VANISHING THEOREMS FOR QUASI-PROJECTIVE VARIETIES

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1. Vanishing and torsion-free theorems

In this section, we prove the following theorem. It was proved for *embedded simple normal crossing pairs* in [F1, Theorem 2.39]. Here, we prove it without assuming the existence of ambient spaces. However, we need the assumption that X is quasi-projective.

Theorem 1.1 (cf. [F1, Theorem 2.39])**.** *Let* (X, B) *be a* quasi-projective *simple normal crossing pair. Let* $f : X \to Y$ *be a proper morphism between algebraic varieties and let* L *be a Cartier divisor on* X*. Let* q *be an arbitrary integer. Then we have the following properties.*

- (i) *Assume that* $L - (K_X + B)$ *is* f*-semi-ample. Then every associated prime of* $R^q f_* \mathcal{O}_X(L)$ *is the generic point of the* f*-image of some stratum of* (X, B)*.*
- (ii) *Let* $\pi : Y \to Z$ *be a projective morphism. We assume that* $L - (K_X + B) \sim_{\mathbb{R}} f^*A$ *for some* π*-ample* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor* A *on* Y*. Then* $R^q f_* \mathcal{O}_X(L)$ *is* π_**-acyclic, that is,* $R^p \pi_* R^q f_* \mathcal{O}_X(L) = 0$ *for every* $p > 0$*.*

Proof. Since X is quasi-projective, we can embed X into a smooth projective variety V. By Lemma 1.2 below, we can replace (X, B) and L with (X_k, B_k) and $\sigma^* L$ and assume that there exists an $\mathbb{R}$-divisor D on V such that $B = D|_X$. Then, by using Bertini's theorem, we can take a general complete intersection $W \subset V$ such that $\dim W = \dim X + 1$, $X \subset W$, and W is smooth at the generic point of any stratum of (X, B).

We take a suitable resolution $\varphi : M \to W$ which is an isomorphism outside the singular locus of W with the following properties.

- (A) The strict transform X' of X is a simple normal crossing divisor on M.
- (B) We can write

$$K_{X'} + B' = \varphi^*(K_X + B) + E$$

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such that $(X', \mathrm{Supp}(B'+E))$ is a global embedded simple normal crossing pair, B' is a boundary $\mathbb{R}$-divisor on X', the φ-image of any stratum of (X', B') is a stratum of (X, B), $\ulcorner E \urcorner$ is effective and φ-exceptional.

Then

$$K_{X'} + B' + \{-E\} = \varphi^*(K_X + B) + \ulcorner E \urcorner,$$
$$\varphi_* \mathcal{O}_{X'}(\varphi^* L + \ulcorner E \urcorner) \simeq \mathcal{O}_X(L),$$

and

$$R^q \varphi_* \mathcal{O}_{X'}(\varphi^* L + \ulcorner E \urcorner) = 0$$

for every $q > 0$ (cf. [F1, Theorem 2.39 (i)]). We note that

$$\varphi^* L + \ulcorner E \urcorner - (K_{X'} + B' + \{-E\}) = \varphi^*(L - (K_X + B))$$

and that we can assume that φ is an isomorphism at the generic point of any stratum of $(X', B' + \{-E\})$.

Therefore, by replacing (X, B) and L with $(X', B' + \{-E\})$ and $\varphi^* L + \ulcorner E \urcorner$, we can assume that (X, B) is a quasi-projective global embedded simple normal crossing pair. In this case, the claims have already been established by [F1, Theorem 2.39]. □

By direct calculations, we can obtain the following elementary lemma.

Lemma 1.2 (cf. [F1, Lemma 3.60])**.** *Let (X, B) be a simple normal crossing pair such that B is a boundary $\mathbb{R}$-divisor. Let V be a smooth variety such that $X \subset V$. Then we can construct a sequence of blow-ups*

$$V_k \to V_{k-1} \to \cdots \to V_0 = V$$

with the following properties.

(1) $\sigma_{i+1} : V_{i+1} \to V_i$ *is the blow-up along a smooth irreducible component of* $\mathrm{Supp} B_i$ *for every* $i \geq 0$.
(2) *We put* $X_0 = X$, $B_0 = B$, *and* X_{i+1} *is the strict transform of* X_i *for every* $i \geq 0$.
(3) *We put* $K_{X_{i+1}} + B_{i+1} = \sigma_{i+1}^*(K_{X_i} + B_i)$ *for every* $i \geq 0$.
(4) *There exists an* $\mathbb{R}$*-divisor* D *on* V_k *such that* $D|_{X_k} = B_k$.
(5) $\sigma_* \mathcal{O}_{X_k} \simeq \mathcal{O}_X$ *and* $R^q \sigma_* \mathcal{O}_{X_k} = 0$ *for every* $q > 0$, *where* $\sigma : V_k \to V_{k-1} \to \cdots \to V_0 = V$.

Proof. All we have to do is to check the property (5). We note that $\sigma_{i+1*}\mathcal{O}_{V_{i+1}}(K_{V_{i+1}}) \simeq \mathcal{O}_{V_{i+1}}(K_{V_{i+1}})$ and $R^q \sigma_{i+1*}\mathcal{O}_{V_{i+1}}(K_{V_{i+1}}) = 0$ for every q and for each step $\sigma_{i+1} : V_{i+1} \to V_i$ (cf. [F1, Lemma 2.33]). Therefore we obtain $R^q \sigma_* \mathcal{O}_{X_k}(K_{X_k}) = 0$ for every $q > 0$ and $\sigma_* \mathcal{O}_{X_k}(K_{X_k}) \simeq \mathcal{O}_X(K_X)$. Thus by the Grothendieck duality we obtain $R^q \sigma_* \mathcal{O}_{X_k} = 0$ for every $q > 0$ and $\sigma_* \mathcal{O}_{X_k} \simeq \mathcal{O}_X$. □

References

[F1] O. Fujino, Introduction to the log minimal model program for log canonical pairs, preprint (2009).

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