ON QUASI-LOG SCHEMES

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ABSTRACT. The notion of quasi-log schemes was first introduced by Florin Ambro in his epoch-making paper: Quasi-log varieties. In this paper, we establish the basepoint-free theorem of Reid–Fukuda type for quasi-log schemes in full generality. Roughly speaking, it means that all the results for quasi-log schemes claimed in Ambro's paper hold true. The proof is Kawamata's X-method with the aid of the theory of basic slc-trivial fibrations. For the reader's convenience, we make many comments on the theory of quasi-log schemes in order to make it more accessible.

CONTENTS

1. INTRODUCTION

Let us start with a simple situation. Let X be a normal projective variety defined over $\mathbb{C}$, the complex number field. Then there exist a proper birational morphism $f\colon Y \to X$ from a smooth variety Y and a $\mathbb{Q}$-divisor B_Y on Y such that $\operatorname{Supp} B_Y$ is a simple normal crossing divisor with $\lfloor B_Y \rfloor \leq 0$ satisfying

(1) $K_Y + B_Y \sim_{\mathbb{Q}} f^*\omega$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor ω on X, and
(2) the natural map

$$\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -B_Y \rceil)$$

is an isomorphism,

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if and only if (X, B), where $B = f_*B_Y$, is kawamata log terminal in the usual sense. Even if we replace the assumption that f is proper birational with one that f is only proper, many results, for example, the cone and contraction theorem, still hold true for X with respect to ω (see [F1]). This observation plays a crucial role in [F12] and [F16] (see also Section 4). Hence it is natural to consider the following more general setting.

Let X be a scheme with an $\mathbb{R}$-Cartier divisor ω on X. Note that X may be reducible and may have non-reduced components. Of course, X is not necessarily equidimensional. Assume that there exists a proper morphism $f\colon Y \to X$ from a globally embedded simple normal crossing pair (Y, B_Y) such that

    (a) $K_Y + B_Y \sim_{\mathbb{R}} f^*\omega$, and

    (b) the natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil)$ induces an isomorphism

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\;\simeq\;} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor),$$

        where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of a closed subscheme $X_{-\infty} \subsetneq X$.

Then we call

$$(X, \omega, f\colon (Y, B_Y) \to X)$$

or simply $[X, \omega]$ a *quasi-log scheme* (see [A]). We can prove various Kodaira type vanishing theorems, the cone and contraction theorem, and so on, for quasi-log schemes (see [F23, Chapter 6]). If $X_{-\infty}$ is empty, then we say that

$$(X, \omega, f\colon (Y, B_Y) \to X)$$

or $[X, \omega]$ is a *quasi-log canonical pair*. In general, quasi-log canonical pairs are reducible and are not equidimensional. However, it is surprising that they have only Du Bois singularities (see [FLh3]). Note that a log canonical pair can be seen as a quasi-log canonical pair by considering a suitable resolution of singularities. More generally, let (X, B) be a quasi-projective semi-log canonical pair. Then $[X, K_X + B]$ naturally becomes a quasi-log canonical pair (see [F17]). Hence any union of some slc strata of (X, B), which is denoted by W, has a natural quasi-log canonical structure induced from the one on $[X, K_X + B]$ by adjunction. Therefore, W has only Du Bois singularities, the cone and contraction theorem holds for W with respect to $(K_X + B)|_W$, and various Kodaira type vanishing theorems can be formulated on W.

One of the main purposes of this paper is to establish the following theorem, which is [A, Theorem 7.2]. We note that there exists no detail of the proof of Theorem 1.1 in [A].

