APPENDIX: RATIONAL SINGULARITIES

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0.1. **Appendix: Rational singularities.** In this subsection, we give a proof of the following well-known theorem again (see Theorem 49-dlt ??).

z-rational **Theorem 0.1.** *Let (X, D) be a dlt pair. Then X has only rational singularities.*

Our proof is a combination of the proofs in km [?, Theorem 5.22] and ko-sing [?, Section 11]. We need no difficult duality theorems. The argument here will be used in Section sec-alex ??.

First, let us recall the definition of the rational singularities.

Definition 0.2 (Rational singularities)**.** A variety X has *rational singularities* if there is a resolution $f : Y \to X$ such that $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$ and $R^i f_*\mathcal{O}_Y = 0$ for all $i > 0$.

Next, we give a dual form of the Grauert–Riemenschneider vanishing theorem.

lem-gr **Lemma 0.3.** *Let $f : Y \to X$ be a proper birational morphism from a smooth variety Y to a variety X. Let $x \in X$ be a closed point. We put $F = f^{-1}(x)$. Then we have*

$$H^i_F(Y, \mathcal{O}_Y) = 0$$

for every $i < n = \dim X$.

Proof. We take a proper birational morphism $g : Z \to Y$ from a smooth variety Z such that $f \circ g$ is projective. We consider the following spectral sequence

$$E_2^{pq} = H^p_F(Y, R^q g_*\mathcal{O}_Z) \Rightarrow H^{p+q}_E(Z, \mathcal{O}_Z),$$

where $E = g^{-1}(F) = (f \circ g)^{-1}(x)$. Since $R^q g_*\mathcal{O}_Z = 0$ for $q > 0$ and $g_*\mathcal{O}_Z \simeq \mathcal{O}_Y$, we have $H^p_F(Y, \mathcal{O}_Y) \simeq H^p_E(Z, \mathcal{O}_Z)$ for every p. Therefore, we can replace Y with Z and assume that $f : Y \to X$ is projective. Without loss of generality, we can assume that X is affine. Then we

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compactify X and assume that X and Y are projective. It is well known that

$$H^i_F(Y, \mathcal{O}_Y) \simeq \varinjlim_m \operatorname{Ext}^i(\mathcal{O}_{mF}, \mathcal{O}_Y)$$

(see hartshorne-local [?, Theorem 2.8]) and that

$$\operatorname{Hom}(\operatorname{Ext}^i(\mathcal{O}_{mF}, \mathcal{O}_Y), \mathbb{C}) \simeq H^{n-i}(Y, \mathcal{O}_{mF} \otimes \omega_Y)$$

by duality on a smooth projective variety Y (see hartshorne-ag [?, Theorem 7.6 (a)]). Therefore,

$$\begin{aligned} \operatorname{Hom}(H^i_F(Y, \mathcal{O}_Y), \mathbb{C}) &\simeq \operatorname{Hom}(\varinjlim_m \operatorname{Ext}^i(\mathcal{O}_{mF}, \mathcal{O}_Y), \mathbb{C}) \\ &\simeq \varprojlim_m H^{n-i}(Y, \mathcal{O}_{mF} \otimes \omega_Y) \\ &\simeq (R^{n-i} f_* \omega_Y)^\wedge_x \end{aligned}$$

by the theorem on formal functions (see hartshorne-ag [?, Theorem 11.1]), where $(R^{n-i} f_* \omega_Y)^\wedge_x$ is the completion of $R^{n-i} f_* \omega_Y$ at $x \in X$. On the other hand, $R^{n-i} f_* \omega_Y = 0$ for $i < n$ by the Grauert–Riemenschneider vanishing theorem. Thus, $H^i_F(Y, \mathcal{O}_Y) = 0$ for $i < n$. □

lem-gr-ho **Remark 0.4.** Lemma lem-gr 0.3 holds true even when Y has rational singularities. It is because $R^q g_* \mathcal{O}_Z = 0$ for $q > 0$ and $g_* \mathcal{O}_Z \simeq \mathcal{O}_Y$ holds in the proof of Lemma lem-gr 0.3.

Let us start the proof of Theorem z-rational 0.1.

