ON QUASI-ALBANESE MAPS (PRELIMINARY VERSION)

OSAMU FUJINO

Abstract. We discuss Iitaka's theory of quasi-Albanese maps in details. We also give a detailed proof of Kawamata's theorem on the quasi-Albanese maps for varieties of the logarithmic Kodaira dimension zero. Note that Iitaka's theory is an application of Deligne's mixed Hodge theory for smooth algebraic varieties.

Contents

1. Introduction

In this paper, we discuss Iitaka's theory of quasi-Albanese maps. We give a detailed proof of:

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Theorem 1.1 (see [I1] and Theorem 3.16)**.** *Let X be a smooth algebraic variety defined over $\mathbb{C}$. Then there exists a morphism $\alpha : X \to A$ to a quasi-abelian variety A such that*

(i) *for any other morphism $\beta : X \to B$ to a quasi-abelian variety B, there is a morphism $f : A \to B$ such that $\beta = f \circ \alpha$*

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[d, "\alpha"'] & B \\ A \arrow[ur, "f"'] & \end{tikzcd}$$

and

(ii) *f is uniquely determined.*

A quasi-abelian variety in Theorem 1.1 is sometimes called a semi-abelian variety in the literature, which is an extension of an abelian variety by an algebraic torus as an algebraic group. Note that if X is complete in Theorem 1.1 then A is nothing but the Albanese variety of X. Theorem 1.1 depends on Deligne's theory of mixed Hodge structures for smooth complex algebraic varieties.

We also give a detailed proof of Kawamata's theorem on the quasi-Albanese maps for varieties of the logarithmic Kodaira dimension zero.

Theorem 1.2 (see [K2] and Theorem 10.1)**.** *Let X be a smooth variety such that the logarithmic Kodaira dimension $\overline{\kappa}(X)$ of X is zero. Then the quasi-Albanese map $\alpha : X \to A$ is dominant and has irreducible general fibers.*

The original proof of Theorem 1.2 in [K2] needs some deep results on the theory of variations of (mixed) Hodge structure. They are the hardest parts of [K2] to follow. In Section 7, we give many supplementary comments on various semipositivity theorems, which clarify Kawamata's original approach to Theorem 1.2 in [K2]. In Section 8, we explain how to avoid using the theory of variations of (mixed) Hodge structure for the proof of Theorem 1.2. A vanishing theorem in [F2] related to the theory of mixed Hodge structures is sufficient for the proof of Theorem 1.2.

One of the main motivations of this paper is to understand Theorem 1.2 in detail. The original proof of Theorem 1.2 in [K2] looks inaccessible because the theory of variations of (mixed) Hodge structure was not fully matured when [K2] was written around 1980. In [K2], Kawamata could and did use only [D1], [Gri], and [S] for the Hodge theory. Although the semipositivity theorem in [F2] (see also [FF] and [FFS]) does not recover Kawamata's statement on semipositivity (see [K2, Theorem 32]), it is natural and is sufficient for us to carry out Kawamata's proof of Theorem 1.2 in [K2] with some suitable modifications. The author has been unable to follow [K2, Theorem 32]. Moreover, the vanishing theorem in [F2] gives a more elementary approach to Theorem 1.2 and makes Theorem 1.2 independent of the theory of variations of (mixed) Hodge structure. The author hopes that this paper will make Iitaka's theory of quasi-Albanese maps and Kawamata's result on the quasi-Albanese maps of varieties of the logarithmic Kodaira dimension zero accessible.

We summarize the contents of this paper. In Section 2, we collect some basic definitions and results of the logarithmic Kodaira dimensions and the quasi-abelian varieties in the sense of Iitaka. Section 3 is devoted to the theory of quasi-Albanese maps and varieties due to Shigeru Iitaka. We explain it in details following Iitaka's paper [I1] with many supplementary arguments. Theorem 3.16, which is Theorem 1.1, is the main result of this section. In Section 4, we prove some basic properties of quasi-abelian varieties for the reader's convenience. In Section 5, we quickly explain a birational characterization of abelian varieties and a bimeromorphic characterization of complex tori without proof. In Section 6, we recall the subadditivity of the logarithmic Kodaira dimensions in some special cases. We use them for the proof of Theorem 1.2. Section 7 is devoted to the explanation of some semipositivity theorems related to the theory of variations of (mixed) Hodge structure. We hope that this section will help the reader to understand [K2]. In Section 8, we discuss some weak positivity theorems. Our approach in Section 8 does not use the theory of variations of (mixed) Hodge structure. We use a generalization of the Kollár vanishing theorem. This section makes Theorem 1.2 independent of the theory of variations of (mixed) Hodge structure. In Section 9, we discuss finite covers of quasi-abelian varieties. We need them for the proof of Theorem 1.2. In Section 10, we prove Theorem 1.2 in details.

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We will work over $\mathbb{C}$, the complex number filed, throughout this paper. We will use the standard notation as in [F4]. The theory of algebraic groups which are not affine nor projective is not so easy to access. So we make efforts to minimize the use of the general theory of algebraic groups for the reader's convenience. In this paper, we do not even use [D2, Lemme (10.1.3.3)].

2. Preliminaries

In this section, we collect some basic definitions and results on the logarithmic Kodaira dimensions and the quasi-abelian varieties (see, for example, [I1], [I2], [I4], [I5], and so on). For the basic properties of the Kodaira dimensions and some related topics, see, for example, [U] and [Mo].

2.1 (Logarithmic Kodaira dimensions and irregularities)**.** First, we recall the logarithmic Kodaira dimensions and the logarithmic irregularities following Iitaka. For the details, see [I1], [I2], [I4], and [I5].

Definition 2.2 (Logarithmic Kodaira dimension)**.** Let X be an algebraic variety. By Nagata, we have a complete algebraic variety $\overline{X}$ which contains X as a dense Zariski open subset. By Hironaka, we have a smooth projective variety $\overline{W}$ and a projective birational morphism $\mu : \overline{W} \to \overline{X}$ such that if $W = \mu^{-1}(X)$, then $\overline{D} = \overline{W} - W = \mu^{-1}(\overline{X} - X)$ is a simple normal crossing divisor on $\overline{W}$. The *logarithmic Kodaira dimension* $\overline{\kappa}(X)$ of X is defined as

$$\overline{\kappa}(X) = \kappa(\overline{W}, K_{\overline{W}} + \overline{D})$$

where κ denotes Iitaka's D-dimension.

Definition 2.3 (Logarithmic irregularity)**.** Let X be an algebraic variety. We take $(\overline{W}, \overline{D})$ as in Definition 2.2. Then we put

$$\overline{q}(X) = \dim_{\mathbb{C}} H^0(\overline{W}, \Omega^1_{\overline{W}}(\log \overline{D}))$$

and call it the *logarithmic irregularity* of X. We put

$$T_1(X) = H^0(\overline{W}, \Omega^1_{\overline{W}}(\log \overline{D}))$$

following Iitaka [I1].

It is easy to see:

Lemma 2.4. *$\overline{\kappa}(X)$, $\overline{q}(X)$, and $T_1(X)$ are well-defined, that is, they are independent of the choice of the pair $(\overline{W}, \overline{D})$.*

This lemma is well known. We give a proof for the reader's convenience.

Proof. By Hironaka's resolution, it is sufficient to prove that

$$\kappa(\overline{W}, K_{\overline{W}} + \overline{D}) = \kappa(\overline{W}_1, K_{\overline{W}_1} + \overline{D}_1)$$

and

$$H^0(\overline{W}, \Omega^1_{\overline{W}}(\log \overline{D})) = H^0(\overline{W}_1, \Omega^1_{\overline{W}_1}(\log \overline{D}_1))$$

where $f : \overline{W}_1 \to \overline{W}$ is a projective birational morphism from a smooth projective variety $\overline{W}_1$ and $\overline{D}_1 = \mathrm{Supp} f^*\overline{D}$. By the local calculation, we see that

$$f^*\Omega^1_{\overline{W}}(\log \overline{D}) \subset \Omega^1_{\overline{W}_1}(\log \overline{D}_1). \tag{$\heartsuit$}$$

Therefore, we obtain

$$H^0(\overline{W}, \Omega^1_{\overline{W}}(\log \overline{D})) \subset H^0(\overline{W}_1, \Omega^1_{\overline{W}_1}(\log \overline{D}_1)).$$

On the other hand, it is obvious that

$$H^0(\overline{W}_1, \Omega^1_{\overline{W}_1}(\log \overline{D}_1)) \subset H^0(\overline{W}, \Omega^1_{\overline{W}}(\log \overline{D}))$$

since $\Omega^1_{\overline{W}}(\log \overline{D})$ is locally free. Thus, we have

$$H^0(\overline{W}_1, \Omega^1_{\overline{W}_1}(\log \overline{D}_1)) = H^0(\overline{W}, \Omega^1_{\overline{W}}(\log \overline{D})).$$

By ($\heartsuit$), we have

$$K_{\overline{W}_1} + \overline{D}_1 = f^*(K_{\overline{W}} + \overline{D}) + E$$

where E is an effective f-exceptional divisor on $\overline{W}_1$. Therefore, it is obvious that

$$\kappa(\overline{W}, K_{\overline{W}} + \overline{D}) = \kappa(\overline{W}_1, K_{\overline{W}_1} + \overline{D}_1)$$

holds. $\square$

Lemma 2.5. *Let X be a variety and let U be a nonempty Zariski open set of X. Then we have*

$$\overline{\kappa}(X) \le \overline{\kappa}(U)$$

and

$$\overline{q}(X) \le \overline{q}(U).$$

Proof. It is obvious by the definitions of $\overline{\kappa}$ and $\overline{q}$. $\square$

We may sometimes use Lemma 2.5 implicitly.

Lemma 2.6. *Let X be a smooth variety. Assume that F is a closed subset of X with $\mathrm{codim}_X F \ge 2$. Then we have*

$$\overline{\kappa}(X) = \overline{\kappa}(X - F).$$

Proof. We take a smooth complete algebraic variety $\overline{X}$ such that $\overline{D} = \overline{X} - X$ is a simple normal crossing divisor on $\overline{X}$. Then we have

$$\overline{\kappa}(X) = \kappa(\overline{X}, K_{\overline{X}} + \overline{D})$$

by definition. Let $\overline{F}$ be the closure of F in $\overline{X}$. Note that $\mathrm{codim}_{\overline{X}}\overline{F} \geq 2$. We take a resolution

$$f : Y \to \overline{X}$$

such that f is an isomorphism over $\overline{X} - \overline{F}$ and that $\mathrm{Supp} f^{-1}(\overline{F})$ and $\mathrm{Supp}\left(f^{-1}(\overline{F}) \cup f^*\overline{D}\right)$ are simple normal crossing divisors on Y. We put $\Delta_1 = \mathrm{Supp} f^*\overline{D}$ and $\Delta_2 = \mathrm{Supp}\left(f^{-1}(\overline{F}) \cup f^*\overline{D}\right)$. Then we have

$$K_Y + \Delta_1 = f^*(K_{\overline{X}} + \overline{D}) + E$$

where E is an effective f-exceptional divisor on Y. Therefore, we obtain

$$\overline{\kappa}(X) = \kappa(\overline{X}, K_{\overline{X}} + \overline{D}) = \kappa(Y, K_Y + \Delta_1).$$

By definition,

$$\overline{\kappa}(X - F) = \kappa(Y, K_Y + \Delta_2).$$

Since $\Delta_2 - \Delta_1$ is an effective f-exceptional divisor on Y by $\mathrm{codim}_{\overline{X}}\overline{F} \geq 2$, we have

$$\overline{\kappa}(X - F) = \kappa(Y, K_Y + \Delta_2) = \kappa(\overline{X}, K_{\overline{X}} + \overline{D}) = \overline{\kappa}(X).$$

This is the desired equality. □

2.7 (Quasi-abelian varieties in the sense of Iitaka)**.** From now on, we quickly recall the basic properties of quasi-abelian varieties in the sense of Iitaka (see [I1] and [I2]). This paper shows that Iitaka's definition of quasi-abelian varieties is reasonable and natural from the viewpoint of the birational geometry.

Definition 2.8 (Quasi-abelian varieties in the sense of Iitaka)**.** Let G be a connected algebraic group. Then we have the Chevalley decomposition (see, for example, [C, Theorem 1.1]):

$$1 \to \mathcal{G} \to G \to \mathcal{A} \to 1$$

in which $\mathcal{G}$ is the maximal affine algebraic subgroup of G and $\mathcal{A}$ is an abelian variety. If $\mathcal{G}$ is an algebraic torus $\mathbb{G}_m^d$ of dimension d, then G is called a *quasi-abelian* variety (in the sense of Iitaka).

Note that the definition of quasi-abelian varieties in the sense of Iitaka (see Definition 2.8) is different from the definition in [AK, 3. Quasi-Abelian Varieties]. We also note that if G is a quasi-abelian variety in the sense of Iitaka then G is a quasi-abelian variety in the sense of [AK] (see, for example, [AK, 3.2.21 Main Theorem]).

