RELATIVE BERTINI TYPE THEOREM FOR MULTIPLIER IDEAL SHEAVES

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Abstract. We establish a relative Bertini type theorem for multiplier ideal sheaves. Then we prove a relative version of the Kollár–Nadel type vanishing theorem as an application.

1. Introduction

Let X be a smooth complex projective variety and let D be an effective $\mathbb{Q}$-divisor on X. Let H be a general member of a free linear system Λ on X. Then it is well known and is easy to see that the equality

$$\mathscr{J}(H, D|_H) = \mathscr{J}(X, D)|_H$$

holds and that there exists the following short exact sequence

$$0 \to \mathscr{J}(X, D) \otimes \mathscr{O}_X(-H) \to \mathscr{J}(X, D) \to \mathscr{J}(H, D|_H) \to 0,$$

where $\mathscr{J}(X, D)$ (resp. $\mathscr{J}(H, D|_H)$) is the multiplier ideal sheaf associated to D (resp. $D|_H$). Let φ be a quasi-plurisubharmonic function on X. Then, for every smooth subvariety H, the inclusion

$$\mathscr{J}(\varphi|_H) \subset \mathscr{J}(\varphi)|_H$$

follows from the Ohsawa–Takegoshi L^2 extension theorem. Note that $\mathscr{J}(\varphi)$ (resp. $\mathscr{J}(\varphi|_H)$) is the multiplier ideal sheaf associated to φ (resp. $\varphi|_H$). However, the equality

$$\mathscr{J}(\varphi|_H) = \mathscr{J}(\varphi)|_H$$

does not always hold.

Furthermore, we think that the existence of a smooth member H_0 of Λ such that the equality

$$\mathscr{J}(\varphi|_{H_0}) = \mathscr{J}(\varphi)|_{H_0}$$

holds and that there exists the following natural short exact sequence

$$0 \to \mathscr{J}(\varphi) \otimes \mathscr{O}_X(-H_0) \to \mathscr{J}(\varphi) \to \mathscr{J}(\varphi|_{H_0}) \to 0$$

is highly nontrivial. In [9, Theorem 1.10], we established that there are many members of Λ satisfying the above good properties. The main purpose of this paper is to prove the following theorem.

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Theorem 1.1 (Relative Bertini type theorem for multiplier ideal sheaves)**.** *Let $f : X \to S$ be a proper surjective morphism from a complex manifold X to a complex analytic space S. Let Π be a set of at most countably many quasi-plurisubharmonic functions on X. We consider the following commutative diagram:*

$$\begin{tikzcd} X \arrow[dd, "f"'] \arrow[dr, "g"] \arrow[drr, "h"] & & \\ & S \times \mathbb{P}^N \arrow[r, "p_2"'] \arrow[dl, "p_1"] & \mathbb{P}^N \\ S & & \end{tikzcd}$$

where p_i is the i-th projection for $i = 1, 2$. We consider the complete linear system

$$\Lambda := |\mathscr{O}_{\mathbb{P}^N}(1)| \simeq \mathbb{P}^N$$

on $\mathbb{P}^N$. Let $S^\dagger$ be any relatively compact open subset of S. We put $X^\dagger := f^{-1}(S^\dagger)$, $h^\dagger := h|_{X^\dagger}$, and consider

$$\mathscr{G} := \left\{ H' \in \Lambda \;\middle|\; \begin{array}{l} H^\dagger := (h^\dagger)^* H' \text{ is well-defined and is a smooth divisor on } X^\dagger, \\ \text{and } \mathscr{J}(\varphi|_{H^\dagger}) = \mathscr{J}(\varphi)|_{H^\dagger} \text{ holds for every } \varphi \in \Pi \end{array} \right\} \subset \Lambda.$$

We note that $H^\dagger$ is well-defined if and only if the image of every connected component of $X^\dagger$ by $h^\dagger$ is not contained in H'. We also note that $\mathscr{J}(\varphi)$ (resp. $\mathscr{J}(\varphi|_{H^\dagger})$) is the multiplier ideal sheaf on X (resp. $H^\dagger$) associated to φ (resp. $\varphi|_{H^\dagger}$) for every $\varphi \in \Pi$. Then $\mathscr{G}$ is dense in Λ ($\simeq \mathbb{P}^N$) in the classical topology. Furthermore, we put

$$\mathscr{H} := \left\{ H' \in \mathscr{G} \;\middle|\; \begin{array}{l} H^\dagger := (h^\dagger)^* H' \text{ contains no associated primes of } \mathscr{O}_X/\mathscr{J}(\varphi) \\ \text{on } X^\dagger \text{ for every } \varphi \in \Pi \end{array} \right\} \subset \mathscr{G}.$$

Then $\mathscr{H}$ is also dense in Λ in the classical topology. More generally, $\mathscr{G} \setminus \mathscr{S}$ and $\mathscr{H} \setminus \mathscr{S}$ are dense in Λ in the classical topology for any analytically meagre subset $\mathscr{S}$ of Λ (see Definition 3.1 below for analytically meagre subsets). We note that there exists the following natural short exact sequence

$$0 \to \mathscr{J}(\varphi|_{X^\dagger}) \otimes \mathscr{O}_{X^\dagger}(-H^\dagger) \to \mathscr{J}(\varphi|_{X^\dagger}) \to \mathscr{J}(\varphi|_{H^\dagger}) \to 0$$

for every $H' \in \mathscr{H}$ and every $\varphi \in \Pi$, where $H^\dagger = (h^\dagger)^* H'$.

In Theorem 1.1, we are mainly interested in the case where there exists $s_0 \in S^\dagger$ such that $\dim g(f^{-1}(s_0)) > 0$. We want to cut down $g(f^{-1}(s_0))$ by the linear system Λ in some applications (see the proof of Theorem 1.4 below). It may happen that $H^\dagger = 0$ for some $H' \in \mathscr{G}$ when $\dim g(f^{-1}(s)) = 0$ holds for every $s \in S^\dagger$. If S is a point and $\#\Pi = 1$, then Theorem 1.1 is essentially the same as [9, Theorem 1.10] (see also [9, Corollary 3.11]). Therefore, Theorem 1.1 can be seen as a relative generalization of [9, Theorem 1.10]. We note that $\Lambda \setminus \mathscr{G}$ is not always analytically meagre in the sense of Definition 3.1 (see, for example, [9, Example 3.10]).

Let φ be a quasi-plurisubharmonic function on a compact complex manifold X. We put $\Pi = \{m\varphi\}_{m \in \mathbb{N}}$. Then, as a very special case of Theorem 1.1, we have:

Corollary 1.2. *Let X be a compact complex manifold and let φ be a quasi-plurisubharmonic function on X. Let W be a free linear system on X. Then there exists a dense subset V of W such that every element H of V is smooth and that*

$$0 \to \mathscr{J}(m\varphi) \otimes \mathscr{O}_X(-H) \to \mathscr{J}(m\varphi) \to \mathscr{J}(m\varphi|_H) \to 0$$

is exact for every positive integer m.

Let us recall one of the main results of [15], which is a relative version of [9, Theorem A]. Of course, the proof of Theorem 1.3 in [15] is much harder than that of [9, Theorem A]. Theorem 1.3 is a generalization of the Enoki injectivity theorem in [4], which is an analytic counterpart of the Kollár injectivity theorem (see [11]).

Theorem 1.3 ([15, Theorem 1.3])**.** *Let $\pi : X \to S$ be a proper surjective locally Kähler morphism from a complex manifold X to a complex analytic space S. Let F be a holomorphic line bundle on X equipped with a singular hermitian metric h and let M be a holomorphic line bundle on X with a smooth hermitian metric h_M. Assume that*

$$\sqrt{-1}\Theta_{h_M}(M) \geq 0 \quad \text{and} \quad \sqrt{-1}\Theta_h(F) - \varepsilon\sqrt{-1}\Theta_{h_M}(M) \geq 0$$

for some $\varepsilon > 0$. Then, for any non-zero holomorphic section s of M, the map

$$\times s : R^q\pi_*(\omega_X \otimes F \otimes \mathscr{J}(h)) \to R^q\pi_*(\omega_X \otimes F \otimes \mathscr{J}(h) \otimes M)$$

induced by the tensor product with s is injective for every q, where ω_X is the canonical bundle of X and $\mathscr{J}(h)$ is the multiplier ideal sheaf associated to the singular hermitian metric h.

By using Theorems 1.1 and 1.3, we prove a relative version of the Kollár–Nadel type vanishing theorem (see [7]).

Theorem 1.4 (Relative Kollár–Nadel type vanishing theorem)**.** *Let $f : X \to Y$ be a proper surjective locally Kähler morphism from a complex manifold X to a complex analytic space Y. Let $\pi : Y \to Z$ be a projective surjective morphism between complex analytic spaces. Let F be a holomorphic line bundle on X equipped with a singular hermitian metric h. Let H be a π-ample holomorphic line bundle on Y. Assume that there exists a smooth hermitian metric g on f^*H such that*

$$\sqrt{-1}\Theta_g(f^*H) \geq 0 \quad \text{and} \quad \sqrt{-1}\Theta_h(F) - \varepsilon\sqrt{-1}\Theta_g(f^*H) \geq 0$$

for some $\varepsilon > 0$. Then we have

$$R^i\pi_*R^jf_*(\omega_X \otimes F \otimes \mathscr{J}(h)) = 0$$

for every $i > 0$ and j, where ω_X is the canonical bundle of X and $\mathscr{J}(h)$ is the multiplier ideal sheaf associated to the singular hermitian metric h.

