A REMARK ON THE LOG MMP (PRIVATE NOTE)

OSAMU FUJINO

Abstract. In this short note, we treat the log MMP without the assumption that the variety is $\mathbb{Q}$-factorial.

This short note is an answer to Takagi's question. We will work over $\mathbb{C}$ throughout this note. For simplicity, we treat only klt pairs and $\mathbb{Q}$-divisors in this note.

Theorem 1. *Assume that the log MMP holds for $\mathbb{Q}$-factorial klt paris in dimension n. Then the following modified version of the log MMP works for (not necessarily $\mathbb{Q}$-factorial) klt pairs in dimension n.*

Proof and explanation of the log MMP without $\mathbb{Q}$-factoriality. We start with a projective morphism $f : X \longrightarrow Y$, where $X_0 := X$ is a (not necessarily $\mathbb{Q}$-factorial) normal variety, and a $\mathbb{Q}$-divisor $D_0 := D$ on X such that (X, D) is klt. The aim is to set up a recursive procedure which creates intermediate $f_i : X_i \longrightarrow Y$ and D_i. After finitely many steps, we obtain a finial objects $\widetilde{f} : \widetilde{X} \longrightarrow Y$ and $\widetilde{D}$. Assume that we already constructed $f_i : X_i \longrightarrow Y$ and D_i with the following properties:

(i) f_i is projective,
(ii) D_i is a $\mathbb{Q}$-divisor on X_i,
(iii) (X_i, D_i) is klt.

If $K_{X_i} + D_i$ is f_i-nef, then we set $\widetilde{X} := X_i$ and $\widetilde{D} := D_i$. Assume that $K_{X_i} + D_i$ is not f_i-nef. Then we can take a $(K_{X_i} + D_i)$-negative extremal ray R (or, more generally, a $(K_{X_i} + D_i)$-negative extremal face F) of $\overline{NE}(X_i/Y)$. Thus we have a contraction morphism $\varphi_R : X_i \longrightarrow W_i$ over Y. If $\dim W_i < \dim X_i$, then we set $\widetilde{X} := X_i$ and $\widetilde{D} := D_i$ and stop the process. If φ_R is birational, then we put $X_{i+1} := \mathrm{Proj}_{W_i} \bigoplus_{m \geq 0} \varphi_{R*}\mathcal{O}_{X_i}(m(K_{X_i} + D_i))$, $D_{i+1} :=$ the proper transform of $\varphi_{R*}D_i$ on X_{i+1} and repeat this process. We note that (X_{i+1}, D_{i+1}) is the log canonical model of (X_i, D_i) over W_i. If $K_{W_i} + \varphi_{R*}D_i$ is $\mathbb{Q}$-Cartier, then $X_{i+1} \simeq W_i$. So, this process coincides with the usual one if the varieties X_i are $\mathbb{Q}$-factorial. It is not difficult to see that

Date: 2003/6/11.
This note was written in order to answer Takagi's question.

$X_i \longrightarrow W_i \longleftarrow X_{i+1}$ is of type (DS) or (SS) (for the definitions of (DS) and (SS), see Definition 6 in [F]). Then, this process always terminates by the same proof as in [F]. For the details, see the final part of Step 2 in the proof of Theorem 1 in [F]. □

References

[F] O. Fujino, On special termination, preprint, 2002.

Graduate School of Mathematics, Nagoya University, Chikusa-ku Nagoya 464-8602 Japan

E-mail address: fujino@math.nagoya-u.ac.jp