Theorem 1.1 (Basepoint-free theorem of Reid–Fukuda type for quasi-log schemes)**.** *Let* $[X, \omega]$ *be a quasi-log scheme, let* $\pi\colon X \to S$ *be a* proper *morphism between schemes, and let* L *be a* π-*nef Cartier divisor on* X *such that* $qL - \omega$ *is nef and log big over* S *with respect to* $[X, \omega]$ *for some positive real number* q*. Assume that* $\mathcal{O}_{X_{-\infty}}(mL)$ *is* π-*generated for every* $m \gg 0$*. Then* $\mathcal{O}_X(mL)$ *is* π-*generated for every* $m \gg 0$*.*

This type of basepoint-free theorem was first considered by Reid in [S, 10.4. Eventual freedom] in order to avoid Zariski's famous counterexample (see [KMM, Remark 3-1-2.(2)]). Note that Section 10 in [S] was written by Reid when he translated Shokurov's paper from Russian to English (see [S, §10. Commentary by M. Reid]). Then Fukuda treated this problem in a series of papers (see [Fk1], [Fk2], and [Fk3]). We know that the basepoint-free theorem of Reid–Fukuda type holds true for divisorial log terminal pairs of arbitrary dimension (see [F2]). In [A], Ambro claims Theorem 1.1 without proof. In [F23, Section 6.9], we proved Theorem 1.1 under the extra assumption that $X_{-\infty} = \emptyset$ and π is projective. In [F20], we treated Theorem 1.1 when π is projective and $X_{-\infty}$ may be nonempty. In [F23, Section 6.9] and [F20], we use Kodaira's lemma for big divisors. Hence the projectivity of π is indispensable in [F23, Section 6.9] and [F20]. Our approach to Theorem 1.1 in this paper

is completely different from the one in [F23, Section 6.9] and [F20]. We use the theory of basic $\mathbb{R}$-slc-trivial fibrations in order to prove Theorem 1.1. This means that we use the theory of variations of mixed Hodge structure on cohomology with compact support for the proof of Theorem 1.1 (see [FF] and [FFS]). We think that the main importance of Theorem 1.1 is not in the statement but in the techniques in the proof. By Theorem 1.1 and the author's series of papers, we can recover all the results for quasi-log schemes in [A]. As an obvious corollary of Theorem 1.1, we have:

Corollary 1.2 (Basepoint-free theorem of Reid–Fukuda type for log canonical pairs)**.** *Let* (X, Δ) *be a log canonical pair and let* $\pi \colon X \to S$ *be a proper morphism to a scheme* S. *Let* L *be a* π-*nef Cartier divisor on* X *such that* $qL - (K_X + \Delta)$ *is nef and log big over* S *with respect to* (X, Δ) *for some positive real number* q. *Then* $\mathcal{O}_X(mL)$ *is* π-*generated for every* $m \gg 0$.

When π is projective, Corollary 1.2 is nothing but [F23, Corollary 6.9.4]. As far as we know, there is no approach to Corollary 1.2 without using the theory of quasi-log schemes. The main ingredient of the proof of Theorem 1.1 is the following theorem, which is a generalization of [F30, Theorem 1.7]. As is well known, some generalizations of Kodaira's canonical bundle formula are very useful for various geometric problems (see [FMo], [F3], [F4], [FG1], [FG2], and so on). Theorem 1.3 below is a kind of canonical bundle formula. The proof depends on [FH2, Corollary 5.2], that is, the theory of basic $\mathbb{R}$-slc-trivial fibrations.

Theorem 1.3 (Normal irreducible quasi-log schemes)**.** *Let*
$$(X, \omega, f \colon (Y, B_Y) \to X)$$
be a quasi-log scheme such that X *is a normal variety,* f *is projective, and every stratum of* Y *is dominant onto* X. *Then* $f \colon (Y, B_Y) \to X$ *is a basic* $\mathbb{R}$-*slc-trivial fibration. Let* $\mathbf{B}$ *and* $\mathbf{M}$ *be the discriminant and moduli* $\mathbb{R}$-*b-divisors associated to* $f \colon (Y, B_Y) \to X$, *respectively. Then there exists a projective birational morphism* $p \colon X' \to X$ *from a smooth quasi-projective variety* X' *such that*