Remark 2.9. It is well known that every algebraic group is quasi-projective (see, for example, [C, Corollary 1.2]).

Although we do not need the following fact, we can easily check:

Remark 2.10. Let G be a connected algebraic group. Then G is a quasi-abelian variety if and only if G contains no $\mathbb{G}_a$ as an algebraic subgroup (see [I2, Lemma 3]).

We note the following important property.

Lemma 2.11 (see [I2, Lemma 4])**.** *A quasi-abelian variety is a commutative algebraic group.*

Proof. We take $\tau \in G$ and consider the group homomorphism:

$$\Psi_\tau(\sigma) = \tau\sigma\tau^{-1} : \mathcal{G} \to G.$$

Since $\mathcal{G}$ is rational and $\mathcal{A}$ is an abelian variety, we see that $\Psi_\tau : \mathcal{G} \to \mathcal{G}$. Therefore, we obtain

$$G \ni \tau \mapsto \Psi_\tau \in \operatorname{Hom}(\mathcal{G}, \mathcal{G}).$$

Note that $\operatorname{Hom}(\mathcal{G}, \mathcal{G})$ is discrete because $\mathcal{G}$ is an algebraic torus. Thus, we obtain $\Psi_1 = \Psi_\tau$. Therefore, $\mathcal{G}$ is contained in the center of G. Moreover, if $\sigma, \tau \in G$, then we have

$$[\tau, \sigma] = \tau\sigma\tau^{-1}\sigma^{-1} \in \mathcal{G}$$

since $\mathcal{A}$ is commutative. Let ρ be any element of $\mathcal{G}$. Then it is easy to see that

$$[\tau\rho, \sigma] = [\tau, \sigma]$$

since $\mathcal{G}$ is contained in the center of G. Note that G is a principal $\mathcal{G}$-bundle over $\mathcal{A}$ as a complex manifold. Therefore, the morphism

$$G \ni \tau \mapsto [\tau, \sigma] \in \mathcal{G}$$

factors through a holomorphic map

$$\mathcal{A} \to \mathcal{G},$$

which is obviously trivial since $\mathcal{A}$ is complete. Hence, we obtain

$$[\tau, \sigma] = 1$$

for every $\sigma, \tau \in G$. This implies that G is commutative. □

Remark 2.12. Let G be a quasi-abelian variety. By Lemma 2.11, G is a commutative group. Therefore, from now on, we write the group law in G additively if there is no danger of confusion. The unit element of G is denoted by 0. Note that an algebraic torus $\mathbb{G}_m^d$ is a quasi-abelian variety in the sense of Iitaka.

We sometimes treat quasi-abelian varieties as commutative complex Lie groups.

Lemma 2.13. *Let G be a quasi-abelian variety. Then the universal cover of G is $\mathbb{C}^{\dim G}$ and G is $\mathbb{C}^{\dim G}/L$ for some lattice L as a complex Lie group. Of course, L is nothing but the topological fundamental group $\pi_1(G)$ of G. Note that the group law of G is induced by the usual addition of $\mathbb{C}^{\dim G}$.*

Proof. By Lemma 2.11, G is a commutative complex Lie group. Therefore, the universal cover is $\mathbb{C}^{\dim X}$ and there is a discrete subgroup L of $\mathbb{C}^{\dim G}$ such that $G = \mathbb{C}^{\dim G}/L$ as a complex Lie group. By construction, the group law in G is induced by the usual addition of $\mathbb{C}^{\dim G}$. See also Example 2.14 and the proof of Lemma 3.8. □

In this paper, we mainly treat non-projective algebraic groups as complex Lie groups. We note the following famous example by Serre. It says that two different algebraic groups may be analytically isomorphic. Of course, we can not directly use Serre's GAGA principle for non-projective varieties.

Example 2.14 (Serre, see [Mu, Footnote in page 33] and [Har, Chapter VI, Example 3.2])**.** Let E be a smooth elliptic curve and let $\mathcal{E}$ be a vector bundle of rank 2 on E, which is a non-trivial extension of $\mathcal{O}_E$ by itself:

$$0 \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{E} \xrightarrow{p} \mathcal{O}_E \longrightarrow 0.$$

We put $X = \mathbb{P}_E(\mathcal{E})$. Let D be the section of $\pi : X \to E$ corresponding to $\mathcal{E} \xrightarrow{p} \mathcal{O}_E \to 0$. We put $G = X - D$. Then G is an algebraic group and its Chevalley decomposition is

$$0 \longrightarrow \mathbb{G}_a \longrightarrow G \xrightarrow{\pi} E \longrightarrow 0.$$

Note that G is analytically isomorphic to $\mathbb{G}_m^2$ (see [Har, Chapter VI, Example 3.2]). Therefore, $\mathbb{G}_m^2$ and G are analytically isomorphic but are two different algebraic groups.

In Section 3, we will discuss Iitaka's quasi-Albanese maps and prove the existence of quasi-Albanese maps and varieties in details.

Definition 2.15 (Quasi-Albanese maps)**.** Let X be a smooth variety. The *quasi-Albanese map* $\alpha : X \to A$ is a morphism to a quasi-abelian variety A such that

(i) for any other morphism $\beta : X \to B$ to a quasi-abelian variety B, there is a morphism $f : A \to B$ such that $\beta = f \circ \alpha$

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[d, "\alpha"'] & B \\ A \arrow[ur, "f"'] & \end{tikzcd}$$

and

(ii) f is uniquely determined.

Note that A is usually called the *quasi-Albanese variety* of X.

If X is complete in Definition 2.15, then A is nothing but the Albanese variety of X.

3. Quasi-Albanese maps due to Iitaka

In this section, we discuss Iitaka's quasi-Albanese maps and varieties following [I1] and [I2]. We recommend the reader to study the basic results on the Albanese maps and varieties before reading this section (see, for example, [B, V.11–14], [U, §9], [GH, Chapter 2, Section 6], and so on).

Let us start with the following easy lemma on singular homology groups. In this section, X is a smooth complete algebraic variety and D is a simple normal crossing divisor on X. The Zariski open set $X \setminus D$ of X is denoted by V.

Lemma 3.1. *Let X be a smooth complete algebraic variety and let D be a simple normal crossing divisor on X. We put $V = X \setminus D$. Then the map*

$$\iota_* : H_1(V, \mathbb{Z}) \to H_1(X, \mathbb{Z})$$

is surjective, where $\iota : V \hookrightarrow X$ is the natural open immersion.

Proof. We put $n = \dim X$. We have the following long exact sequence:

$$\cdots \longrightarrow H^{2n-1}(X, D; \mathbb{Z}) \xrightarrow{p} H^{2n-1}(X, \mathbb{Z}) \longrightarrow H^{2n-1}(D, \mathbb{Z}) \longrightarrow \cdots .$$

Note that $H^{2n-1}(D, \mathbb{Z}) = 0$ since D is an $(n-1)$-dimensional simple normal crossing variety. Therefore, p is surjective. We also note that

$$H^{2n-1}(X, D; \mathbb{Z}) \simeq H_c^{2n-1}(V, \mathbb{Z}).$$

We have the following commutative diagram:

$$\begin{tikzcd} H_c^{2n-1}(V, \mathbb{Z}) \arrow[r, "p"] \arrow[d, "D_V"', "\simeq"] & H^{2n-1}(X, \mathbb{Z}) \arrow[d, "\simeq"', "D_X"] \\ H_1(V, \mathbb{Z}) \arrow[r, "\iota_*"'] & H_1(X, \mathbb{Z}). \end{tikzcd}$$

Note that the duality maps D_V and D_X are both isomorphisms by Poincaré duality. Since p is surjective, we see that ι_* is also surjective. For the details of Poincaré duality, see, for example, [Hat, Section 3.3]. $\square$

Lemma 3.2. *The natural injection*

$$\iota^* : H^1(X, \mathbb{C}) \to H^1(V, \mathbb{C})$$

is nothing but

$$a_1 \oplus a_2 : H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1_X) \to H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1_X(\log D))$$

where a_1 is the identity on $H^1(X, \mathcal{O}_X)$ and a_2 is the natural inclusion

$$H^0(X, \Omega^1_X) \hookrightarrow H^0(X, \Omega^1_X(\log D))$$

by Deligne's theory of mixed Hodge structures. Note that we have

$$b_1(V) - b_1(X) = \overline{q}(V) - q(X)$$

where

$$\overline{q}(V) = \dim H^0(X, \Omega^1_X(\log D)) \quad \textit{and} \quad q(X) = \dim H^0(X, \Omega^1_X).$$

Of course,

$$b_1(V) = \dim_{\mathbb{C}} H^1(V, \mathbb{C}) \quad \textit{and} \quad b_1(X) = \dim_{\mathbb{C}} H^1(X, \mathbb{C}).$$

Proof. By Lemma 3.1, ι^* is injective. By Deligne's mixed Hodge theory (see [D1]), we have

$$H^1(V, \mathbb{C}) = H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1_X(\log D)).$$

By the Hodge decomposition, we have

$$H^1(X, \mathbb{C}) = H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1_X).$$

Since $\iota^* : H^1(X, \mathbb{C}) \to H^1(V, \mathbb{C})$ is a morphism of mixed Hodge structures (see [D1]), we obtain the desired description of ι^*. $\square$

Let us describe the theory of quasi-Albanese maps and varieties due to Shigeru Iitaka (see [I1]).

3.3 (Quasi-Albanese maps and varieties)**.** We take a basis

$$\{\omega_1, \cdots, \omega_q\}$$

of $H^0(X, \Omega^1_X)$, where $q = q(X) = \dim H^0(X, \Omega^1_X)$. Note that $b_1(X) = 2q$ by the Hodge theory. We take

$$\varphi_1, \cdots, \varphi_d \in H^0(X, \Omega^1_X(\log D))$$

with $d = \overline{q}(V) - q(X)$ such that

$$\{\omega_1, \cdots, \omega_q, \varphi_1, \cdots, \varphi_d\}$$

is a basis of $H^0(X, \Omega^1_X(\log D))$. Let

$$\{\xi_1, \cdots, \xi_{2q}\}$$

be a basis of the free part of $H_1(X, \mathbb{Z})$. We take

$$\eta_1, \cdots, \eta_d \in \mathrm{Ker}\iota_* \subset H_1(V, \mathbb{Z})$$

such that

$$\{\xi_1, \cdots, \xi_{2q}, \eta_1, \cdots, \eta_d\}$$

is a basis of the free part of $H_1(V, \mathbb{Z})$ (see Lemma 3.1). We put $\overline{q} = \overline{q}(V)$,

$$A_i = \left(\int_{\xi_i} \omega_1, \cdots \int_{\xi_i} \omega_q, \int_{\xi_i} \varphi_1, \cdots, \int_{\xi_i} \varphi_d\right) \in \mathbb{C}^{\overline{q}}$$

for $1 \leq i \leq 2q$, and

$$B_j = \left(\int_{\eta_j} \omega_1, \cdots \int_{\eta_j} \omega_q, \int_{\eta_j} \varphi_1, \cdots, \int_{\eta_j} \varphi_d\right) \in \mathbb{C}^{\overline{q}}$$

for $1 \leq j \leq d$.

Lemma 3.4. *Let γ be a torsion element of $H_1(V, \mathbb{Z})$. Then we have*

$$\int_\gamma \omega = 0$$

for every $\omega \in H^0(X, \Omega^1_X(\log D))$.

Proof. It is obvious since

$$m \int_\gamma \omega = \int_{m\gamma} \omega = 0$$

if $m\gamma = 0$ in $H_1(V, \mathbb{Z})$. □

Lemma 3.5. *We have*

$$\int_{\eta_j} \omega_k = 0$$

for every j and k.

Proof. We see that

$$\int_{\eta_j} \omega_k = \int_{\eta_j} \iota^* \omega_k = \int_{\iota_* \eta_j} \omega_k = 0$$

since $\iota_* \eta_j = 0$. □

Lemma 3.6 (see [I1, Lemma 2]). *Let φ be an arbitrary element of $H^0(X, \Omega^1_X(\log D))$. Assume that*

$$\int_\eta \varphi = 0$$

for every $\eta \in \mathrm{Ker}\iota_* \subset H_1(V, \mathbb{Z})$. Then we have $\varphi \in H^0(X, \Omega^1_X)$.