As an application of Theorem 1.4 and the strong openness in [10], we have:

Corollary 1.5. *Let $f : X \to Y$ be a proper surjective locally Kähler morphism from a complex manifold X to a complex analytic space Y. Let $\pi : Y \to Z$ be a locally projective surjective morphism between complex analytic spaces. Let F be a holomorphic line bundle on X equipped with a singular hermitian metric h such that $\sqrt{-1}\Theta_h(F) \geq 0$. Let M be a π-nef and π-big holomorphic line bundle on Y. Then we have*

$$R^i\pi_*(M \otimes R^jf_*(\omega_X \otimes F \otimes \mathscr{J}(h))) = 0$$

for every $i > 0$ and j, where ω_X is the canonical bundle of X and $\mathscr{J}(h)$ is the multiplier ideal sheaf associated to the singular hermitian metric h.

For related vanishing theorems, see [6], [7], [11], [14], [17], [18], and so on. We recommend the reader to see [8, Chapters 5 and 6], where we discuss various Kollár type vanishing theorems by using the theory of mixed Hodge structures on cohomology with compact support and explain their applications to the minimal model program for higher-dimensional complex algebraic varieties.

We give a remark on Nakano semipositive vector bundles.

Remark 1.6 (Twists by Nakano semipositive vector bundles)**.** Let E be a Nakano semipositive holomorphic vector bundle on X. Then it is not difficult to see that Theorems 1.3, 1.4, and Corollary 1.5 hold even when ω_X is replaced by $\omega_X \otimes E$. We leave the details as an exercise for the reader (see [9, Section 6]).

The following example may help the reader understand Theorem 1.1 and its proof given in this paper.

Example 1.7. We put

$$\Delta^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \,|\, |z_1| < 1, \cdots, |z_n| < 1\}.$$

Let $\pi : \Delta^n \to \Delta = \{z \in \mathbb{C} \,|\, |z| < 1\}$ be the projection given by $(z_1, \dots, z_n) \mapsto z_n$. Let φ be a quasi-plurisubharmonic function in a neighborhood of $\overline{\Delta^n}$, that is, the closure of Δ^n in $\mathbb{C}^n$. Then, by the Ohsawa–Takegoshi L^2 extension theorem, we have the following inclusion

$$\mathscr{J}(\varphi|_{H_s}) \subset \mathscr{J}(\varphi)|_{H_s}$$

for every $s \in \Delta$, where $H_s = \pi^{-1}(s)$. Since $\mathscr{J}(\varphi)$ is a coherent ideal sheaf, it is locally finitely generated. By applying Fubini's theorem to each local generator of $\mathscr{J}(\varphi)$, we get the opposite inclusion

$$\mathscr{J}(\varphi|_{H_s}) \supset \mathscr{J}(\varphi)|_{H_s}$$

for almost all $s \in \Delta$. Therefore, the equality

$$\mathscr{J}(\varphi|_{H_s}) = \mathscr{J}(\varphi)|_{H_s}$$

holds for almost all $s \in \Delta$.

Recently, Xiankui Meng and Xiangyu Zhou established a simpler and more natural approach to the Bertini type theorem for multiplier ideal sheaves in [16]. We strongly recommend the interested reader to see [16]. Moreover, Mingchen Xia answered [9, Problem 1.11] affirmatively by a very clever argument.

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2. Preliminaries

For the basic results of the theory of complex analytic spaces, see [2] and [5]. For various analytic methods used in this paper, we recommend the reader to see [3].

Definition 2.1 (Singular hermitian metrics and curvatures)**.** Let F be a holomorphic line bundle on a complex manifold X. A *singular hermitian metric* on F is a metric h which is given in every trivialization $\theta : F|_U \simeq U \times \mathbb{C}$ by

$$|\xi|_h = |\theta(\xi)| e^{-\varphi} \text{ on } U,$$

where ξ is a section of F on U and $\varphi \in L^1_{\mathrm{loc}}(U)$ is an arbitrary function. Here $L^1_{\mathrm{loc}}(U)$ is the space of locally integrable functions on U. We usually call φ the weight function of the metric with respect to the trivialization θ. The curvature of a singular hermitian metric h is defined by

$$\Theta_h(F) := 2\partial\overline{\partial}\varphi,$$

where φ is a weight function and $\partial\overline{\partial}\varphi$ is taken in the sense of currents. It is easy to see that the right hand side does not depend on the choice of trivializations. Therefore, we get a global closed $(1,1)$-current $\Theta_h(F)$ on X.

Definition 2.2 ((Quasi-)plurisubharmonic functions and multiplier ideal sheaves)**.** A function $\varphi : U \to [-\infty, \infty)$ defined on an open set $U \subset \mathbb{C}^n$ is called *plurisubharmonic* if

(i) φ is upper semicontinuous, and
(ii) for every complex line $L \subset \mathbb{C}^n$, $\varphi|_{U\cap L}$ is subharmonic on $U \cap L$, that is, for every $a \in U$ and $\xi \in \mathbb{C}^n$ satisfying $|\xi| < d(a, U^c) = \inf\{|a - x| \mid x \in U^c\}$, the function φ satisfies the mean inequality

$$\varphi(a) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi(a + e^{i\theta}\xi)d\theta.$$

Let X be an n-dimensional complex manifold. A function $\varphi : X \to [-\infty, \infty)$ is said to be *plurisubharmonic* if there exists an open cover $X = \bigcup_{i\in I} U_i$ such that $\varphi|_{U_i}$ is plurisubharmonic on U_i ($\subset \mathbb{C}^n$) for every i. A *quasi-plurisubharmonic* function is a function φ which is locally equal to the sum of a plurisubharmonic function and of a smooth function.

Let φ be a quasi-plurisubharmonic function on a complex manifold X. Then the *multiplier ideal sheaf* $\mathscr{J}(\varphi) \subset \mathscr{O}_X$ is defined by

$$\Gamma(U, \mathscr{J}(\varphi)) = \{f \in \mathscr{O}_X(U) \mid |f|^2 e^{-2\varphi} \in L^1_{\mathrm{loc}}(U)\}$$

for every open set $U \subset X$. It is well known that $\mathscr{J}(\varphi)$ is a coherent ideal sheaf on X.

Let S be a complex submanifold of X. Then the restriction $\mathscr{J}(\varphi)|_S$ of the multiplier ideal sheaf $\mathscr{J}(\varphi)$ to S is defined by the image of $\mathscr{J}(\varphi)$ under the natural surjective morphism $\mathscr{O}_X \to \mathscr{O}_S$, that is,

$$\mathscr{J}(\varphi)|_S = \mathscr{J}(\varphi)/\mathscr{J}(\varphi) \cap \mathscr{I}_S,$$

where $\mathscr{I}_S$ is the defining ideal sheaf of S on X. We note that the restriction $\mathscr{J}(\varphi)|_S$ does not always coincide with $\mathscr{J}(\varphi) \otimes \mathscr{O}_S = \mathscr{J}(\varphi)/\mathscr{J}(\varphi)\mathscr{I}_S$.

Definition 2.3 (Multiplier ideal sheaves associated to singular hermitian metrics)**.** Let F be a holomorphic line bundle on a complex manifold X and let h be a singular hermitian metric on F. We assume $\sqrt{-1}\Theta_h(F) \geq \gamma$ for some smooth $(1,1)$-form γ on X. We fix a smooth hermitian metric h_∞ on F. Then we can write $h = h_\infty e^{-2\psi}$ for some $\psi \in L^1_{\mathrm{loc}}(X)$. Then ψ coincides with a quasi-plurisubharmonic function φ on X almost everywhere. In this situation, we put $\mathscr{J}(h) := \mathscr{J}(\varphi)$. We note that $\mathscr{J}(h)$ is independent of h_∞ and is well-defined.

3. Bertini type theorem revisited

In this section, we will reformulate some results in [12] for our purposes. Let us recall the definition of *analytically meagre* subsets.

Definition 3.1. A subset $\mathscr{S}$ of a complex analytic space X is said to be *analytically meagre* if

$$\mathscr{S} \subset \bigcup_{n\in\mathbb{N}} Y_n$$

where each Y_n is a locally closed analytic subset of X of codimension ≥ 1.

The following result is a slight reformulation of [12, (II.5) Theorem and (II.7) Corollary]. We need it for the proof of Theorem 1.1 in Section 4.

Theorem 3.2 (Bertini type theorem for complex manifolds)**.** *Let M be a complex manifold which has a countable base of open subsets and let $\mathscr{L}$ be a holomorphic line bundle on M. Assume that M has only finitely many connected components. Let t_l be an element of $H^0(M, \mathscr{L})$ for every $1 \leq l \leq N+1$ such that $\{t_1, \ldots, t_{N+1}\}$ generates $\mathscr{L}$, that is, $W \otimes_{\mathbb{C}} \mathscr{O}_M \to \mathscr{L}$ is surjective, where W is the linear subspace of $H^0(M, \mathscr{L})$ spanned by $\{t_1, \ldots, t_{N+1}\}$. We consider an $(N+1)$-dimensional vector space $V = \bigoplus_{l=1}^{N+1} \mathbb{C}t_l$. Then there exists a dense subset $\mathscr{D}$ of $\Lambda = (V - \{0\})/\mathbb{C}^\times (\simeq \mathbb{P}^N)$ such that $\Lambda \setminus \mathscr{D}$ is analytically meagre and that for each element of $\mathscr{D}$ the corresponding divisor on M is smooth.*

In Theorem 3.2, we do not assume that $\{t_1, \ldots, t_{N+1}\}$ is linearly independent.

Proof of Theorem 3.2. If $N = 0$, then the statement is trivial. Therefore, we may assume that $N \geq 1$.

Step 1. In this step, we will prove that there exists a dense subset $\mathscr{E}$ of V, which is a countable intersection of dense open subsets of V, such that for every $s \in V$ the zero set $(s = 0)$ is a smooth divisor on M if and only if $s \in \mathscr{E}$.