(i) $\mathbf{K} + \mathbf{B} = \overline{K_{X'} + \mathbf{B}_{X'}}$ *holds, where* $\mathbf{K}$ *is the canonical b-divisor of* X,
(ii) $\operatorname{Supp} \mathbf{B}_{X'}$ *is a simple normal crossing divisor on* X',
(iii) $\mathbf{M} = \overline{\mathbf{M}_{X'}}$ *holds such that* $\mathbf{M}_{X'}$ *is a potentially nef* $\mathbb{R}$-*divisor on* X',
(iv) $p\left(\mathbf{B}_{X'}^{\geq 1}\right) = \operatorname{Nqklt}(X, \omega)$ *holds set theoretically, and*
(v) $p\left(\mathbf{B}_{X'}^{> 1}\right) = \operatorname{Nqlc}(X, \omega)$ *holds set theoretically.*

Note that
$$\mathbf{K} + \mathbf{B} + \mathbf{M} = \overline{\omega}$$
holds by definition. Moreover, if we put
$$\mathcal{J}_{\operatorname{Ngklt}} := p_* \mathcal{O}_{X'}\left(-\lfloor \mathbf{B}_{X'} \rfloor\right)$$
and
$$\mathcal{J}_{\operatorname{Nglc}} := p_* \mathcal{O}_{X'}\left(-\lfloor \mathbf{B}_{X'} \rfloor + \mathbf{B}_{X'}^{=1}\right) = p_* \mathcal{O}_{X'}\left(\lceil -(\mathbf{B}_{X'}^{\leq 1}) \rceil - \lfloor \mathbf{B}_{X'}^{> 1} \rfloor\right),$$
then $\mathcal{J}_{\operatorname{Ngklt}} = p_* \mathcal{O}_{X'}\left(-\lfloor \mathbf{B}_{X'}^{\geq 1} \rfloor\right)$ *and* $\mathcal{J}_{\operatorname{Nglc}} = p_* \mathcal{O}_{X'}\left(-\lfloor \mathbf{B}_{X'}^{> 1} \rfloor\right)$ *are ideal sheaves on* X *such that the following inclusions*
$$\mathcal{J}_{\operatorname{Ngklt}} \subset \mathcal{I}_{\operatorname{Nqklt}(X, \omega)} \quad and \quad \mathcal{J}_{\operatorname{Nglc}} \subset \mathcal{I}_{\operatorname{Nqlc}(X, \omega)}$$
hold, where $\mathcal{I}_{\operatorname{Nqklt}(X, \omega)}$ *and* $\mathcal{I}_{\operatorname{Nqlc}(X, \omega)}$ *are the defining ideal sheaves of* $\operatorname{Nqklt}(X, \omega)$ *and* $\operatorname{Nqlc}(X, \omega)$, *respectively.*

We note that in the above statement $\mathbf{B}$ and $\mathbf{M}$ become $\mathbb{Q}$-*b-divisors when*
$$(X, \omega, f \colon (Y, B_Y) \to X)$$
has a $\mathbb{Q}$-*structure.*

We will prove Theorem 1.1 by combining Kawamata's X-method with Theorem 1.3 in the framework of quasi-log schemes. We note that we do not use the minimal model program in this paper.

