ON STRONG $\mathbb{Q}$-FACTORIALITY

OSAMU FUJINO

1. Strong $\mathbb{Q}$-factoriality

Definition 1.1 (Divisorial sheaves)**.** Let X be a normal complex variety. A coherent sheaf $\mathcal{L}$ on X is said to be *divisorial* if it is reflexive of rank one.

Lemma 1.2. *Let $\mathcal{L}$ be a divisorial sheaf on a smooth complex variety. Then $\mathcal{L}$ is a line bundle, that is, $\mathcal{L}$ is invertible.*

Proof. This fact is very well known. For the proof, see, for example, [OSS, Chapter 2, Lemma 1.1.15]. □

Following [DH], we define *strongly $\mathbb{Q}$-factorial* normal complex varieties.

Definition 1.3 (Strong $\mathbb{Q}$-factoriality)**.** Let $\mathcal{L}$ be a divisorial sheaf on a normal complex variety X. Let x be a point of X. If there exists a positive integer m_x such that

$$\mathcal{L}^{[m_x]} := \left(\mathcal{L}^{\otimes m_x}\right)^{**}$$

is a line bundle on some open neighborhood of $x \in X$, then $\mathcal{L}$ is said to be *$\mathbb{Q}$-Cartier at x*. Let K be a subset of X. If $\mathcal{L}$ is $\mathbb{Q}$-Cartier at every $x \in K$, then $\mathcal{L}$ is said to be *$\mathbb{Q}$-Cartier at K*. When $K = X$, we simply say that $\mathcal{L}$ is *$\mathbb{Q}$-Cartier*.

We say that X is *strongly $\mathbb{Q}$-factorial at K* if every divisorial sheaf $\mathcal{L}$ defined over some open neighborhood $\mathcal{U}_{\mathcal{L}}$ of K is $\mathbb{Q}$-Cartier at K. As above, when $K = X$, we simply say that X is *strongly $\mathbb{Q}$-factorial*.

Let $\pi\colon X \to Y$ be a morphism of complex analytic spaces and let W be a subset of Y. If X is strongly $\mathbb{Q}$-factorial at $\pi^{-1}(W)$, then we say that X is *strongly $\mathbb{Q}$-factorial over W*.

Remark 1.4. Let X be a normal complex variety and let K and L be subsets of X with $L \subsetneq K$. We note that X is not necessarily strongly $\mathbb{Q}$-factorial at L even when X is strongly $\mathbb{Q}$-factorial at K.

We are mainly interested in the case where K is a compact subset of X in Definition 1.3.

Lemma 1.5. *Let $\mathcal{L}$ be a divisorial sheaf on a normal complex variety X and let K be a compact subset of X. If $\mathcal{L}$ is $\mathbb{Q}$-Cartier at K, then there exists a positive integer m such that $\mathcal{L}^{[m]}$ is a line bundle on some open neighborhood of K.*

Proof. By definition, we can take a finite open cover $\{\mathcal{U}_i\}_{i\in I}$ of K such that $\mathcal{L}^{[m_i]}$ is a line bundle on $\mathcal{U}_i$, where m_i is a positive integer, for every $i \in I$ since K is compact. We put $m := \mathrm{lcm}_{i\in I}\{m_i\}$. Then it is easy to see that $\mathcal{L}^{[m]}$ is a line bundle on $\mathcal{U} := \bigcup_{i\in I}\mathcal{U}_i$. This is what we wanted. □

Date: 2024/3/28, version 0.00.
2020 *Mathematics Subject Classification.* Primary 14E30; Secondary 32C15.
Key words and phrases. $\mathbb{Q}$-factorial, divisorial sheaves, minimal model program.

Lemma 1.6. *A smooth complex variety is strongly $\mathbb{Q}$-factorial.*

Proof. This is obvious by Lemma 1.2. $\square$

The following lemma is more or less well known. We state it here for the sake of completeness.

Lemma 1.7. *Let $f\colon X \to Y$ be a small projective bimeromorphic morphism of normal complex varieties. Assume that Y is $\mathbb{Q}$-factorial at $y \in Y$. Then f is an isomorphism over some open neighborhood of $y \in Y$.*

Proof. We take an f-ample line bundle $\mathcal{A}$ on X. We put $\mathcal{L} := (f_*\mathcal{A})^{**}$. Then $\mathcal{L}$ is a divisorial sheaf on Y. By assumption, there exists a positive integer m such that $\mathcal{L}^{[m]}$ is a line bundle on some open neighborhood U of $y \in Y$. Since f is small, we can easily see that $\mathcal{A}^{\otimes m} = f^*\mathcal{L}^{[m]}$ holds over U. This implies that f is an isomorphism over U because $\mathcal{A}$ is f-ample. $\square$

When we run the minimal model program, we need the following two lemmas. The proof is essentially the same as for algebraic varieties. We explain it in detail for the sake of completeness.

