VANISHING AND INJECTIVITY THEOREMS FOR LMMP

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Abstract. We discuss cohomology injectivity and vanishing theorems for the LMMP. This paper contains a completely final form of the Kawamata–Viehweg vanishing theorem for log canonical pairs. The results in this paper are indispensable for the theory of quasi-log varieties.

Contents

1. Introduction

The following diagram is well known and described, for example, in [KM, §3.1].

Kawamata–Viehweg vanishing theorem for klt pairs	$\Longrightarrow$	Cone, contraction, rationality, and base point free theorems for klt pairs

This means that the Kawamata–Viehweg vanishing theorem produces the fundamental theorems of the log minimal model program (LMMP, for short) for klt pairs. This method is sometimes called X-method and now classical. It is sufficient for the LMMP for $\mathbb{Q}$-factorial

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dlt pairs. In [A1], Ambro obtained the same diagram for *quasi-log varieties*. Note that the class of quasi-log varieties naturally contains lc pairs. Ambro introduced the notion of quasi-log varieties for the inductive treatments of lc pairs.

Kollár's torsion-free and vanishing theorems for embedded normal crossing pairs	$\Longrightarrow$	Cone, contraction, rationality, and base point free theorems for quasi-log varieties

Namely, if we obtain Kollár's torsion-free and vanishing theorems for *embedded normal crossing pairs*, then X-method works and we obtain the fundamental theorems of the LMMP for quasi-log varieties. Unfortunately, the proofs of torsion-free and vanishing theorems in [A1, Section 3] contains various gaps. So, there exists an important open problem for the LMMP for lc paris.

Problem 1.1. Are the injectivity, torsion-free and vanishing theorems for embedded normal crossing pairs true?

Once this question is solved affirmatively, we can obtain the fundamental theorems of the LMMP for lc pairs. The X-method, which was explained in [A1, Section 5], is essentially the same as the klt case. It may be more or less a routine work for the experts (see [F12]). In this paper, we give an affirmative answer to Problem 1.1.

Theorem 1.2. *Ambro's formulation of Kollár's injectivity, torsion-free, and vanishing theorems for embedded normal crossing pairs hold true.*

Ambro's proofs in [A1] do not work even for smooth varieties. So, we need new ideas to prove the desired injectivity, torsion-free, vanishing theorems. It is the main subject of this paper. We will explain various troubles in the proofs in [A1, Section 3] below. Here, we give an application of Ambro's theorems to motivate the reader. It is the culmination of the works of several authors: Kawamata, Viehweg, Nadel, Reid, Fukuda, Ambro, and many others. It is the first time that the following theorem is stated explicitly in the literature.

Theorem 1.3 (cf. Theorem 5.17)**.** *Let (X, B) be a proper lc pair such that B is a boundary $\mathbb{R}$-divisor and let L be a $\mathbb{Q}$-Cartier Weil divisor on X. Assume that $L-(K_X+B)$ is nef and log big. Then $H^q(X, \mathcal{O}_X(L)) = 0$ for any $q > 0$.*

It also contains a complete form of Kovács' Kodaira vanishing theorem for lc pairs (see Corollary 5.11). Let us explain the main trouble in [A1, Section 3] by the following simple example.

Example 1.4. Let X be a smooth projective variety and H a Cartier divisor on X. Let A be a smooth member of $|2H|$ and S a smooth divisor on X such that S and A are disjoint. We put $B = \frac{1}{2}A + S$ and $L = H + K_X + S$. Then $L \sim_{\mathbb{Q}} K_X + B$ and $2L \sim 2(K_X + B)$. We define $\mathcal{E} = \mathcal{O}_X(-L + K_X)$ as in the proof of [A1, Theorem 3.1]. Apply the argument in the proof of [A1, Theorem 3.1]. Then we have a double cover $\pi : Y \to X$ corresponding to $2B \in |\mathcal{E}^{-2}|$. Then $\pi_*\Omega^p_Y(\log \pi^*B) \simeq \Omega^p_X(\log B) \oplus \Omega^p_X(\log B) \otimes \mathcal{E}(S)$. Note that $\Omega^p_X(\log B) \otimes \mathcal{E}$ is not a direct summand of $\pi_*\Omega^p_Y(\log \pi^*B)$. Theorem 3.1 in [A1] claims that the homomorphisms $H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D))$ are injective for all q. Here, we used the notation in [A1, Theorem 3.1]. In our case, $D = mA$ for some positive integer m. However, Ambro's argument just implies that $H^q(X, \mathcal{O}_X(L - \llcorner B \lrcorner)) \to H^q(X, \mathcal{O}_X(L - \llcorner B \lrcorner + D))$ is injective for any q. Therefore, his proof works only for the case when $\llcorner B \lrcorner = 0$ even if X is smooth.

This trouble is crucial in several applications on the LMMP. Ambro's proof is based on the mixed Hodge structure of $H^i(Y - \pi^*B, \mathbb{Z})$. It is a standard technique for vanishing theorems in the LMMP. In this paper, we use the mixed Hodge structure of $H^i_c(Y - \pi^*S, \mathbb{Z})$, where $H^i_c(Y - \pi^*S, \mathbb{Z})$ is the cohomology group with compact support. Let us explain the main idea of this paper. Let X be a smooth projective variety with $\dim X = n$ and D a simple normal crossing divisor on X. The main ingredient of our arguments is the decomposition

$$H^i_c(X - D, \mathbb{C}) = \bigoplus_{p+q=i} H^q(X, \Omega^p_X(\log D) \otimes \mathcal{O}_X(-D)).$$

The dual statement

$$H^{2n-i}(X - D, \mathbb{C}) = \bigoplus_{p+q=i} H^{n-q}(X, \Omega^{n-p}_X(\log D)),$$

which is well known and is commonly used for vanishing theorems, is not useful for our purposes. To solve Problem 1.1, we have to carry out this simple idea for reducible varieties.

Remark 1.5. In the proof of [A1, Theorem 3.1], if we assume that X is smooth, $B' = S$ is a reduced smooth divisor on X, and $T \sim 0$, then we need the E_1-degeneration of

$$E^{pq}_1 = H^q(X, \Omega^p_X(\log S) \otimes \mathcal{O}_X(-S)) \Longrightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_X(\log S) \otimes \mathcal{O}_X(-S)).$$

However, Ambro seemed to confuse it with the E_1-degeneration of

$$E^{pq}_1 = H^q(X, \Omega^p_X(\log S)) \Longrightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_X(\log S)).$$

Some problems on the Hodge theory seem to exist in the proof of [A1, Theorem 3.1].

Remark 1.6. In [A2, Theorem 3.1], Ambro reproved his theorem under some extra assumptions. Here, we use the notation in [A2, Theorem 3.1]. In the last line of the proof of [A2, Theorem 3.1], he used the E_1-degeneration of some spectral sequence. It seems to be the E_1-degeneration of

$$E_1^{pq} = H^q(X', \widetilde{\Omega}^p_{X'}(\log \sum_{i'} E'_{i'})) \Longrightarrow \mathbb{H}^{p+q}(X', \widetilde{\Omega}^\bullet_{X'}(\log \sum_{i'} E'_{i'}))$$

since he cited [D1, Corollary 3.2.13]. Or, he applied the same type of E_1-degeneration to a desingularization of X'. However, we think that the E_1-degeneration of

$$\begin{aligned} E_1^{pq} &= H^q(X', \widetilde{\Omega}^p_{X'}(\log(\pi^*R + \sum_{i'} E'_{i'})) \otimes \mathcal{O}_{X'}(-\pi^*R)) \\ &\Longrightarrow \mathbb{H}^{p+q}(X', \widetilde{\Omega}^\bullet_{X'}(\log(\pi^*R + \sum_{i'} E'_{i'})) \otimes \mathcal{O}_{X'}(-\pi^*R)) \end{aligned}$$

is the appropriate one in his proof. If we assume that $T \sim 0$ in [A2, Theorem 3.1], then Ambro's proof seems to imply that the E_1-degeneration of

$$E_1^{pq} = H^q(X, \Omega^p_X(\log R) \otimes \mathcal{O}_X(-R)) \Longrightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_X(\log R) \otimes \mathcal{O}_X(-R))$$

follows from the usual E_1-degeneration of

$$E_1^{pq} = H^q(X, \Omega^p_X) \Longrightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_X).$$

Anyway, there are some problems in the proof of [A2, Theorem 3.1]. In this paper, we adopt the following spectral sequence

$$\begin{aligned} E_1^{pq} &= H^q(X', \widetilde{\Omega}^p_{X'}(\log \pi^*R) \otimes \mathcal{O}_{X'}(-\pi^*R)) \\ &\Longrightarrow \mathbb{H}^{p+q}(X', \widetilde{\Omega}^\bullet_{X'}(\log \pi^*R) \otimes \mathcal{O}_{X'}(-\pi^*R)) \end{aligned}$$

and prove its E_1-degeneration. For the details, see Sections 3 and 4.

One of the main contributions of this paper is the rigorous proof of Proposition 3.2, which we call a fundamental injectivity theorem. Even if we prove this proposition, there are still several technical difficulties to recover Ambro's theorems: Theorems 6.1 and 6.2. Some important arguments are missing in [A1]. We will discuss the other troubles on the arguments in [A1] throughout Sections 5 and 6.

1.7 (Background, history, and related topics)**.** The standard references for vanishing and injectivity theorems for the LMMP are [Ko, Part III

Vanishing Theorems] and the first half of the book [EV]. In this paper, we closely follow the presentation of [EV] and that of [A1]. Some special cases of Ambro's theorems were proved in [F1, Section 2]. The vanishing and injectivity theorems for the LMMP are treated from a transcendental viewpoint in [F3] and [F4]. The reader who reads Japanese can find [F5] useful. It is a survey article. Chapter 1 in [KMM] is still a good source for vanishing theorems for the LMMP. We note that one of the origins of Ambro's results is [Ka, Section 4]. However, we do not treat Kawamata's vanishing and injectivity theorems for *generalized* normal crossing varieties. It is mainly because we can quickly reprove the main theorem of [Ka] without appealing these difficult vanishing and injectivity theorems once we know a generalized version of Kodaira's canonical bundle formula. For the details, see my recent preprint [F6] or [F8].

We summarize the contents of this paper. In Section 2, we collect basic definitions and fix some notations. In Section 3, we prove a fundamental cohomology injectivity theorem for simple normal crossing pairs. It is a very special case of Ambro's theorem. Our proof heavily depends on the E_1-degeneration of a certain Hodge to de Rham type spectral sequence. We postpone the proof of the E_1-degeneration in Section 4 since it is a purely Hodge theoretic argument. Section 4 consists of a short survey of mixed Hodge structures on various objects and the proof of the key E_1-degeneration. We could find no references on mixed Hodge structures which are appropriate for our purposes. So, we write it for the reader's convenience. Section 5 is devoted to the proofs of Ambro's theorems for embedded simple normal crossing pairs. We discuss various problems in [A1, Section 3] and give the first rigorous proofs to [A1, Theorems 3.1, 3.2] for embedded simple normal crossing pairs. We think that several indispensable arguments such as Lemmas 5.1, 5.2, and 5.4 are missing in [A1, Section 3]. We treat some new generalizations of vanishing and torsion-free theorems in 5.14. In Section 6, we recover Ambro's theorems in full generality. We recommend the reader to compare this paper with [A1]. We note that Section 6 seems to be unnecessary for applications. In 6.8, we will quickly review the structure of our proofs of the injectivity and vanishing theorems. It may help the reader to understand the reason why our proofs are much longer than the original proofs in [A1, Section 3]. We think that the proofs of injectivity and vanishing theorems and their applications are completely different topics. So, we do not treat any applications for quasi-log varieties in this paper. We recommend the interested reader to see [A1, Sections 4 and 5] and [F7]. The reader can find various other applications of our new cohomological results in [F9], [F10], and [F11]. To tell the truth, we do not need the notion of normal crossing pairs for the theory of quasi-log varieties. For the details, see [F7].

Notation. For an $\mathbb{R}$-Weil divisor $D = \sum_{j=1}^{r} d_j D_j$ such that $D_i \neq D_j$ for $i \neq j$, we define the *round-up* $\ulcorner D \urcorner = \sum_{j=1}^{r} \ulcorner d_j \urcorner D_j$ (resp. the *round-down* $\llcorner D \lrcorner = \sum_{j=1}^{r} \llcorner d_j \lrcorner D_j$), where for any real number x, $\ulcorner x \urcorner$ (resp. $\llcorner x \lrcorner$) is the integer defined by $x \leq \llcorner x \lrcorner < x+1$ (resp. $x - 1 < \llcorner x \lrcorner \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \llcorner D \lrcorner$. We call D a *boundary* (resp. *subboundary*) $\mathbb{R}$-divisor if $0 \leq d_j \leq 1$ (resp. $d_j \leq 1$) for any j.