Let us explain the idea of the proof of Theorem 1.1. For simplicity, we assume that S is a point. By the standard argument in the theory of quasi-log schemes, we can reduce the problem to the case where X is irreducible and $\mathcal{O}_{\mathrm{Nqklt}(X,\omega)}(mL)$ is generated by global sections for every $m \gg 0$. This implies that $\mathrm{Bs}\,|mL| \cap \mathrm{Nqklt}(X,\omega) = \emptyset$ for every $m \gg 0$. Let $\nu \colon Z \to X$ be the normalization. Then $[Z, \nu^*\omega]$ has a natural quasi-log scheme structure with $\nu_*\mathcal{I}_{\mathrm{Nqklt}(Z,\nu^*\omega)} = \mathcal{I}_{\mathrm{Nqklt}(X,\omega)}$. Moreover, we may assume that $[Z, \nu^*\omega]$ satisfies Theorem 1.3. By the classical X-method, we can prove that there are many global sections of $\mathcal{O}_Z(m\nu^*L) \otimes \mathcal{J}_{\mathrm{Ngklt}}$ for $m \gg 0$, where $\mathcal{J}_{\mathrm{Ngklt}}$ is the ideal sheaf defined in Theorem 1.3. By $\mathcal{J}_{\mathrm{Ngklt}} \subset \mathcal{I}_{\mathrm{Nqklt}(Z,\nu^*\omega)}$ and $\nu_*\mathcal{I}_{\mathrm{Nqklt}(Z,\nu^*\omega)} = \mathcal{I}_{\mathrm{Nqklt}(X,\omega)}$,

$$H^0\left(Z, \mathcal{O}_Z(m\nu^*L) \otimes \mathcal{J}_{\mathrm{Ngklt}}\right) \subset H^0\left(X, \mathcal{O}_X(mL) \otimes \mathcal{I}_{\mathrm{Nqklt}(X,\omega)}\right)$$

holds for every m. Hence $\mathcal{O}_X(mL) \otimes \mathcal{I}_{\mathrm{Nqklt}(X,\omega)}$ has many global sections such that $\mathrm{Bs}\,|mL| \cap (X \setminus \mathrm{Nqklt}(X,\omega)) = \emptyset$. Therefore, we obtain $\mathrm{Bs}\,|mL| = \emptyset$ for $m \gg 0$.

This paper is organized as follows. In Section 2, we make some comments on base fields and the Lefschetz principle for the reader's convenience. In Section 3, we collect some basic definitions. In Section 4, we slightly reformulate the Kawamata–Shokurov basepoint-free theorem. The results in this section can be proved by Kawamata's X-method without difficulties. In Section 5, we quickly recall the definition of quasi-log schemes and basic slc-trivial fibrations and explain some fundamental results. In Section 6, we prove Theorem 1.3, which is one of the main results of this paper. Section 7 is devoted to the proof of Theorem 1.1. In Section 8, we make many comments on [A] to help the reader understand differences between Ambro's original approach in [A] and our framework of quasi-log schemes.

We will work over $\mathbb{C}$, the complex number field, throughout this paper. A *scheme* means a separated scheme of finite type over $\mathbb{C}$. A *variety* means an integral scheme, that is, an irreducible and reduced separated scheme of finite type over $\mathbb{C}$. We will freely use the framework of quasi-log schemes established in [F23, Chapter 6]. We note that $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the set of *intergers*, *rational numbers*, and *real numbers*, respectively. We also note that $\mathbb{Q}_{>0}$ and $\mathbb{R}_{>0}$ are the set of *positive rational numbers* and *positive real numbers*, respectively. In this paper, the expression '... for every $m \gg 0$' means that 'there exists a positive number m_0 such that ... for every $m \geq m_0$.'

2. On the Lefschetz principle

In this short section, before starting the main contents of this paper, we make some comments on base fields and the Lefschetz principle for the reader's convenience.

Remark 2.1 (On the base field k)**.** We mainly work over $\mathbb{C}$, the complex number field, in the papers on quasi-log schemes (see [F23, Chapter 6]). This is because the author's approach depends on the theory of mixed Hodge structures on cohomology with compact support. However, almost all results on quasi-log schemes hold true over any algebraically closed field k of characteristic zero. For example, we can prove the vanishing theorems for quasi-log schemes over k by the Lefschetz principle. Hence the cone and contraction theorem for quasi-log schemes can be proved as an application of some vanishing theorems. When we treat *sufficiently general fibers*, *uniruledness*, *rationally chain connectedness*, and so on, we have to take care of the base field k if the cardinality of k is countable (see

[F35]). It is obvious that some results, for example, the simple connectedness of quasi-log canonical Fano pairs, make sense only over $\mathbb{C}$ (see [FLw] and [F31]).