Proof. Assume that $\varphi \in H^0(X, \Omega^1_X(\log D)) \setminus H^0(X, \Omega^1_X)$. Then φ has a pole along some D_a, where D_a is an irreducible component of D. Let p be a general point of D_a. We take a local holomorphic coordinate system $(z_1, \cdots, z_n)$ around p such that D_a is defined by $z_1 = 0$. In this case, we can write

$$\varphi = \alpha(z)\frac{dz_1}{z_1} + \beta(z)$$

around p, where $\beta(z)$ is a holomorphic 1-form. We may assume that $\alpha(z) = \alpha(z_2, \cdots, z_n)$. Since $d\varphi = 0$, we obtain

$$d\varphi = d\alpha \wedge \frac{dz_1}{z_1} + d\beta = 0.$$

Thus we have $d\alpha = 0$. This means that α is a constant. Let us consider a circle γ_a around D_a at p. Then we obtain $\iota_*\gamma_a = 0$ in $H_1(X, \mathbb{Z})$ and

$$0 = \int_{\gamma_a} \varphi = \alpha \int_{\gamma_a} \frac{dz_1}{z_1} = \alpha 2\pi\sqrt{-1}.$$

This implies that $\alpha = 0$. Thus, φ is holomorphic at p. This is a contradiction. Therefore, we have $\varphi \in H^0(X, \Omega^1_X)$. $\square$

Lemma 3.7 (see [I2, Proposition 2]). *The above vectors $A_1, \cdots, A_{2q}$, $B_1, \cdots, B_d$ are $\mathbb{R}$-$\mathbb{C}$ linearly independent. This means that if*

$$\sum_{i=1}^{2q} a_i A_i + \sum_{j=1}^{d} b_j B_j = 0$$

for $a_i \in \mathbb{R}$ and $b_j \in \mathbb{C}$ then $a_i = 0$ for every i and $b_j = 0$ for every j.

Proof. We put

$$\widehat{A}_i = \left(\int_{\xi_i} \omega_1, \cdots, \int_{\xi_i} \omega_q\right)$$

for $1 \leq i \leq 2q$. Then $\widehat{A}_1, \cdots, \widehat{A}_{2q}$ are $\mathbb{R}$-linearly independent, which is well known by the Hodge theory. By Lemma 3.5, we have $a_i = 0$ for every i. We put

$$\widehat{B}_j = \left(\int_{\eta_j} \varphi_1, \cdots, \int_{\eta_j} \varphi_d\right)$$

for $1 \leq j \leq d$. It is sufficient to prove that $\widehat{B}_1, \cdots, \widehat{B}_d$ are $\mathbb{C}$-linearly independent. If $\widehat{B}_1, \cdots, \widehat{B}_d$ are $\mathbb{C}$-linearly dependent, then the rank of the $d \times d$ matrix

$$\left(\int_{\eta_j} \varphi_i \right)_{i,j}$$

is less than d. This means that there is $(c_1, \cdots, c_d) \neq 0$ such that

$$\int_{\eta_j} \sum_{i=1}^{d} c_i \varphi_i = 0$$

for every j. Therefore, we see that

$$\sum_{i=1}^{d} c_i \varphi_i \in H^0(X, \Omega_X^1)$$

by Lemma 3.6. This contradicts the choice of $\{\varphi_1, \cdots, \varphi_d\}$. Thus, $\widehat{B}_1, \cdots, \widehat{B}_d$ are $\mathbb{C}$-linearly independent. □

By the proof of Lemma 3.7, we can choose $\varphi_1, \cdots, \varphi_d$ such that

$$\int_{\eta_j} \varphi_k = \delta_{jk}.$$

We put

$$L = \sum_i \mathbb{Z} A_i + \sum_j \mathbb{Z} B_j,$$

$$L_1 = \sum_i \mathbb{Z} \widehat{A}_i,$$

and

$$L_0 = \sum_j \mathbb{Z} \widehat{B}_j.$$

Then we get the following short exact sequence of complex Lie groups:

$$(\spadesuit) \qquad 0 \longrightarrow \mathbb{C}^d / L_0 \longrightarrow \mathbb{C}^{\overline{q}} / L \longrightarrow \mathbb{C}^q / L_1 \longrightarrow 0.$$

Note that $T = \mathbb{C}^d / L_0$ is an algebraic torus $\mathbb{G}_m^d$ and that $\mathcal{A}_X = \mathbb{C}^q / L_1$ is the Albanese variety of X. More explicitly, if $(z_1, \cdots, z_d)$ is the standard coordinate system of $\mathbb{C}^d$, then the isomorphism

$$\mathbb{C}^d / L_0 \xrightarrow{\sim} \mathbb{G}_m^d$$

is given by

$$(z_1, \cdots, z_d) \mapsto (\exp 2\pi\sqrt{-1} z_1, \cdots, \exp 2\pi\sqrt{-1} z_d).$$

We call

$$\widetilde{\mathcal{A}}_V = \mathbb{C}^{\overline{q}} / L$$

the quasi-Albanese variety of V. By the above description, we see that $\widetilde{\mathcal{A}}_V$ is a principal $\mathbb{G}_m^d$-bundle over an abelian variety $\mathcal{A}_X$ as a complex manifold. We have to check:

Lemma 3.8. *The quasi-Albanese variety $\widetilde{\mathcal{A}}_V$ is a quasi-abelian variety.*

Proof. We put $A = \widetilde{\mathcal{A}}_V$ and $B = \mathcal{A}_X$ for simplicity. Note that A is a principal $\mathbb{G}_m^d$-bundle over B as a complex manifold. We consider the following group homomorphism:

$$\rho : \mathbb{G}_m^d \to \mathrm{PGL}(d, \mathbb{C})$$

given by

$$\rho(\lambda_1, \cdots, \lambda_d) = \begin{pmatrix} 1 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_d \end{pmatrix}.$$

By ρ, we obtain $\mathbb{P}^d$-bundle $Z = A \times_\rho \mathbb{P}^d$ over $B = A/\mathbb{G}_m^d$ which is a compactification of A. It is easy to see that the divisor $\Delta = Z \setminus A$ is a simple normal crossing divisor on Z and is ample over B. Moreover, we can easily see that $Z \to B$ and $A \to B$ are locally trivial in the Zariski topology. We will see that the group law

$$\psi : A \times A \longrightarrow A$$

in A as a complex Lie group is algebraic. By construction, the map ψ can be extended to holomorphic maps

$$g_1 : Z \times A \longrightarrow Z \quad \text{and} \quad g_2 : A \times Z \longrightarrow Z$$

since Z is a $\mathbb{G}_m^d$-equivariant embedding of A. Therefore, we obtain a holomorphic map

$$g : Z \times Z \setminus \Sigma \longrightarrow Z \hookrightarrow \mathbb{P}^N,$$

where $\Sigma = (\Delta \times Z) \cap (Z \times \Delta)$. Of course, g is an extension of $\psi : A \times A \to A$. Note that $\mathrm{codim}_{Z\times Z}\Sigma \geq 2$. We consider $g^*\mathcal{O}_{\mathbb{P}^N}(1)$. This line bundle can be extended to a line bundle $\mathcal{L}$ on $Z \times Z$. Moreover, we can see

$$l_i := g^* X_i \in H^0(Z \times Z, \mathcal{L})$$

for $0 \leq i \leq N$, where $[X_0 : \cdots : X_N]$ are homogeneous coordinates of $\mathbb{P}^N$. Therefore, we obtain a rational map $h : Z \times Z \dashrightarrow Z$, which is given by the linear system spanned by $\{l_0, \cdots, l_N\}$ and is an extension of g. Thus, the group law

$$\psi : A \times A \to A$$

is algebraic since $\psi = h|_{A\times A}$. This means that $A = \widetilde{\mathcal{A}}_V$ is an algebraic group. So, $\widetilde{\mathcal{A}}_V$ is a quasi-abelian variety. Note that the short exact sequence (♠) is nothing but the Chevalley decomposition. $\square$

Lemma 3.9. *Let ω be an element of $H^0(X, \Omega^1_X(\log D))$. We fix a point $0 \in V$. Then we have a multivalued holomorphic function*

$$\int_0^p \omega$$

on V. For a point $p \in V$, we can define $\alpha_V : V \to \widetilde{\mathcal{A}}_V$ by

$$\alpha_V(p) = \left(\int_0^p \omega_1, \cdots, \int_0^p \omega_q, \int_0^p \varphi_1, \cdots, \int_0^p \varphi_d\right) \in \widetilde{\mathcal{A}}_V.$$

This map is independent of the choice of the path from 0 to p in V. Thus we get a quasi-Albanese map:

$$\alpha_V : V \to \widetilde{\mathcal{A}}_V.$$

It is a holomorphic map.

Proof. Let γ be a 2-cycle on V. Then

$$\int_{\partial\gamma} \omega = \int_\gamma d\omega = 0$$

for every $\omega \in H^0(X, \Omega^1_X(\log D))$. This is because ω is d-closed by Deligne (see [D1]). Therefore, α_V is well-defined. $\square$

Lemma 3.10 (see [I1, Proposition 3])**.** *The map α_V in Lemma 3.9 is algebraic.*

Proof. Note that $A = \widetilde{\mathcal{A}}_V$ is a principal $\mathbb{G}_m^d$-bundle over $B = \mathcal{A}_X$ as a complex manifold. We consider the group homomorphism

$$\rho' : \mathbb{G}_m^d \to \mathrm{PGL}(2, \mathbb{C}) \times \mathrm{PGL}(2, \mathbb{C}) \times \cdots \times \mathrm{PGL}(2, \mathbb{C})$$

given by

$$\rho'(\lambda_1, \cdots, \lambda_d) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 & 0 \\ 0 & \lambda_d \end{pmatrix}.$$

Then we obtain a $\mathbb{G}_m^d$-equivariant embedding $Z' = A \times_{\rho'} (\mathbb{P}^1 \times \cdots \times \mathbb{P}^1)$ of A over B.

Claim. *The holomorphic map*

$$\alpha_V : V \to \widetilde{\mathcal{A}}_V$$

given in Lemma 3.9 can be extended to a rational map

$$\beta_X : X \dashrightarrow Z'.$$

Proof of Claim. We note that it is sufficient to prove that there exists a meromorphic extension β_X of α_V since X and Z' are projective. Let p be a point of $D \subset X$. Let $(z_1, \cdots, z_n)$ be a local holomorphic coordinate system of X at p such that D is defined by $z_1 \cdots z_r = 0$. In this case, we can write

$$\varphi_i = \sum_{b=1}^{r} \alpha_{ib} \frac{dz_b}{z_b} + \widetilde{\varphi}_i$$

where $\alpha_{ib} \in \mathbb{C}$ and $\widetilde{\varphi}_i$ is a holomorphic 1-form for every i around p (see the proof of Lemma 3.6). Let δ_a be a circle around $D_a = (z_a = 0)$ near p. Then $\iota_* \delta_a = 0$. Therefore, we have

$$\delta_a = \sum_j m_{ja} \eta_j + \widetilde{\delta}_a,$$

where $m_{ja} \in \mathbb{Z}$ and $\widetilde{\delta}_a$ is a torsion element. Thus we have

$$\alpha_{ia} = \frac{1}{2\pi\sqrt{-1}} \int_{\delta_a} \varphi_i = \frac{1}{2\pi\sqrt{-1}} \sum_j m_{ja} \int_{\eta_j} \varphi_i = \frac{m_{ia}}{2\pi\sqrt{-1}}.$$

Without loss of generality, we may assume that $0 \in V$ is near p. For a point $p' \in V$ near p, we have

(♣)

$$\begin{aligned} \exp\left(2\pi\sqrt{-1}\int_0^{p'} \varphi_i\right) &= c \exp\left(\sum_b m_{ib} \log z_b(p')\right) \cdot \exp\left(2\pi\sqrt{-1}\int_0^{p'} \widetilde{\varphi}_i\right) \\ &= c \prod_b z_b(p')^{m_{ib}} \cdot \exp\left(2\pi\sqrt{-1}\int_0^{p'} \widetilde{\varphi}_i\right) \end{aligned}$$

for some constant c. We consider the following commutative diagram:

$$\begin{tikzcd} V \arrow[r, "\alpha_V"] \arrow[d, hook] & \widetilde{\mathcal{A}}_V \arrow[d, hook] \\ X \arrow[r, dashed] \arrow[d] & Z' \arrow[d, "\pi_{Z'}"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

Note that $X \to B$ is nothing but the Albanese map of X. Let U be a small open set of B in the classical topology. Then

$$\pi^{-1}(U) \simeq U \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times,$$

where $\pi : \widetilde{\mathcal{A}}_V \to B = \mathcal{A}_X$, and

$$\pi_{Z'}^{-1}(U) \simeq U \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$$

over U. Over U, it is easy to see that α_V can be extended to a meromorphic map $X \dashrightarrow Z'$ in the sense of Remmmert by (♣) (see [GR, Chapter 10, §6, 3. Graph of a Finite System of Meromorphic Functions]). For the definition of meromorphic mappings in the sense of Remmert, see, for example, [U, Definition 2.2]. Therefore, α_V can be extended to a meromorphic map β_X from X to Z' in the sense of Remmert. By Serre's GAGA, a meromorphic map $\beta_X : X \dashrightarrow Z'$ is a rational map between smooth projective varieties. $\square$

Thus we obtain that α_V in Lemma 3.9 is algebraic. $\square$

Lemma 3.11. *We have that*

$$(\alpha_V)_* : H_1(V, \mathbb{Z}) \to H_1(\widetilde{\mathcal{A}}_V, \mathbb{Z})$$

is surjective. Moreover, we have

$$\operatorname{Ker}(\alpha_V)_* = H_1(V, \mathbb{Z})_{\mathrm{tor}},$$

where $H_1(V, \mathbb{Z})_{\mathrm{tor}}$ is the torsion part of $H_1(V, \mathbb{Z})$.