Proof of Theorem z-rational 0.1. Without loss of generality, we can assume that X is affine. Moreover, by taking generic hyperplane sections of X, we can also assume that X has only rational singularities outside a closed point $x \in X$. By the definition of dlt pairs, we can take a resolution $f : Y \to X$ such that $\operatorname{Exc}(f)$ and $\operatorname{Exc}(f) \cup \operatorname{Supp} f^{-1}_* D$ are both simple normal crossing divisors on Y, $K_Y + f^{-1}_* D = f^*(K_X + D) + E$ with $\ulcorner E \urcorner \geq 0$, and that f is projective. Moreover, we can make f an isomorphism over the generic point of any lc center of (X, D). Therefore, by Lemma vani-rf-le ??, we can check that $R^i f_* \mathcal{O}_Y(\ulcorner E \urcorner) = 0$ for every $i > 0$. See also the proof of Theorem 49-dlt ??. We note that $f_* \mathcal{O}_Y(\ulcorner E \urcorner) \simeq \mathcal{O}_X$ since $\ulcorner E \urcorner$ is effective and f-exceptional. For every $i > 0$, by the above assumption, $R^i f_* \mathcal{O}_Y$ is supported at a point $x \in X$ if it ever has a non-empty support at all. We put $F = f^{-1}(x)$. Then we have a spectral sequence

$$E^{ij}_2 = H^i_x(X, R^j f_* \mathcal{O}_Y(\ulcorner E \urcorner)) \Rightarrow H^{i+j}_F(Y, \mathcal{O}_Y(\ulcorner E \urcorner)).$$

By the above vanishing result, we have

$$H^i_x(X, \mathcal{O}_X) \simeq H^i_F(Y, \mathcal{O}_Y(\ulcorner E \urcorner))$$

for every $i \geq 0$. We obtain a commutative diagram

$$\begin{array}{ccc} H^i_F(Y, \mathcal{O}_Y) & \longrightarrow & H^i_F(Y, \mathcal{O}_Y(\ulcorner E \urcorner)) \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\beta} \\ H^i_x(X, \mathcal{O}_X) & = & H^i_x(X, \mathcal{O}_X). \end{array}$$

We have already checked that β is an isomorphism for every i and that $H^i_F(Y, \mathcal{O}_Y) = 0$ for $i < n$ (see Lemma 0.3). Therefore, $H^i_x(X, \mathcal{O}_X) = 0$ for every $i < n = \dim X$. Thus, X is Cohen–Macaulay. For $i = n$, we obtain that

$$\alpha : H^n_x(X, \mathcal{O}_X) \to H^n_F(Y, \mathcal{O}_Y)$$

is injective. We consider the following spectral sequence

$$E^{ij}_2 = H^i_x(X, R^j f_* \mathcal{O}_Y) \Rightarrow H^{i+j}_F(Y, \mathcal{O}_Y).$$

We note that $H^i_x(X, R^j f_* \mathcal{O}_Y) = 0$ for every $i > 0$ and $j > 0$ since X is affine, $\mathrm{Supp} R^j f_* \mathcal{O}_Y \subset \{x\}$ for $j > 0$, and

$$\begin{aligned} \cdots \to H^{i-1}(X \setminus \{x\}, R^j f_* \mathcal{O}_Y) \to H^i_x(X, R^j f_* \mathcal{O}_Y) \\ \to H^i(X, R^j f_* \mathcal{O}_Y) \to \cdots . \end{aligned}$$

On the other hand, we have already obtained $E^{i0}_2 = H^i_x(X, \mathcal{O}_X) = 0$ for every $i < n$. Therefore, $H^0_x(X, R^j f_* \mathcal{O}_Y) \simeq H^j_F(Y, \mathcal{O}_Y) = 0$ for all $j \leq n-2$. Thus, $R^j f_* \mathcal{O}_Y = 0$ for $1 \leq j \leq n-2$. Since $H^{n-1}_x(X, \mathcal{O}_X) = 0$, we obtain that

$$0 \to H^0_x(X, R^{n-1} f_* \mathcal{O}_Y) \to H^n_x(X, \mathcal{O}_X) \xrightarrow{\alpha} H^n_F(Y, \mathcal{O}_Y) \to 0$$

is exact. We have already checked that α is injective. So, we obtain that $H^0_x(X, R^{n-1} f_* \mathcal{O}_Y) = 0$. This means that $R^{n-1} f_* \mathcal{O}_Y = 0$. Thus, we have $R^i f_* \mathcal{O}_Y = 0$ for every $i > 0$. We complete the proof. □

References

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