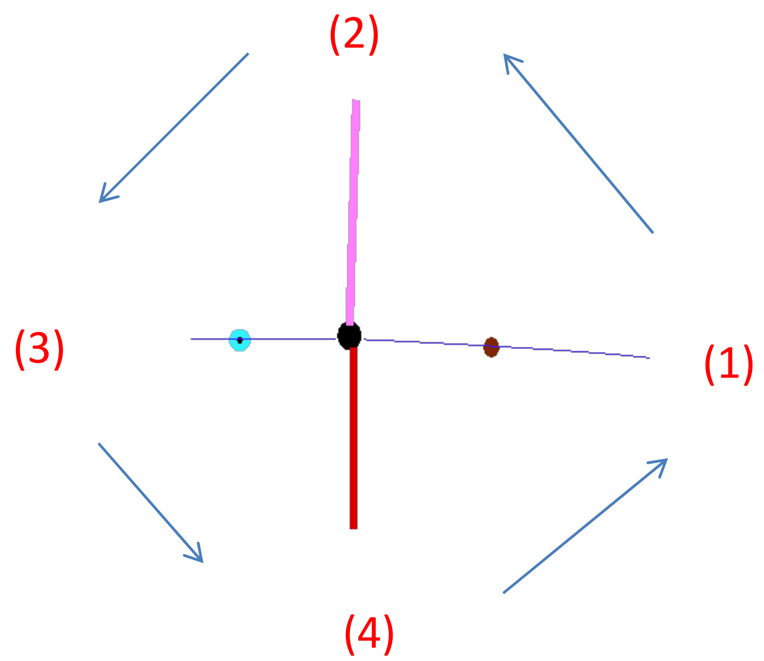

$T^{\beta\cap\omega} x_1^{k_1} x_2^{k_2} \mapsto T^{\beta\cap\omega} x_1$



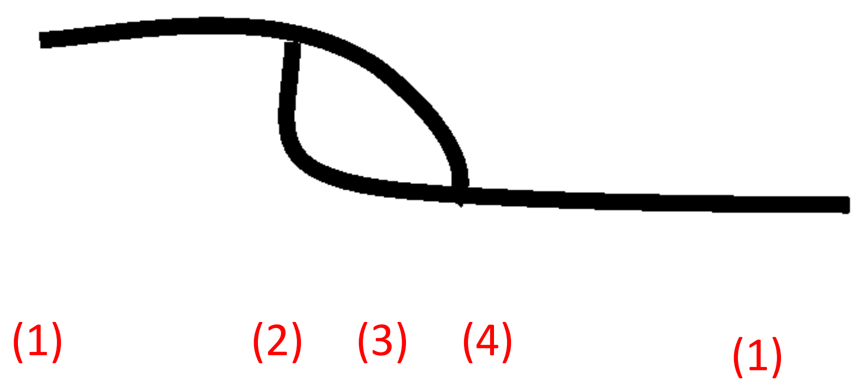
$T^{\beta\cap\omega}(y_1 \pm y_1 y_2)$

Two discs

Bifurcation of
the moduli space of holomorphic strip
around `type one' singular fiber
(F-Oh-Ohta-Ono 2000 version of [FOOO])