Proof. Let $H_1(V, \mathbb{Z})_{\mathrm{free}}$ be the free part of $H_1(V, \mathbb{Z})$. Note that $\widetilde{\mathcal{A}}_V = \mathbb{C}^{\overline{q}}/L$ by construction, where

$$\mathbb{C}^{\overline{q}} = (H^0(X, \Omega^1_X(\log D)))^*$$

and L is an embedding of $H_1(V, \mathbb{Z})_{\mathrm{free}}$ into $(H^0(X, \Omega^1_X(\log D)))^*$. On the other hand,

$$H_1(\widetilde{\mathcal{A}}_V, \mathbb{Z}) = \pi_1(\widetilde{\mathcal{A}}_V) = L$$

by construction. By the construction of the lattice L, it is obvious that

$$(\alpha_V)_* : H_1(V, \mathbb{Z}) \to H_1(\widetilde{\mathcal{A}}_V, \mathbb{Z})$$

is surjective and that

$$\operatorname{Ker}(\alpha_V)_* = H_1(V, \mathbb{Z})_{\mathrm{tor}}.$$

This is the desired property. $\square$

Lemma 3.12. *We have that*

$$\alpha_V^* : T_1(\widetilde{\mathcal{A}}_V) \to T_1(V)$$

is an isomorphism.

Proof. By Lemma 3.11,

$$(\alpha_V)_*: H_1(V,\mathbb{Q}) \to H_1(\widetilde{\mathcal{A}}_V,\mathbb{Q})$$

is an isomorphism. Therefore, we obtain

$$(\alpha_V)^*: H^1(\widetilde{\mathcal{A}}_V,\mathbb{Q}) \to H^1(V,\mathbb{Q})$$

is also an isomorphism. Moreover, it is an isomorphism of mixed Hodge structures (see [D1]). Therefore, we have an isomorphism

$$\alpha_V^*: T_1(\widetilde{\mathcal{A}}_V) \to T_1(V)$$

by Deligne (see [D1]). □

The following lemma is useful and important.

Lemma 3.13. *Let W be a quasi-abelian variety. Then the quasi-Albanese map*

$$\alpha_W: W \to \widetilde{\mathcal{A}}_W$$

is an isomorphism.

Proof. By translation, we may assume that $\alpha_W(0)=0$. Note that α_W induces a complex linear map

$$(\alpha_W)_*: T_{W,0} \to T_{\widetilde{\mathcal{A}}_W,0},$$

where $T_{W,0}$ is the tangent space of W at 0 and $T_{\widetilde{\mathcal{A}}_W,0}$ is the tangent space of $\widetilde{\mathcal{A}}_W$ at 0. By considering the exponential maps, we can recover α_W by $(\alpha_W)_*$. In particular, α_W is a homomorphism of complex Lie groups. Since

$$(\alpha_W)_*: H_1(W,\mathbb{Z}) \to H_1(\widetilde{\mathcal{A}}_W,\mathbb{Z})$$

is an isomorphism by Lemma 3.11, α_W is an isomorphism of complex Lie groups. Note that α_W is algebraic. Therefore, α_W is an isomorphism between smooth algebraic varieties. □

Lemma 3.14. *Let $f: V \to T$ be a morphism to a quasi-abelian variety T. Then there exists a unique algebraic morphism $\widetilde{f}: \widetilde{\mathcal{A}}_V \to T$ such that $f = \widetilde{f}\circ\alpha_V$*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\alpha_V"'] & T \\ \widetilde{\mathcal{A}}_V \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

where $\alpha_V: V \to \widetilde{\mathcal{A}}_V$ is a quasi-Albanese map of V.

Proof. We take a point $0 \in V$. By translations, we may assume that $\alpha_V(0) = 0$ and $f(0) = 0$. Let $\{u_1, \cdots, u_k\}$ be a basis of $T_1(T)$. We may assume that

$$f^*u_1, \cdots, f^*u_l$$

are linearly independent, where

$$l = \dim_{\mathbb{C}}\langle f^*u_1, \cdots, f^*u_k\rangle.$$

We take $v_1, \cdots, v_m \in T_1(V)$ such that

$$\{v_1, \cdots, v_m, f^*u_1, \cdots, f^*u_l\}$$

is a basis of $T_1(V)$. Since $f_* : H_1(V, \mathbb{Z}) \to H_1(T, \mathbb{Z})$, by using the basis $\{v_1, \cdots, v_m, f^*u_1, \cdots, f^*u_l\}$ of $T_1(V)$, we can easily construct a holomorphic map

$$\widetilde{f} : \widetilde{\mathcal{A}}_V \longrightarrow \widetilde{\mathcal{A}}_T \xrightarrow[\alpha_T^{-1}]{\sim} T$$

(see Lemma 3.13) satisfying $f = \widetilde{f} \circ \alpha_V$. Therefore, there is a commutative diagram:

$$\begin{tikzcd}
T_1(V) & T_1(T) \arrow[l, "f^*"'] \arrow[dl, "\widetilde{f}^*"] \\
T_1(\widetilde{\mathcal{A}}_V) \arrow[u, "\alpha_V^*"] &
\end{tikzcd}$$

which determines $\widetilde{f}^*$ uniquely. This is because α_V^* is an isomorphism (see Lemma 3.12). Thus, $\widetilde{f}$ is unique. This is because $\widetilde{f}$ can be uniquely recovered by $\widetilde{f}^*$ (cf. the proof of Lemma 3.13). Therefore, all we have to do is to prove that $\widetilde{f}$ is algebraic. It is sufficient to prove that the graph

$$\Gamma = \{(x, \widetilde{f}(x)) \,|\, x \in \widetilde{\mathcal{A}}_V\} \subset \widetilde{\mathcal{A}}_V \times T$$

is an algebraic variety. We consider the map

$$\alpha_n : V^{2n} = V \times \cdots \times V \to \widetilde{\mathcal{A}}_V$$

given by

$$\alpha_n(z_1, \cdots, z_{2n}) = \alpha_V(z_1) + \cdots + \alpha_V(z_n) - \alpha_V(z_{n+1}) - \cdots - \alpha_V(z_{2n}).$$

We put

$$F_n = \overline{\mathrm{Im}\alpha_n},$$

that is, the Zariski closure of $\mathrm{Im}\alpha_n$. Then F_n is an irreducible algebraic subvariety of $\widetilde{\mathcal{A}}_V$ for every n such that

$$F_1 \subset F_2 \subset \cdots \subset F_k \subset \cdots .$$

Therefore, there is a positive integer n_0 such that

$$F_{n_0} = F_{n_0+1} = \cdots .$$

Note that F_{n_0} is a quasi-abelian subvariety of $\widetilde{\mathcal{A}}_V$ because it is closed under the group law of $\widetilde{\mathcal{A}}_V$. Moreover, by the universality of $\widetilde{\mathcal{A}}_V$ proved above, F_{n_0} is not contained in a quasi-abelian proper subvariety of $\widetilde{\mathcal{A}}_V$. This implies that $F_{n_0} = \widetilde{\mathcal{A}}_V$. Note that $\widetilde{f}$ is a homomorphism of complex Lie groups. We consider the following commutative diagram:

$$\begin{tikzcd} V^{2n_0} \arrow[r, "f_{n_0}"] \arrow[d, "\alpha_{n_0}"'] & T \\ \widetilde{\mathcal{A}}_V \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

where

$$f_{n_0}(z_1, \cdots, z_{2n_0}) = f(z_1) + \cdots + f(z_{n_0}) - f(z_{n_0+1}) - \cdots - f(z_{2n_0}).$$

We consider the Zariski closure of

$$\{(\alpha_{n_0}(x), f_{n_0}(x)) \,|\, x \in V^{2n_0}\} \subset \Gamma \subset \widetilde{\mathcal{A}}_V \times T.$$

Then it is an algebraic subvariety of $\widetilde{\mathcal{A}}_V \times T$ and coincides with the graph Γ. This implies that $\widetilde{f}$ is algebraic. $\square$

Lemma 3.15. *Let $f : V_1 \to V_2$ be a morphism between smooth algebraic varieties. Then f induces an algebraic morphism $f_* : \widetilde{\mathcal{A}}_{V_1} \to \widetilde{\mathcal{A}}_{V_2}$ which satisfies the following commutative diagram.*

$$\begin{tikzcd} V_1 \arrow[r, "f"] \arrow[d, "\alpha_{V_1}"'] & V_2 \arrow[d, "\alpha_{V_2}"] \\ \widetilde{\mathcal{A}}_{V_1} \arrow[r, "f_*"'] & \widetilde{\mathcal{A}}_{V_2} \end{tikzcd}$$

Moreover, f_* is unique.

Proof. It is almost obvious by Lemma 3.14. We apply Lemma 3.14 to the map $\alpha_{V_2} \circ f : V_1 \to \widetilde{\mathcal{A}}_{V_2}$. Then we obtain the desired map $f_* : \widetilde{\mathcal{A}}_{V_1} \to \widetilde{\mathcal{A}}_{V_2}$ uniquely. $\square$

We summarize:

Theorem 3.16 (Iitaka's quasi-Albanese varieties and maps)**.** *Let V be a smooth algebraic variety. Then there exists a quasi-abelian variety $\widetilde{\mathcal{A}}_V$ and a morphism $\alpha_V : V \to \widetilde{\mathcal{A}}_V$ with the following property:*

for any quasi-abelian variety T and any morphism $f : V \to T$, there exists a unique morphism $\widetilde{f} : \widetilde{\mathcal{A}}_V \to T$ such that $\widetilde{f} \circ \alpha_V = f$.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\alpha_V"'] & T \\ \widetilde{\mathcal{A}}_V \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

The quasi-abelian variety $\widetilde{\mathcal{A}}_V$, determined up to isomorphism by this condition, is called the quasi-Albanese variety of V. The map $\alpha_V : V \to \widetilde{\mathcal{A}}_V$ is called the quasi-Albanese map of V. By the construction of $\widetilde{\mathcal{A}}_V$, $\widetilde{\mathcal{A}}_V$ is nothing but the Albanese variety of V when V is complete.

Anyway, Theorem 3.16 is a generalization of the theory of Albanese maps and varieties for non-compact smooth complex algebraic varieties. We close this section with an easy corollary of Theorem 3.16.

Corollary 3.17 (cf. Remark 2.10)**.** *Let $f : \mathbb{A}^1 \to G$ be an algebraic morphism from $\mathbb{A}^1$ to a quasi-abelian variety G. Then $f(\mathbb{A}^1)$ is a point.*

Proof. Note that $T_1(\mathbb{A}^1) = 0$. Thus the quasi-Albanese variety $\widetilde{\mathcal{A}}_{\mathbb{A}^1}$ is a point. Since f factors through $\widetilde{\mathcal{A}}_{\mathbb{A}^1}$ by Theorem 3.16, $f(\mathbb{A}^1)$ is a point. □

4. Basic properties of quasi-abelian varieties

In this section, we collect some basic properties of quasi-abelian varieties for the reader's convenience. We will use them in the proof of Theorem 1.2.

4.1. Let G be a quasi-abelian variety and let

$$0 \longrightarrow \mathcal{G} \longrightarrow G \longrightarrow \mathcal{A} \longrightarrow 0 \tag{1}$$

be the Chevalley decomposition such that $\mathcal{G} = \mathbb{G}_m^d$. We put $\dim \mathcal{A} = q$ and $n = \dim G = q + d$. Then there exists a $(2q + d) \times n$ matrix M with

$$M = \begin{pmatrix} P & Q \\ 0 & I_d \end{pmatrix}$$

where P is a $2q \times q$ matrix. The lattice spanned by the row vectors of M (resp. P) is denoted by L (resp. L_1). Then we have the short exact sequence of complex Lie groups:

$$0 \longrightarrow \mathbb{G}_m^d \longrightarrow \mathbb{C}^n/L \longrightarrow \mathbb{C}^q/L_1 \longrightarrow 0. \tag{2}$$

Note that

$$\mathbb{C}^d/\mathbb{Z}^d \simeq \mathbb{G}_m^d$$

by

$$(z_{q+1}, \cdots, z_n) \mapsto (\exp 2\pi\sqrt{-1}z_{q+1}, \cdots, \exp 2\pi\sqrt{-1}z_n),$$

where $(z_1, \cdots, z_n)$ is the standard coordinate system of $\mathbb{C}^n$. By the descriptions in Section 3, the short exact sequence of complex Lie groups (2) is isomorphic to the short exact sequence (1). The description $\mathbb{C}^n/L$ for G is useful for various computations in the following theorems.

We will repeatedly use the following theorem implicitly.