We will work over $\mathbb{C}$, the complex number field, throughout this paper. I hope I will make no new mistakes in this paper.

2. Preliminaries

We explain basic notion according to [A1, Section 2].

Definition 2.1 (Normal and simple normal crossing varieties)**.** A variety X has *normal crossing* singularities if, for every closed point $x \in X$,

$$\widehat{\mathcal{O}}_{X,x} \simeq \frac{\mathbb{C}[[x_0, \cdots, x_N]]}{(x_0 \cdots x_k)}$$

for some $0 \leq k \leq N$, where $N = \dim X$. Furthermore, if each irreducible component of X is smooth, X is called a *simple normal crossing* variety. If X is a normal crossing variety, then X has only Gorenstein singularities. Thus, it has an invertible dualizing sheaf ω_X. So, we can define the *canonical divisor* K_X such that $\omega_X \simeq \mathcal{O}_X(K_X)$. It is a Cartier divisor on X and is well defined up to linear equivalence.

Definition 2.2 (Mayer–Vietoris simplicial resolution)**.** Let X be a simple normal crossing variety with the irreducible decomposition $X = \bigcup_{i\in I} X_i$. Let I_n be the set of strictly increasing sequences $(i_0, \cdots, i_n)$ in I and $X^n = \coprod_{I_n} X_{i_0} \cap \cdots \cap X_{i_n}$ the disjoint union of the intersections of X_i. Let $\varepsilon_n : X^n \to X$ be the disjoint union of the natural inclusions. Then $\{X^n, \varepsilon_n\}_n$ has a natural semi-simplicial scheme structure. The face operator is induced by $\lambda_{j,n}$, where $\lambda_{j,n} : X_{i_0} \cap \cdots \cap X_{i_n} \to X_{i_0} \cap \cdots \cap X_{i_{j-1}} \cap X_{i_{j+1}} \cap \cdots \cap X_{i_n}$ is the natural closed embedding for $j \leq n$ (cf. [E2, 3.5.5]). We denote it by $\varepsilon : X^{\bullet} \to X$ and call it the *Mayer–Vietoris simplicial resolution* of X. The complex

$$0 \to \varepsilon_{0*}\mathcal{O}_{X^0} \to \varepsilon_{1*}\mathcal{O}_{X^1} \to \cdots \to \varepsilon_{k*}\mathcal{O}_{X^k} \to \cdots,$$

where the differential $d_k : \varepsilon_{k*}\mathcal{O}_{X^k} \to \varepsilon_{k+1*}\mathcal{O}_{X^{k+1}}$ is $\sum_{j=0}^{k+1}(-1)^j\lambda^*_{j,k+1}$ for any $k \geq 0$, is denoted by $\mathcal{O}_{X^\bullet}$. It is easy to see that $\mathcal{O}_{X^\bullet}$ is quasi-isomorphic to $\mathcal{O}_X$. By tensoring $\mathcal{L}$, any line bundle on X, to $\mathcal{O}_{X^\bullet}$, we obtain a complex

$$0 \to \varepsilon_{0*}\mathcal{L}^0 \to \varepsilon_{1*}\mathcal{L}^1 \to \cdots \to \varepsilon_{k*}\mathcal{L}^k \to \cdots,$$

where $\mathcal{L}^n = \varepsilon_n^*\mathcal{L}$. It is denoted by $\mathcal{L}^\bullet$. Of course, $\mathcal{L}^\bullet$ is quasi-isomorphic to $\mathcal{L}$. We note that $H^q(X^\bullet, \mathcal{L}^\bullet)$ is $\mathbb{H}^q(X, \mathcal{L}^\bullet)$ by the definition and it is obviously isomorphic to $H^q(X, \mathcal{L})$ for any $q \geq 0$ because $\mathcal{L}^\bullet$ is quasi-isomorphic to $\mathcal{L}$.

Definition 2.3. Let X be a simple normal crossing variety. A *stratum* of X is the image on X of some irreducible component of $X^\bullet$. Note that an irreducible component of X is a stratum of X.

Definition 2.4 (Permissible and normal crossing divisors)**.** Let X be a simple normal crossing variety. A Cartier divisor D on X is called *permissible* if it induces a Cartier divisor $D^\bullet$ on $X^\bullet$. This means that $D^n = \varepsilon_n^*D$ is a Cartier divisor on X_n for any n. It is equivalent to the condition that D contains no strata of X in its support. We say that D is a *normal crossing* divisor on X if, in the notation of Definition 2.1, we have

$$\widehat{\mathcal{O}}_{D,x} \simeq \frac{\mathbb{C}[[x_0, \cdots, x_N]]}{(x_0 \cdots x_k, x_{i_1} \cdots x_{i_l})}$$

for some $\{i_1, \cdots, i_l\} \subset \{k+1, \cdots, N\}$. It is equivalent to the condition that D^n is a normal crossing divisor on X^n for any n in the usual sense. Furthermore, let D be a normal crossing divisor on a simple normal crossing variety X. If D^n is a simple normal crossing divisor on X^n for any n, then D is called a *simple normal crossing divisor* on X.

The following lemma is easy but important. We will repeatedly use it in Sections 3 and 5.

Lemma 2.5. *Let X be a simple normal crossing variety and B a permissible $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, that is, B is an $\mathbb{R}$-linear combination of permissible Cartier divisor on X, such that $\llcorner B \lrcorner = 0$. Let A be a Cartier divisor on X. Assume that $A \sim_{\mathbb{R}} B$. Then there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor C on X such that $A \sim_{\mathbb{Q}} C$, $\llcorner C \lrcorner = 0$, and* $\mathrm{Supp} C = \mathrm{Supp} B$.

Sketch of the proof. We can write $B = A + \sum_i r_i(f_i)$, where $f_i \in \Gamma(X, \mathcal{K}_X^*)$ and $r_i \in \mathbb{R}$ for any i. Here, $\mathcal{K}_X$ is the sheaf of total quotient ring of $\mathcal{O}_X$. First, we assume that X is smooth. In this case, the claim is well known and easy to check. Perturb r_i's suitably. Then we obtain a desired $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor C on X. It is an elementary problem of the linear algebra. In the general case, we take the normalization $\varepsilon_0 : X^0 \to X$ and apply the above result to X^0, ε_0^*A, ε_0^*B, and $\varepsilon_0^*(f_i)$'s. We note that $\varepsilon_0 : X_i \to X$ is a closed embedding for any irreducible component X_i of X^0. So, we get a desired $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor C on X. □

Definition 2.6 (Simple normal crossing pair)**.** We say that the pair (X, B) is a *simple normal crossing pair* if the following conditions are satisfied.

(1) X is a simple normal crossing variety, and
(2) B is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor whose support is a simple normal crossing divisor on X.

We say that a simple normal crossing pair (X, B) is *embedded* if there exists a closed embedding $\iota : X \to M$, where M is a smooth variety of dimension $\dim X + 1$. We put $K_{X^0} + \Theta = \varepsilon_0^*(K_X + B)$, where $\varepsilon_0 : X^0 \to X$ is the normalization of X. From now on, we assume that B is a subboundary $\mathbb{R}$-divisor. A *stratum* of (X, B) is an irreducible component of X or the image of some lc center of (X^0, Θ) on X. It is compatible with Definition 2.3 when $B = 0$. A Cartier divisor D on a simple normal crossing pair (X, B) is called *permissible with respect to* (X, B) if D contains no strata of the pair (X, B).

Remark 2.7. Let (X, B) be a simple normal crossing pair. Assume that X is smooth. Then (X, B) is embedded. It is because X is a divisor on $X \times C$, where C is a smooth curve.

We give a typical example of embedded simple normal crossing pairs.

Example 2.8. Let M be a smooth variety and X a simple normal crossing divisor on M. Let A be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on M such that $\mathrm{Supp}(X + A)$ is simple normal crossing on M and that X and A have no common irreducible components. We put $B = A|_X$. Then (X, B) is an embedded simple normal crossing pair.

The following lemma is obvious.

Lemma 2.9. *Let $(X, S + B)$ be an embedded simple normal crossing pair such that $S+B$ is a boundary $\mathbb{R}$-divisor, S is reduced, and $\llcorner B \lrcorner = 0$. Let M be the ambient space of X and $f : N \to M$ the blow-up along a smooth irreducible component C of* $\mathrm{Supp}(S + B)$*. Let Y be the strict transform of X on N. Then Y is a simple normal crossing divisor on N. We can write $K_Y + S_Y + B_Y = f^*(K_X + S + B)$, where $S_Y + B_Y$ is a boundary $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y such that S_Y is reduced and $\llcorner B_Y \lrcorner = 0$. In particular, $(Y, S_Y + B_Y)$ is an embedded simple normal* *crossing pair. By the construction, we can easily check the following properties.*

(i) S_Y *is the strict transform of* S *on* Y *if* $C \subset \mathrm{Supp} B$,
(ii) B_Y *is the strict transform of* B *on* Y *if* $C \subset \mathrm{Supp} S$,
(iii) f*-image of any stratum of* $(Y, S_Y + B_Y)$ *is a stratum of* $(X, S + B)$*, and*
(iv) $R^i f_* \mathcal{O}_Y = 0$ *for* $i > 0$ *and* $f_* \mathcal{O}_Y \simeq \mathcal{O}_X$.

As a consequence of Lemma 2.9, we obtain a very useful lemma.

Lemma 2.10. *Let* (X, B_X) *be an embedded simple normal crossing pair,* B_X *a boundary* $\mathbb{R}$*-divisor, and* M *the ambient space of* X*. Then there is a projective birational morphism* $f : N \to M$*, which is a sequence of blow-ups as in* Lemma 2.9*, with the following properties.*

(i) *Let* Y *be the strict transform of* X *on* N*. We put* $K_Y + B_Y = f^*(K_X + B_X)$*. Then* (Y, B_Y) *is an embedded simple normal crossing pair. Note that* B_Y *is a boundary* $\mathbb{R}$*-divisor.*
(ii) $f : Y \to X$ *is an isomorphism at any generic points of strata of* Y*.* f*-image of any stratum of* (Y, B_Y) *is a stratum of* (X, B_X)*.*
(iii) $R^i f_* \mathcal{O}_Y = 0$ *for any* $i > 0$ *and* $f_* \mathcal{O}_Y \simeq \mathcal{O}_X$.
(iv) *There exists an* $\mathbb{R}$*-divisor* D *on* N *such that* D *and* Y *have no common irreducible components and* $\mathrm{Supp}(D + Y)$ *is simple normal crossing on* N*, and* $B_Y = D|_Y$.

In general, normal crossing varieties are much more difficult than *simple* normal crossing varieties. We postpone the definition of *normal crossing pairs* in Section 6 to avoid unnecessary confusion. Let us recall the notion of semi-ample $\mathbb{R}$-divisors since we often use it in this paper.

2.11 (Semi-ample $\mathbb{R}$-divisor)**.** Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on a variety X and $\pi : X \to S$ a proper morphism. Then, D is π-semi-ample if $D \sim_{\mathbb{R}} f^*H$, where $f : X \to Y$ is a proper morphism over S and H a relatively ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y. It is not difficult to see that D is π-semi-ample if and only if $D \sim_{\mathbb{R}} \sum_i a_i D_i$, where a_i is a positive real number and D_i is a π-semi-ample Cartier divisor on X for any i.

In the following sections, we have to treat algebraic varieties with quotient singularities. All the V-manifolds in this paper are obtained as cyclic covers of smooth varieties whose ramification loci are contained in simple normal crossing divisors. So, they also have toroidal structures. We collect basic definitions according to [S, Section 1], which is the best reference for our purposes.