We take a countable covering $\{K_i\}_{i \in \mathbb{N}}$ of M such that K_i is compact for every i. We may assume that K_i is contained in an open subset U_i of M such that there exists $s_i \in V$ which is never zero on U_i for every i. We put

$$\mathscr{E}_i := \left\{ s \in V \;\middle|\; \begin{array}{l} (s=0) \text{ contains no connected components of } M \\ \text{and is smooth at every point of } K_i \cap (s=0) \end{array} \right\}.$$

Then $\mathscr{E}_i$ is open in V by [12, I Step in the proof of (II.5) Theorem] and is dense in V by [12, II Step in the proof of (II.5) Theorem]. We put $\mathscr{E} = \bigcap_{i \in N} \mathscr{E}_i$. Then $\mathscr{E}$ is dense in V by the Baire category theorem. By definition, for every $s \in V$, $(s = 0)$ is a smooth divisor on M if and only if $s \in \mathscr{E}$. By definition, $\mathscr{E}_i$ is $\mathbb{C}^\times$-invariant and $\mathscr{E}_i \subset V - \{0\}$. We put $p(\mathscr{E}) = \mathscr{D}$, where $p : V - \{0\} \to \Lambda$ is the natural projection. Of course, $\mathscr{D}$ is dense in Λ.

Step 2. In this step, we will prove that $\Lambda \setminus \mathscr{D}$ is a countable union of locally closed analytic subsets of Λ.

Let $\{U_i\}_{i \in \mathbb{N}}$ be an open covering of M on which $\mathscr{L}$ is trivial as in Step 1. With respect to this trivialization of $\mathscr{L}$, we can see that every $s \in V$ is a holomorphic function on each U_i. Since the number of connected components of M is finite, we can take a finite number of linear subspaces $\{V_j\}_{j=1}^k$ of V such that $V_j \subsetneq V$ for every j and that $s \in V$ is not identically zero on any connected component of M if and only if $s \in V \setminus \bigcup_{j=1}^k V_j$. For each i, we can consider the holomorphic map

$$F_i : U_i \times V \to \mathbb{C} \times V$$

defined by $F_i(x, s) = (s(x), s)$. Since every $s \in V^\dagger := V \setminus \bigcup_{j=1}^k V_j$ is not identically zero on any connected component of M, F_i is flat on $U_i \times V^\dagger$ (see [12, (II.1) Lemma]). We consider

$$\begin{aligned} A_i &:= \left\{ (x, s) \in U_i \times V^\dagger \;\middle|\; s(x) = 0 \text{ and } (s = 0) \text{ is not smooth at } x \right\} \\ &= F_i^{-1}(\{0\} \times V^\dagger) \cap \{(x, s) \in U_i \times V^\dagger \,|\, F_i^{-1}(F_i(x, s)) \text{ is not smooth at } (x, s)\}. \end{aligned}$$

Then, by [12, (0.3) a) Proposition], A_i is an analytic subset of $U_i \times V^\dagger$ for every i. Therefore,

$$A := \bigcup_{i\in\mathbb{N}} A_i \cup \left(M \times \bigcup_{j=1}^{k} V_j \right)$$

is a countable union of locally closed analytic subsets of $M \times V$. By construction, $V \setminus \mathscr{E} = q(A)$, where $q : M \times V \to V$ is the natural projection. Therefore, $V \setminus \mathscr{E}$ is a countable union of locally closed analytic subsets by [12, Lemma in (0.2)]. Thus we see that $\Lambda \setminus \mathscr{D} = p(q(A) - \{0\})$ is also a countable union of locally closed analytic subsets by [12, Lemma in (0.2)].

Hence $\Lambda \setminus \mathscr{D}$ is analytically meagre since $\Lambda \setminus \mathscr{D}$ is a countable union of locally closed analytic subsets by Step 2 and $\mathscr{D}$ is dense by Step 1. $\square$

Although Theorem 3.2 is sufficient for the proof of Theorem 1.1 in Section 4, we add some remarks for the reader's convenience.

Remark 3.3. The proof of Theorem 3.2 says that we can take $\mathscr{D}$ such that $\Lambda \setminus \mathscr{D}$ is a countable union of locally closed analytic subsets of Λ of codimension ≥ 1 and that for every $s \in \Lambda$ the zero set $(s = 0)$ defines a smooth divisor on M if and only if $s \in \mathscr{D}$.

Remark 3.4. Theorem 3.2 and Remark 3.3 hold true without assuming that M has only finitely many connected components. We assume that M has infinitely many connected components. Then we have the irreducible decomposition $M = \bigcup_{n\in\mathbb{N}} M_n$ since M has a countable base of open subsets. By applying Theorem 3.2 and Remark 3.3 to each M_n, we get a dense subset $\mathscr{D}_n$ of Λ with the desired properties for every n. We put $\mathscr{D} = \bigcap_{n\in\mathbb{N}} \mathscr{D}_n$. Then $\Lambda \setminus \mathscr{D}$ is a countable union of locally closed analytic subsets of Λ of codimension ≥ 1, and for every $s \in \Lambda$ the zero set $(s = 0)$ defines a smooth divisor on M if and only if $s \in \mathscr{D}$.

We prepare easy lemmas for the proof of Theorem 1.1 in Section 4.

Lemma 3.5. *Let $\mathscr{S}$ be an analytically meagre subset of $\mathbb{P}^N$. Let $p : \mathbb{P}^N - \{P\} \to \mathbb{P}^{N-1}$ be the linear projection from $P \in \mathbb{P}^N$. Then there exists an analytically meagre subset $\mathscr{S}'$ of $\mathbb{P}^{N-1}$ such that $p^{-1}(x) \cap \mathscr{S}$ is an analytically meagre subset of $p^{-1}(x) \simeq \mathbb{C}$ for every $x \in \mathbb{P}^{N-1} \setminus \mathscr{S}'$.*

Proof. We may assume that $\mathscr{S}$ is a countable union of locally closed analytic subsets of $\mathbb{P}^N$. We note that $p(V - \{P\})$ is a countable union of locally closed analytic subsets of $\mathbb{P}^{N-1}$, where V is any locally closed analytic subset of $\mathbb{P}^N$ (see, for example, [12, Lemma in (0.2)]). By taking a suitable subdivision of $\mathscr{S}$ into locally closed analytic subsets of $\mathbb{P}^N$, we can write

$$\mathscr{S} = \left(\bigcup_{j\in\mathbb{N}} Y_j \right) \cup \left(\bigcup_{k\in\mathbb{N}} Z_k \right),$$

where $\dim Y_j = N - 1$ such that $p : Y_j - \{P\} \to \mathbb{P}^{N-1}$ has no positive dimensional fibers for every j, and any irreducible component of $p(Z_k - \{P\})$ has dimension $\leq N - 2$ for every k. We put $\mathscr{S}' = \bigcup_{k\in\mathbb{N}} p(Z_k - \{P\})$. Then $\mathscr{S}'$ satisfies the desired properties. $\square$

Lemma 3.6 will play an important role in the induction on N.

Lemma 3.6. *Let $\mathscr{G}_N$ be a subset of $\mathbb{P}^N$ and let Σ be an analytically meagre subset of $\mathbb{P}^N$. Let $\mathscr{G}_{N-1}$ be a subset of $\mathbb{P}^{N-1}$ such that $\mathscr{G}_{N-1} \setminus \mathscr{S}_{N-1}$ is dense in $\mathbb{P}^{N-1}$ in the classical* *topology for any analytically meagre subset* $\mathscr{S}_{N-1}$ *of* $\mathbb{P}^{N-1}$. *Let* $p : \mathbb{P}^N - \{P\} \to \mathbb{P}^{N-1}$ *be the linear projection from* $P \in \mathbb{P}^N$. *Assume that almost all points of* $p^{-1}(x)$ *is contained in* $\mathscr{G}_N$ *for every* $x \in \mathscr{G}_{N-1}$ *with* $p^{-1}(x) \setminus \Sigma \neq \emptyset$. *Then* $\mathscr{G}_N \setminus \mathscr{S}_N$ *is dense in* $\mathbb{P}^N$ *in the classical topology for any analytically meagre subset* $\mathscr{S}_N$ *of* $\mathbb{P}^N$.

Proof. We put $\mathscr{S} = \Sigma \cup \mathscr{S}_N$. Then $\mathscr{S}$ is an analytically meagre subset of $\mathbb{P}^N$. We can define an analytically meagre subset $\mathscr{S}'$ of $\mathbb{P}^{N-1}$ as in the proof of Lemma 3.5. Then $\mathscr{G}_{N-1} \setminus \mathscr{S}'$ is dense in $\mathbb{P}^{N-1}$ in the classical topology by assumption. By assumption again, almost all points of $p^{-1}(x)$ is contained in $\mathscr{G}_N \setminus \mathscr{S}_N$ for every $x \in \mathscr{G}_{N-1} \setminus \mathscr{S}'$. Therefore, we can easily see that $\mathscr{G}_N \setminus \mathscr{S}_N$ is dense in $\mathbb{P}^N$ in the classical topology. $\square$

We will use Lemma 3.6 in order to prove the density of $\mathscr{G}$ in Theorem 1.1 by induction on N.

Remark 3.7. In Lemma 3.6, we assume that $\mathbb{P}^N$ is the linear system $\Lambda = |\mathscr{O}_{\mathbb{P}^N}(1)|$ as in Theorem 1.1. We assume that $P \in \mathbb{P}^N = \Lambda$ corresponds to a hyperplane H'_0 on the original projective space $\mathbb{P}^N$. Let $p : \mathbb{P}^N - \{P\} \to \mathbb{P}^{N-1}$ be the linear projection as in Lemma 3.6. Then we can regard $\mathbb{P}^{N-1}$ as the linear system $\Lambda|_{H'_0}$.

4. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. We prepare some lemmas before we start the proof of Theorem 1.1. Lemma 4.1 is essentially the same as [9, Lemma 3.2]. Note that a main ingredient of Lemma 4.1 is the Ohsawa–Takegoshi L^2 extension theorem.