Theorem 4.2. *Let G be a quasi-abelian variety. Assume that $\pi : G' \to G$ is a finite étale morphism. Then G' is a quasi-abelian variety.*

Proof. We use the notation in 4.1. By 4.1, $G = \mathbb{C}^n/L$. Then we have a sublattice L' of L such that $[L : L'] < \infty$ and that $G' = \mathbb{C}^n/L'$. By a translation of G, we may assume that $\pi(0) = 0$. Then we can easily construct a commutative diagram of complex Lie groups:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{G}_m^d \arrow[r] \arrow[d, "\pi_2"] & \mathbb{C}^n/L' \arrow[r] \arrow[d, "\pi"] & \mathbb{C}^q/L'_1 \arrow[r] \arrow[d, "\pi_1"] & 0 \\
0 \arrow[r] & \mathbb{G}_m^d \arrow[r] & \mathbb{C}^n/L \arrow[r] & \mathbb{C}^q/L_1 \arrow[r] & 0
\end{tikzcd}$$

such that π, π_1, and π_2 are finite. Since π_1 is finite, $\mathbb{C}^q/L'_1$ is an abelian variety. Note that $G' = \mathbb{C}^n/L'$ is a principal $\mathbb{G}_m^d$-bundle over $\mathbb{C}^q/L'_1$ as a complex manifold. By the proof of Lemma 3.8, the group law of $G' = \mathbb{C}^n/L'$ as a complex Lie group is algebraic. This means that G' is a quasi-abelian variety and that $\pi : G' \to G$ is a group homomorphism between quasi-abelian varieties. $\square$

Theorem 4.3 ([I1, 10.]). *Let G be a quasi-abelian variety. Then we have $\overline{\kappa}(G) = 0$ and $\overline{q}(G) = \dim G$.*

Proof. Note that G is a principal $\mathbb{G}_m^d$-bundle over an abelian variety $\mathcal{A}$ as a complex manifold. As in the proof of Lemma 3.8, we have a $\mathbb{P}^d$-bundle $\overline{G}$ over $\mathcal{A}$ such that $\overline{G}$ is a $\mathbb{G}_m^d$-equivariant embedding of G over $\mathcal{A}$. We put $D = \overline{G} - G$. Then D is a simple normal crossing divisor on $\overline{G}$. We can easily check that

$$\Omega^1_{\overline{G}}(\log D) \simeq \oplus \mathcal{O}_{\overline{G}}.$$

More explicitly, $\Omega^1_{\overline{G}}(\log D)$ is isomorphic to $\oplus_{i=1}^n \mathcal{O}_{\overline{G}} dz_i$ in the notation of 4.1. Therefore, we obtain that $\overline{q}(G) = \dim G$ and $K_{\overline{G}} + D \sim 0$. In particular, we have $\overline{\kappa}(G) = \kappa(\overline{G}, K_{\overline{G}} + D) = 0$. $\square$

Theorem 4.4 (cf. [I1, Theorem 4.1])**.** *Let G be a quasi-abelian variety. Let W be a closed subvariety of G. Then $\overline{\kappa}(W) \geq 0$. Moreover, $\overline{\kappa}(W) = 0$ if and only if W is a translation of a quasi-abelian subvariety of G.*

Proof. We take a general point $p \in W$, around which we take a system of local analytic coordinates $(\zeta_1, \cdots, \zeta_n)$ such that

$$W = (\zeta_{r+1} = \cdots = \zeta_n = 0).$$

Let $\pi : \mathbb{C}^n \to G$ be the universal cover. We take $q \in \pi^{-1}(p)$ and assume that $z_1(a) = \cdots = z_n(q) = 0$, where $(z_1, \cdots, z_n)$ is a system of global coordinates of $\mathbb{C}^n$. Note that $(\zeta_1, \cdots, \zeta_n)$ can be regarded as a system of local analytic coordinates around q. By taking a suitable linear transformation of $\mathbb{C}^n$, we have

$$\zeta_j = z_j - \varphi_j(z_1, \cdots, z_n),$$

where $\varphi_j(0) = 0$ and

$$\frac{\partial \varphi_j}{\partial z_k}(0) = 0$$

for every j and k around q. The dz_j defines a logarithmic 1-form on G, that is, $dz_j \in T_1(G)$ for every j (see the proof of Theorem 4.3). Let $f : V \to W$ be a resolution and let $\overline{V}$ be a smooth projective variety such that $\Delta = \overline{V} - V$ is a simple normal crossing divisor on $\overline{V}$. Without loss of generality, we may assume that f is an isomorphism over a neighborhood of p. Thus, we see that $f^*(dz_j|_W)$ is an element of $T_1(W)$. Since $d\zeta_1, \cdots, d\zeta_r$ are linearly independent holomorphic 1-forms on W around p, $f^*(dz_1|_W), \cdots, f^*(dz_r|_W)$ are also linearly independent. Thus we have

$$0 \neq f^*(dz_1 \wedge \cdots \wedge dz_r)|_W \in H^0(\overline{V}, \mathcal{O}_{\overline{V}}(K_{\overline{V}} + \Delta)).$$

This means that $\overline{\kappa}(W) \geq 0$. For $r + 1 \leq j \leq n$, we have

$$dz_j|_{\pi^{-1}(W)} - \sum_{k=1}^{n} \left.\frac{\partial \varphi_j}{\partial z_k}\right|_{\pi^{-1}(W)} \cdot dz_k|_{\pi^{-1}(W)} = 0$$

around q. Therefore, we obtain

$$\sum_{k=r+1}^{n} \left(\delta_{jk} - \left.\frac{\partial \varphi_j}{\partial z_k}\right|_{\pi^{-1}(W)}\right) dz_k|_{\pi^{-1}(W)} = \sum_{i=1}^{r} \left.\frac{\partial \varphi_j}{\partial z_i}\right|_{\pi^{-1}(W)} \cdot dz_i|_{\pi^{-1}(W)}$$

in a neighborhood of q. Thus, for $r + 1 \leq j \leq n$, we have

$$dz_j|_W = \sum_{i=1}^{r} A_{ji}(\zeta_1, \cdots, \zeta_r) \cdot dz_i|_W$$

around p, where A_{ji} is a holomorphic function for every i and j such that $A_{ji}(0) = 0$. Note that

$$f^*(dz_1|_W), \cdots, f^*(dz_r|_W) \in T_1(W),$$

which are linearly independent. Assume that $\overline{\kappa}(W) = 0$. Then we have $\kappa(\overline{V}, K_{\overline{V}} + \Delta) = 0$. We note that $f^*(dz_1 \wedge \cdots \wedge dz_r)|_W$ is a nonzero element of $H^0(\overline{V}, \mathcal{O}_{\overline{V}}(K_{\overline{V}} + \Delta))$. Therefore, $H^0(\overline{V}, \mathcal{O}_{\overline{V}}(K_{\overline{V}} + \Delta)) = \mathbb{C}$ is spanned by $f^*(dz_1 \wedge \cdots \wedge dz_r)|_W$. Thus, we obtain

$$f^*(dz_2 \wedge \cdots \wedge dz_r \wedge dz_j)|_W = \alpha_{1j} f^*(dz_1 \wedge \cdots \wedge dz_r)|_W,$$

where $\alpha_{1j} \in \mathbb{C}$ for every j. On the other hand,

$$f^*(dz_2 \wedge \cdots \wedge dz_r \wedge dz_j)|_W = \pm f^* \left(A_{j1}(dz_1 \wedge \cdots \wedge dz_r)|_W\right)$$

over a neighborhood of p. Hence we obtain $\pm f^* A_{j1} = \alpha_{1j}$ for every j. Note that f is an isomorphism over a neighborhood of p. From this, $A_{j1} = 0$ because $A_{j1}(0) = 0$ for every j. By the same arguments, we get $A_{ji} = 0$ for $1 \leq i \leq r$ and every j. Thus, we obtain that

$$dz_{r+1}|_W = \cdots = dz_n|_W = 0$$

around p. This means that

$$\pi^{-1}(W) \subset \{z_{r+1} = \cdots = z_n = 0\}$$

near q. Note that $\{z_{r+1} = \cdots = z_n = 0\}$ is of dimension r and is irreducible. Thus

$$\pi^{-1}(W) = \{z_{r+1} = \cdots = z_n = 0\}.$$

Therefore, W is a quasi-abelian subvariety of G. On the other hand, if W is a translation of a quasi-abelian subvariety of G, then $\overline{\kappa}(W) = 0$ by Theorem 4.3. □

The following theorem is almost obvious by the description in 4.1.

Theorem 4.5. *Let G be a quasi-abelian variety. Then there are at most countably many quasi-abelian subvarieties of G.*

Proof. Let H be a quasi-abelian subvariety of G. Then we obtain

$$\iota : H = \mathbb{C}^{\dim H}/H_1(H, \mathbb{Z}) \hookrightarrow \mathbb{C}^{\dim G}/H_1(G, \mathbb{Z}),$$

where ι is the natural inclusion. Anyway, ι is determined by the subgroup $\mathrm{Im}\iota_*$ of $H_1(G, \mathbb{Z})$, where $\iota_* : H_1(H, \mathbb{Z}) \to H_1(G, \mathbb{Z})$. Therefore, there are at most countably many quasi-abelian subvarieties of G. □

5. Characterizations of abelian varieties and complex tori

In this section, we quickly recall an important property of the Albanese map of varieties of the Kodaira dimension zero for the reader's convenience. It is well known that Kawamata established the following theorem in [K2], which is his doctoral thesis.

Theorem 5.1 (see [K2, Theorem 1]). *Let X be a smooth projective variety with $\kappa(X)=0$. Then the Albanese map*

$$\alpha : X \to A$$

is surjective and has connected fibers.

As an obvious corollary of Theorem 5.1, we obtain a birational characterization of abelian varieties.

Corollary 5.2. *Let X be a smooth projective variety. Then X is birationally equivalent to an abelian variety if and only if the Kodaira dimension $\kappa(X)=0$ and the irregularity $q(X)=\dim X$.*

In this paper, we will use Theorem 5.1 and Corollary 5.2 for the proof of Theorem 1.2. For compact Kähler manifolds, we have:

Theorem 5.3 (see [K2, Theorem 24]). *Let X be a compact Kähler manifold with $\kappa(X)=0$. Then the Albanese map*

$$\alpha : X \to A$$

is surjective and has connected fibers.

Therefore, we have:

Corollary 5.4. *Let X be a compact Kähler manifold. Then X is bimeromorphic to a complex torus if and only if the Kodaira dimension $\kappa(X)=0$ and the irregularity $q(X)=\dim X$.*

Kawamata's original arguments in [K2] heavily depends on the theory of variations of Hodge structure (see Section 7 below). In [EL, Section 2], Ein and Lazarsfeld give a new proof of the above results. Their arguments are based on the generic vanishing theorem due to Green–Lazarsfeld. Anyway, the results in this section can be proved without using [K2] now. Note that Theorem 1.2 is a generalization of Theorem 5.1. We will give a detailed proof of Theorem 1.2 in Section 10 (see Theorem 10.1) following [K2]. The author does not know any proofs of Theorem 1.2 which are independent of Theorem 6.1 below and only depend on the generic vanishing theorem due to Green–Lazarsfeld.

6. On subadditivity of the logarithmic Kodaira dimensions

In this section, we explain some known results on the subadditivity of the logarithmic Kodaira dimensions.

Theorem 6.1. *Let $f : X \to Y$ be a dominant morphism between smooth varieties with irreducible general fibers. Assume that the logarithmic Kodaira dimension $\overline{\kappa}(Y) = \dim Y$. Then we have*

$$\begin{aligned}\overline{\kappa}(X) &= \overline{\kappa}(F) + \overline{\kappa}(Y) \\ &= \overline{\kappa}(F) + \dim Y\end{aligned}$$

where F is a sufficiently general fiber of $f : X \to Y$.

Remark 6.2. Theorem 6.1 is a generalization of [K2, Theorem 30]. In [K2], Kawamata claimed Theorem 6.1 under the extra assumption that $\overline{\kappa}(X) \geq 0$. Theorem 6.1 was first obtained by Maehara (see [Ma, Corollary 2]). Note that the arguments in [K2] and [Ma] heavily depend on [K2, Theorem 32]. Since the author has been unable to follow [K2, Theorem 32], he gave a proof of Theorem 6.1 which is independent of [K2, Theorem 32]. For the details, see [F5, Theorem 1.9] (see also Section 7).

Theorem 6.3 is the main theorem of [K1] (see [K1, Theorem 1]). For the proof, we recommend the reader to see [F1].

Theorem 6.3. *Let $f : X \to Y$ be a dominant morphism between smooth varieties whose general fibers are irreducible curves. Then we have*

$$\overline{\kappa}(X) \geq \overline{\kappa}(F) + \overline{\kappa}(Y)$$

where F is a general fiber of $f : X \to Y$.

Theorem 6.1 and Theorem 6.3 will play crucial roles in Section 9 and Section 10. In general, we have:

Conjecture 6.4. *Let $f : X \to Y$ be a dominant morphism between smooth varieties whose general fibers are irreducible. Then we have*

$$\overline{\kappa}(X) \geq \overline{\kappa}(F) + \overline{\kappa}(Y)$$

where F is a sufficiently general fiber of $f : X \to Y$.

By [F6], we see that Conjecture 6.4 follows from the minimal model program and the abundance conjecture. For the details, see [F6].