Moduli of holomorphic strips



Vanishing cycle is realized by
Holomorphic disc

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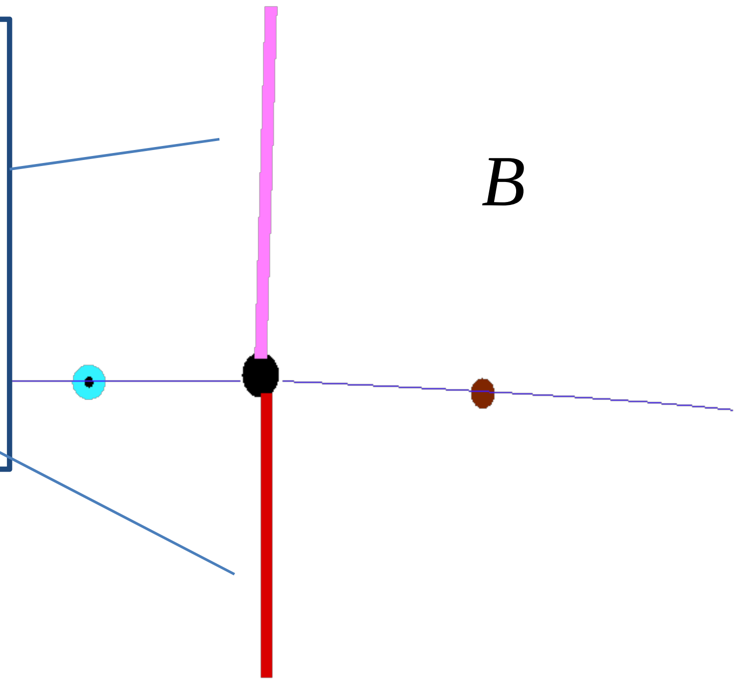
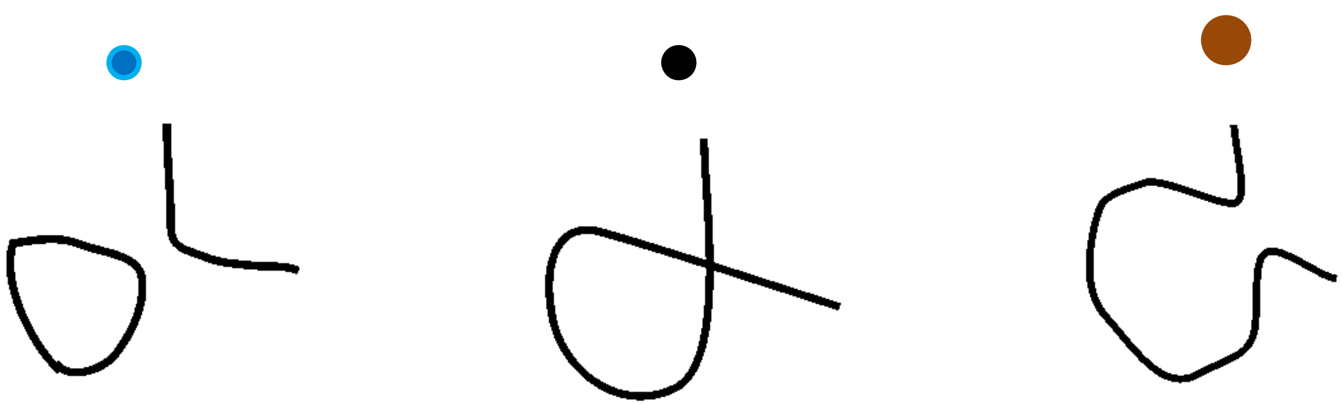
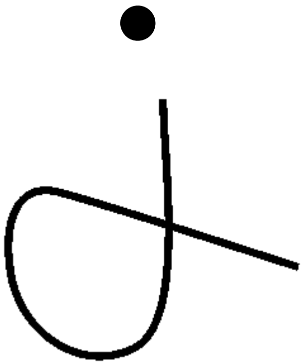
Wall crossing line

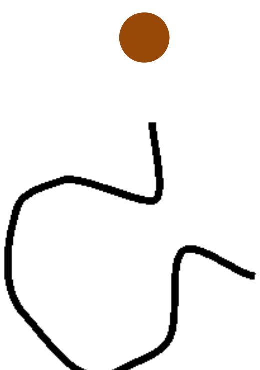


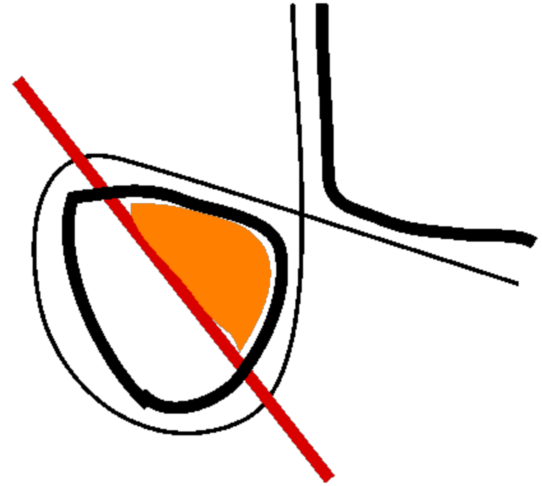


$$\left\langle \partial_{(x_1,x_2)} p, q \right\rangle = \sum_{(\beta,k_1,k_2)} T^{\beta\cap\omega} x_1^{k_1} x_2^{k_2} \# M((\beta,k_1,k_2); p,q)$$

$$\left\langle \partial_{(y_1,y_2)} p, q \right\rangle = \sum_{(\beta,k_1,k_2)} T^{\beta\cap\omega} y_1^{k_1} (y_1^{-1} \pm y_1^{-1} y_2)^{k_2} \# M((\beta,k_1,k_2); p,q)$$