2.12 (V-manifold)**.** A *V-manifold* of dimension N is a complex analytic space that admits an open covering $\{U_i\}$ such that each U_i is analytically isomorphic to V_i/G_i, where $V_i \subset \mathbb{C}^N$ is an open ball and G_i is a finite subgroup of $\mathrm{GL}(N,\mathbb{C})$. In this paper, G_i is always a cyclic group for any i. Let X be a V-manifold and Σ its singular locus. Then we define $\widetilde{\Omega}^{\bullet}_X = j_*\Omega^{\bullet}_{X-\Sigma}$, where $j : X - \Sigma \to X$ is the natural open immersion. A divisor D on X is called a *divisor with V-normal crossings* if locally on X we have $(X, D) \simeq (V, E)/G$ with $V \subset \mathbb{C}^N$ an open domain, $G \subset \mathrm{GL}(N, \mathbb{C})$ a small subgroup acting on V, and $E \subset V$ a G-invariant divisor with only normal crossing singularities. We define $\widetilde{\Omega}^{\bullet}_X(\log D) = j_*\Omega^{\bullet}_{X-\Sigma}(\log D)$. Furthermore, if D is Cartier, then we put $\widetilde{\Omega}^{\bullet}_X(\log D)(-D) = \widetilde{\Omega}^{\bullet}_X(\log D) \otimes \mathcal{O}_X(-D)$. This complex will play crucial roles in Sections 3 and 4.

3. Fundamental injectivity theorems

The following theorem is a reformulation of the well-known result by Esnault–Viehweg (cf. [EV, 3.2. Theorem. c), 5.1. b)]). Their proof in [EV] depends on the characteristic p methods obtained by Deligne and Illusie. Here, we give another proof for the later usage. Note that all we want to do in this section is to generalize the following theorem for simple normal crossing pairs.

Proposition 3.1 (Fundamental injectivity theorem I)**.** *Let X be a proper smooth variety and $S + B$ a boundary $\mathbb{R}$-divisor on X such that the support of $S + B$ is simple normal crossing, S is reduced, and $\llcorner B \lrcorner = 0$. Let L be a Cartier divisor on X and let D be an effective Cartier divisor whose support is contained in* Supp$B$. *Assume that $L \sim_{\mathbb{R}} K_X + S + B$. Then the natural homomorphisms*

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D)),$$

which are induced by the inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$, are injective for all q.

Proof. We can assume that B is a $\mathbb{Q}$-divisor and $L \sim_{\mathbb{Q}} K_X + S + B$ by Lemma 2.5. We put $\mathcal{L} = \mathcal{O}_X(L - K_X - S)$. Let ν be the smallest positive integer such that $\nu L \sim \nu(K_X + S + B)$. In particular, νB is an integral Weil divisor. We take the ν-fold cyclic cover $\pi' : Y' = \mathrm{Spec}_X \bigoplus_{i=0}^{\nu-1} \mathcal{L}^{-i} \to X$ associated to the section $\nu B \in |\mathcal{L}^{\nu}|$. More precisely, let $s \in H^0(X, \mathcal{L}^{\nu})$ be a section whose zero divisor is νB. Then the dual of $s : \mathcal{O}_X \to \mathcal{L}^{\nu}$ defines a $\mathcal{O}_X$-algebra structure on $\bigoplus_{i=0}^{\nu-1} \mathcal{L}^{-i}$. For the details, see, for example, [EV, 3.5. Cyclic covers]. Let $Y \to Y'$ be the normalization and $\pi : Y \to X$ the composition morphism. Then Y has only quotient singularities because the support of νB is simple normal crossing (cf. [EV, 3.24. Lemma]). We put $T = \pi^*S$. The usual differential $d : \mathcal{O}_Y \to \widetilde{\Omega}^1_Y \subset \widetilde{\Omega}^1_Y(\log T)$ gives the differential $d : \mathcal{O}_Y(-T) \to \widetilde{\Omega}^1_Y(\log T)(-T)$. This induces a natural connection $\pi_*(d) : \pi_*\mathcal{O}_Y(-T) \to \pi_*(\widetilde{\Omega}^1_Y(\log T)(-T))$. It is easy to see that $\pi_*(d)$ decomposes into ν eigen components. One of them is $\nabla : \mathcal{L}^{-1}(-S) \to \Omega^1_X(\log(S+B))\otimes\mathcal{L}^{-1}(-S)$ (cf. [EV, 3.2. Theorem. c)]). It is well known and easy to check that the inclusion $\Omega^\bullet_X(\log(S+B)) \otimes \mathcal{L}^{-1}(-S-D) \to \Omega^\bullet_X(\log(S+B)) \otimes \mathcal{L}^{-1}(-S)$ is a quasi-isomorphism (cf. [EV, 2.9. Properties]). On the other hand, the following spectral sequence

$$\begin{aligned} E_1^{pq} &= H^q(X, \Omega^p_X(\log(S+B)) \otimes \mathcal{L}^{-1}(-S)) \\ &\Longrightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_X(\log(S+B)) \otimes \mathcal{L}^{-1}(-S)) \end{aligned}$$

degenerates in E_1. This follows from the E_1-degeneration of

$$H^q(Y, \widetilde{\Omega}^p_Y(\log T)(-T)) \Longrightarrow \mathbb{H}^{p+q}(Y, \widetilde{\Omega}^\bullet_Y(\log T)(-T))$$

where the right hand side is isomorphic to $H^{p+q}_c(Y-T, \mathbb{C})$. We will discuss this E_1-degeneration in Section 4. For the details, see 4.5 in Section 4 below. We note that $\Omega^\bullet_X(\log(S+B)) \otimes \mathcal{L}^{-1}(-S)$ is a direct summand of $\pi_*(\widetilde{\Omega}^\bullet_Y(\log T)(-T))$. We consider the following commutative diagram for any q.

$$\begin{array}{ccc} \mathbb{H}^q(X, \Omega^\bullet_X(\log(S+B)) \otimes \mathcal{L}^{-1}(-S)) & \xrightarrow{\alpha} & H^q(X, \mathcal{L}^{-1}(-S)) \\ \uparrow{\scriptstyle\gamma} & & \uparrow{\scriptstyle\beta} \\ \mathbb{H}^q(X, \Omega^\bullet_X(\log(S+B)) \otimes \mathcal{L}^{-1}(-S-D)) & \longrightarrow & H^q(X, \mathcal{L}^{-1}(-S-D)) \end{array}$$

Since γ is an isomorphism by the above quasi-isomorphism and α is surjective by the E_1-degeneration, we obtain that β is surjective. By the Serre duality, we obtain $H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}(S)) \to H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}(S+D))$ is injective for any q. This means that $H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D))$ is injective for any q. □

The next result is a key result of this paper.

Proposition 3.2 (Fundamental injectivity theorem II). *Let $(X, S+B)$ be a simple normal crossing pair such that X is proper, $S+B$ is a boundary $\mathbb{R}$-divisor, S is reduced, and $\llcorner B \lrcorner = 0$. Let L be a Cartier divisor on X and let D be an effective Cartier divisor whose support is contained in* Supp$B$. *Assume that $L \sim_{\mathbb{R}} K_X + S + B$. Then the natural homomorphisms*

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D)),$$

which are induced by the inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$, are injective for all q.

Proof. We can assume that B is a $\mathbb{Q}$-divisor and $L \sim_{\mathbb{Q}} K_X + S + B$ by Lemma 2.5. Without loss of generality, we can assume that X is connected. Let $\varepsilon : X^\bullet \to X$ be the Mayer–Vietoris simplicial resolution of X. Let ν be the smallest positive integer such that $\nu L \sim \nu(K_X + S+B)$. We put $\mathcal{L} = \mathcal{O}_X(L-K_X-S)$. We take the ν-fold cyclic cover $\pi' : Y' \to X$ associated to $\nu B \in |\mathcal{L}^\nu|$ as in the proof of Proposition 3.1. Let $\widetilde{Y} \to Y'$ be the normalization of Y'. We can glue $\widetilde{Y}$ naturally along the inverse image of $\varepsilon_1(X^1) \subset X$ and then obtain a connected reducible variety Y and a finite morphism $\pi : Y \to X$. For a supplementary argument, see Remark 3.3 below. We can construct the Mayer–Vietoris simplicial resolution $\varepsilon : Y^\bullet \to Y$ and a natural morphism $\pi_\bullet : Y^\bullet \to X^\bullet$. Note that Definition 2.2 makes sense without any modifications though Y has singularities. The finite morphism $\pi_0 : Y^0 \to X^0$ is essentially the same as the finite cover constructed in Proposition 3.1. Note that the inverse image of an irreducible component X_i of X by π_0 may be a disjoint union of copies of the finite cover constructed in the proof of Proposition 3.1. More precisely, let V be any stratum of X. Then $\pi^{-1}(V)$ is not necessarily connected and $\pi : \pi^{-1}(V) \to V$ may be a disjoint union of copies of the finite cover constructed in the proof of the Proposition 3.1. Since $H^q(X^\bullet, (\mathcal{L}^{-1}(-S-D))^\bullet) \simeq H^q(X, \mathcal{L}^{-1}(-S-D))$ and $H^q(X^\bullet, (\mathcal{L}^{-1}(-S))^\bullet) \simeq H^q(X, \mathcal{L}^{-1}(-S))$, it is sufficient to see that $H^q(X^\bullet, (\mathcal{L}^{-1}(-S-D))^\bullet) \to H^q(X^\bullet, (\mathcal{L}^{-1}(-S))^\bullet)$ is surjective. First, we note that the natural inclusion

$$\Omega^\bullet_{X^n}(\log(S^n+B^n)) \otimes (\mathcal{L}^{-1}(-S-D))^n \to \Omega^\bullet_{X^n}(\log(S^n+B^n)) \otimes (\mathcal{L}^{-1}(-S))^n$$

is a quasi-isomorphism for any $n \geq 0$ (cf. [EV, 2.9. Properties]). So,

$$\Omega^\bullet_{X^\bullet}(\log(S^\bullet+B^\bullet)) \otimes (\mathcal{L}^{-1}(-S-D))^\bullet \to \Omega^\bullet_{X^\bullet}(\log(S^\bullet+B^\bullet)) \otimes (\mathcal{L}^{-1}(-S)^\bullet)$$

is a quasi-isomorphism. We put $T = \pi^* S$. Then $\Omega^\bullet_{X^n}(\log(S^n+B^n)) \otimes (\mathcal{L}^{-1}(-S))^n$ is a direct summand of $\pi_{n*}\widetilde{\Omega}^\bullet_Y(\log T^n)(-T^n)$ for any $n \geq 0$. Next, we can check that

$$E_1^{pq} = H^q(Y^\bullet, \widetilde{\Omega}^p_{Y^\bullet}(\log T^\bullet)(-T^\bullet)) \Longrightarrow \mathbb{H}^{p+q}(Y^\bullet, \widetilde{\Omega}^\bullet_{Y^\bullet}(\log T^\bullet)(-T^\bullet))$$

degenerates in E_1. We will discuss this E_1-degeneration in Section 4. See 4.6 in Section 4. The right hand side is isomorphic to $H_c^{p+q}(Y - T, \mathbb{C})$. Therefore,

$$\begin{aligned} E_1^{pq} &= H^q(X^\bullet, \Omega^p_{X^\bullet}(\log(S^\bullet+B^\bullet)) \otimes (\mathcal{L}^{-1}(-S))^\bullet) \\ &\Longrightarrow \mathbb{H}^{p+q}(X^\bullet, \Omega^\bullet_{X^\bullet}(\log(S^\bullet+B^\bullet)) \otimes (\mathcal{L}^{-1}(-S))^\bullet) \end{aligned}$$

degenerates in E_1. Thus, we have the following commutative diagram.

$$\begin{array}{ccc}
\mathbb{H}^q(X^\bullet, \Omega^\bullet_{X^\bullet}(\log(S^\bullet + B^\bullet)) \otimes (\mathcal{L}^{-1}(-S))^\bullet) & \xrightarrow{\alpha} & H^q(X^\bullet, (\mathcal{L}^{-1}(-S))^\bullet) \\
\uparrow \gamma & & \uparrow \beta \\
\mathbb{H}^q(X^\bullet, \Omega^\bullet_{X^\bullet}(\log(S^\bullet + B^\bullet)) \otimes (\mathcal{L}^{-1}(-S-D))^\bullet) & \longrightarrow & H^q(X^\bullet, (\mathcal{L}^{-1}(-S-D))^\bullet)
\end{array}$$

As in the proof of Proposition 3.1, γ is an isomorphism and α is surjective. Thus, β is surjective. This implies the desired injectivity results. □

Remark 3.3. For simplicity, we assume that $X = X_1 \cup X_2$, where X_1 and X_2 are smooth, and that $V = X_1 \cap X_2$ is irreducible. We consider the natural projection $p : \widetilde{Y} \to X$. We note that $\widetilde{Y} = \widetilde{Y}_1 \coprod \widetilde{Y}_2$, where $\widetilde{Y}_i$ is the inverse image of X_i by p for $i = 1$ and 2. We put $p_i = p|_{\widetilde{Y}_i}$ for $i = 1$ and 2. It is easy to see that $p_1^{-1}(V)$ is isomorphic to $p_2^{-1}(V)$ over V. We denote it by W. We consider the following surjective $\mathcal{O}_X$-module homomorphism $\mu : p_*\mathcal{O}_{\widetilde{Y}_1} \oplus p_*\mathcal{O}_{\widetilde{Y}_2} \to p_*\mathcal{O}_W : (f, g) \mapsto f|_W - g|_W$. Let $\mathcal{A}$ be the kernel of μ. Then $\mathcal{A}$ is an $\mathcal{O}_X$-algebra and $\pi : Y \to X$ is nothing but $\mathrm{Spec}_X\mathcal{A} \to X$. We can check that $\pi^{-1}(X_i) \simeq \widetilde{Y}_i$ for $i = 1$ and 2 and that $\pi^{-1}(V) \simeq W$.