7. Remarks on semipositivity theorems

In this section, we make some comments on the semipositivity theorems in [K2] for the reader's convenience. We recommend the reader to skip this section if he is only interested in Theorem 1.2. The arguments in Section 8 are sufficient for the proof of Theorem 1.2 and are more elementary. Let us recall Kawamata's famous result in [K2]. It is one of the main ingredients of Kawamata's proof of Theorem 5.1.

Theorem 7.1 ([K2, Theorem 5=Main Lemma])**.** *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers which satisfies the following conditions:*

- (i) *There is a Zariski open dense subset Y_0 of Y such that $\Sigma = Y - Y_0$ is a simple normal crossing divisor on Y.*
- (ii) *Put $X_0 = f^{-1}(Y_0)$ and $f_0 = f|_{X_0}$. Then f_0 is smooth.*
- (iii) *The local monodromies of $R^n f_{0*}\mathbb{C}_{X_0}$ around Σ are unipotent, where $n = \dim X - \dim Y$.*

Then $f_*\mathcal{O}_X(K_{X/Y})$ is a locally free sheaf and semipositive, where $K_{X/Y} = K_X - f^*K_Y$.

Remark 7.2. In [K2, §4. Semi-positivity (1)], Kawamata proved that $f_*\mathcal{O}_X(K_{X/Y})$ coincides with the canonical extension of the bottom Hodge filtration $\mathcal{F}$. This part was generalized by Nakayama and Kollár independently (see [N, Theorem 1] and [Ko, Theorem 2.6]). They proved that $R^i f_*\mathcal{O}_X(K_{X/Y})$ is locally free and can be characterized as the canonical extension of the bottom Hodge filtration of a suitable variation of Hodge structure.

Remark 7.3. In [K2, §4. Semi-positivity (2)], Kawamata proved that the canonical extension of the bottom Hodge filtration $\mathcal{F}$ is semipositive. This part is not so easy to follow. Note that Kawamata could and did use only [D1], [Gri], and [S] for the Hodge theory when [K2] was written around 1980. Fortunately, [FF, Theorem 1.3] and [FFS, Theorem 3] completely generalize [K2, §4. Semi-positivity (2)] for admissible variations of mixed Hodge structure and clarify Kawamata's proof simultaneously. For Morihiko Saito's comments on Kawamata's arguments in [K2, §4. Semi-positivity (2)], see [FFS, 4.6. Remarks].

Remark 7.4. An approach to the semipositivity of $R^i f_*\mathcal{O}_X(K_{X/Y})$ which dose not use [K2, §4. Semi-positivity (2)] can be found in [F3, Section 4].

Anyway, [K2, Theorem 5] is now clearly understood. Let us go to a mixed generalization of Theorem 7.1, which was used in the proof of Theorem 6.1 in [F5]. In [F2], we obtain:

Theorem 7.5 (see [F2, Theorems 3.1, 3.4, and 3.9])**.** *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties and let D be a simple normal crossing divisor on X such that every stratum of D is dominant onto Y. Let Σ be a simple normal crossing divisor on Y. If f is smooth and D is relatively normal crossing over $Y_0 = Y \setminus \Sigma$ and the local monodromies of $R^{n+i} f_{0*}\mathbb{C}_{X_0 \setminus D_0}$ around Σ are unipotent, where $X_0 = f^{-1}(Y_0)$, $D_0 = D|_{X_0}$, $f_0 = f|_{X_0}$, and $n = \dim X - \dim Y$, then $R^i f_* \mathcal{O}_X(K_{X/Y} + D)$ is locally free and semipositive.*

Theorem 7.5 is obviously a generalization of Theorem 7.1.

Remark 7.6. In [F2], we characterize $R^i f_* \mathcal{O}_X(K_{X/Y} + D)$ as the canonical extension of the bottom Hodge filtration of a suitable variation of mixed Hodge structure. The proof of the semipositivity of $R^i f_* \mathcal{O}_X(K_{X/Y} + D)$ in [F2] used [K2, §4. Semi-positivity (2)]. Now we can use [FF, Theorem 1.3] or [FFS, Theorem 3] for the semipositivity of $R^i f_* \mathcal{O}_X(K_{X/Y} + D)$ in place of [K2, §4. Semi-positivity (2)].

Remark 7.7. As we pointed out in [F5, Remark 6.5], Kawamata seems to misuse Schmid's nilpotent orbit theorem in [K3] and [K4]. Therefore, we do not use the papers [K3] and [K4]. Moreover, the main theorem of [K3] (see [K3, Theorem 1.1]) is weaker than [FF, Theorem 1.1].

Remark 7.8. The main theorem in [K3] (see [K3, Theorem 1.1]) does not cover Theorem 7.5 nor [K2, Theorem 32]. We also note that [K5] does not cover Theorem 7.5 nor [K2, Theorem 32]. In [K5], Kawamata treats *well prepared fiber spaces*. For the details, see [K5]. The author did not find any proofs of [K2, Theorem 32] except the original one in [K2].

8. Weak positivity theorems revisited

In this section, we explain how to avoid using the theory of variations of mixed Hodge structure for the proof of Theorem 6.1. Let us recall the definition of weakly positive sheaves. Note that the theory of weakly positive sheaves is due to Viehweg. Roughly speaking, Viehweg treated only the *pure* case. For the details of the *mixed* case, we recommend the reader to see [F5].

Definition 8.1 (Weak positivity)**.** Let W be a smooth projective variety and let $\mathcal{F}$ be a torsion-free coherent sheaf on W. We call $\mathcal{F}$ *weakly positive*, if for every ample line bundle $\mathcal{H}$ on W and every positive integer α there exists some positive integer β such that $\widehat{S}^{\alpha\beta}(\mathcal{F}) \otimes \mathcal{H}^{\otimes\beta}$ is generically generated by global sections. This means that the natural map

$$H^0(W, \widehat{S}^{\alpha\beta}(\mathcal{F}) \otimes \mathcal{H}^{\otimes\beta}) \otimes \mathcal{O}_W \to \widehat{S}^{\alpha\beta}(\mathcal{F}) \otimes \mathcal{H}^{\otimes\beta}$$

is generically surjective.

Remark 8.2. In Definition 8.1, let $\widehat{W}$ be the largest Zariski open subset of W such that $\mathcal{F}|_{\widehat{W}}$ is locally free. Then we put

$$\widehat{S}^k(\mathcal{F}) = i_*S^k(i^*\mathcal{F})$$

where $i : \widehat{W} \to W$ is the natural open immersion and S^k denotes the k-th symmetric product. Note that $\mathrm{codim}_W(W \setminus \widehat{W}) \geq 2$ since $\mathcal{F}$ is torsion-free.

The following theorem, which is due to Viehweg, Campana, and others, is useful and is very important.

Theorem 8.3 (Twisted weak positivity, see [F5, Theorem 1.1])**.** *Let X be a normal projective variety and let Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is log canonical. Let $f : X \to Y$ be a surjective morphism onto a smooth projective variety Y with connected fibers. Assume that $k(K_X + \Delta)$ is Cartier. Then, for every positive integer m,*

$$f_*\mathcal{O}_X(mk(K_{X/Y} + \Delta))$$

is weakly positive.

Once we establish Theorem 8.3, we can prove Theorem 6.1 without any difficulties. For the details, see [F5, Section 10]. Theorem 8.3 is sufficient for [F5, Section 9 and Section 10]. A key ingredient of Theorem 8.3 is the following result.

Theorem 8.4 (see [F5, Corollary 7.11])**.** *Let $f : V \to W$ be a surjective morphism between smooth projective varieties. Let D be a simple normal crossing divisor on V. Then*

$$f_*\mathcal{O}_V(K_{V/W} + D)$$

is weakly positive.

By Theorem 8.4, the arguments in [F5, Section 8] work without any modifications and produce Theorem 8.3. We recommend the reader to see [F5, Section 8]. In [F5, Section 7], we give a proof of Theorem 8.4 based on the theory of variations of mixed Hodge structure (cf. Theorem 7.5). Here, we give a more elementary proof based on the following easy observation.

Lemma 8.5. *Let $f : X \to Y$ be a surjective morphism from a smooth projective variety X to a projective variety Y and let D be a simple normal crossing divisor on X. Let $\mathcal{A}$ be an ample line bundle on Y*

such that $|\mathcal{A}|$ *is free and let* $\mathcal{B}$ *be a line bundle on* Y *such that* $\mathcal{A}^{\otimes a}\otimes\mathcal{B}$ *is nef for some positive integer* a*. Then*

$$R^if_*\mathcal{O}_X(K_X+D)\otimes\mathcal{B}\otimes\mathcal{A}^{\otimes m}$$

is generated by global sections for every i *and every positive integer* $m\geq\dim Y+1+a$*.*

Proof of Lemma 8.5. By [F2, Theorem 2.6] (see also [F4, Theorem 6.3 (ii)]), we obtain that

$$H^p(Y, R^if_*\mathcal{O}_X(K_X+D)\otimes\mathcal{B}\otimes\mathcal{A}^{\otimes a}\otimes\mathcal{A}^{m-a-p})=0$$

for $p>0$. By Mumford's regularity, we see that $R^if_*\mathcal{O}_X(K_X+D)\otimes\mathcal{B}\otimes\mathcal{A}^{\otimes m}$ is generated by global sections for every i and $m\geq\dim Y+1+a$. □

Let us start the proof of Theorem 8.4.

Proof of Theorem 8.4. In Step 1, we reduce the problem to a simpler case. In Step 2, we use Viehweg's clever trick and obtain the desired weak positivity.

Step 1. By replacing D with its horizontal part, we may assume that every irreducible component of D is dominant onto W (see [F5, Lemma 7.7]). If there is a log canonical center C of (V,D) such that $f(C)\subsetneq W$, then we take the blow-up $h:V'\to V$ along C. We put

$$K_{V'}+D'=h^*(K_V+D).$$

Then D' is a simple normal crossing divisor on V' and

$$f_*\mathcal{O}_V(K_{V/W}+D)\simeq(f\circ h)_*\mathcal{O}_{V'}(K_{V'/W}+D').$$

Therefore, we can replace (V,D) with (V',D'). Then we replace D with its horizontal part (see [F5, Lemma 7.7]). By repeating this process finitely many times, we may assume that every stratum of D is dominant onto W. Now we take a closed subset Σ of W such that f is smooth over $W\setminus\Sigma$ and that D is relatively normal crossing over $W\setminus\Sigma$. Let $g:W'\to W$ be a birational morphism from a smooth projective variety W' such that $\Sigma'=g^{-1}(\Sigma)$ is a simple normal crossing divisor. By taking some suitable blow-ups of V in $f^{-1}(\Sigma)$ and replacing D with its strict transform, we may further assume the following conditions:

(i) $f'=g^{-1}\circ f:V\to W'$ is a morphism,
(ii) f' is smooth over $W'\setminus\Sigma'$ and D is relatively normal crossing over $W'\setminus\Sigma'$, and
(iii) every irreducible component of D is dominant onto W and $\mathrm{Supp}(f'^*\Sigma'+D)$ is a simple normal crossing divisor on V.

$$\begin{tikzcd} V \arrow[d, "f'"'] \arrow[dr, "f"] & \\ W' \arrow[r, "g"'] & W \end{tikzcd}$$

Here we used Szabó's resolution lemma. We assume that $f'_*\mathcal{O}_V(K_{V/W'}+D)$ is weakly positive. Note that

$$f'_*\mathcal{O}_V(K_{V/W}+D) \simeq f'_*\mathcal{O}_V(K_{V/W'}+D)\otimes \mathcal{O}_{W'}(E)$$

where E is a g-exceptional effective divisor such that $K_{W'}=g^*K_W+E$. Thus $f'_*\mathcal{O}_V(K_{V/W}+D)$ is weakly positive. We note that

$$g_*f'_*\mathcal{O}_V(K_{V/W}+D)\simeq f_*\mathcal{O}_V(K_{V/W}+D).$$

We can take an effective g-exceptional divisor F on W' such that $-F$ is g-ample. Let H be an ample Cartier divisor on W. Then there exists a positive integer k such that kg^*H-F is ample. Let α be a positive integer. Since $f'_*\mathcal{O}_V(K_{V/W}+D)$ is weakly positive,

$$\widehat{S}^{k\alpha\beta}(f'_*\mathcal{O}_V(K_{V/W}+D))\otimes\mathcal{O}_{W'}(\beta(kg^*H-F))$$

is generically generated by global sections for some positive integer β. By taking g_*,

$$\widehat{S}^{\alpha k\beta}(f_*\mathcal{O}_V(K_{V/W}+D))\otimes\mathcal{O}_W(k\beta H)$$

is generically generated by global sections. This means that $f_*\mathcal{O}_V(K_{V/W}+D)$ is weakly positive. Therefore, all we have to do is to prove that $f'_*\mathcal{O}_V(K_{V/W'}+D)$ is weakly positive.