Let us quickly see how to use the Lefschetz principle. Let X be a projective scheme defined over an algebraically closed field k of characteristic zero. Let L be a Cartier divisor on X and let H be an ample Cartier divisor on X. We can take a subfield k_0 of k, which is finitely generated over $\mathbb{Q}$, and a scheme X_0 defined over k_0, a Cartier divisor L_0 on X_0, and an ample Cartier divisor H_0 on X_0 such that $X \simeq X_0 \times_{\operatorname{Spec} k_0} \operatorname{Spec} k$, $L \simeq L_0 \times_{\operatorname{Spec} k_0} \operatorname{Spec} k$, and $H \simeq H_0 \times_{\operatorname{Spec} k_0} \operatorname{Spec} k$. We consider some embedding $k_0 \subset \mathbb{C}$ and the induced morphism $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} k_0$. Then we put $X_{\mathbb{C}} := X_0 \times_{\operatorname{Spec} k_0} \operatorname{Spec} \mathbb{C}$, $L_{\mathbb{C}} := L_0 \times_{\operatorname{Spec} k_0} \operatorname{Spec} \mathbb{C}$, and $H_{\mathbb{C}} := H_0 \times_{\operatorname{Spec} k_0} \operatorname{Spec} \mathbb{C}$. Since H is ample, H_0 and $H_{\mathbb{C}}$ are both ample. Note that L, L_0, and $L_{\mathbb{C}}$ are nef if and only if $L + rH$, $L_0 + rH_0$, and $L_{\mathbb{C}} + rH_{\mathbb{C}}$ are ample for every rational number r with $0 < r \ll 1$, respectively. Hence, L is nef if and only if $L_{\mathbb{C}}$ is nef.

3. Preliminaries

Here, we collect some basic definitions for the reader's convenience. Let X be a scheme and let $\operatorname{Pic}(X)$ be the group of line bundles on X, that is, the *Picard group* of X. An element of $\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. $\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$) is called an $\mathbb{R}$-*line bundle* (resp. a $\mathbb{Q}$-*line bundle*) on X. We write the group law of $\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ additively for simplicity of notation. Let $\operatorname{Div}(X)$ be the group of Cartier divisors on X. An element of $\operatorname{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. $\operatorname{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$) is called an $\mathbb{R}$-*Cartier divisor* (resp. a $\mathbb{Q}$-*Cartier divisor*) on X. Let Δ_1 and Δ_2 be $\mathbb{R}$-Cartier (resp. $\mathbb{Q}$-Cartier) divisors on X. Then $\Delta_1 \sim_{\mathbb{R}} \Delta_2$ (resp. $\Delta_1 \sim_{\mathbb{Q}} \Delta_2$) means that Δ_1 is $\mathbb{R}$-*linearly* (resp. $\mathbb{Q}$-*linearly*) *equivalent* to Δ_2. There exists a natural group homomorphism $\operatorname{Div}(X) \to \operatorname{Pic}(X)$ given by $A \mapsto \mathcal{O}_X(A)$, where A is a Cartier divisor on X. It induces a homomorphism $\delta_X \colon \operatorname{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R} \to \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. We sometimes write $A + \mathcal{L} \sim_{\mathbb{R}} B + \mathcal{M}$ for $A, B \in \operatorname{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathcal{L}, \mathcal{M} \in \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. This means that $\delta_X(A) + \mathcal{L} = \delta_X(B) + \mathcal{M}$ holds in $\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. We usually use this type of abuse of notation, that is, the confusion of $\mathbb{R}$-line bundles with $\mathbb{R}$-Cartier divisors. In the theory of minimal models for higher-dimensional algebraic varieties, we sometimes use $\mathbb{R}$-Cartier divisors for ease of notation even when they should be $\mathbb{R}$-line bundles.