Remark 3.4. As pointed out in the introduction, the proof of [A1, Theorem 3.1] only implies that the homomorphisms $H^q(X, \mathcal{O}_X(L - S)) \to H^q(X, \mathcal{O}_X(L - S + D))$ are injective for all q. When $S = 0$, we do not need the mixed Hodge structure on the cohomology with compact support. The mixed Hodge structure on the usual singular cohomology is sufficient for the case when $S = 0$.

We close this section with an easy application of Proposition 3.2. The following vanishing theorem is the Kodaira vanishing theorem for simple normal crossing varieties.

Corollary 3.5. *Let X be a projective simple normal crossing variety and $\mathcal{L}$ an ample line bundle on X. Then $H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}) = 0$ for any $q > 0$.*

Proof. We take a general member $B \in |\mathcal{L}^l|$ for some $l \gg 0$. Then we can find a Cartier divisor M such that $M \sim_{\mathbb{Q}} K_X + \frac{1}{l}B$ and $\mathcal{O}_X(K_X) \otimes \mathcal{L} \simeq \mathcal{O}_X(M)$. By Proposition 3.2, we obtain injections $H^q(X, \mathcal{O}_X(M)) \to H^q(X, \mathcal{O}_X(M + mB))$ for any q and any positive integer m. Since B is ample, Serre's vanishing theorem implies the desired vanishing theorem. □

4. E_1-degenerations of Hodge to de Rham type spectral sequences

From 4.1 to 4.3, we recall some well-known results on mixed Hodge structures. We use the notations in [D2] freely. The basic references on this topic are [D2, Section 8], [E1, Part II], and [E2, Chapitres 2 and 3]. The starting point is the pure Hodge structures on proper smooth algebraic varieties.

4.1. (Hodge structures for proper smooth varieties). Let X be a proper smooth algebraic variety over $\mathbb{C}$. Then the triple $(\mathbb{Z}_X, (\Omega^\bullet_X, F), \alpha)$, where $\Omega^\bullet_X$ is the holomorphic de Rham complex with the filtration bête F and $\alpha : \mathbb{C}_X \to \Omega^\bullet_X$ is the inclusion, is a cohomological Hodge complex (CHC, for short) of weight zero.

The next one is also a fundamental example. For the details, see [E1, I.1.] or [E2, 3.5].

4.2. (Mixed Hodge structures for proper simple normal crossing varieties). Let D be a proper simple normal crossing algebraic variety over $\mathbb{C}$. Let $\varepsilon : D^\bullet \to D$ be the Mayer–Vietoris simplicial resolution. The following complex of sheaves, denoted by $\mathbb{Q}_{D^\bullet}$,

$$0 \to \varepsilon_{0*}\mathbb{Q}_{D^0} \to \varepsilon_{1*}\mathbb{Q}_{D^1} \to \cdots \to \varepsilon_{k*}\mathbb{Q}_{D^k} \to \cdots ,$$

is a resolution of $\mathbb{Q}_D$. More explicitly, the differential $d_k : \varepsilon_{k*}\mathbb{Q}_{D^k} \to \varepsilon_{k+1*}\mathbb{Q}_{D^{k+1}}$ is $\sum_{j=0}^{k+1}(-1)^j\lambda^*_{j,k+1}$ for any $k \geq 0$. For the details, see [E1, I.1.] or [E2, 3.5.3]. We obtain the resolution $\Omega^\bullet_{D^\bullet}$ of $\mathbb{C}_D$ as follows,

$$0 \to \varepsilon_{0*}\Omega^\bullet_{D^0} \to \varepsilon_{1*}\Omega^\bullet_{D^1} \to \cdots \to \varepsilon_{k*}\Omega^\bullet_{D^k} \to \cdots .$$

Of course, $d_k : \varepsilon_{k*}\Omega^\bullet_{D^k} \to \varepsilon_{k+1*}\Omega^\bullet_{D^{k+1}}$ is $\sum_{j=0}^{k+1}(-1)^j\lambda^*_{j,k+1}$. Let $s(\Omega^\bullet_{D^\bullet})$ be the simple complex associated to the double complex $\Omega^\bullet_{D^\bullet}$. The Hodge filtration F on $s(\Omega^\bullet_{D^\bullet})$ is defined by $F^p = s(0 \to \cdots \to 0 \to \varepsilon_*\Omega^p_{D^\bullet} \to \varepsilon_*\Omega^{p+1}_{D^\bullet} \to \cdots)$. We note that $\varepsilon_*\Omega^p_{D^\bullet} = (0 \to \varepsilon_{0*}\Omega^p_{D^0} \to \varepsilon_{1*}\Omega^p_{D^1} \to \cdots \to \varepsilon_{k*}\Omega^p_{D^k} \to \cdots)$. There exist natural weight filtrations W's on $\mathbb{Q}_{D^\bullet}$ and $s(\Omega^\bullet_{D^\bullet})$. We omit the definition of the weight filtrations W's on $\mathbb{Q}_{D^\bullet}$ and $s(\Omega^\bullet_{D^\bullet})$ since we do not need their explicit descriptions. See [E1, I.1.] or [E2, 3.5.6]. Then $(\mathbb{Z}_D, (\mathbb{Q}_{D^\bullet}, W), (s(\Omega^\bullet_{D^\bullet}), W, F))$ is a cohomological mixed Hodge complex (CMHC, for short). This CMHC induces a natural mixed Hodge structure on $H^\bullet(D, \mathbb{Z})$.

For the precise definitions of CHC and CMHC (CHMC, in French), see [D2, Section 8] or [E2, Chapitre 3]. The third example is not so standard but is indispensable for our injectivity theorems.

4.3. (Mixed Hodge structure on the cohomolgy with compact support). Let X be a proper smooth algebraic variety over $\mathbb{C}$ and D a simple normal crossing divisor on X. We consider the mixed cone of $\mathbb{Q}_X \to \mathbb{Q}_{D^\bullet}$ with suitable shifts of complexes and weight filtrations (for the details, see [E1, I.3.] or [E2, 3.7.14]). We obtain a complex $\mathbb{Q}_{X-D^\bullet}$, which is quasi-isomorphic to $j_!\mathbb{Q}_{X-D}$, where $j : X - D \to X$ is the natural open immersion, and a weight filtration W on $\mathbb{Q}_{X-D^\bullet}$. We define in the same way, that is, by taking a cone of a morphism of complexes $\Omega^\bullet_X \to \Omega^\bullet_{D^\bullet}$, a complex $\Omega^\bullet_{X-D^\bullet}$ with filtrations W and F. Then we obtain that the triple $(j_!\mathbb{Z}_{X-D}, (\mathbb{Q}_{X-D^\bullet}, W), (\Omega^\bullet_{X-D^\bullet}, W, F))$ is a CMHC. It defines a natural mixed Hodge structure on $H^\bullet_c(X-D, \mathbb{Z})$. Since we can check that the complex

$$\begin{aligned}0 \to \Omega^\bullet_X(\log D)(-D) \to \Omega^\bullet_X \to \varepsilon_{0*}\Omega^\bullet_{D^0}\\ \to \varepsilon_{1*}\Omega^\bullet_{D^1} \to \cdots \to \varepsilon_{k*}\Omega^\bullet_{D^k} \to \cdots\end{aligned}$$

is exact by direct local calculations, we see that $(\Omega^\bullet_{X-D^\bullet}, F)$ is quasi-isomorphic to $(\Omega^\bullet_X(\log D)(-D), F)$ in $D^+F(X, \mathbb{C})$, where

$$\begin{aligned}&F^p\Omega^\bullet_X(\log D)(-D)\\ &\quad = (0 \to \cdots \to 0 \to \Omega^p_X(\log D)(-D) \to \Omega^{p+1}_X(\log D)(-D) \to \cdots).\end{aligned}$$

Therefore, the spectral sequence

$$E_1^{pq} = H^q(X, \Omega^p_X(\log D)(-D)) \Longrightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_X(\log D)(-D))$$

degenerates in E_1 and the right hand side is isomorphic to $H^{p+q}_c(X - D, \mathbb{C})$.

From here, we treat mixed Hodge structures on much more complicated algebraic varieties.

4.4. (Mixed Hodge structures for proper simple normal crossing pairs). Let (X, D) be a proper simple normal crossing pair over $\mathbb{C}$ such that D is reduced. Let $\varepsilon : X^\bullet \to X$ be the Mayer–Vietoris simplicial resolution of X. As we saw in the previous step, we have a CHMC $(j_{n!}\mathbb{Z}_{X^n-D^n}, (\mathbb{Q}_{X^n-(D^n)^\bullet}, W), (\Omega^\bullet_{X^n-(D^n)^\bullet}, W, F))$ on X^n, where $j_n : X^n - D^n \to X^n$ is the natural open immersion, and that $(\Omega^\bullet_{X^n-(D^n)^\bullet}, F)$ is quasi-isomorphic to $(\Omega^\bullet_{X^n}(\log D^n)(-D^n), F)$ in $D^+F(X^n, \mathbb{C})$ for any $n \geq 0$. Therefore, by using the Mayer–Vietoris simplicial resolution $\varepsilon : X^\bullet \to X$, we can construct a CMHC $(j_!\mathbb{Z}_{X-D}, (K_\mathbb{Q}, W), (K_\mathbb{C}, W, F))$ on X that induces a natural mixed Hodge structure on $H^\bullet_c(X-D, \mathbb{Z})$. We can see that $(K_\mathbb{C}, F)$ is quasi-isomorphic to $(s(\Omega^\bullet_{X^\bullet}(\log D^\bullet)(-D^\bullet)), F)$

in $D^+F(X,\mathbb{C})$, where

$$F^p = s(0\to\cdots\to 0\to \varepsilon_*\Omega^p_{X^\bullet}(\log D^\bullet)(-D^\bullet) \\ \to \varepsilon_*\Omega^{p+1}_{X^\bullet}(\log D^\bullet)(-D^\bullet)\to\cdots).$$

We note that $\Omega^\bullet_{X^\bullet}(\log D^\bullet)(-D^\bullet)$ is the double complex

$$0\to \varepsilon_{0*}\Omega^\bullet_{X^0}(\log D^0)(-D^0)\to \varepsilon_{1*}\Omega^\bullet_{X^1}(\log D^1)(-D^1)\to\cdots \\ \to \varepsilon_{k*}\Omega^\bullet_{X^k}(\log D^k)(-D^k)\to\cdots.$$

Therefore, the spectral sequence

$$E_1^{pq} = H^q(X^\bullet, \Omega^p_{X^\bullet}(\log D^\bullet)(-D^\bullet)) \Longrightarrow \mathbb{H}^{p+q}(X^\bullet, \Omega^\bullet_{X^\bullet}(\log D^\bullet)(-D^\bullet))$$

degenerates in E_1 and the right hand side is isomorphic to $H^{p+q}_c(X-D,\mathbb{C})$.

Let us go to the proof of the E_1-degeneration that we already used in the proof of Proposition 3.1.

