Minimal model program for projective morphisms between complex analytic spaces

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By BCHM, we have the following theorem.

Theorem 1 (BCHM)

- X, Y: quasi-projective *algebraic* varieties
- $\pi\colon X \to Y$: projective morphism
- (X, Δ): $\mathbb{Q}$-factorial klt such that Δ: π-big
- $C \geq 0$ such that $K_X + \Delta + C$: π-nef and $(X, \Delta + C)$: klt

$\Longrightarrow$ we can run the $(K_X + \Delta)$-MMP/Y with scaling of C

$\Longrightarrow$ we finally get a *minimal model*/Y or a *Mori fiber space structure*/Y

As an application of Theorem 1, we have:

Theorem 2 (BCHM)

- X, Y: quasi-projective algebraic varieties,
- $\pi\colon X \to Y$: projective morphism,
- (X, Δ): klt,

Assume that

- Δ is π-big and $K_X + \Delta$ is π-pseudo-effective, or
- $K_X + \Delta$ is π-big

$\Longrightarrow$

(1) $K_X + \Delta$ has a minimal model over Y

(2) $K_X + \Delta$: π-big $\Longrightarrow$ $K_X + \Delta$ has a log canonical model over Y

(3) if $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then

$$R(X/Y, K_X + \Delta) := \bigoplus_{m \in \mathbb{N}} \pi_* \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor)$$

is finitely generated as an $\mathcal{O}_Y$-algebra

Comments on BCHM

- It is well known that Theorems 1 and 2 give many non-trivial applications. We do not repeat them here.
- Theorems 1 and 2 follow from Hironaka's resolution of singularities and the Kawamata–Viehweg vanishing theorem.
- Hence there is no obstruction to generalize Theorems 1 and 2 for projective morphisms between complex analytic spaces.

Algebraic vs Analytic

There are differences between birational geometry and bimeromorphic geometry.

Example 3 (Serre)

Let C be an elliptic curve and let $\mathcal{E}$ be the rank two vector bundle on C which is defined by the unique non-splitting extension

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_C \to 0$$

$\mathbb{C}^\times \times \mathbb{C}^\times$ is a complex manifold which is Stein, where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. We have the following two compactifications of $\mathbb{C}^\times \times \mathbb{C}^\times$:

$$\mathbb{P}_C(\mathcal{E}) \xleftarrow[\text{ana}]{} \mathbb{C}^\times \times \mathbb{C}^\times \xhookrightarrow[\text{alg}]{} \mathbb{P}^1 \times \mathbb{P}^1$$

Note that $\mathbb{P}_C(\mathcal{E})$ is not bimeromorphically equivalent to $\mathbb{P}^1 \times \mathbb{P}^1$.

How to set up

We obtain an analytic version of BCHM. One of the most difficult problems is how to formulate it.

4 (Setting)

- X, Y: complex *analytic* spaces
- $\pi\colon X \to Y$: projective morphism
- W: *Stein compact subset* of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian

This is the standard setting in our complex analytic BCHM. We will see this is a correct setting for our purpose.

Theorem 5

- X, Y, $\pi\colon X \to Y$, and W: as in 4
- (X, Δ): klt, Δ: π-big
- X: $\mathbb{Q}$-factorial over W
- $C \geq 0$ such that $K_X + \Delta + C$ is klt and π-nef over W

$\Longrightarrow$ we can run the $(K_X + \Delta)$-MMP with scaling of C over Y

Hence we have a finite sequence of flips and divisorial contractions over Y

$$(X, \Delta) =: (X_0, \Delta_0) \dashrightarrow \cdots \dashrightarrow (X_i, \Delta_i) \dashrightarrow \cdots \dashrightarrow (X_m, \Delta_m)$$

as usual such that (X_m, Δ_m) is a minimal model$/Y$ or has a Mori fiber space structure$/Y$

Note that each step exists only after shrinking Y around W suitably.

As in the algebraic case, we have:

Theorem 6

- X, Y, $\pi\colon X \to Y$, and W: as in 4, and (X, Δ): klt

Assume that

- Δ is π-big and $K_X + \Delta$ is π-pseudo-effective, or
- $K_X + \Delta$ is π-big

$\Longrightarrow$

(1) $K_X + \Delta$ has a minimal model over some open neighborhood of W

(2) $K_X + \Delta$: π-big $\Longrightarrow$ $K_X + \Delta$ has a log canonical model over some open neighborhood of W

(3) if $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then

$$R(X/Y, K_X + \Delta) := \bigoplus_{m \in \mathbb{N}} \pi_* \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor)$$

is a locally finitely generated graded $\mathcal{O}_Y$-algebra

There are some subtle problems. Let (X, Δ) be klt or lc.