Lemma 4.1. *Let* $f\colon X \to S$, Π, Λ, $S^\dagger$, $X^\dagger$, *and* $h^\dagger$ *be as in Theorem 1.1. Let* H'_i *be an element of* Λ *for* $1 \leq i \leq k$. *We assume the following condition:*

- ♠ $H^\dagger_i := (h^\dagger)^* H'_i$ *is a well-defined smooth divisor on* $X^\dagger$ *for every* $1 \leq i \leq k$ *and* $\sum_{i=1}^k H^\dagger_i$ *is a simple normal crossing divisor on* $X^\dagger$. *Moreover, for every* $1 \leq i_1 < i_2 < \cdots < i_l \leq k$ *and any* $P \in H^\dagger_{i_1} \cap H^\dagger_{i_2} \cap \cdots \cap H^\dagger_{i_l}$, *the set* $\{f_{i_1}, f_{i_2}, \ldots, f_{i_l}\}$ *is a regular sequence for* $\mathscr{O}_{X,P}/\mathscr{J}(\varphi)_P$ *for every* $\varphi \in \Pi$, *where* f_i *is a* (*local*) *defining equation of* $H^\dagger_i$ *for every* i.

We put $F_i := H^\dagger_1 \cap H^\dagger_2 \cap \cdots \cap H^\dagger_i$ *for* $1 \leq i \leq k$. *Let* F *be an irreducible component of* F_k. *We assume that the equality*

$$\mathscr{J}(\varphi|_F) = \mathscr{J}(\varphi)|_F$$

holds for some $\varphi \in \Pi$. *Then*

$$\mathscr{J}(\varphi|_{F_j}) = \mathscr{J}(\varphi)|_{F_j}$$

holds in a neighborhood of F *in* F_j *for every* j.

Remark 4.2. (1) Let $(A, \mathfrak{m})$ be a local ring and let M be a finitely generated non-zero A-module. Let $\{x_1, \ldots, x_r\}$ be a sequence of elements of $\mathfrak{m}$. We put $M_0 = M$ and $M_i = M/x_1M + \cdots + x_iM$. Then $\{x_1, \ldots, x_r\}$ is said to be a *regular sequence* for M if $\times x_{i+1}\colon M_i \to M_i$ is injective for every $0 \leq i \leq r-1$.

(2) Condition ♠ in Lemma 4.1 does not depend on the order of $\{H'_1, H'_2, \cdots, H'_k\}$ (see, for example, [13, Theorem 16.3] and [1, Chapter III, Corollary (3.5)]).

(3) Let $\mathscr{F}$ be a coherent analytic sheaf on a complex manifold X. Then there exists a locally finite family $\{Y_i\}_{i \in I}$ of irreducible analytic subsets of X such that

$$\mathrm{Ass}_{\mathscr{O}_{X,x}}(\mathscr{F}_x) = \{\mathfrak{p}_{x,1}, \ldots, \mathfrak{p}_{x,r(x)}\},$$

where $\mathfrak{p}_{x,1}, \ldots, \mathfrak{p}_{x,r(x)}$ are prime ideals of $\mathscr{O}_{X,x}$ associated to the irreducible components of the germs $x \in Y_i$ (see, for example, [12, (I.6) Lemma]). Note that Y_i is called an analytic subset associated with $\mathscr{F}$. In this paper, we simply say that Y_i is an *associated prime* of $\mathscr{F}$ if there is no risk of confusion. Then we can check that condition ♠ is equivalent to the following condition:

- $H_i^\dagger := (h^\dagger)^* H_i'$ is a well-defined smooth divisor on $X^\dagger$ for every $1 \leq i \leq k$ and $\sum_{i=1}^k H_i^\dagger$ is a simple normal crossing divisor on $X^\dagger$. Moreover, for every $1 \leq i_1 < i_2 < \cdots < i_{l-1} < i_l \leq k$, the divisor $H_{i_l}^\dagger$ contains no associated primes of $\mathscr{O}_X/\mathscr{J}(\varphi)$ and $\mathscr{O}_{H_{i_1}^\dagger \cap \cdots \cap H_{i_{l-1}}^\dagger}/\mathscr{J}(\varphi)|_{H_{i_1}^\dagger \cap \cdots \cap H_{i_{l-1}}^\dagger}$ for every $\varphi \in \Pi$.

For the proof and the details of Lemma 4.1, see [9, Lemmas 3.1 and 3.2, Remark 3.3, and Lemma 3.4]. Lemma 4.3 below is similar to [9, Lemma 3.5].

Lemma 4.3. *Let $f\colon X \to S$, Π, Λ, $S^\dagger$, $X^\dagger$, and $h^\dagger$ be as in Theorem 1.1. Let Λ_0 be an m-dimensional sublinear system of Λ spanned by $\{H_1', \ldots, H_m', H_{m+1}'\}$ such that $\{H_1', \ldots, H_m', H_{m+1}'\}$ satisfies ♠. We put*

$$\mathscr{F}_0 = \{H' \in \Lambda_0 \,|\, \{H_1', \ldots, H_m', H'\} \text{ satisfies } \spadesuit\}.$$

Then $\Lambda_0 \setminus \mathscr{F}_0$ is analytically meagre.

Moreover, we assume that $\mathscr{J}(\varphi|_F) = \mathscr{J}(\varphi)|_F$ holds for some $\varphi \in \Pi$, where F is an irreducible component of $H_1^\dagger \cap \cdots \cap H_{m+1}^\dagger$. Let H' be a member of $\mathscr{F}_0$. Then

$$\mathscr{J}(\varphi|_{H^\dagger}) = \mathscr{J}(\varphi)|_{H^\dagger}$$

holds in a neighborhood of F in $H^\dagger$, where $H^\dagger = (h^\dagger)^* H'$.

Proof. Let $\widetilde{\Lambda}_0$ be the sublinear system of Λ_0 spanned by $\{H_1', \ldots, H_m'\}$. Then we see that

$$H_1^\dagger \cap \cdots \cap H_m^\dagger \cap H_{m+1}^\dagger = H_1^\dagger \cap \cdots \cap H_m^\dagger \cap H^\dagger$$

holds for every $H' \in \Lambda_0 \setminus \widetilde{\Lambda}_0$. We note that the number of irreducible components of $H_{i_1}^\dagger \cap H_{i_2}^\dagger \cap \cdots \cap H_{i_l}^\dagger$ with $1 \leq i_1 < i_2 < \cdots < i_l \leq m$ is finite. We also note that for every $\varphi \in \Pi$ the number of the associated primes of $\mathscr{O}_{X^\dagger}/\mathscr{J}(\varphi)|_{X^\dagger}$ and the number of the associated primes of

$$\mathscr{O}_{H_{i_1}^\dagger \cap \cdots \cap H_{i_l}^\dagger}/\mathscr{J}(\varphi)|_{H_{i_1}^\dagger \cap \cdots \cap H_{i_l}^\dagger}$$

with $1 \leq i_1 < i_2 < \cdots < i_l \leq m$ are finite (see Remark 4.2 (3) and [12, (I.6) Lemma]). Moreover, $\#\Pi$ is at most countable by assumption. On the other hand, it is obvious that $H_1^\dagger \cap \cdots \cap H_{m+1}^\dagger$ is empty on $X^\dagger \setminus H_1^\dagger \cap \cdots \cap H_{m+1}^\dagger$. Therefore, by applying Theorem 3.2 to $X^\dagger \setminus H_1^\dagger \cap \cdots \cap H_{m+1}^\dagger$ and $H_{i_1}^\dagger \cap \cdots \cap H_{i_l}^\dagger \setminus H_1^\dagger \cap \cdots \cap H_{m+1}^\dagger$ for every $1 \leq i_1 < i_2 < \cdots < i_l \leq m$, we can easily check that $\Lambda_0 \setminus \mathscr{F}_0$ is analytically meagre.

Let H' be a member of $\mathscr{F}_0$. Then

$$H_1^\dagger \cap \cdots \cap H_m^\dagger \cap H_{m+1}^\dagger = H_1^\dagger \cap \cdots \cap H_m^\dagger \cap H^\dagger$$

always holds. Therefore, F is an irreducible component of $H_1^\dagger \cap \cdots \cap H_m^\dagger \cap H^\dagger$. Thus, by Lemma 4.1 and Remark 4.2, the equality $\mathscr{J}(\varphi|_{H^\dagger}) = \mathscr{J}(\varphi)|_{H^\dagger}$ holds in a neighborhood of F in $H^\dagger$ for every $H' \in \mathscr{F}_0$. □

Let us prove Theorem 1.1.

Proof of Theorem 1.1. Without loss of generality, we may assume that S has a countable base of open subsets by shrinking S suitably. Moreover, by replacing S with its smaller relatively compact open subset if necessary, we may further assume that S is a relatively compact open subset of a complex analytic space throughout the proof of Theorem 1.1.

Of course, we may assume that every connected component of X intersects with $X^\dagger$ by abandoning unnecessary connected components of X. We may assume that $\varphi \not\equiv -\infty$ for every $\varphi \in \Pi$.

Step 1. In this step, we will prove that $\mathscr{G}$ is dense in Λ in the classical topology under the assumption that $N=1$. More generally, we will see that $\mathscr{H}$, $\mathscr{G}\setminus\mathscr{S}$, and $\mathscr{H}\setminus\mathscr{S}$ are dense in Λ in the classical topology for any analytically meagre subset $\mathscr{S}$ of Λ under the assumption that $N=1$.