Step 2. By replacing W with W', we may assume that $W'=W$. Let s be an arbitrary positive integer. We take the s-times fiber product

$$V^s=V\times_W V\times_W\cdots\times_W V.$$

We put $f^s: V^s\to W$. Let $p_i: V^s\to V$ be the i-th projection for $1\leq i\leq s$. Let $W^\dagger$ be a Zariski open set of W such that f is flat over $W^\dagger$ and that $\operatorname{codim}_W(W\setminus W^\dagger)\geq 2$. We may assume that $W_0=W\setminus\Sigma\subset W^\dagger\subset W$. We put $V^\dagger=f^{-1}(W^\dagger)$. By the flat base change theorem (see, for example, [Mo, Section 4]), we obtain an isomorphism

$$f^s_*\omega_{V^{\dagger s}/W^\dagger}\simeq\bigotimes_{i=1}^{s} f_*\omega_{V^\dagger/W^\dagger},$$

where $V^{\dagger s}$ is the s-times fiber product

$$V^\dagger\times_{W^\dagger}\cdots\times_{W^\dagger}V^\dagger.$$

We put $D^s=\sum_{i=1}^s p_i^*D$. Then, by the same argument, we have an isomorphism

$$f^s_*(\omega_{V^{\dagger s}/W^\dagger}\otimes \mathcal{O}_{V^{\dagger s}}(D^s))\simeq \bigotimes_{i=1}^{s} f_*(\omega_{V^\dagger/W^\dagger}\otimes \mathcal{O}_{V^\dagger}(D)). \tag{A}$$

Let $\pi: V^{(s)}\to V^s$ be a resolution such that π is an isomorphism over $(f^s)^{-1}(W_0)$ with the following properties:

(i) $V^{(s)}$ is a smooth projective variety,
(ii) $f^{(s)}=f^s\circ\pi: V^{(s)}\to W$ is smooth over W_0,
(iii) $D^{(s)}$ is a simple normal crossing divisor on $V^{(s)}$,
(iv) $\mathrm{Supp}(D^{(s)}+(f^{(s)})^*\Sigma)$ is a simple normal crossing divisor on $V^{(s)}$,
(v) every irreducible component of $D^{(s)}$ is dominant onto W, and
(vi) $D^{(s)}$ coincides with D^s over W_0.

Note that $V^{\dagger s}$ is Gorenstein. We have

$$\pi_*\mathcal{O}_{V^{\dagger(s)}}(K_{V^{\dagger(s)}})\subset \omega_{V^{\dagger s}},$$

where $V^{\dagger(s)}=\pi^{-1}(V^{\dagger s})$. Therefore, we obtain

$$\pi_*\mathcal{O}_{V^{\dagger(s)}}(K_{V^{\dagger(s)}}+D^{(s)}-\pi^*D^s)\subset \omega_{V^{\dagger s}} \tag{B}$$

since $D^{(s)}-\pi^*D^s\leq 0$. Thus we have a natural inclusion

$$f^{(s)}_*\mathcal{O}_{V^{(s)}}(K_{V^{(s)}/W}+D^{(s)})\hookrightarrow \left(\bigotimes_{i=1}^{s} f_*\mathcal{O}_V(K_{V/W}+D)\right)^{**}$$

which is an isomorphism over W_0 by (A) and (B). Let $\mathcal{H}$ be an ample line bundle on W. Then

$$f^{(s)}_*\mathcal{O}_{V^{(s)}}(K_{V^{(s)}/W}+D^{(s)})\otimes\mathcal{H}^{\otimes m}$$

is generated by global sections for every positive integer s and for every $m\geq b(\dim W+1)+a$, where a is a positive integer such that $\mathcal{O}_W(-K_W)\otimes\mathcal{H}^{\otimes a}$ is nef and b is a positive integer such that $|\mathcal{H}^{\otimes b}|$ is free by Lemma 8.5. Therefore, we obtain that

$$\left(\bigotimes_{i=1}^{s} f_*\mathcal{O}_V(K_{V/W}+D)\right)^{**}\otimes\mathcal{H}^{\otimes m}$$

is generated by global sections over W_0, where s and m are as above. This means that

$$\widehat{S}^{\alpha\beta}(f_*\mathcal{O}_V(K_{V/W}+D))\otimes\mathcal{H}^{\otimes\beta}$$

is generated by global sections over W_0 for every $\alpha\geq 1$ and $\beta\geq b(\dim W+1)+a$. Therefore, $f_*\mathcal{O}_V(K_{V/W}+D)$ is weakly positive.

We complete the proof of Theorem 8.4. □

Remark 8.6. Note that $f_*\mathcal{O}_V(K_{V/W}+D)$ is locally free in Step 2 in the proof of Theorem 8.4. This is because $f_*\mathcal{O}_V(K_{V/W}+D)$ is the upper canonical extension of the bottom Hodge filtration of a suitable variation of mixed Hodge structure (cf. Theorem 7.5).

Anyway, by this section, Theorem 6.1 is now released from the deep results of the theory of variations of mixed Hodge structure. This means that Theorem 1.2 is also independent of the theory of variations of mixed Hodge structure.

9. Finite covers of quasi-abelian varieties

In this section, we discuss finite covers of abelian and quasi-abelian varieties. Let us start with the following well known theorem due to Kawamata–Viehweg.

Theorem 9.1 (see [KV, Main Theorem] and [K2, Theorem 4])**.** *Let $f: X \to A$ be a finite surjective morphism from a normal complete variety X to an abelian variety A. Assume that the Kodaira dimension $\kappa(X)$ of X is zero. Then f is an étale morphism.*

Proof. Let $\pi: \widetilde{X} \to X$ be a resolution of singularities from a smooth projective variety $\widetilde{X}$. Then $q(\widetilde{X}) \geq \dim \widetilde{X}$ since $f \circ \pi : \widetilde{X} \to A$ is surjective. Therefore, $\widetilde{X}$ is birationally equivalent to an abelian variety by Theorem 5.1 and Corollary 5.2 since $\kappa(\widetilde{X}) = 0$. We consider the following commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\alpha"] \arrow[d, "\alpha_{\widetilde{X}}"'] & A \\ \mathcal{A}_{\widetilde{X}} \arrow[ur, "g"'] & \end{tikzcd}$$

where $\alpha_{\widetilde{X}} : \widetilde{X} \to \mathcal{A}_{\widetilde{X}}$ is the Albanese map of $\widetilde{X}$. Of course, $\alpha_{\widetilde{X}}$ is birational and g is a finite étale morphism between abelian varieties. Note that both X and $\mathcal{A}_{\widetilde{X}}$ are the normalization of A in $\mathbb{C}(\widetilde{X})$. Therefore, X is isomorphic to $\mathcal{A}_{\widetilde{X}}$ over A. This means that $f: X \to A$ is an étale morphism. □

Remark 9.2. Kawamata's original proof of Theorem 5.1, which is [K2, Theorem 1], in [K2] uses Theorem 9.1 (see [K2, Theorem 4] and [KV, Main Theorem]). However, Ein–Lazarsfeld's approach in [EL, Section 2] does not need Theorem 9.1 (see [K2, Theorem 4] and [KV, Main Theorem]) for the proof of Theorem 5.1 (see [K2, Theorem 1]).

Therefore, there are no problems if we use Theorem 5.1 for the proof of Theorem 9.1.

We can generalize Theorem 9.1 as follows. Theorem 9.3 is nothing but [K2, Theorem 26].

Theorem 9.3 (Finite covers of quasi-abelian varieties)**.** *Let $f : X \to A$ be a finite surjective morphism from a normal variety X to a quasi-abelian variety A. Assume that the logarithmic Kodaira dimension $\overline{\kappa}(X)$ of X is zero. Then f is an étale morphism.*

Proof. Let

$$0 \longrightarrow \mathcal{G}_A \longrightarrow A \longrightarrow \mathcal{A}_A \longrightarrow 0$$

be the Chevalley decomposition. We will prove that f is étale by induction on $d = \dim \mathcal{G}_A$. If $d = 0$, then it is Theorem 9.1. So, we assume that $d > 0$. We take a subgroup

$$G_1 = \mathbb{G}_m \times \{1\} \times \cdots \times \{1\} \subset \mathbb{G}_m^d = \mathcal{G}_A.$$

We consider

$$0 \longrightarrow G_1 \longrightarrow A \xrightarrow{\pi_1} A_1 \longrightarrow 0.$$

Note that A is a principal G_1-bundle over A_1 as a complex manifold. We have a comactification $\overline{\pi_1} : \overline{A} \to \overline{A_1}$ of $\pi_1 : A \to A_1$, where $\overline{A}$ is a $(\mathbb{P}^1)^d$-bundle over $\mathcal{A}_A$ and $\overline{A_1}$ is a $(\mathbb{P}^1)^{d-1}$-bundle over $\mathcal{A}_A$ as in the proof of Lemma 3.10. Let $\overline{X}$ be the normalization of $\overline{A}$ in $\mathbb{C}(X)$ and $\overline{f} : \overline{X} \to \overline{A}$ is the natural map. Let $\overline{X} \to \overline{X_1} \to \overline{A_1}$ be the Stein factorization of $\overline{\pi_1} \circ \overline{f} : \overline{X} \to \overline{A_1}$. Then we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "p_1"] \arrow[d, "f"'] & X_1 \arrow[d, "f_1"] \\ A \arrow[r, "\pi_1"'] & A_1, \end{tikzcd}$$

where $f_1 : X_1 \to A_1$ is a finite morphism from a normal variety X_1. Since f_1 is finite and A_1 is a quasi-abelian variety, we have $\overline{\kappa}(X_1) \geq 0$. On the other hand, by Theorem 6.3, we have

$$0 = \overline{\kappa}(X) \geq \overline{\kappa}(F) + \overline{\kappa}(X_1),$$

where F is a general fiber of p_1. Note that $\overline{\kappa}(F) \geq 0$ since $\overline{\kappa}(X) = 0$. Therefore, we obtain $\overline{\kappa}(X_1) = \overline{\kappa}(F) = 0$. By induction on d, f_1 is étale. By replacing A_1 (resp. A) with X_1 (resp. $A \times_{A_1} X_1$), we may assume that f_1 is the identity.

$$\begin{tikzcd} X \arrow[r, "p_1"] \arrow[d, "f"'] & A_1 \arrow[d, equal] \\ A \arrow[r, "\pi_1"'] & A_1, \end{tikzcd}$$

Let x be a general point of A_1. Then

$$f|_{X_x} : X_x \simeq \mathbb{G}_m \to A_x \simeq \mathbb{G}_m$$

is étale. We put $e = \deg f$. By construction, there are prime divisors H_1 and H_2 on $\overline{A}$ such that $H_1, H_2 \subset \overline{A} \setminus A$, $H_1 \sim_{\overline{\pi_1}} H_2$, $H_1 \neq H_2$, and H_i is a section of $\overline{\pi_1}$ for $i = 1, 2$. We can take a nonempty Zariski open set U of $\overline{A_1}$ such that

(i) $\overline{p_1} : \overline{X} \to \overline{A_1}$ is smooth over U.
(ii) every fiber of $\overline{p_1}$ is $\mathbb{P}^1$ over U.
(iii) there are prime divisors D_1 and D_2 on $\overline{X}$ such that $D_1, D_2 \subset \overline{X} \setminus X$, $D_1 \sim D_2$ over U, $D_1 \neq D_2$, and D_i is a section of $\overline{p_1} : \overline{X} \to \overline{A_1}$ over U for $i = 1, 2$.
(iv) $\overline{f}^* H_i = eD_i$ over U for $i = 1, 2$.

Therefore, we see that $f : X \to A$ is

$$\mathbb{G}_m \times U \to \mathbb{G}_m \times U$$

given by

$$(a, b) \mapsto (a^e, b)$$

over U. On the other hand, we can construct a quasi-abelian variety A' such that

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "h"'] & A_1 \arrow[d, equal] \\ A \arrow[r, "\pi_1"'] & A_1, \end{tikzcd}$$

where h is étale with $\deg h = e$ (see the description of quasi-abelian varieties in 4.1) and that $h : A' \to A_1$ is

$$\mathbb{G}_m \times U \to \mathbb{G}_m \times U$$

given by

$$(a, b) \mapsto (a^e, b)$$

over U, that is, h coincides with f over U. Note that X is normal and both f and h are finite. Thus X is isomorphic to A' over A. Hence, we obtain that $f : X \to A$ is étale. $\square$

We will use Theorem 9.3 in the proof of Theorem 1.2 (see the proof of Theorem 10.1 in Section 10).

10. Quasi-Albanese maps for varieties with $\overline{\kappa}=0$

In this final section, we give a detailed proof of Kawamata's theorem on quasi-Albanese maps for varieties with $\overline{\kappa}=0$.

Theorem 10.1 (see [K2, Theorem 28]). *Let X be a smooth variety such that the logarithmic Kodaira dimension $\overline{\kappa}(X)$ of X is zero. Then the quasi-Albanese map $\alpha: X\to A$ is dominant and has irreducible general fibers.*

As an easy consequence of Theorem 10.1, we have:

Corollary 10.2 (see [K2, Corollary 29] and [I3, Theorem I]). *Let X be a smooth variety such that the logarithmic Kodaira dimension $\overline{\kappa}(X)$ of X is zero. Then we have $\overline{q}(X)\leq \dim X$, where $\overline{q}(X)$ is the logarithmic irregularity of X. Moreover, the equality holds if and only if the quasi-Albanese map $\alpha: X\to A$ is birational.*

Before we prove Theorem 10.1, we have to prove an important lemma.