4.5 (E_1-degeneration for Proposition 3.1)**.** In this section, we use the notation in the proof of Proposition 3.1. In this case, Y has only quotient singularities. Then $(\mathbb{Z}_Y, (\widetilde{\Omega}^\bullet_Y, F), \alpha)$ is a CHC, where F is the filtration bête and $\alpha : \mathbb{C}_Y \to \widetilde{\Omega}^\bullet_Y$ is the inclusion. For the details, see [S, (1.6)]. It is easy to see that T is a divisor with V-normal crossings on Y (see 2.12 or [S, (1.16) Definition]). We can easily check that Y is singular only over the singular locus of Supp$B$. Let $\varepsilon : T^\bullet \to T$ be the Mayer–Vietoris simplicial resolution. Though T has singularities, Definition 2.2 makes sense without any modifications. We note that T^n has only quotient singularities for any $n\geq 0$ by the construction of $\pi : Y\to X$. We can also check that the same construction in 4.2 works with minor modifications and we have a CMHC $(\mathbb{Z}_T, (\mathbb{Q}_{T^\bullet}, W), (s(\widetilde{\Omega}^\bullet_{T^\bullet}), W, F))$ that induces a natural mixed Hodge structure on $H^\bullet(T,\mathbb{Z})$. By the same arguments as in 4.3, we can construct a triple $(j_!\mathbb{Z}_{Y-T}, (\mathbb{Q}_{Y-T^\bullet}, W), (K_{\mathbb{C}}, W, F))$, where $j : Y-T\to Y$ is the natural open immersion. It is a CHMC that induces a natural mixed Hodge structure on $H^\bullet_c(Y-T,\mathbb{Z})$ and $(K_{\mathbb{C}}, F)$ is quasi-isomorphic to $(\widetilde{\Omega}^\bullet_Y(\log T)(-T), F)$ in $D^+F(Y,\mathbb{C})$, where

$$F^p\widetilde{\Omega}^\bullet_Y(\log T)(-T) \\ = (0\to\cdots\to 0\to \widetilde{\Omega}^p_Y(\log T)(-T)\to \widetilde{\Omega}^{p+1}_Y(\log T)(-T)\to\cdots).$$

Therefore, the spectral sequence

$$E_1^{pq} = H^q(Y, \widetilde{\Omega}^p_Y(\log T)(-T)) \Longrightarrow \mathbb{H}^{p+q}(Y, \Omega^\bullet_Y(\log T)(-T))$$

degenerates in E_1 and the right hand side is isomorphic to $H^{p+q}_c(Y-T,\mathbb{C})$.

The final one is the E_1-degeneration that we used in the proof of Proposition 3.2. It may be one of the main contributions of this paper.

4.6 (E_1-degeneration for Proposition 3.2)**.** We use the notation in the proof of Proposition 3.2. Let $\varepsilon : Y^\bullet \to Y$ be the Mayer–Vietoris simplicial resolution. By the previous step, we can obtain a CHMC $(j_{n!}\mathbb{Z}_{Y^n-T^n}, (\mathbb{Q}_{Y^n-(T^n)^\bullet}, W), (K_\mathbb{C}, W, F))$ for each $n \geq 0$. Of course, $j_n : Y^n - T^n \to Y^n$ is the natural open immersion for any $n \geq 0$. Therefore, we can construct a CMHC $(j_!\mathbb{Z}_{Y-T}, (K_\mathbb{Q}, W), (K_\mathbb{C}, W, F))$ on Y. It induces a natural mixed Hodge structure on $H^\bullet_c(Y-T, \mathbb{Z})$. We note that $(K_\mathbb{C}, F)$ is quasi-isomorphic to $(s(\widetilde{\Omega}^\bullet_{Y^\bullet}(\log T^\bullet)(-T^\bullet)), F)$ in $D^+F(Y, \mathbb{C})$, where

$$F^p = s(0 \to \cdots \to 0 \to \varepsilon_*\widetilde{\Omega}^p_{Y^\bullet}(\log T^\bullet)(-T^\bullet) \\ \to \varepsilon_*\widetilde{\Omega}^{p+1}_{Y^\bullet}(\log T^\bullet)(-T^\bullet) \to \cdots).$$

See 4.4 above. Thus, the desired spectral sequence

$$E_1^{pq} = H^q(Y^\bullet, \widetilde{\Omega}^p_{Y^\bullet}(\log T^\bullet)(-T^\bullet)) \Longrightarrow \mathbb{H}^{p+q}(Y^\bullet, \widetilde{\Omega}^\bullet_{Y^\bullet}(\log T^\bullet)(-T^\bullet))$$

degenerates in E_1. It is what we need in the proof of Proposition 3.2. Note that $\mathbb{H}^{p+q}(Y^\bullet, \widetilde{\Omega}^\bullet_{Y^\bullet}(\log T^\bullet)(-T^\bullet)) \simeq H^{p+q}_c(Y-T, \mathbb{C})$.

5. Vanishing and injectivity theorems

The main purpose of this section is to prove Ambro's theorems (cf. [A1, Theorems 3.1 and 3.2]) for embedded simple normal crossing pairs. The next lemma (cf. [F1, Proposition 1.11]) is missing in the proof of [A1, Theorem 3.1]. It justifies the first three lines in the proof of [A1, Theorem 3.1].

Lemma 5.1 (Relative vanishing lemma)**.** *Let $f : Y \to X$ be a proper morphism from a simple normal crossing pair $(Y, T+D)$ such that $T+D$ is a boundary $\mathbb{R}$-divisor, T is reduced, and $\llcorner D \lrcorner = 0$. We assume that f is an isomorphism at any generic points of strata of the pair $(Y, T+D)$. Let L be a Cartier divisor on Y such that $L \sim_\mathbb{R} K_Y + T + D$. Then $R^q f_*\mathcal{O}_Y(L) = 0$ for $q > 0$.*

Proof. By Lemma 2.5, we can assume that D is a $\mathbb{Q}$-divisor and $L \sim_\mathbb{Q} K_Y + T + D$. We divide the proof into two steps.

Step 1. We assume that Y is irreducible. In this case, $L-(K_Y+T+D)$ is nef and log big over X with respect to the pair $(Y, T+D)$. Therefore, $R^q f_*\mathcal{O}_Y(L) = 0$ for any $q > 0$ by the vanishing theorem.

Step 2. Let Y_1 be an irreducible component of Y and Y_2 the union of the other irreducible components of Y. Then we have a short exact sequence $0 \to i_*\mathcal{O}_{Y_1}(-Y_2|_{Y_1}) \to \mathcal{O}_Y \to \mathcal{O}_{Y_2} \to 0$, where $i : Y_1 \to Y$ is the natural closed immersion (cf. [A1, Remark 2.6]). We put $L' = L|_{Y_1} - Y_2|_{Y_1}$. Then we have a short exact sequence $0 \to i_*\mathcal{O}_{Y_1}(L') \to \mathcal{O}_Y(L) \to \mathcal{O}_{Y_2}(L|_{Y_2}) \to 0$ and $L' \sim_{\mathbb{Q}} K_{Y_1} + T|_{Y_1} + D|_{Y_1}$. On the other hand, we can check that $L|_{Y_2} \sim_{\mathbb{Q}} K_{Y_2} + Y_1|_{Y_2} + T|_{Y_2} + D|_{Y_2}$. We have already known that $R^q f_*\mathcal{O}_{Y_1}(L') = 0$ for any $q > 0$ by Step 1. By the induction on the number of the irreducible components of Y, we have $R^q f_*\mathcal{O}_{Y_2}(L|_{Y_2}) = 0$ for any $q > 0$. Therefore, $R^q f_*\mathcal{O}_Y(L) = 0$ for any $q > 0$ by the exact sequence: $\cdots \to R^q f_*\mathcal{O}_{Y_1}(L') \to R^q f_*\mathcal{O}_Y(L) \to R^q f_*\mathcal{O}_{Y_2}(L|_{Y_2}) \to \cdots$.

So, we finish the proof of Lemma 5.1. $\square$

The following lemma is a variant of Szabó's resolution lemma (cf. [F2, 3.5. Resolution lemma]).

Lemma 5.2. *Let (X, B) be an embedded simple normal crossing pair and D a permissible Cartier divisor on X. Let M be an ambient space of X. Assume that there exists an $\mathbb{R}$-divisor A on M such that* $\mathrm{Supp}(A+X)$ *is simple normal crossing on M and that $B = A|_X$. Then there exists a projective birational morphism $g : N \to M$ from a smooth variety N with the following properties. Let Y be the strict transform of X on N and $f = g|_Y : Y \to X$. Then we have*

- (i) *$g^{-1}(D)$ is a divisor on N.* $\mathrm{Exc}(g) \cup g_*^{-1}(A+X)$ *is simple normal crossing on N, where* $\mathrm{Exc}(g)$ *is the exceptional locus of g. In particular, Y is a simple normal crossing divisor on N.*
- (ii) *g and f are isomorphisms outside D, in particular, $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$.*
- (iii) *$f^*(D + B)$ has a simple normal crossing support on Y. More precisely, there exists an $\mathbb{R}$-divisor A' on N such that* $\mathrm{Supp}(A' + Y)$ *is simple normal crossing on N, A' and Y have no common irreducible components, and that $A'|_Y = f^*(D + B)$.*

Proof. First, we take a blow-up $M_1 \to M$ along D. Apply Hironaka's desingularization theorem to M_1 and obtain a projective birational morphism $M_2 \to M_1$ from a smooth variety M_2. Let F be the reduced divisor that coincides with the support of the inverse image of D on M_2. Apply Szabó's resolution lemma to $\mathrm{Supp}\sigma^*(A+X) \cup F$ on M_2 (see, for example, [F2, 3.5. Resolution lemma]), where $\sigma : M_2 \to M$. Then, we obtain desired projective birational morphisms $g : N \to M$ from a smooth variety N, and $f = g|_Y : Y \to X$, where Y is the strict transform of X on N, such that Y is a simple normal crossing divisor on N, g and f are isomorphisms outside D, and $f^*(D+B)$ has a simple normal crossing support on Y. Since f is an isomorphism outside D and D is permissible on X, f is an isomorphism at any generic points of strata of Y. Therefore, every fiber of f is connected and then $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$. □

Remark 5.3. In Lemma 5.2, we can directly check that $f_*\mathcal{O}_Y(K_Y) \simeq \mathcal{O}_X(K_X)$. By Lemma 5.1, $R^qf_*\mathcal{O}_Y(K_Y) = 0$ for $q > 0$. Therefore, we obtain $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$ and $R^qf_*\mathcal{O}_Y = 0$ for any $q > 0$ by the Grothendieck duality.

Here, we treat the compactification problem. It is because we can use the same technique as in the proof of Lemma 5.2. This lemma plays important roles in this section.

Lemma 5.4. *Let $f : Z \to X$ be a proper morphism from an embedded simple normal crossing pair (Z, B). Let M be the ambient space of Z. Assume that there is an $\mathbb{R}$-divisor A on M such that* $\mathrm{Supp}(A + Z)$ *is simple normal crossing on M and that $B = A|_Z$. Let $\overline{X}$ be a projective variety such that $\overline{X}$ contains X as a Zariski open set. Then there exist a proper embedded simple normal crossing pair $(\overline{Z}, \overline{B})$ that is a compactification of (Z, B) and a proper morphism $\overline{f} : \overline{Z} \to \overline{X}$ that compactifies $f : Z \to X$. Moreover,* $\mathrm{Supp}\overline{B} \cup \mathrm{Supp}(\overline{Z} \setminus Z)$ *is a simple normal crossing divisor on $\overline{Z}$, and $\overline{Z} \setminus Z$ has no common irreducible components with $\overline{B}$. We note that $\overline{B}$ is $\mathbb{R}$-Cartier. Let $\overline{M}$, which is a compactification of M, be the ambient space of $(\overline{Z}, \overline{B})$. Then, by the construction, we can find an $\mathbb{R}$-divisor $\overline{A}$ on $\overline{M}$ such that* $\mathrm{Supp}(\overline{A}+\overline{Z})$ *is simple normal crossing on $\overline{M}$ and that $\overline{B} = \overline{A}|_{\overline{Z}}$.*

Proof. Let $\overline{Z}, \overline{A} \subset \overline{M}$ be any compactification. By blowing up $\overline{M}$ inside $\overline{Z} \setminus Z$, we can assume that $f : Z \to X$ extends to $\overline{f} : \overline{Z} \to \overline{X}$. By Hironaka's desingularization and the resolution lemma, we can assume that $\overline{M}$ is smooth and $\mathrm{Supp}(\overline{Z}+\overline{A}) \cup \mathrm{Supp}(\overline{M} \setminus M)$ is a simple normal crossing divisor on $\overline{M}$. It is not difficult to see that the above compactification has the desired properties. □

Remark 5.5. There exists a big trouble to compactify normal crossing varieties. When we treat normal crossing varieties, we can not directly compactify them. For the details, see [F2, 3.6. Whitney umbrella], especially, Corollary 3.6.10 and Remark 3.6.11 in [F2]. Therefore, the first two lines in the proof of [A1, Theorem 3.2] is nonsense.

It is the time to state the main injectivity theorem (cf. [A1, Theorem 3.1]) for embedded *simple* normal crossing pairs. For applications, this formulation seems to be sufficient. We note that we will recover [A1, Theorem 3.1] in full generality in Section 6 (see Theorem 6.1).