By Sard's theorem (see, for example, [12, (I.1) Theorem]), there exists a countable subset Σ of $\mathbb{P}^1$ such that $X_x = h^*x$ is a smooth divisor on X for every $x \in \mathbb{P}^1\setminus\Sigma$. Of course, it may happen that $h^{-1}(x)$ is empty. By the Ohsawa–Takegoshi L^2 extension theorem, we have

$$\mathscr{J}(\varphi|_{X_x}) \subset \mathscr{J}(\varphi)|_{X_x}$$

for every $x \in \mathbb{P}^1\setminus\Sigma$. On the other hand, for every $\varphi\in\Pi$, by Fubini's theorem, we see that

$$\mathscr{J}(\varphi|_{X_x^\dagger}) \supset \mathscr{J}(\varphi)|_{X_x^\dagger}$$

holds for almost all $x\in\mathbb{P}^1\setminus\Sigma$, where $X_x^\dagger := X_x\cap X^\dagger$ (see Example 1.7). Note that $\#\Pi$ is at most countable by assumption. This means that $\mathscr{G}$ is dense in $\Lambda\simeq\mathbb{P}^1$ in the classical topology. Since there are only finitely many associated primes of $\mathscr{O}_X/\mathscr{J}(\varphi)$ on $X^\dagger$ for every $\varphi\in\Pi$ (see Remark 4.2 (3) and [12, (I.6) Lemma]), $\mathscr{G}\setminus\mathscr{H}$ is an analytically meagre subset of Λ. We note that $(\Lambda\setminus\mathscr{G})\cup\mathscr{S}$ has measure zero for any analytically meagre subset $\mathscr{S}$ of $\Lambda\simeq\mathbb{P}^1$. Therefore, we see that $\mathscr{H}$, $\mathscr{G}\setminus\mathscr{S}$, and $\mathscr{H}\setminus\mathscr{S}$ are dense in Λ in the classical topology for any analytically meagre subset $\mathscr{S}$ of Λ.

Step 2. By Step 1, we can prove the following lemma.

Lemma 4.4. *Let H'_1 and H'_2 be two members of Λ such that $\{H'_1, H'_2\}$ satisfies $\spadesuit$. Let $\mathscr{P}$ be the pencil spanned by H'_1 and H'_2, that is, $\mathscr{P}$ is the sublinear system of Λ spanned by H'_1 and H'_2. Then, for almost all $H'\in\mathscr{P}$, $\{H'\}$ satisfies $\spadesuit$, and*

$$\mathscr{J}(\varphi|_{H^\dagger}) = \mathscr{J}(\varphi)|_{H^\dagger}$$

holds for every $\varphi\in\Pi$ outside $H_1^\dagger\cap H_2^\dagger$, where $H^\dagger=(h^\dagger)^*H'$, $H_1^\dagger=(h^\dagger)^*H'_1$, and $H_2^\dagger=(h^\dagger)^*H'_2$.

Proof of Lemma 4.4. First, by Lemma 4.3, for almost all $H'\in\mathscr{P}$, $\{H'\}$ satisfies $\spadesuit$. Next, we consider the following commutative diagram.

$$\begin{tikzcd}
\widetilde{X} \arrow[r] \arrow[d] & S\times\mathbb{P}_{\mathbb{P}^1}(\mathscr{E}) \arrow[r] \arrow[d] & \mathbb{P}_{\mathbb{P}^1}(\mathscr{E}) \arrow[r] \arrow[d] & \mathbb{P}^1 \\
X \arrow[r] & S\times\mathbb{P}^N \arrow[r] & \mathbb{P}^N \arrow[ur, dashed] &
\end{tikzcd}$$

Note that $\mathscr{E}=\mathscr{O}_{\mathbb{P}^1}^{\oplus N-1}\oplus\mathscr{O}_{\mathbb{P}^1}(1)$, $\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})\to\mathbb{P}^N$ is the blow-up along $H'_1\cap H'_2$, and $\mathbb{P}^N \dashrightarrow \mathbb{P}^1$ is the projection from $H'_1\cap H'_2$. In the above diagram, $\widetilde{X}$ is a resolution of the blow-up of X along $h^*H'_1\cap h^*H'_2$ such that $\widetilde{X}$ is nothing but the blow-up of $X^\dagger$ along $H_1^\dagger\cap H_2^\dagger$ over $X^\dagger$. We apply the argument in Step 1 to $\widetilde{X}\to S\times\mathbb{P}^1\to\mathbb{P}^1$ and get the desired property, that is, $\mathscr{J}(\varphi|_{H^\dagger})=\mathscr{J}(\varphi)|_{H^\dagger}$ holds for every $\varphi\in\Pi$ outside $H_1^\dagger\cap H_2^\dagger$ for almost all $H'\in\mathscr{P}$. Note that a point of $\mathbb{P}^1$ corresponds to a hyperplane of $\mathbb{P}^N$ containing $H'_1\cap H'_2$ by the projection $\mathbb{P}^N\dashrightarrow\mathbb{P}^1$. $\square$

Step 3. In this step, we will prove the following lemma, which is the most difficult part of the proof of Theorem 1.1.

Lemma 4.5. *There exists some $H' \in \mathscr{G}$ such that $\{H'\}$ satisfies $\spadesuit$, equivalently, $H' \in \mathscr{H}$.*

Proof of Lemma 4.5. If $N = 1$, then this lemma follows from Step 1. From now on, we assume that $N \geq 2$. We take two general hyperplanes H_1' and H_2' of $\mathbb{P}^N$. We can choose H_1' and H_2' such that $\{H_1', H_2'\}$ satisfies $\spadesuit$ since Λ is free. By Lemma 4.4, we can take a hyperplane A_1 of $\mathbb{P}^N$ such that $X_1 = h^*A_1$ is smooth, $\{A_1\}$ satisfies $\spadesuit$, and the equality

$$\mathscr{J}(\varphi|_{X_1^\dagger}) = \mathscr{J}(\varphi)|_{X_1^\dagger}$$

holds for every $\varphi \in \Pi$ outside $H_1^\dagger \cap H_2^\dagger$, where $X_1^\dagger = X_1 \cap X^\dagger = (h^\dagger)^*A_1$. More precisely, if $S^\dagger$ is not compact, then we take a strictly larger open subset $\widetilde{S}$ with $S^\dagger \Subset \widetilde{S} \Subset S$ and apply everything to $\widetilde{S}$ instead of $S^\dagger$. Then we replace S with $\widetilde{S}$. By this argument, we can make $X_1 = h^*A_1$ smooth on X (not on $X^\dagger$). By applying the induction hypothesis to $\Lambda|_{A_1}$, we see that

$$\{H' \in \Lambda \,|\, X_1 \cap H^\dagger \text{ is smooth and } \mathscr{J}(\varphi|_{X_1 \cap H^\dagger}) = \mathscr{J}(\varphi|_{X_1})|_{X_1 \cap H^\dagger} \text{ holds for every } \varphi \in \Pi\}$$

is dense in Λ in the classical topology, where $H^\dagger = (h^\dagger)^*H'$.

We can take general hyperplanes $A_2, \ldots, A_N$ of $\mathbb{P}^N$ such that $Q = A_1 \cap \cdots \cap A_N$, $X_Q^\dagger := X_Q \cap X^\dagger$ is smooth, where $X_Q = h^{-1}(Q)$, and the equality

$$\mathscr{J}(\varphi|_{X_Q^\dagger}) = \mathscr{J}(\varphi|_{X_1})|_{X_Q^\dagger}$$

holds for every $\varphi \in \Pi$ by using the induction hypothesis repeatedly. As we saw above, if necessary, we apply everything to a strictly larger open subset $\widetilde{S}$ instead of $S^\dagger$ with $S^\dagger \Subset \widetilde{S} \Subset S$ and replace S with $\widetilde{S}$ in each step. Without loss of generality, we may assume that $X_Q^\dagger \cap H_1^\dagger \cap H_2^\dagger = \emptyset$. Since $\mathscr{J}(\varphi|_{X_1^\dagger}) = \mathscr{J}(\varphi)|_{X_1^\dagger}$ outside $H_1^\dagger \cap H_2^\dagger$,

$$\mathscr{J}(\varphi|_{X_1})|_{X_Q^\dagger} = \mathscr{J}(\varphi|_{X_1^\dagger})|_{X_Q^\dagger} = \mathscr{J}(\varphi)|_{X_Q^\dagger}$$

holds for every $\varphi \in \Pi$. Therefore, we obtain

$$\mathscr{J}(\varphi|_{X_Q^\dagger}) = \mathscr{J}(\varphi|_{X_1})|_{X_Q^\dagger} = \mathscr{J}(\varphi)|_{X_Q^\dagger}$$

for every $\varphi \in \Pi$. Of course, we can choose $A_2, A_3, \ldots, A_N$ such that

$$\{A_1, A_2, \ldots, A_N\}$$

satisfies $\spadesuit$ with the aid of Lemma 3.6 (see also Remark 3.7) since Λ is a free linear system. We put

$$\Lambda_0 = \{A \,|\, Q \in A \in |\mathscr{O}_{\mathbb{P}^N}(1)|\} \subset \Lambda,$$

equivalently, Λ_0 is the sublinear system of Λ spanned by $\{A_1, \ldots, A_N\}$. Then

$$\mathscr{F}_0 = \{H' \in \Lambda_0 \,|\, \{H', A_2, \ldots, A_N\} \text{ satisfies } \spadesuit\}$$

is non-empty by $A_1 \in \mathscr{F}_0$ and $\Lambda_0 \setminus \mathscr{F}_0$ is analytically meagre by Lemma 4.3. Thus, by Lemma 4.3, we have:

Claim. *The equality $\mathscr{J}(\varphi|_{X_g^\dagger}) = \mathscr{J}(\varphi)|_{X_g^\dagger}$ holds in a neighborhood of $X_Q^\dagger$ for every $\varphi \in \Pi$ and for every $A_g \in \mathscr{F}_0$, where $X_g := h^*A_g$.*