Lemma 10.3 ([K2, Theorem 27]). *Let X be a normal algebraic variety, let A be a quasi-abelian variety, and let $f: X\to A$ be a finite morphism. Then $\overline{\kappa}(X)\geq 0$ and there are a quasi-abelian subvariety B of A, finite étale covers $\widetilde{X}$ and $\widetilde{B}$ of X and B respectively, and a normal algebraic variety $\widetilde{Y}$ such that:*

(i) *$\widetilde{Y}$ is finite over A/B.*
(ii) *$\widetilde{X}$ is a principal $\widetilde{B}$-bundle over $\widetilde{Y}$ as a complex manifold.*
(iii) *$\overline{\kappa}(\widetilde{Y})=\dim \widetilde{Y}=\overline{\kappa}(X)$.*

In [K2], Kawamata claims this statement without proof. So we give a detailed proof for the reader's convenience.

Proof of Lemma 10.3. We divide the proof into several steps.

Step 1. Let

$$\begin{tikzcd} Z \arrow[r, "\Phi"] \arrow[d, "g"'] & Y \\ X \arrow[ur, dashed] \arrow[d, "f"'] & \\ A & \end{tikzcd}$$

be the logarithmic Iitaka fibration of X, that is, we take a smooth complete variety $\overline{X}$ such that $D=\overline{X}-X$ is a simple normal crossing divisor, $\overline{X}\dashrightarrow Y$ is a rational map to a normal projective variety Y associated to $|m(K_{\overline{X}}+D)|$ for a sufficiently large and divisible positive integer m, and $g : Z \to X$ is an elimination of indeterminacy of $X \dashrightarrow Y$. Let y be a sufficiently general point of Y. Then we have $\overline{\kappa}(Z_y) = 0$. Since $f \circ g : Z_y \to f \circ g(Z_y)$ is generically finite and $f \circ g(Z_y) \subset A$, $f \circ g(Z_y)$ is a translation of a quasi-abelian subvariety B_y of A by Theorem 4.4. By Theorem 9.3, Z'_y is a quasi-abelian variety where $Z_y \to Z'_y \to B_y$ is the Stein factorization. Let y be a general point of Y. Note that

$$H_1(Z_y, \mathbb{Z}) \to H_1(A, \mathbb{Z})$$

does not depend on y by discreteness. Therefore, the image of Z_y by $f \circ g$ in A does not depend on y up to translation by the following commutative diagram

$$\begin{tikzcd} Z_y \arrow[d] \arrow[dr, "f \circ g"] & \\ \mathcal{A}_{Z_y} \arrow[r] & A \end{tikzcd}$$

where $Z_y \to \mathcal{A}_{Z_y}$ is the quasi-Albanese map. Therefore, we obtain a quasi-abelian subvariety B of A such that $B_y = B$ for sufficiently general $y \in Y$. Moreover, we have an étale cover $\widetilde{B}$ of B such that $Z'_y = \widetilde{B}$ for sufficiently general $y \in Y$.

Step 2. In this step, we prove the following lemma.

Lemma 10.4. *Let A be a quasi-abelian variety and let B be a quasi-abelian subvariety of A. Let $B^\dagger \to B$ be a finite étale cover. Then we can construct a finite étale cover $A^\dagger \to A$ such that $B^\dagger$ is a quasi-abelian subvariety of $A^\dagger$ satisfying*

$$\begin{tikzcd} B^\dagger \arrow[r, hook] \arrow[d] & A^\dagger \arrow[d] \\ B \arrow[r, hook] & A. \end{tikzcd}$$

Proof of Lemma 10.4. We consider the Chevalley decompositions:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G}_B \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & \mathcal{A}_B \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{G}_A \arrow[r] & A \arrow[r] & \mathcal{A}_A \arrow[r] & 0 \end{tikzcd}$$

and

$$0 \longrightarrow \mathcal{G}_{B^\dagger} \longrightarrow B^\dagger \longrightarrow \mathcal{A}_{B^\dagger} \longrightarrow 0.$$

By Poincaré reducibility (see, for example, [Mu]), we have an étale morphism

$$a : \mathcal{A}_{B^\dagger} \times \mathcal{A}' \to \mathcal{A}_A$$

for some abelian variety $\mathcal{A}'$. By taking the base change of $A \to \mathcal{A}_A$ by a, we obtain an étale cover $A_1 \to A$ and

$$\begin{tikzcd} B^\dagger \arrow[r] \arrow[d] & A_1 \arrow[d] \\ B \arrow[r, hook] & A. \end{tikzcd}$$

We note the Chevalley decompositions:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G}_{B^\dagger} \arrow[r] \arrow[d] & B^\dagger \arrow[r] \arrow[d] & \mathcal{A}_{B^\dagger} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{G}_{A_1} \arrow[r] & A_1 \arrow[r] & \mathcal{A}_{B^\dagger} \times \mathcal{A}' \arrow[r] & 0. \end{tikzcd}$$

By replacing the lattice corresponding to $\mathcal{G}_{A_1}$ with a suitable sublattice, we can construct a finite étale morphism

$$A^\dagger \to A_1$$

over $\mathcal{A}_{B^\dagger} \times \mathcal{A}'$ such that

$$\begin{tikzcd} B^\dagger \arrow[r, hook] \arrow[d] & A^\dagger \arrow[d] \\ B \arrow[r, hook] & A \end{tikzcd}$$

(see the description of quasi-abelian varieties in 4.1). We obtain a desired finite étale cover $A^\dagger \to A$. $\square$

Step 3. By Lemma 10.4, we take a finite étale cover $\widetilde{A} \to A$ such that

$$\begin{tikzcd} \widetilde{B} \arrow[r, hook] \arrow[d] & \widetilde{A} \arrow[d] \\ B \arrow[r, hook] & A. \end{tikzcd}$$

By taking base changes, we obtain $\widetilde{X}$, $\widetilde{Z}$, and the following commutative diagram:

$$\begin{tikzcd} \widetilde{B} \arrow[r, hook] \arrow[d] & \widetilde{A} \arrow[d] & \widetilde{X} \arrow[l, "\widetilde{f}"'] \arrow[d] & \widetilde{Z} \arrow[l, "\widetilde{g}"'] \arrow[d] \\ B \arrow[r, hook] & A & X \arrow[l, "f"] & Z. \arrow[l, "g"] \end{tikzcd}$$

By construction, we can construct a logarithmic Iitaka fibration $\widetilde{\Phi} : \widetilde{Z} \to Y'$ such that $\widetilde{g} \circ \widetilde{f}(\widetilde{Z}_y)$ is a translation of $\widetilde{B}$ for every sufficiently general $y \in Y'$. More explicitly, we can construct $\widetilde{\Phi} : \widetilde{Z} \to Y'$ as follows. Without loss of generality, we may assume that there are a smooth projective variety $\overline{Z}$ such that $\overline{Z}-Z$ is a simple normal crossing divisor on $\overline{Z}$ and a morphism $a : \overline{Z} \to Y$ such that $\Phi = a|_Z : Z \to Y$ in Step 1. Let $Z^\dagger$ be the normalization of $\overline{Z}$ in $\mathbb{C}(\widetilde{Z})$ and let $b : Z^\dagger \to \overline{Z}$ be the natural map. Let H be a very ample Cartier divisor on Y. We consider $\Phi_{|mb^*a^*H|} : Z^\dagger \to Y'$ for a sufficiently large and divisible positive integer m. We put $\widetilde{\Phi} := \Phi_{|mb^*a^*H|}|_{\widetilde{Z}} : \widetilde{Z} \to Y'$. Then this $\widetilde{\Phi} : \widetilde{Z} \to Y'$ is the desired logarithmic Iitaka fibration. Therefore, there is a rational map $Y' \dashrightarrow \widetilde{A}/\widetilde{B}$ such that

$$\begin{tikzcd}
\widetilde{Z} \arrow[r, "\widetilde{\Phi}"] \arrow[d, "\widetilde{f}\circ\widetilde{g}"'] & Y' \arrow[d, dashed] \\
\widetilde{A} \arrow[r] & \widetilde{A}/\widetilde{B}
\end{tikzcd}$$

(see, for example, [K2, Lemma 14]). Let $\widetilde{Y}$ be the normalization of $\widetilde{A}/\widetilde{B}$ in $\mathbb{C}(Y')$. We put $X' = \widetilde{A} \times_{\widetilde{A}/\widetilde{B}} \widetilde{Y}$. Then X' is normal and is birationally equivalent to $\widetilde{X}$. We note that $\widetilde{X}$ and X' are both finite over $\widetilde{A}$. Thus X' is isomorphic to $\widetilde{X}$ over $\widetilde{A}$. We also note that $\widetilde{Y}$ is finite over A/B since $\widetilde{A}/\widetilde{B}$ is finite over A/B. By construction, $\widetilde{X}$ is a principal $\widetilde{B}$-bundle over $\widetilde{Y}$.

Step 4. All we have to show is $\overline{\kappa}(\widetilde{Y}) = \dim \widetilde{Y} = \overline{\kappa}(X)$. Since $\widetilde{Y}$ is finite over $\widetilde{A}/\widetilde{B}$, we have $\overline{\kappa}(\widetilde{Y}) \geq 0$. We assume that $\overline{\kappa}(Y) < \dim \widetilde{Y}$. By applying the results obtained in Steps 1 and 3 to $\widetilde{Y} \to A/B$, we obtain an étale cover $\widetilde{Y}'$ with the following commutative diagram:

$$\begin{tikzcd}
\widetilde{X}' = \widetilde{X} \times_{\widetilde{Y}} \widetilde{Y}' \arrow[r] \arrow[d] & \widetilde{Y}' \arrow[r] \arrow[d] & W \arrow[dd] \\
\widetilde{X} \arrow[r] \arrow[d] & \widetilde{Y} \arrow[d] & \\
A \arrow[r] & A/B \arrow[r] & A/C,
\end{tikzcd}$$

where C is a quasi-abelian subvariety of A such that $B \subset C$. Note that W is finite over A/C and that $\dim W = \overline{\kappa}(Y)$. We can easily see that $\widetilde{X}'$ is a principal G-bundle for some quasi-abelian variety G and

$$\overline{\kappa}(\widetilde{X}') = \overline{\kappa}(\widetilde{X}) = \overline{\kappa}(X).$$

By the easy addition formula, we obtain

$$\overline{\kappa}(\widetilde{X}') \leq \dim W < \dim Y = \overline{\kappa}(X).$$

This is a contradiction. Therefore, we have $\dim \widetilde{Y} = \overline{\kappa}(\overline{Y})$.

We have desired $\widetilde{X}$, $\widetilde{B}$, and $\widetilde{Y}$. □

Let us start the proof of Theorem 10.1.

Proof of Theorem 10.1. By using the Stein factorization, we obtain

$$\alpha : X \xrightarrow{q} Z \xrightarrow{p} A$$

where q is dominant, q has irreducible general fibers, p is finite, and Z is normal. It is sufficient to prove that p is an isomorphism. We assume that $\overline{\kappa}(Z) > 0$. Then, by Lemma 10.3, we obtain an étale cover $\widetilde{Z} \to Z$ such that $\widetilde{Z} \to W$ is a principal G-bundle for some quasi-abelian variety G with $\overline{\kappa}(W) = \dim W = \overline{\kappa}(Z) > 0$. We consider $r : \widetilde{X} = X \times_Z \widetilde{Z} \to \widetilde{Z} \to W$. Since $\overline{\kappa}(\widetilde{X}) = \overline{\kappa}(X) = 0$, $\overline{\kappa}(F) \geq 0$ for a sufficiently general fiber F of r. By Theorem 6.1, we obtain

$$0 = \overline{\kappa}(X) = \overline{\kappa}(\widetilde{X}) \geq \overline{\kappa}(W) + \overline{\kappa}(F) \geq \overline{\kappa}(Z) > 0.$$

This is a contradiction. Therefore, we obtain $\overline{\kappa}(Z) = 0$. By Theorem 9.3, we obtain that $p : Z \to A$ is étale. In particular, Z is a quasi-abelian variety. This means that p is an isomorphism since $\alpha : X \to A$ is a quasi-Albanese map of X. This means that $\alpha : X \to A$ is dominant and has irreducible general fibers. □

We close this paper with the proof of Corollary 10.2.

Proof of Corollary 10.2. Let $\alpha : X \to A$ be a quasi-Albanese map. By Theorem 10.1, α is surjective. Note that $\dim A = \overline{q}(X)$. Therefore, we have $\overline{q}(X) \leq \dim X$. By Theorem 10.1, the general fibers of α are irreducible. Thus, α is birational if and only if $\dim X = \dim A = \overline{q}(X)$. □

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Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan

E-mail address: `fujino@math.kyoto-u.ac.jp`