Theorem 5.6 (cf. [A1, Theorem 3.1]). *Let $(X, S+B)$ be an embedded simple normal crossing pair such that X is proper, $S+B$ is a boundary $\mathbb{R}$-divisor, S is reduced, and $\llcorner B \lrcorner = 0$. Let L be a Cartier divisor on X and D an effective Cartier divisor that is permissible with respect to $(X, S+B)$. Assume the following conditions.*

(i) $L \sim_{\mathbb{R}} K_X + S + B + H$,
(ii) *H is a semi-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, and*
(iii) *$tH \sim_{\mathbb{R}} D + D'$ for some positive real number t, where D' is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor that is permissible with respect to $(X, S+B)$.*

Then the homomorphisms

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D)),$$

which are induced by the inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$, are injective for all q.

Proof. First, we use Lemma 2.10. Thus, we can assume that there exists a divisor A on M, where M is the ambient space of X, such that $\mathrm{Supp}(A+X)$ is simple normal crossing on M and that $A|_X = S$. Apply Lemma 5.2 to an embedded simple normal crossing pair (X, S) and a divisor $\mathrm{Supp}(D+D'+B)$ on X. Then we obtain a projective birational morphism $f : Y \to X$ from an embedded simple normal crossing variety Y such that f is an isomorphism outside $\mathrm{Supp}(D+D'+B)$, and that the union of the support of $f^*(S+B+D+D')$ and the exceptional locus of f has a simple normal crossing support on Y. Let B' be the strict transform of B on Y. We can assume that $\mathrm{Supp}B'$ is disjoint from any strata of Y that are not irreducible components of Y by taking blow-ups. We write $K_Y + S' + B' = f^*(K_Y + S + B) + E$, where S' is the strict transform of S, and E is f-exceptional. By the construction of $f : Y \to X$, S' is Cartier and B' is $\mathbb{R}$-Cartier. Therefore, E is also $\mathbb{R}$-Cartier. It is easy to see that $E_+ = \ulcorner E \urcorner \geq 0$. We put $L' = f^*L + E_+$ and $E_- = E_+ - E \geq 0$. We note that E_+ is Cartier and E_- is $\mathbb{R}$-Cartier because $\mathrm{Supp}E$ is simple normal crossing on Y. Since f^*H is an $\mathbb{R}_{>0}$-linear combination of semi-ample Cartier divisors, we can write $f^*H \sim_{\mathbb{R}} \sum_i a_i H_i$, where $0 < a_i < 1$ and H_i is a general Cartier divisor on Y for any i. We put $B'' = B' + E_- + \frac{\varepsilon}{t} f^*(D+D') + (1-\varepsilon)\sum_i a_i H_i$ for some $0 < \varepsilon \ll 1$. Then $L' \sim_{\mathbb{R}} K_Y + S' + B''$. By the construction, $\llcorner B'' \lrcorner = 0$, the support of $S' + B''$ is simple normal crossing on Y, and $\mathrm{Supp}B'' \supset \mathrm{Supp}f^*D$. So, Proposition 3.2 implies that the homomorphisms $H^q(Y, \mathcal{O}_Y(L')) \to H^q(Y, \mathcal{O}_Y(L' + f^*D))$ are injective for all q. By Lemma 5.1, $R^qf_*\mathcal{O}_Y(L')=0$ for any $q>0$ and it is easy to see that $f_*\mathcal{O}_Y(L')\simeq\mathcal{O}_X(L)$. By the Leray spectral sequence, the homomorphisms $H^q(X,\mathcal{O}_X(L))\to H^q(X,\mathcal{O}_X(L+D))$ are injective for all q. □

The following theorem is another main theorem of this section. It is essentially the same as [A1, Theorem 3.2]. We note that we assume that $(Y,S+B)$ is a *simple* normal crossing pair. It is a small but technically important difference. For the full statement, see Theorem 6.2 below.

Theorem 5.7 (cf. [A1, Theorem 3.2])**.** *Let $(Y,S+B)$ be an embedded simple normal crossing pair such that $S+B$ is a boundary $\mathbb{R}$-divisor, S is reduced, and $\llcorner B\lrcorner=0$. Let $f:Y\to X$ be a proper morphism and L a Cartier divisor on Y such that $H\sim_{\mathbb{R}}L-(K_Y+S+B)$ is f-semi-ample.*

(i) *every non-zero local section of $R^qf_*\mathcal{O}_Y(L)$ contains in its support the f-image of some strata of $(Y,S+B)$.*
(ii) *let $\pi:X\to V$ be a projective morphism and assume that $H\sim_{\mathbb{R}}f^*H'$ for some π-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor H' on X. Then $R^qf_*\mathcal{O}_Y(L)$ is π_*-acyclic, that is, $R^p\pi_*R^qf_*\mathcal{O}_Y(L)=0$ for any $p>0$.*

Proof. Let M be the ambient space of Y. Then, by Lemma 2.10, we can assume that there exists an $\mathbb{R}$-divisor D on M such that $\mathrm{Supp}(D+Y)$ is simple normal crossing on M and that $D|_Y=S+B$. Therefore, we can use Lemma 5.4 in Step 3 of (i) and (ii) below.

(i) We have already proved a very spacial case in Lemma 5.1. The argument in Step 1 is not new and it is well known.

Step 1. First, we assume that X is projective. We can assume that H is semi-ample by replacing L (resp. H) with $L+f^*A'$ (resp. $H+f^*A'$), where A' is a very ample Cartier divisor. Assume that $R^qf_*\mathcal{O}_Y(L)$ has a local section whose support does not contain any image of the $(Y,S+B)$-strata. Then we can find a very ample Cartier divisor A with the following properties.

(a) f^*A is permissible with respect to $(Y,S+B)$, and
(b) $R^qf_*\mathcal{O}_Y(L)\to R^qf_*\mathcal{O}_Y(L)\otimes\mathcal{O}_X(A)$ is not injective.

We can assume that $H-f^*A$ is semi-ample by replacing L (resp. H) with $L+f^*A$ (resp. $H+f^*A$). If necessary, we replace L (resp. H) with $L+f^*A''$ (resp. $H+f^*A''$), where A'' is a very ample Cartier divisor. Then, we have $H^0(X,R^qf_*\mathcal{O}_Y(L))\simeq H^q(Y,\mathcal{O}_Y(L))$ and $H^0(X,R^qf_*\mathcal{O}_Y(L)\otimes\mathcal{O}_X(A))\simeq H^q(Y,\mathcal{O}_Y(L+f^*A))$. We obtain that $H^0(X,R^qf_*\mathcal{O}_Y(L))\to$

$H^0(X, R^qf_*\mathcal{O}_Y(L)\otimes\mathcal{O}_X(A))$ is not injective by (b) if A'' is sufficiently ample. So, $H^q(Y,\mathcal{O}_Y(L))\to H^q(Y,\mathcal{O}_Y(L+f^*A))$ is not injective. It contradicts Theorem 5.1. We finish the proof when X is projective.

Step 2. Next, we assume that X is not projective. Note that the problem is local. So, we can shrink X and assume that X is affine. By the argument similar to the one in Step 1 in the proof of (ii) below, we can assume that H is a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. We compactify X and apply Lemma 5.4. Then we obtain a compactification $\overline{f}:\overline{Y}\to\overline{X}$ of $f:Y\to X$. Let $\overline{H}$ be the closure of H on $\overline{Y}$. If $\overline{H}$ is not a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, then we take blowing-ups of $\overline{Y}$ inside $\overline{Y}\setminus Y$ and obtain a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\overline{H}$ on $\overline{Y}$ such that $\overline{H}|_Y=H$. Let $\overline{L}$ (resp. $\overline{B}$, $\overline{S}$) be the closure of L (resp. B, S) on $\overline{Y}$. We note that $\overline{H}\sim_{\mathbb{R}}\overline{L}-(K_{\overline{Y}}+\overline{S}+\overline{B})$ does not necessarily hold. We can write $H+\sum_i a_i(f_i)=L-(K_Y+S+B)$, where a_i is a real number and $f_i\in\Gamma(Y,\mathcal{K}_Y^*)$ for any i. We put $E=\overline{H}+\sum_i a_i(f_i)-(\overline{L}-(K_{\overline{Y}}+\overline{S}+\overline{B}))$. We replace $\overline{L}$ (resp. $\overline{B}$) with $\overline{L}+\ulcorner E\urcorner$ (resp. $\overline{B}+\{-E\}$). Then we obtain the desired property of $R^q\overline{f}_*\mathcal{O}_{\overline{Y}}(\overline{L})$ since $\overline{X}$ is projective. We note that $\mathrm{Supp}E$ is in $\overline{Y}\setminus Y$. So, we finish the whole proof.

(ii) We divide the proof into three steps.

Step 1. We assume that $\dim V=0$. The following arguments are well known and standard. We describe them for the reader's convenience. In this case, we can write $H'\sim_{\mathbb{R}}H'_1+H'_2$, where H'_1 (resp. H'_2) is a π-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor (resp. π-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor) on X. So, we can write $H'_2\sim_{\mathbb{R}}\sum_i a_iH_i$, where $0<a_i<1$ and H_i is a general very ample Cartier divisor on X for any i. Replacing B (resp. H') with $B+\sum_i a_if^*H_i$ (resp. H'_1), we can assume that H' is a π-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. We take a general member $A\in|mH'|$, where m is a sufficiently large and divisible integer, such that $A'=f^*A$ and $R^qf_*\mathcal{O}_Y(L+A')$ is π_*-acyclic for all q. By (i), we have the following short exact sequences,

$$0\to R^qf_*\mathcal{O}_Y(L)\to R^qf_*\mathcal{O}_Y(L+A')\to R^qf_*\mathcal{O}_{A'}(L+A')\to 0.$$

for any q. Note that $R^qf_*\mathcal{O}_{A'}(L+A')$ is π_*-acyclic by induction on $\dim X$ and $R^qf_*\mathcal{O}_Y(L+A')$ is also π_*-acyclic by the above assumption. Thus, $E_2^{pq}=0$ for $p\geq 2$ in the following commutative diagram of spectral sequences.

$$\begin{tikzcd}
E_2^{pq}=R^p\pi_*R^qf_*\mathcal{O}_Y(L) \arrow[r, Rightarrow] \arrow[d, "\varphi^{pq}"'] & R^{p+q}(\pi\circ f)_*\mathcal{O}_Y(L) \arrow[d, "\varphi^{p+q}"'] \\
\overline{E}_2^{pq}=R^p\pi_*R^qf_*\mathcal{O}_Y(L+A') \arrow[r, Rightarrow] & R^{p+q}(\pi\circ f)_*\mathcal{O}_Y(L+A')
\end{tikzcd}$$

Since φ^{1+q} is injective by Theorem 5.6, $E_2^{1q} \to R^{1+q}(\pi \circ f)_*\mathcal{O}_Y(L)$ is injective, and $\overline{E}_2^{1q} = 0$ by the above assumption, we have $E_2^{1q} = 0$. This implies that $R^p\pi_*R^qf_*\mathcal{O}_Y(L) = 0$ for any $p > 0$.

Step 2. We assume that V is projective. By replacing H' (resp. L) with $H' + \pi^*G$ (resp. $L + (\pi \circ f)^*G$), where G is a very ample Cartier divisor on V, we can assume that H' is an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. By the same argument as in Step 1, we can assume that H' is ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and $H \sim_{\mathbb{Q}} f^*H'$. If G is a sufficiently ample Cartier divisor on V, $H^k(V, R^p\pi_*R^qf_*\mathcal{O}_Y(L) \otimes G) = 0$ for any $k \geq 1$, $H^0(V, R^p\pi_*R^qf_*\mathcal{O}_Y(L) \otimes G) \simeq H^p(X, R^qf_*\mathcal{O}_Y(L) \otimes \pi^*G)$, and $R^p\pi_*R^qf_*\mathcal{O}_Y(L) \otimes G$ is generated by its global sections. Since $H + f^*\pi^*G \sim_{\mathbb{R}} L + f^*\pi^*G - (K_Y + S + B)$, $H + f^*\pi^*G \sim_{\mathbb{Q}} f^*(H' + \pi^*G)$, and $H' + \pi^*G$ is ample, we can apply Step 1 and obtain $H^p(X, R^qf_*\mathcal{O}_Y(L + f^*\pi^*G)) = 0$ for any $p > 0$. Thus, $R^p\pi_*R^qf_*\mathcal{O}_Y(L) = 0$ for any $p > 0$ by the above arguments.