Let $\pi : \widetilde{X} \to X$ be a proper bimeromorphic morphism from a complex manifold $\widetilde{X}$ such that $\pi : \widetilde{X} \to X$ is nothing but the blow-up of $X^\dagger$ along $X^\dagger_Q$ over $X^\dagger$. Then we have the following commutative diagram.

$$\begin{tikzcd}
\widetilde{X} \arrow[r] \arrow[d, "\pi"'] & S \times \mathbb{P}(\mathscr{E}) \arrow[r] \arrow[d] & \mathbb{P}(\mathscr{E}) \arrow[r] \arrow[d] & \mathbb{P}^{N-1} \\
X \arrow[r] & S \times \mathbb{P}^N \arrow[r] & \mathbb{P}^N \arrow[ur, dashed, "p_Q"'] &
\end{tikzcd}$$

Of course, $p_Q : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1}$ is the linear projection from Q and $\mathbb{P}(\mathscr{E})$ is the blow-up of $\mathbb{P}^N$ at Q, where $\mathbb{P}(\mathscr{E}) = \mathbb{P}_{\mathbb{P}^{N-1}}(\mathscr{O}_{\mathbb{P}^{N-1}} \oplus \mathscr{O}_{\mathbb{P}^{N-1}}(1))$. We consider the following commutative diagram.

$$\begin{tikzcd}
\widetilde{X} \arrow[dd, "\widetilde{f}"'] \arrow[dr, "\widetilde{g}"'] \arrow[drr, "\widetilde{h}"] & & \\
& S \times \mathbb{P}^{N-1} \arrow[r, "p_2"'] \arrow[dl, "p_1"] & \mathbb{P}^{N-1} \\
S & &
\end{tikzcd}$$

We put $\widetilde{X}^\dagger = \widetilde{f}^{-1}(S^\dagger)$. By induction on N, we can take a general hyperplane B of $\mathbb{P}^{N-1}$ such that $\widetilde{h}^*B \cap \widetilde{X}^\dagger$ is smooth and that

$$\mathscr{J}(\pi^*\varphi|_{\widetilde{h}^*B\cap\widetilde{X}^\dagger}) = \mathscr{J}(\pi^*\varphi)|_{\widetilde{h}^*B\cap\widetilde{X}^\dagger} \tag{4.1}$$

holds for every $\varphi \in \Pi$. Let H' be the hyperplane of $\mathbb{P}^N$ spanned by Q and B. Note that, by induction on N, we can choose B such that

$$\{A_2, \ldots, A_N, H'\}$$

satisfies ♠ since $\Lambda_0 \setminus \mathscr{F}_0$ is analytically meagre. Therefore, we obtain that the equality

$$\mathscr{J}(\varphi|_{H^\dagger}) = \mathscr{J}(\varphi)|_{H^\dagger}$$

holds for every $\varphi \in \Pi$ by Claim and (4.1), where $H^\dagger = (h^\dagger)^*H'$ as usual. More precisely, (4.1) implies that the equality

$$\mathscr{J}(\varphi|_{H^\dagger}) = \mathscr{J}(\varphi)|_{H^\dagger}$$

holds outside $X^\dagger_Q$ and Claim implies that the equality

$$\mathscr{J}(\varphi|_{H^\dagger}) = \mathscr{J}(\varphi)|_{H^\dagger}$$

holds in a neighborhood of $X^\dagger_Q$. Hence this H' is a desired divisor. □

Step 4. In this step, we will see that $\mathscr{G} \setminus \mathscr{S}$ is dense in Λ in the classical topology for any analytically meagre subset $\mathscr{S}$ of Λ.

By Step 1, we may assume that $N \geq 2$. By Lemma 4.5, we can take a member $H'_0 \in \mathscr{G}$ such that $\{H'_0\}$ satisfies ♠. By the same argument as before, if $S^\dagger$ is not compact, then we take a strictly larger open subset $\widetilde{S}$ with $S^\dagger \Subset \widetilde{S} \Subset S$. Then we apply everything to $\widetilde{S}$ instead of $S^\dagger$. By replacing S with $\widetilde{S}$, we may assume that $h^*H'_0$ is smooth on X. By applying the induction hypothesis to $\Lambda|_{H'_0}$, we see that

$$\mathscr{G}' := \left\{ H' \in \Lambda \;\middle|\; \begin{array}{l} H^\dagger_0 \cap H^\dagger \text{ is smooth and } \mathscr{J}(\varphi|_{H^\dagger_0\cap H^\dagger}) = \mathscr{J}(\varphi|_{H^\dagger_0})|_{H^\dagger_0\cap H^\dagger} \\ \text{holds for every } \varphi \in \Pi \end{array} \right\}$$

is dense in Λ in the classical topology, where $H_0^\dagger = (h^\dagger)^* H_0'$ and $H^\dagger = (h^\dagger)^* H'$ as usual. Since Λ is free,

$$\mathscr{F} := \{H' \in \Lambda \,|\, \{H_0', H'\} \text{ satisfies } \spadesuit\}$$

is non-empty and $\Lambda \setminus \mathscr{F}$ is analytically meagre. Therefore, we see that

$$\mathscr{G}'' := \{H' \in \mathscr{G}' \,|\, \{H_0', H'\} \text{ satisfies } \spadesuit\}$$

is also dense in Λ in the classical topology with the aid of Lemma 3.6. We note that

$$\mathscr{J}(\varphi|_{H_0^\dagger \cap H_1^\dagger}) = \mathscr{J}(\varphi|_{H_0^\dagger})|_{H_0^\dagger \cap H_1^\dagger} = \mathscr{J}(\varphi)|_{H_0^\dagger \cap H_1^\dagger} \tag{4.2}$$

with $H_1^\dagger = (h^\dagger)^* H_1'$ for every $H_1' \in \mathscr{G}'$ since

$$\mathscr{J}(\varphi|_{H_0^\dagger}) = \mathscr{J}(\varphi)|_{H_0^\dagger}.$$

Here, we used the fact that $H_0' \in \mathscr{G}$. We consider the pencil $\mathscr{P}$ spanned by H_0' and $H_1' \in \mathscr{G}''$, that is, the sublinear system of Λ spanned by H_0' and H_1'. By Lemma 4.4, for almost all $H' \in \mathscr{P}$, $H^\dagger = (h^\dagger)^* H'$ is smooth and the equality

$$\mathscr{J}(\varphi|_{H^\dagger}) = \mathscr{J}(\varphi)|_{H^\dagger}$$

holds for every $\varphi \in \Pi$ outside $H_0^\dagger \cap H^\dagger = H_0^\dagger \cap H_1^\dagger$. For almost all $H' \in \mathscr{P}$, $\{H_0', H'\}$ satisfies $\spadesuit$ by Lemma 4.3. Therefore, the equality

$$\mathscr{J}(\varphi|_{H^\dagger}) = \mathscr{J}(\varphi)|_{H^\dagger}$$

holds for every $\varphi \in \Pi$ in a neighborhood of $H_0^\dagger \cap H^\dagger = H_0^\dagger \cap H_1^\dagger$ for almost all $H' \in \mathscr{P}$ by Lemma 4.3 and (4.2). Therefore, for almost all $H' \in \mathscr{P}$, $H^\dagger$ is smooth and the equality

$$\mathscr{J}(\varphi|_{H^\dagger}) = \mathscr{J}(\varphi)|_{H^\dagger}$$

holds for every $\varphi \in \Pi$. This means that almost all members of $\mathscr{P}$ are contained in $\mathscr{G}$.

Let P be a point of $\mathbb{P}^N \simeq \Lambda$ corresponding to H_0'. We put $\Sigma = \Lambda \setminus \mathscr{F}$, $\mathscr{G}_{N-1} = \mathscr{G}'|_{H_0'} \subset \Lambda|_{H_0'}$, and $\mathscr{G}_N = \mathscr{G}$. Then we can apply Lemma 3.6 (see also Remark 3.7). Therefore, $\mathscr{G} \setminus \mathscr{S}$ is dense in Λ in the classical topology for any analytically meagre subset $\mathscr{S}$ of Λ.

Step 5. In this step, we will see that $\mathscr{H}$ is dense in Λ in the classical topology.

We put

$$\mathscr{K} := \left\{ H' \in \Lambda \;\middle|\; \begin{array}{l} H^\dagger := (h^\dagger)^* H' \text{ is well-defined and contains no associated primes of} \\ \mathscr{O}_X/\mathscr{J}(\varphi) \text{ on } X^\dagger \text{ for every } \varphi \in \Pi \end{array} \right\}.$$

Then $\Lambda \setminus \mathscr{K}$ is analytically meagre. Note that Λ is free and the number of the associated primes of $\mathscr{O}_X/\mathscr{J}(\varphi)$ on $X^\dagger$ is finite for every $\varphi \in \Pi$ and that $\#\Pi$ is at most countable. Therefore, by Step 4, $\mathscr{H}$ is dense in Λ in the classical topology because $\mathscr{G} \setminus \mathscr{H}$ is contained in an analytically meagre subset of Λ.