Lemma 1.8 (Divisorial contractions)**.** *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y. Let (X, Δ) be a log canonical pair. Let $\varphi_R\colon X \to Z$ be the divisorial contraction associated to a $(K_X + \Delta)$-negative extremal ray R of $\overline{\mathrm{NE}}(X/Y; W)$. Assume that X is strongly $\mathbb{Q}$-factorial over W. Then Z is also strongly $\mathbb{Q}$-factorial over W.*

Proof. Throughout this proof, we will freely shrink Y around W without mentioning it explicitly. Let $\mathcal{L}$ be a divisorial sheaf on Z. We put $\mathcal{M} := (\varphi_R^*\mathcal{L})^{**}$. Then $\mathcal{M}$ is a divisorial sheaf on X. Let E be the φ_R-exceptional divisor on X. We can take a positive integer m and an integer r such that $\mathcal{M}^{[m]}$ and $\mathcal{O}_X(rE)$ are line bundles on X and $\big(\mathcal{M}^{[m]} \otimes \mathcal{O}_X(rE)\big) \cdot C = 0$ for every $[C] \in R$. By the cone and contraction theorem for log canonical pairs (see [F]), there exists a line bundle $\mathcal{N}$ on Z such that $\mathcal{M}^{[m]} \otimes \mathcal{O}_X(rE) \simeq \varphi_R^*\mathcal{N}$. This implies that $\mathcal{L}^{[m]} \simeq \mathcal{N}$. Hence $\mathcal{L}$ is $\mathbb{Q}$-Cartier. Thus Z is strongly $\mathbb{Q}$-factorial over W. We finish the proof. $\square$

Lemma 1.9 (Flips)**.** *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y. Let (X, Δ) be a log canonical pair. Assume that X is strongly $\mathbb{Q}$-factorial over W. Let $\varphi_R\colon X \to Z$ be the flipping contraction associated to a $(K_X + \Delta)$-negative extremal ray R of $\overline{\mathrm{NE}}(X/Y; W)$ and let $\varphi_R^+\colon X^+ \to Z$ be the flip of φ_R.*

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr] \arrow[ddr] & & X^+ \arrow[dl] \arrow[ddl] \\ & Z \arrow[d] & \\ & Y & \end{tikzcd}$$

Then X^+ is also strongly $\mathbb{Q}$-factorial over W.

Proof. Throughout this proof, we will freely shrink Y around W without mentioning it explicitly. By perturbing the coefficients of Δ, we may assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\mathcal{L}$ be a divisorial sheaf on X^+. Since $\phi\colon X \dashrightarrow X^+$ is an isomorphism in codimension one, we can define a divisorial sheaf $\mathcal{M}$ on X which is the strict transform of $\mathcal{L}$. Then there exist a positive integer m and an integer r such that $\mathcal{M}^{[m]}$ and $\mathcal{O}_X(r(K_X+\Delta))$ are line bundles and $\left(\mathcal{M}^{[m]}\otimes\mathcal{O}_X(r(K_X+\Delta))\right)\cdot C=0$ holds for every $[C]\in R$. By the cone and contraction theorem for log canonical pairs (see [F]), this implies that there exists a line bundle $\mathcal{N}$ on Z such that $\mathcal{M}^{[m]}\otimes\mathcal{O}_X(r(K_X+\Delta))\simeq\varphi_R^*\mathcal{N}$. Hence we have $\mathcal{L}^{[m]}\otimes\mathcal{O}_X(r(K_{X^+}+\Delta^+))\simeq(\varphi_R^+)^*\mathcal{N}$, where Δ^+ is the strict transform of Δ on X^+. Therefore, $\mathcal{L}$ is $\mathbb{Q}$-Cartier over W. Thus, X^+ is strongly $\mathbb{Q}$-factorial over W. We finish the proof. □

By Lemmas 1.8 and 1.9, we see that the strong $\mathbb{Q}$-factoriality is preserved under the minimal model program starting from a strongly $\mathbb{Q}$-factorial log canonical pairs.

References

[DH] O. Das, C. Hacon, The log minimal model program for Kähler 3-folds, preprint (2020). arXiv:2009.05924 [math.AG]

[F] O. Fujino, Cone and contraction theorem for projective morphisms between complex analytic spaces, preprint (2022).

[OSS] C. Okonek, M. Schneider, H. Spindler, *Vector bundles on complex projective spaces*, Corrected reprint of the 1988 edition. With an appendix by S. I. Gelfand. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 2011.

Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan

Email address: `fujino@math.kyoto-u.ac.jp`