Step 3. When V is not projective, we shrink V and assume that V is affine. By the same argument as in Step 1 above, we can assume that H' is $\mathbb{Q}$-Cartier. We compactify V and X, and can assume that V and X are projective. By Lemma 5.4, we can reduce it to the case when V is projective. This step is essentially the same as Step 2 in the proof of (i). So, we omit the details here.

We finish the whole proof of (ii). $\square$

Remark 5.8. In Theorem 5.6, if X is smooth, then Proposition 3.1 is enough for the proof of Theorem 5.6. In the proof of Theorem 5.7, if Y is smooth, then Theorem 5.6 for a smooth X is sufficient. Lemmas 5.1, 5.2, and 5.4 are easy and well known for smooth varieties. Therefore, the reader can find that our proof of Theorem 5.7 becomes much easier if we assume that Y is smooth. Ambro's original proof of [A1, Theorem 3.2 (ii)] used embedded simple normal crossing pairs even when Y is smooth (see (b) in the proof of [A1, Theorem 3.2 (ii)]). It may be a technically important difference. I could not follow Ambro's original argument in (a) in the proof of [A1, Theorem 3.2 (ii)].

Remark 5.9. It is easy to see that Theorem 5.6 is a generalization of Kollár's injectivity theorem. Theorem 5.7 (i) (resp. (ii)) is a generalization of Kollár's torsion-free (resp. vanishing) theorem.

We treat an easy vanishing theorem for lc pairs as an application of Theorem 5.7 (ii). It seems to be buried in [A1]. We note that we do not need the notion of embedded simple normal crossing pairs to prove Theorem 5.10. See Remark 5.8.

Theorem 5.10 (Kodaira vanishing theorem for lc pairs). *Let (X, B) be an lc pair such that B is a boundary $\mathbb{R}$-divisor. Let L be a $\mathbb{Q}$-Cartier Weil divisor on X such that $L-(K_X+B)$ is π-ample, where $\pi : X \to V$ is a projective morphism. Then $R^q\pi_*\mathcal{O}_X(L) = 0$ for any $q > 0$.*

Proof. Let $f : Y \to X$ be a log resolution of (X, B) such that $K_Y = f^*(K_X + B) + \sum_i a_iE_i$ with $a_i \geq -1$ for any i. We can assume that $\sum_i E_i \cup \mathrm{Supp} f^*L$ is a simple normal crossing divisor on Y. We put $E = \sum_i a_iE_i$ and $F = \sum_{a_j=-1}(1 - b_j)E_j$, where $b_j = \mathrm{mult}_{E_j}\{f^*L\}$. We note that $A = L - (K_X + B)$ is π-ample by the assumption. So, we have $f^*A = f^*L - f^*(K_X+B) = \ulcorner f^*L+E+F\urcorner - (K_Y+F+\{-(f^*L+E + F)\})$. We can easily check that $f_*\mathcal{O}_Y(\ulcorner f^*L + E + F\urcorner) \simeq \mathcal{O}_X(L)$ and that $F + \{-(f^*L + E + F)\}$ has a simple normal crossing support and is a boundary $\mathbb{R}$-divisor on Y. By Theorem 5.7 (ii), we obtain that $\mathcal{O}_X(L)$ is π_*-acyclic. Thus, we have $R^q\pi_*\mathcal{O}_X(L) = 0$ for any $q > 0$. □

We note that Theorem 5.10 contains a complete form of [Kv, Theorem 0.3] as a corollary.

Corollary 5.11 (Kodaira vanishing theorem for lc varieties). *Let X be a projective lc variety and L an ample Cartier divisor on X. Then $H^q(X, \mathcal{O}_X(K_X+L)) = 0$ for any $q > 0$. Furthermore, if we assume that X is Cohen-Macaulay, then $H^q(X, \mathcal{O}_X(-L)) = 0$ for any $q < \dim X$.*

Remark 5.12. We can see that Corollary 5.11 is contained in [F1, Theorem 2.6], which is a very special case of Theorem 5.7 (ii). I forgot to state Corollary 5.11 explicitly in [F1]. There, we do not need embedded simple normal crossing pairs. We note that there are typos in the proof of [F1, Theorem 2.6]. In the commutative diagram, $R^if_*\omega_X(D)$'s should be replaced by $R^jf_*\omega_X(D)$'s.

Example 5.13. Let X be a projective lc threefold which has the following properties: (i) there exists a projective birational morphism $f : Y \to X$ from a smooth projective threefold, and (ii) the exceptional locus E of f is an Abelian surface with $K_Y = f^*K_X - E$. For example, X is a cone over an Abelian surface and $f : Y \to X$ is the blow-up at the vertex of X. Let L be an ample Cartier divisor on X. By the Leray spectral sequence, we have $0 \to H^1(X, f_*f^*\mathcal{O}_X(-L)) \to H^1(Y, f^*\mathcal{O}_X(-L)) \to H^0(X, R^1f_*f^*\mathcal{O}_X(-L)) \to H^2(X, f_*f^*\mathcal{O}_X(-L)) \to H^2(Y, f^*\mathcal{O}_X(-L)) \to \cdots$. Therefore, $H^2(X, \mathcal{O}_X(-L)) \simeq H^0(X, \mathcal{O}_X(-L)\otimes R^1f_*\mathcal{O}_Y)$. On the other hand, we have $R^qf_*\mathcal{O}_Y \simeq H^q(E, \mathcal{O}_E)$ for any $q > 0$ since $R^qf_*\mathcal{O}_Y(-E) = 0$ for every $q > 0$. Thus, $H^2(X, \mathcal{O}_X(-L)) \simeq \mathbb{C}^2$. In particular, $H^2(X, \mathcal{O}_X(-L)) \neq 0$. We note that X is not Cohen-Macaulay. In the above example, if we assume that E is a $K3$-surface, then $H^q(X, \mathcal{O}_X(-L)) = 0$ for $q < 3$ and X is Cohen-Macaulay.

5.14 (Some further generalizations)**.** Here, we treat some generalizations of Theorem 5.7. First, we introduce the notion of nef and log big (resp. nef and log abundant) divisors.

Definition 5.15. Let $f : (Y, B) \to X$ be a proper morphism from an embedded simple normal crossing pair (Y, B) such that B is a subboundary. Let $\pi : X \to V$ be a proper morphism and H an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. We say that H is *nef and log big* (resp. *nef and log abundant*) over V if and only if $H|_C$ is nef and big (resp. nef and abundant) over V for any qlc center C. We note that a qlc center C is the image of a stratum of (Y, B). When (X, B_X) is an lc pair, we choose a log resolution of (X, B_X) to be $f : (Y, B) \to X$, where $K_Y + B = f^*(K_X + B_X)$.

We can generalize Theorem 5.7 as follows. It is [A1, Theorem 7.4] for embedded *simple* normal crossing pairs.

Theorem 5.16 (cf. [A1, Theorem 7.4])**.** *Let $f : (Y, S + B) \to X$ be a proper morphism from an embedded simple normal crossing pair such that $S + B$ is a boundary $\mathbb{R}$-divisor, S is reduced, and $\llcorner B \lrcorner = 0$. Let L be a Cartier divisor on Y and $\pi : X \to V$ a proper morphism. Assume that $f^*H \sim_{\mathbb{R}} L - (K_Y + S + B)$, where H is nef and log big over V. Then*

(i) *every non-zero local section of $R^q f_* \mathcal{O}_Y(L)$ contains in its support the f-image of some strata of $(Y, S + B)$, and*
(ii) *$R^q f_* \mathcal{O}_Y(L)$ is π_*-acyclic.*

For the proof, see the proof of [A1, Theorem 7.4]. Ambro cleverly reduced Theorem 5.16 to Theorem 5.7. In the second step (2) in the proof of [A1, Theorem 7.4], Ambro used "embedded log transformation"(cf. Lemmas 6.4 and 6.6 below) and the dévissage (see [A1, Remark 2.6]). So, we need the notion of embedded simple normal crossing pairs to prove Theorem 5.16 even when Y is smooth. It is a key point. As a corollary of Theorem 5.16, we can prove the following vanishing theorem, which is stated implicitly in the introduction of [A1]. It is the culmination of the works of several authors: Kawamata, Viehweg, Nadel, Reid, Fukuda, Ambro, and many others (cf. [KMM, Theorem 1-2-5]).

Theorem 5.17. *Let (X, B) be an lc pair such that B is a boundary $\mathbb{R}$-divisor and let L be a $\mathbb{Q}$-Cartier Weil divisor on X. Assume that $L - (K_X + B)$ is nef and log big over V, where $\pi : X \to V$ is a proper morphism. Then $R^q \pi_* \mathcal{O}_X(L) = 0$ for any $q > 0$.*

The proof of Theorem 5.10 works for Theorem 5.17 without any changes if we adopt Theorem 5.16. We add one example.

Example 5.18. Let Y be a projective surface which has the following properties: (i) there exists a projective birational morphism $f : X \to Y$ from a smooth projective surface X, and (ii) the exceptional locus E of f is an elliptic curve with $K_X + E = f^*K_Y$. For example, Y is a cone over a smooth plane cubic curve and $f : X \to Y$ is the blow-up at the vertex of Y. We note that (X, E) is a plt pair. Let H be an ample Cartier divisor on Y. We consider a Cartier divisor $L = f^*H + K_X + E$ on X. Then $L - (K_X + E)$ is nef and big, but not log big. By the short exact sequence $0 \to \mathcal{O}_X(f^*H + K_X) \to \mathcal{O}_X(f^*H + K_X + E) \to \mathcal{O}_E(K_E) \to 0$, we obtain $R^1f_*\mathcal{O}_X(f^*H + K_X + E) \simeq H^1(E, \mathcal{O}_E(K_E)) \simeq \mathbb{C}(P)$, where $P = f(E)$. By the Leray spectral sequence, we have $0 \to H^1(Y, f_*\mathcal{O}_X(K_X + E) \otimes \mathcal{O}_Y(H)) \to H^1(X, \mathcal{O}_X(L)) \to H^0(Y, \mathbb{C}(P)) \to H^2(Y, f_*\mathcal{O}_X(K_X + E) \otimes \mathcal{O}_Y(H)) \to \cdots$. If H is sufficiently ample, then $H^1(X, \mathcal{O}_X(L)) \simeq H^0(Y, \mathbb{C}(P)) \simeq \mathbb{C}(P)$. In particular, $H^1(X, \mathcal{O}_X(L)) \neq 0$.

Remark 5.19. In Example 5.18, there exists an effective $\mathbb{Q}$-divisor B on X such that $L - \frac{1}{k}B$ is ample for any $k > 0$ by Kodaira's lemma. Since $L \cdot E = 0$, we have $B \cdot E < 0$. In particular, $(X, E + \frac{1}{k}B)$ is not lc for any $k > 0$. This is the main reason why $H^1(X, \mathcal{O}_X(L)) \neq 0$. If $(X, E + \frac{1}{k}B)$ were lc, then the ampleness of $L - (K_X + E + \frac{1}{k}B)$ would imply $H^1(X, \mathcal{O}_X(L)) = 0$.

We modify the proof of [A1, Theorem 7.4]. Then we can easily obtain the following generalization of Theorem 5.7 (i). We leave the details for the reader's exercise.

Theorem 5.20. *Let $f : (Y, S+B) \to X$ be a proper morphism from an embedded simple normal crossing pair such that $S + B$ is a boundary, S is reduced, and $\llcorner B \lrcorner = 0$. Let L be a Cartier divisor on Y and $\pi : X \to V$ a proper morphism. Assume that $f^*H \sim_{\mathbb{R}} L-(K_Y+S+B)$, where H is nef and log abundant over V. Then, every non-zero local section of $R^qf_*\mathcal{O}_Y(L)$ contains in its support the f-image of some strata of $(Y, S + B)$.*

6. From SNC pairs to NC pairs

In this final section, we recover Ambro's theorems from Theorems 5.6 and 5.7. We repeat Ambro's statements for the reader's convenience.