Step 6. Let H' be a member of $\mathscr{H}$ and let φ be any member of Π. We consider the following big commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathscr{J}(\varphi|_{X^\dagger}) \otimes \mathscr{O}_{X^\dagger}(-H^\dagger) \arrow[r, "\alpha"] \arrow[d] & \mathscr{J}(\varphi|_{X^\dagger}) \arrow[r] \arrow[d] & \operatorname{Coker}\alpha \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \mathscr{O}_{X^\dagger}(-H^\dagger) \arrow[r] \arrow[d] & \mathscr{O}_{X^\dagger} \arrow[r] \arrow[d] & \mathscr{O}_{H^\dagger} \arrow[r] & 0 \\
 & (\mathscr{O}_{X^\dagger}/\mathscr{J}(\varphi|_{X^\dagger})) \otimes \mathscr{O}_{X^\dagger}(-H^\dagger) \arrow[r, "\gamma"] \arrow[d] & \mathscr{O}_{X^\dagger}/\mathscr{J}(\varphi|_{X^\dagger}) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Of course, $H^\dagger$ is $(h^\dagger)^*H'$ in the above diagram. Since $H' \in \mathscr{H}$, γ is injective. Therefore, β is also injective by the snake lemma. Thus we obtain that

$$\operatorname{Coker}\alpha = \mathscr{J}(\varphi|_{X^\dagger})|_{H^\dagger}$$

by definition. Then we have the following desired short exact sequence

$$0 \to \mathscr{J}(\varphi|_{X^\dagger}) \otimes \mathscr{O}_{X^\dagger}(-H^\dagger) \to \mathscr{J}(\varphi|_{X^\dagger}) \to \mathscr{J}(\varphi|_{H^\dagger}) \to 0$$

because $\mathscr{J}(\varphi|_{H^\dagger}) = \mathscr{J}(\varphi|_{X^\dagger})|_{H^\dagger}$ holds for $H' \in \mathscr{H}$.

We complete the proof of Theorem 1.1. $\square$

Remark 4.6. Theorem 1.1 says that $\mathscr{G}$ is dense in Λ in the classical topology. However, the proof of Theorem 1.1 gives no information on the set $\Lambda \setminus \mathscr{G}$ in Λ ($\simeq \mathbb{P}^N$). This is because we use Lemma 3.6 for induction on N. We recommend the reader to see [9, Examples 3.9 and 3.10].

We close this section with the proof of Corollary 1.2.

Proof of Corollary 1.2. We assume that S is a point. We put $\Pi = \{m\varphi\}_{m\in\mathbb{N}}$ and $h = \Phi_W : X \to \mathbb{P}^N$. Hence, by using Theorem 1.1, we obtain a desired subset V of W. $\square$

5. Proof of Theorem 1.4

In this section, we prove Theorem 1.4 as an application of Theorems 1.1 and 1.3.

Proof of Theorem 1.4. We take an arbitrary point $z \in Z$. Let us prove

$$R^i\pi_*R^jf_*(\omega_X \otimes F \otimes \mathscr{J}(h)) = 0$$

for every $i > 0$ and j in a neighborhood of z by induction on $\dim \pi^{-1}(z)$. Without loss of generality, we may assume that $f_*\mathscr{O}_X \simeq \mathscr{O}_Y$ and $\pi_*\mathscr{O}_Y \simeq \mathscr{O}_Z$ by taking the Stein factorizations of f and π. Since $\pi \circ f$ is locally Kähler (see, for example, [18, Proposition 6.2 (ii)]), we may assume that X is Kähler by shrinking Z around z. If $\dim \pi^{-1}(z) = 0$, then $\pi : Y \to Z$ is finite over a neighborhood of z. In this case, it is obvious that $R^i\pi_*R^jf_*(\omega_X \otimes F \otimes \mathscr{J}(h)) = 0$ holds for every $i > 0$ and j in a neighborhood of z. From now on, we assume that $\dim \pi^{-1}(z) > 0$. By replacing H with $H^{\otimes m}$ for some sufficiently large positive integer m, we may assume that H is π-very ample and

$$R^i\pi_*(H \otimes R^j f_*(\omega_X \otimes F \otimes \mathscr{J}(h))) = 0 \tag{5.1}$$

for every $i > 0$ and j (see, for example, [2, Chapter IV, Theorem 2.1 (B)]). We may further assume that there exists the following commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr] & & \\ Y \arrow[r, hook] \arrow[d, "\pi"'] & Z \times \mathbb{P}^N \arrow[r, "p_2"'] \arrow[dl, "p_1"] & \mathbb{P}^N \\ Z & & \end{tikzcd}$$

such that $H \simeq (p_2^*\mathscr{O}_{\mathbb{P}^N}(1))|_Y$ by shrinking Z around z suitably (see, for example, [2, Chapter IV, §2]). By Theorem 1.1, we can take a general member A' of $|\mathscr{O}_{\mathbb{P}^N}(1)|$ such that $A_Y := (p_2^*A')|_Y$ contains no associated primes of $R^j f_*(\omega_X \otimes F \otimes \mathscr{J}(\varphi))$ for every j, A_Y contains no irreducible components of $\pi^{-1}(z)$, A is smooth, where $A = f^*A_Y$, and

$$0 \to \mathscr{J}(h) \otimes \mathscr{O}_X(-A) \to \mathscr{J}(h) \to \mathscr{J}(h|_A) \to 0$$

is exact after shrinking Z around z suitably. Therefore, by adjunction,

$$0 \to \omega_X \otimes F \otimes \mathscr{J}(h) \to \omega_X \otimes F \otimes \mathscr{J}(h) \otimes \mathscr{O}_X(A) \to \omega_A \otimes F|_A \otimes \mathscr{J}(h|_A) \to 0$$

is exact. Thus, we see that

$$\begin{aligned} 0 \to R^j f_*(\omega_X \otimes F \otimes \mathscr{J}(h)) &\to R^j f_*(\omega_X \otimes F \otimes \mathscr{J}(h) \otimes \mathscr{O}_X(A)) \\ &\to R^j f_*(\omega_A \otimes F|_A \otimes \mathscr{J}(h|_A)) \to 0 \end{aligned}$$

is exact for every j since A_Y contains no associated primes of $R^j f_*(\omega_X \otimes F \otimes \mathscr{J}(h))$ for every j. We note that $R^j f_*(\omega_A \otimes F|_A \otimes \mathscr{J}(h|_A))$ is π_*-acyclic in a neighborhood of z by induction on $\dim \pi^{-1}(z)$ and that $R^j f_*(\omega_X \otimes F \otimes \mathscr{J}(h) \otimes \mathscr{O}_X(A))$ is π_*-acyclic by the above assumption (see (5.1)). We consider the long exact sequence:

$$\begin{aligned} \cdots \to R^i\pi_* R^j f_*(\omega_X \otimes F \otimes \mathscr{J}(h)) &\to R^i\pi_* R^j f_*(\omega_X \otimes F \otimes \mathscr{J}(h) \otimes \mathscr{O}_X(A)) \\ &\to R^i\pi_* R^j f_*(\omega_A \otimes F|_A \otimes \mathscr{J}(h|_A)) \to \cdots . \end{aligned}$$

Thus, if we shrink Z around z suitably, then we have $E_2^{i,j} = 0$ for every $i \geq 2$ and j in the following commutative diagram of spectral sequences.

$$\begin{tikzcd} E_2^{i,j} = R^i\pi_* R^j f_*(\omega_X \otimes F \otimes \mathscr{J}(h)) \arrow[r, Rightarrow] \arrow[d, "\varphi^{i,j}"'] & R^{i+j}(\pi \circ f)_*(\omega_X \otimes F \otimes \mathscr{J}(h)) \arrow[d, "\varphi^{i+j}"] \\ \overline{E}_2^{i,j} = R^i\pi_* R^j f_*(\omega_X \otimes F \otimes \mathscr{J}(h) \otimes \mathscr{O}_X(A)) \arrow[r, Rightarrow] & R^{i+j}(\pi \circ f)_*(\omega_X \otimes F \otimes \mathscr{J}(h) \otimes \mathscr{O}_X(A)) \end{tikzcd}$$

We note that φ^{i+j} is injective by Theorem 1.3. We also note that

$$E_2^{1,j} \xrightarrow{\alpha} R^{1+j}(\pi \circ f)_*(\omega_X \otimes F \otimes \mathscr{J}(h))$$

is injective for every j by the fact that $E_2^{i,j} = 0$ for every $i \geq 2$ and j. By the above assumption (see (5.1)), we have $\overline{E}_2^{1,j} = 0$ for every j. Therefore, we obtain $E_2^{1,j} = 0$ for every j since the injection

$$E_2^{1,j} \xrightarrow{\alpha} R^{1+j}(\pi\circ f)_*(\omega_X\otimes F\otimes \mathscr{J}(h)) \xrightarrow{\varphi^{1+j}} R^{1+j}(\pi\circ f)_*(\omega_X\otimes F\otimes \mathscr{J}(h)\otimes \mathscr{O}_X(A))$$

factors through $\overline{E}_2^{1,j}=0$. This implies that $R^i\pi_*R^jf_*(\omega_X\otimes F\otimes \mathscr{J}(h))=0$ for every $i>0$ and j in a neighborhood of an arbitrary point $z\in Z$. This means that

$$R^i\pi_*R^jf_*(\omega_X\otimes F\otimes \mathscr{J}(h))=0$$

for every $i>0$ and j. □

6. Proof of Corollary 1.5

By using the strong openness in [10], we can prove Corollary 1.5 as an easy application of Theorem 1.4.

Let us prepare a lemma suitable for our application.

Lemma 6.1 (cf. [10, Theorem 1.1]). *Let X be a complex manifold and let φ and ψ be quasi-plurisubharmonic functions on X. Let $X^\dagger$ be a relatively compact open subset of X. Then there exists a small positive number ε such that*

$$\mathscr{J}(\varphi)=\mathscr{J}(\varphi+\varepsilon\psi)$$

holds on $X^\dagger$.

Proof. By definition, it is obvious that the natural inclusion

$$\mathscr{J}(\varphi)\supset \mathscr{J}(\varphi+\varepsilon\psi)$$

holds since ε is positive.