Remark 7 (π-ampleness over W)

$K_X + \Delta$ is π-ample over W

$\underset{\text{def}}{\iff}$ $(K_X + \Delta)|_{\pi^{-1}(w)}$ is ample for every $w \in W$

$\iff$ $K_X + \Delta$ is π-ample over some open neighborhood of W

π-ampleness behaves well. However, π-nefness does not so.

Remark 8 (π-nefness over W)

$K_X + \Delta$ is π-nef over W

$\underset{\text{def}}{\iff}$ $(K_X + \Delta)|_{\pi^{-1}(w)}$ is nef for every $w \in W$

$\underset{?}{\Longrightarrow}$ $K_X + \Delta$ is π-nef over some open neighborhood of W

Conjectures

It is natural to ask:

Conjecture 9

Let $\pi\colon X \to Y$ be a projective morphism between complex analytic spaces. Let (X, Δ) be a klt pair. If $(K_X + \Delta)|_{\pi^{-1}(P)}$ is nef, then $K_X + \Delta$ is π-nef over some open neighborhood of P.

More generally, we ask:

Conjecture 10 (Abundance conjecture)

Let $\pi\colon X \to Y$ be a projective morphism between complex analytic spaces. Let (X, Δ) be a klt pair. If $(K_X + \Delta)|_{\pi^{-1}(P)}$ is nef, then $K_X + \Delta$ is π-semiample over some open neighborhood of P.

The above conjectures are widely open.

BPF theorem

In Theorems 5 and 6, we can use the following BPF theorem.

Theorem 11 (BPF theorem, Nakayama)

- $\pi\colon X \to Y$: projective morphism of complex analytic spaces
- (X, Δ): klt
- L: a Cartier divisor on X
- $P \in Y$: a point

Assume that $L|_{\pi^{-1}(P)}$ is nef and $(aL - (K_X + \Delta))\,|_{\pi^{-1}(P)}$ is ample for some positive real number a.

$\Longrightarrow$

there exists an open neighborhood U of P such that $\mathcal{O}_X(mL)$ is π-generated over U for every $m \gg 0$. In particular, L is π-nef over U.

Motivations

- (Singularities). Let $P \in X$ be an analytic germ. Let $\pi\colon Z \to X$ be a projective resolution. We can apply our analytic MMP to $\pi\colon Z \to X$. Then we get a partial resolution $\pi'\colon Z' \to X$ of $P \in X$. We can use $\pi'\colon Z' \to X$ for the study of $P \in X$.

- (Degenerations). Let $f\colon X \to \Delta$ be a degeneration of projective varieties, where Δ is a unit disk. We can run the K_X-minimal model program/Δ for the study of $f^{-1}(0)$.

Stein compact subsets

Let us see the definition of Stein compact subsets.

Definition 12 (Stein compact subsets)

A compact subset K of a complex analytic space is called *Stein compact* if it admits a fundamental system of Stein open neighborhoods.

We recall a characterization of Stein spaces.

Remark 13 (Stein spaces)

X: Stein space

$\underset{\text{equiv}}{\iff} H^i(X, \mathcal{F}) = 0$ for every coherent sheaf $\mathcal{F}$ and for every $i > 0$

We can find many Stein compact subsets by the following lemma.

Lemma 14

K: compact subset of a Stein space X

$$\widehat{K} := \left\{ x \in X : |f(x)| \leq \sup_{z \in K} |f(z)| \text{ for every } f \in \Gamma(X, \mathcal{O}_X) \right\}$$

$\widehat{K}$ is the holomorphically convex hull of X

$\Longrightarrow$

$\widehat{K}$: Stein compact subset of X

Example 15 (Cantor set)

- $X = \{z \in \mathbb{C} \mid |z| < 2\}$
- C: Cantor set. Note: $C \subset [0, 1] \subset X$.

Then C is a Stein compact subset of X.

Unfortunately,

$$\mathcal{O}_X(C) = \Gamma(C, \mathcal{O}_X) = \varinjlim_{C \subset U} \Gamma(U, \mathcal{O}_X)$$

is not noetherian.

Siu's theorem

The following theorem due to Siu is very important.

Theorem 16 (Siu)

Let K be a Stein compact subset of a complex analytic space X. Then $\mathcal{O}_X(K) = \Gamma(K, \mathcal{O}_X)$ is noetherian if and only if

($\star$) $K \cap Z$ has only finitely many connected components for any analytic subset Z which is defined over an open neighborhood of K.

Note that the Cantor set C has infinitely many connected components.

($\star$) plays a crucial role!

Remark 17

If K is a compact *semianalytic* subset, then K always satisfies $(\star)$.