Theorem 6.1 (cf. [A1, Theorem 3.1])**.** *Let $(X, S + B)$ be an embedded normal crossing pair such that X is proper, $S + B$ is a boundary $\mathbb{R}$-divisor, S is reduced, and $\llcorner B \lrcorner = 0$. Let L be a Cartier divisor on X and D an effective Cartier divisor that is permissible with respect to $(X, S + B)$. Assume the following conditions.*

(i) $L \sim_{\mathbb{R}} K_X + S + B + H$,
(ii) H *is a semi-ample* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor, and*
(iii) $tH \sim_{\mathbb{R}} D + D'$ *for some positive real number* t*, where* D' *is an effective* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor that is permissible with respect to* $(X, S + B)$*.*

Then the homomorphisms

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D)),$$

which are induced by the inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$*, are injective for all* q*.*

Theorem 6.2 (cf. [A1, Theorem 3.2])**.** *Let* $(Y, S + B)$ *be an embedded normal crossing pair such that* $S + B$ *is a boundary* $\mathbb{R}$*-divisor,* S *is reduced, and* $\llcorner B \lrcorner = 0$*. Let* $f : Y \to X$ *be a proper morphism and* L *a Cartier divisor on* Y *such that* $H \sim_{\mathbb{R}} L-(K_Y+S+B)$ *is* f*-semi-ample.*

(i) *every non-zero local section of* $R^q f_*\mathcal{O}_Y(L)$ *contains in its support the* f*-image of some strata of* $(Y, S + B)$*.*
(ii) *let* $\pi : X \to V$ *be a projective morphism and assume that* $H \sim_{\mathbb{R}} f^*H'$ *for some* π*-ample* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor* H' *on* X*. Then* $R^q f_*\mathcal{O}_Y(L)$ *is* π_**-acyclic, that is,* $R^p\pi_*R^q f_*\mathcal{O}_Y(L) = 0$ *for any* $p > 0$*.*

Before we go to the proof, let us recall the definition of *normal crossing pairs*, which is a slight generalization of Definition 2.6. The following definition is the same as [A1, Definition 2.3] though it may look different.

Definition 6.3 (Normal crossing pair)**.** Let X be a normal crossing variety. We say that a reduced divisor D on X is *normal crossing* if, in the notation of Definition 2.1, we have

$$\widehat{\mathcal{O}}_{D,x} \simeq \frac{\mathbb{C}[[x_0, \cdots, x_N]]}{(x_0 \cdots x_k, x_{i_1} \cdots x_{i_l})}$$

for some $\{i_1, \cdots, i_l\} \subset \{k+1, \cdots, N\}$. We say that the pair (X, B) is a *normal crossing pair* if the following conditions are satisfied.

(1) X is a normal crossing variety, and
(2) B is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor whose support is normal crossing on X.

We say that a normal crossing pair (X, B) is *embedded* if there exists a closed embedding $\iota : X \to M$, where M is a smooth variety of dimension $\dim X + 1$. We put $K_{X^0} + \Theta = \eta^*(K_X + B)$, where $\eta : X^0 \to X$ is the normalization of X. From now on, we assume that B is a subboundary $\mathbb{R}$-divisor. A *stratum* of (X, B) is an irreducible component of X or the image of some lc center of (X^0, Θ) on X. A Cartier divisor D on a normal crossing pair (X, B) is called *permissible with respect to* (X, B) if D contains no strata of the pair (X, B).

The following three lemmas are easy to check. So, we omit the proofs.

Lemma 6.4. *Let X be a normal crossing divisor on a smooth variety M. Then there exists a sequence of blow-ups $M_k \to M_{k-1} \to \cdots \to M_0 = M$ with the following properties.*

(i) *$\sigma_{i+1} : M_{i+1} \to M_i$ is the blow-up along a smooth stratum of X_i for any $i \geq 0$,*
(ii) *$X_0 = X$ and X_{i+1} is the inverse image of X_i with the reduced structure for any $i \geq 0$, and*
(iii) *X_k is a simple normal crossing divisor on M_k.*

For each step σ_{i+1}, we can directly check that $\sigma_{i+1*}\mathcal{O}_{X_{i+1}} \simeq \mathcal{O}_{X_i}$ and $R^q\sigma_{i+1*}\mathcal{O}_{X_{i+1}} = 0$ for any $i \geq 0$ and $q \geq 1$. Let B be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $\mathrm{Supp}B$ *is normal crossing. We put $B_0 = B$ and $K_{X_{i+1}} + B_{i+1} = \sigma_{i+1}^*(K_{X_i} + B_i)$ for all $i \geq 0$. Then it is obvious that B_i is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor and* $\mathrm{Supp}B_i$ *is normal crossing on X_i for any $i \geq 0$. We can also check that B_i is a boundary $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor) for any $i \geq 0$ if so is B. If B is a boundary, then the σ_{i+1}-image of any stratum of (X_{i+1}, B_{i+1}) is a stratum of (X_i, B_i).*

Remark 6.5. Each step in Lemma 6.4 is called *embedded log transformation* in [A1, Section 2].

Lemma 6.6. *Let X be a simple normal crossing divisor on a smooth variety M. Let $S + B$ be a boundary $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that* $\mathrm{Supp}(S+B)$ *is normal crossing, S is reduced, and $\llcorner B \lrcorner = 0$. Then there exists a sequence of blow-ups $M_k \to M_{k-1} \to \cdots \to M_0 = M$ with the following properties.*

(i) *$\sigma_{i+1} : M_{i+1} \to M_i$ is the blow-up along a smooth stratum of (X_i, S_i) that is contained in S_i for any $i \geq 0$,*
(ii) *we put $X_0 = X$, $S_0 = S$, and $B_0 = B$, and X_{i+1} is the strict transform of X_i for any $i \geq 0$,*
(iii) *we define $K_{X_{i+1}} + S_{i+1} + B_{i+1} = \sigma_{i+1}^*(K_{X_i} + S_i + B_i)$ for any $i \geq 0$, where B_{i+1} is the strict transform of B_i on X_{i+1},*
(iv) *the σ_{i+1}-image of any stratum of $(X_{i+1}, S_{i+1}+B_{i+1})$ is a stratum of $(X_i, S_i + B_i)$, and*
(v) *S_k is a simple normal crossing divisor on X_k.*

For each step σ_{i+1}, we can easily check that $\sigma_{i+1*}\mathcal{O}_{X_{i+1}} \simeq \mathcal{O}_{X_i}$ and $R^q\sigma_{i+1*}\mathcal{O}_{X_{i+1}} = 0$ for any $i \geq 0$ and $q \geq 1$. We note that X_i is simple normal crossing, $\mathrm{Supp}(S_i + B_i)$ *is normal crossing on X_i, and S_i is reduced for any $i \geq 0$.*

Lemma 6.7. *Let X be a simple normal crossing divisor on a smooth variety M. Let $S+B$ be a boundary $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that* $\mathrm{Supp}(S+B)$ *is normal crossing, S is reduced and simple normal crossing, and $\llcorner B \lrcorner = 0$. Then there exists a sequence of blow-ups $M_k \to M_{k-1} \to \cdots \to M_0 = M$ with the following properties.*

(i) *$\sigma_{i+1} : M_{i+1} \to M_i$ is the blow-up along a smooth stratum of $(X_i, \mathrm{Supp} B_i)$ that is contained in* $\mathrm{Supp} B_i$ *for any $i \geq 0$,*
(ii) *we put $X_0 = X$, $S_0 = S$, and $B_0 = B$, and X_{i+1} is the strict transform of X_i for any $i \geq 0$,*
(iii) *we define $K_{X_{i+1}} + S_{i+1} + B_{i+1} = \sigma_{i+1}^*(K_{X_i} + S_i + B_i)$ for any $i \geq 0$, where S_{i+1} is the strict transform of S_i on X_{i+1}, and*
(iv) $\mathrm{Supp}(S_k + B_k)$ *is a simple normal crossing divisor on X_k.*

We note that X_i is simple normal crossing on M_i and $\mathrm{Supp}(S_i + B_i)$ *is normal crossing on X_i for any $i \geq 0$. We can easily check that $\llcorner B_i \lrcorner \leq 0$ for any $i \geq 0$. The composition morphism $M_k \to M$ is denoted by σ. Let L be any Cartier divisor on X. We put $E = \ulcorner -B_k \urcorner$. Then E is an effective σ-exceptional Cartier divisor on X_k and we obtain $\sigma_* \mathcal{O}_{X_k}(\sigma^* L + E) \simeq \mathcal{O}_X(L)$ and $R^q \sigma_* \mathcal{O}_{X_k}(\sigma^* L + E) = 0$ for any $q \geq 1$ by* Theorem 5.7 (i). *We note that $\sigma^* L + E - (K_{X_k} + S_k + \{B_k\}) = \sigma^* L - \sigma^*(K_X + S + B)$ is $\mathbb{R}$-linearly trivial over X and σ is an isomorphism at any generic points of strata of $(X_k, S_k + \{B_k\})$.*

Let us go to the proofs of Theorems 6.1 and 6.2.

Proof of Theorems 6.1 and 6.2. We take a sequence of blow-ups and obtain a projective morphism $\sigma : X' \to X$ (resp. $\sigma : Y' \to Y$) from an embedded simple normal crossing variety X' (resp. Y') in Theorem 6.1 (resp. Theorem 6.2) by Lemma 6.4. We can replace X (resp. Y) and L with X' (resp. Y') and $\sigma^* L$ by Leray's spectral sequence. So, we can assume that X (resp. Y) is simple normal crossing. Similarly, we can assume that S is simple normal crossing on X (resp. Y) by applying Lemma 6.6. Finally, we use Lemma 6.7 and obtain a birational morphism $\sigma : (X', S' + B') \to (X, S + B)$ (resp. $(Y', S' + B') \to (Y, S+B)$) from an embedded simple normal crossing pair $(X', S'+B')$ (resp. $(Y', S' + B')$) such that $K_{X'} + S' + B' = \sigma^*(K_X + S + B)$ (resp. $K_{Y'}+S'+B' = \sigma^*(K_Y+S+B)$) as in Lemma 6.7. By Lemma 6.7, we can replace $(X, S+B)$ (resp. $(Y, S+B)$) and L with $(X', S'+\{B'\})$ (resp. $(Y', S' + \{B'\})$) and $\sigma^* L + \ulcorner -B' \urcorner$ by Leray's spectral sequence. Then we apply Theorem 5.6 (resp. Theorem 5.7). Thus, we obtain Theorems 6.1 and 6.2. □

We close this paper with the review of our proofs of Theorems 6.1 and 6.2. It may help the reader to compare this paper with [A1, Section

3]. We think that our proofs are not so long. Ambro's proofs seem to be too short.

6.8 (Review)**.** We review our proofs of the injectivity and vanishing theorems.

Step 1. (E_1-degeneration of a certain Hodge to de Rham type spectral sequence). We discuss this E_1-degeneration in 4.6. As we pointed out in the introduction, the appropriate spectral sequence was not chosen in [A1]. It is one of the crucial technical problems in [A1, Section 3]. This step is purely Hodge theoretic. We describe it in Section 4.

Step 2. (Fundamental injectivity theorem: Proposition 3.2). This is a very special case of [A1, Theorem 3.1] and follows from the E_1-degeneration in Step 1 by using covering arguments. This step is in Section 3.

Step 3. (Relative vanishing lemma: Lemma 5.1). This step is missing in [A1]. It is a very special case of [A1, Theorem 3.2 (ii)]. However, we can not use [A1, Theorem 3.2 (ii)] in this stage. Our proof of this lemma does *not* work directly for normal crossing pairs. So, we need to assume that the varieties are *simple* normal crossing pairs.

Step 4. (Injectivity theorem for embedded simple normal crossing pairs: Theorem 5.6). It is [A1, Theorem 3.1] for embedded *simple* normal crossing pairs. It follows easily from Step 2 since we already have the relative vanishing lemma in Step 3. A key point in this step is Lemma 5.2, which is missing in [A1] and works only for embedded *simple* normal crossing pairs.

Step 5. (Torsion-free and vanishing theorems for embedded simple normal crossing pairs: Theorem 5.7). It is [A1, Theorem 3.2] for embedded *simple* normal crossing pairs. The proof uses the lemmas on desingularization and compactification (see Lemmas 5.2 and 5.4), which hold only for embedded *simple* normal crossing pairs, and the injectivity theorem proved for embedded *simple* normal crossing pairs in Step 4. Therefore, this step also works only for embedded *simple* normal crossing pairs. Our proof of the vanishing theorem is slightly different from Ambro's one. Compare Steps 2 and 3 in the proof of Theorem 5.7 with (a) and (b) in the proof of [A1, Theorem 3.2 (ii)]. See Remark 5.8.

Step 6. (Ambro's theorems: Theorems 6.1 and 6.2). In this final step, we recover Ambro's theorems, that is, [A1, Theorems 3.1 and 3.2], in full generality. Since we have already proved [A1, Theorem 3.2 (i)] for embedded *simple* normal crossing pairs in Step 5, we can reduce the problems to the case when the varieties are embedded *simple* normal crossing pairs by blow-ups and Leray's spectral sequences. This step is described in Section 6.

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