Let us see the problem locally. Let $\Delta^n=\{(z_1,\dots,z_n)\,|\,|z_1|<1,\cdots,|z_n|<1\}$ be the unit polydisc and let φ and ψ be plurisubharmonic functions on Δ^n. Let $f_1,\dots,f_k$ be holomorphic functions on Δ^n such that

$$\int_{\Delta^n}|f_i|^2e^{-2\varphi}d\lambda_n<\infty$$

for every i, where $d\lambda_n$ is the Lebesgue measure on $\mathbb{C}^n$, and that $\{f_1,\dots,f_k\}$ generates $\mathscr{J}(\varphi)_0$, the stalk of $\mathscr{J}(\varphi)$ at $0\in\Delta^n$. By [10, Theorem 1.1], we can take $r\in(0,1)$ and $p>1$ such that

$$\int_{\Delta^n_r}|f_i|^2e^{-2p\varphi}d\lambda_n<\infty$$

for every i, where $\Delta^n_r=\{(z_1,\dots,z_n)\,|\,|z_1|<r,\cdots,|z_n|<r\}$. We put $q=\frac{p}{p-1}>0$. Then, by the Hölder inequality, we have

$$\int_{\Delta^n_r}|f_i|^2e^{-2(\varphi+\varepsilon\psi)}d\lambda_n\leq\left(\int_{\Delta^n_r}|f_i|^2e^{-2p\varphi}d\lambda_n\right)^{1/p}\left(\int_{\Delta^n_r}|f_i|^2e^{-2q\varepsilon\psi}d\lambda_n\right)^{1/q}.$$

By replacing r with a smaller positive number, we can take $\varepsilon>0$ such that

$$\int_{\Delta^n_r}e^{-2q\varepsilon\psi}d\lambda_n<\infty$$

by Skoda's theorem (see, for example, [3, (5.6) Lemma]). Then we obtain

$$\int_{\Delta^n_r}|f_i|^2e^{-2(\varphi+\varepsilon\psi)}d\lambda_n<\infty.$$

This implies that $f_i \in \mathscr{J}(\varphi+\varepsilon\psi)_0$ for every i. Therefore, we obtain the inclusion

$$\mathscr{J}(\varphi)_0 \subset \mathscr{J}(\varphi+\varepsilon\psi)_0.$$

Then the equality

$$\mathscr{J}(\varphi)_0 = \mathscr{J}(\varphi+\varepsilon\psi)_0$$

holds. So, the equality

$$\mathscr{J}(\varphi) = \mathscr{J}(\varphi+\varepsilon\psi)$$

holds in a neighborhood of $0 \in \Delta^n$ since $\mathscr{J}(\varphi)$ and $\mathscr{J}(\varphi+\varepsilon\psi)$ are both coherent.

Thus, we can take $\varepsilon > 0$ such that

$$\mathscr{J}(\varphi) = \mathscr{J}(\varphi+\varepsilon\psi)$$

holds on $X^\dagger$ since $X^\dagger$ is a relatively compact open subset of X. □

Let us prove Corollary 1.5.

Proof of Corollary 1.5. It is sufficient to prove that

$$R^i\pi_*(M \otimes R^jf_*(\omega_X \otimes F \otimes \mathscr{J}(h))) = 0$$

holds for every $i > 0$ and j in a neighborhood of any fixed point $z \in Z$. By shrinking Z around z, we may assume that X is Kähler since $\pi \circ f$ is locally Kähler (see, for example, [18, Proposition 6.2 (ii)]). Without loss of generality, we may assume that Z is Stein. By shrinking Z around z, we may further assume that there exists the following commutative diagram since $\pi : Y \to Z$ is locally projective.

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[dr] \arrow[drr, "h"] & & \\
Y \arrow[r, hook] \arrow[d, "\pi"'] & Z \times \mathbb{P}^N \arrow[r, "p_2"'] \arrow[dl, "p_1"] & \mathbb{P}^N \\
Z & &
\end{tikzcd}$$

Then we can take a sufficiently large and divisible positive integer m such that

$$M^{\otimes m} \simeq H \otimes \mathscr{O}_Y(E)$$

where $H \simeq (p_2^*\mathscr{O}_{\mathbb{P}^N}(1))|_Y$ and E is an effective Cartier divisor on Y by Kodaira's lemma. Then we obtain

$$M^{\otimes(2m+k)} \simeq (M^{\otimes k} \otimes H) \otimes H \otimes \mathscr{O}_Y(2E).$$

We note that $M^{\otimes k} \otimes H$ is π-ample for every positive integer k since M is π-nef. Since $f^*H \simeq h^*\mathscr{O}_{\mathbb{P}^N}(1)$, we can construct a smooth hermitian metric g on f^*H such that $\sqrt{-1}\Theta_g(f^*H) \geq 0$. Similarly, $f^*(M^{\otimes k} \otimes H)$ has a smooth hermitian metric g_1 such that $\sqrt{-1}\Theta_{g_1}(f^*(M^{\otimes k} \otimes H)) \geq 0$ after shrinking Z around z suitably because $M^{\otimes k} \otimes H$ is π-ample. Let s be the canonical section of $\mathscr{O}_X(f^*E)$, that is, $s \in \Gamma(X, \mathscr{O}_X(f^*E))$ with $(s = 0) = f^*E$. Let g_2 be any smooth hermitian metric on $\mathscr{O}_X(f^*E)$. We put

$$g_3 = \frac{g_2}{|s|^2_{g_2}}.$$

Then g_3 is a singular hermitian metric on $\mathscr{O}_X(f^*E)$ such that $\sqrt{-1}\Theta_{g_3}(\mathscr{O}_X(f^*E)) \geq 0$ and that g_3 is smooth outside $\operatorname{Supp} f^*E$. We put

$$h' = (g_1 \cdot g \cdot g_3^2)^{\frac{1}{2m+k}}.$$

Then h' is a singular hermitian metric on f^*M, which is smooth outside $\operatorname{Supp} f^*E$. By construction,

$$\sqrt{-1}\Theta_g(f^*H) \geq 0 \quad \text{and} \quad \sqrt{-1}\Theta_{h'}(f^*M) - \varepsilon\sqrt{-1}\Theta_g(f^*H) \geq 0$$

for some $\varepsilon > 0$. If k is sufficiently large, then we can make h' satisfy $\mathscr{J}(hh') = \mathscr{J}(h)$ by Lemma 6.1. We note that we can freely shrink Z around z if necessary. Hence this means that

$$\sqrt{-1}\Theta_g(f^*H) \geq 0 \quad \text{and} \quad \sqrt{-1}\Theta_{hh'}(F \otimes f^*M) - \varepsilon\sqrt{-1}\Theta_g(f^*H) \geq 0$$

for some $\varepsilon > 0$ such that the equality $\mathscr{J}(hh') = \mathscr{J}(h)$ holds. By applying Theorem 1.4, we obtain that

$$R^i\pi_*(M \otimes R^jf_*(\omega_X \otimes F \otimes \mathscr{J}(h))) = R^i\pi_*R^jf_*(\omega_X \otimes F \otimes f^*M \otimes \mathscr{J}(hh')) = 0$$

holds for every $i > 0$ and j. □

References

[1] A. Altman, S. Kleiman, *Introduction to Grothendieck duality theory*, Lecture Notes in Mathematics, **146**, Springer-Verlag, Berlin-New York 1970.

[2] C. Bănică, O. Stănăşilă, *Algebraic methods in the global theory of complex spaces*, Translated from the Romanian. Editura Academiei, Bucharest; John & Wiley Sons, London-New York-Sydney, 1976.

[3] J.-P. Demailly, *Analytic methods in algebraic geometry*, Surveys of Modern Mathematics, **1**. International Press, Somerville, MA; Higher Education Press, Beijing, 2012.

[4] I. Enoki, Kawamata–Viehweg vanishing theorem for compact Kähler manifolds, *Einstein metrics and Yang–Mills connections (Sanda, 1990)*, 59–68, Lecture Notes in Pure and Appl. Math., **145**, Dekker, New York, 1993.

[5] G. Fischer, *Complex analytic geometry*, Lecture Notes in Mathematics, **538**. Springer-Verlag, Berlin-New York, 1976.

[6] O. Fujino, A transcendental approach to Kollár's injectivity theorem II, J. Reine Angew. Math. **681** (2013), 149–174.

[7] O. Fujino, Kollár–Nadel type vanishing theorem, Southeast Asian Bull. Math. **42** (2018), no. 5, 643–646.

[8] O. Fujino, *Foundations of the minimal model program*, MSJ Memoirs, **35**. Mathematical Society of Japan, Tokyo, 2017.

[9] O. Fujino, S. Matsumura, Injectivity theorem for pseudo-effective line bundles and its applications, Trans. Amer. Math. Soc. Ser. B 8 (2021), 849–884.

[10] Q. Guan, X. Zhou, A proof of Demailly's strong openness conjecture, Ann. of Math. (2) **182** (2015), no. 2, 605–616.

[11] J. Kollár, Higher direct images of dualizing sheaves. I, Ann. of Math. (2) **123** (1986), no. 1, 11–42.

[12] M. Manaresi, Sard and Bertini type theorems for complex spaces, Ann. Mat. Pura Appl. (4) **131** (1982), 265–279.

[13] H. Matsumura, *Commutative ring theory*, Translated from the Japanese by M. Reid. Second edition. Cambridge Studies in Advanced Mathematics, **8**. Cambridge University Press, Cambridge, 1989.

[14] S. Matsumura, A vanishing theorem of Kollár–Ohsawa type, Math. Ann. **366** (2016), no. 3-4, 1451–1465

[15] S. Matsumura, Injectivity theorems with multiplier ideal sheaves for higher direct images under Kähler morphisms, to appear in Algebr. Geom.

[16] X. Meng, X. Zhou, On the restriction formula, preprint (2021). arXiv:2101.08120 [math.CV]

[17] T. Ohsawa, Vanishing theorems on complete Kähler manifolds, Publ. Res. Inst. Math. Sci. 20 (1984), no. 1, 21–38.

[18] K. Takegoshi, Higher direct images of canonical sheaves tensorized with semi-positive vector bundles by proper Kähler morphisms, Math. Ann. **303** (1995), no. 3, 389–416.

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