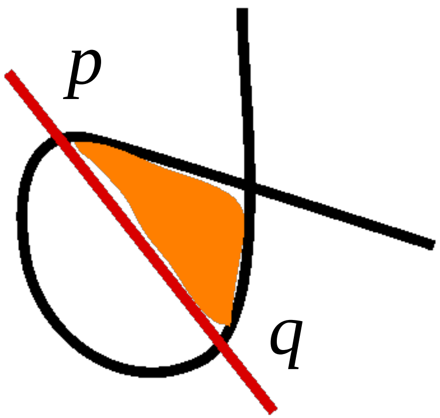


$$\left\langle \partial_{(y_1,y_2)} p, q \right\rangle = \sum_{(\beta,k_1,k_2)} T^{\beta\cap\omega} (y_1 \pm y_1 y_2)^{k_1} y_1^{-k_2} \# M((\beta,k_1,k_2); p,q)$$

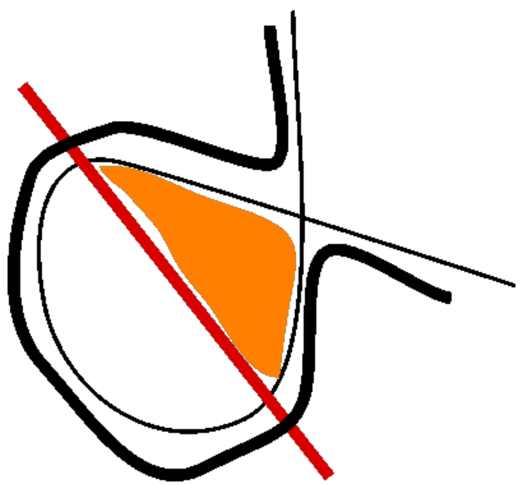


$T^{\beta\cap\omega}y_1$

One disc




$T^{\beta\cap\omega}x_1^{k_1}x_2^{k_2} \mapsto T^{\beta\cap\omega}x_1$

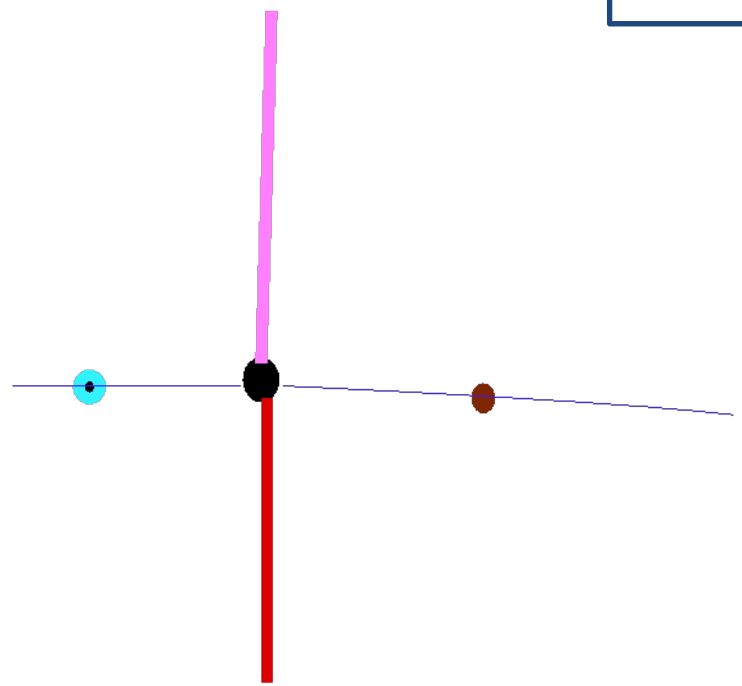


$T^{\beta\cap\omega}(y_1 \pm y_1 y_2)$

Two discs

This coordinate change is the same as one appearing in the work by Gross-Siebert.

$$y_1 = y_1' \pm y_1' y_2$$



$$x_1 = y_1$$
$$x_2 = y_1^{-1} \pm y_1^{-1} y_2$$

$$x_1 = y_1' \pm y_1' y_2$$
$$x_2 = y_1'^{-1}$$

$$y_1'' = y_1 \pm y_1 y_2^{-1}$$

$$y_1'' = \pm y_1 y_2^{-1}$$

This is the monodromy by the Dehn twist.
(= singularity of affine structure).