How to formulate analytic MMP

- $\pi\colon X \to Y$: projective morphism of complex analytic spaces
- W: compact subset of Y
- $Z_1(X/Y;W)$: free abelian group generated by the projective integral curves C on X such that $\pi(C)$ is a point of W

We can consider the following intersection pairing

$$\mathrm{Pic}\left(\pi^{-1}(U)\right) \times Z_1(X/Y;W) \to \mathbb{Z}$$

as usual, where U is an open neighborhood of W.
We put

$$\widetilde{A}(U,W) := \mathrm{Pic}\left(\pi^{-1}(U)\right)/\equiv$$

and

$$A^1(X/Y;W) := \varinjlim_{W\subset U} \widetilde{A}(U,W)$$

In general, $A^1(X/Y;W)$ is not finitely generated!

Nakayama's finiteness

We recall Nakayama's finiteness.

Theorem 18 (Nakayama)

- $\pi\colon X \to Y$: projective morphism of complex analytic spaces
- W: compact subset of Y

Assume that

($\star$) $W \cap Z$ has only finitely many connected components for any analytic subset Z which is defined over an open neighborhood of W.

$\Longrightarrow A^1(X/Y;W)$ is a finitely generated abelian group

($\star$) is very important!

How to formulate analytic MMP, 2

When $A^1(X/Y;W)$ is finitely generated, we can put

$$N^1(X/Y;W) := A^1(X/Y;W) \otimes_{\mathbb{Z}} \mathbb{R}$$

and define the Kleiman–Mori cone

$$\overline{\mathrm{NE}}(X/Y;W),$$

and so on.

We can formulate and prove Kleiman's ampleness criterion and the cone and contraction theorem under the assumption that $A^1(X/Y;W)$ is a finitely generated abelian group.

Therefore, we see that the following setting is reasonable.

19 (Setting, see 4)

- X, Y: complex analytic spaces,
- $\pi\colon X \to Y$: projective morphism,
- W: Stein compact subset of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian

All we have to do is to check that the arguments in BCHM can work in the above setting.

Abundance theorem

On the abundance conjecture for projective morphisms between complex analytic spaces, we have:

Theorem 20 (Abundance theorem)

- X, Y, $\pi\colon X \to Y$, and W: as in 4
- (X, Δ): klt

Assume that the abundance conjecture holds for projective klt pairs in dimension n.

$\Longrightarrow$

If $K_X + \Delta$ is π-nef over Y and $\dim X - \dim Y = n$, then $K_X + \Delta$ is π-semiample over some open neighborhood of W.

When (X, Δ) is algebraic, the following theorem is well known.

Theorem 21 (Basic properties of lc centers)

- (X, Δ): lc

$\Longrightarrow$

(1) The intersection of two lc centers is a union of some lc centers.

(2) Let $x \in X$ be any point such that (X, Δ) is lc but is not klt at x. Then there exists a unique minimal lc center C_x passing through x. Moreover, C_x is normal at x.

It plays a crucial role for the study of lc pairs. We need some vanishing theorems more powerful than the KV vanishing theorem.

It was a big problem to establish necessary vanishing theorems in the complex analytic setting.

Theorem 22

- (X, Δ): an analytic SNC pair, Δ: a boundary $\mathbb{R}$-divisor
- $f\colon X \to Y$: a projective morphism of complex analytic spaces
- $\mathcal{L}$: a line bundle on X
- q: an arbitrary non-negative integer

$\Longrightarrow$

(i) (Strict support condition). If $\mathcal{L} - (\omega_X + \Delta)$ is f-semiample, then every associated subvariety of $R^q f_*\mathcal{L}$ is the f-image of some stratum of (X, Δ).

(ii) (Vanishing theorem). If $\mathcal{L} - (\omega_X + \Delta) \sim_{\mathbb{R}} f^*\mathcal{H}$ holds for some π-ample $\mathbb{R}$-line bundle $\mathcal{H}$ on Y, where $\pi\colon Y \to Z$ is a projective morphism to a complex analytic space Z, then we have $R^p\pi_*R^q f_*\mathcal{L} = 0$ for every $p > 0$.

- once we have Theorem 22, we can prove Theorem 21 for complex analytic spaces.
- we can prove the cone and contraction theorem and so on for lc pairs in the complex analytic setting.
- we can prove Theorem 22 with the aid of Saito's theory of mixed Hodge modules and Takegoshi's result.
- (with Fujisawa) now we have an alternative approach to Theorem 22 without using Saito's theory of MHM.

Although there are still many problems to work out, we established the framework of MMP for projective morphisms between complex analytic spaces.

Thank you very much!