Let us recall the definition of *potentially nef divisors*. We need it in Theorem 1.3.

Definition 3.1 (Potentially nef divisors, see [F30, Definition 2.5]). Let X be a normal variety and let D be a divisor on X. If there exist a completion $X^{\flat}$ of X, that is, $X^{\flat}$ is a complete normal variety and contains X as a dense Zariski open subset, and a nef divisor $D^{\flat}$ on $X^{\flat}$ such that $D = D^{\flat}|_X$, then D is called a *potentially nef* divisor on X. A finite $\mathbb{Q}_{>0}$-linear (resp. $\mathbb{R}_{>0}$-linear) combination of potentially nef divisors is called a *potentially nef* $\mathbb{Q}$-divisor (resp. $\mathbb{R}$-divisor).

Remark 3.2. (i) Let D be a nef $\mathbb{R}$-divisor on a smooth projective variety X. Then D is not necessarily a potentially nef $\mathbb{R}$-divisor. This means that D is not always a finite $\mathbb{R}_{>0}$-linear combination of nef Cartier divisors on X. (ii) Let X be a normal variety and let D be a potentially nef $\mathbb{R}$-divisor on X. Then $D \cdot C \geq 0$ for every projective curve C on X. In particular, D is π-nef for every proper morphism $\pi \colon X \to S$ to a scheme S.

It is convenient to use *b-divisors* to explain several results. We note that the notion of b-divisors was first introduced by Shokurov.

Definition 3.3 (Canonical b-divisors). Let X be a normal variety and let ω be a top rational differential form of X. Then (ω) defines a b-divisor $\mathbf{K}$. We call $\mathbf{K}$ the *canonical b-divisor* of X.

Definition 3.4 (ℝ-Cartier closures)**.** The ℝ-*Cartier closure* of an ℝ-Cartier ℝ-divisor D on a normal variety X is the ℝ-b-divisor $\overline{D}$ with trace

$$\overline{D}_Y = f^*D,$$

where $f \colon Y \to X$ is a proper birational morphism from a normal variety Y.

Definition 3.5 ([F30, Definition 2.12])**.** Let X be a normal variety. An ℝ-b-divisor $\mathbf{D}$ of X is *b-potentially nef* (resp. *b-semi-ample*) if there exists a proper birational morphism $X' \to X$ from a normal variety X' such that $\mathbf{D} = \overline{\mathbf{D}_{X'}}$, that is, $\mathbf{D}$ is the ℝ-Cartier closure of $\mathbf{D}_{X'}$, and that $\mathbf{D}_{X'}$ is potentially nef (resp. semi-ample). An ℝ-b-divisor $\mathbf{D}$ of X is ℝ-*b-Cartier* if there is a proper birational morphism $X' \to X$ from a normal variety X' such that $\mathbf{D} = \overline{\mathbf{D}_{X'}}$.

Definition 3.6. Let X be an equidimensional reduced scheme. Note that X is not necessarily regular in codimension one. Let D be an ℝ-divisor (resp. a ℚ-divisor), that is, D is a finite formal sum $\sum_i d_i D_i$, where D_i is an irreducible reduced closed subscheme of X of pure codimension one and $d_i \in \mathbb{R}$ (resp. $d_i \in \mathbb{Q}$) for every i such that $D_i \neq D_j$ for $i \neq j$. We put

$$D^{<1} = \sum_{d_i<1} d_i D_i, \quad D^{=1} = \sum_{d_i=1} D_i, \quad D^{>1} = \sum_{d_i>1} d_i D_i, \quad \text{and} \quad \lfloor D \rfloor = \sum_i \lfloor d_i \rfloor D_i,$$

where $\lfloor d_i \rfloor$ is the integer defined by $d_i - 1 < \lfloor d_i \rfloor \leq d_i$. We note that $\lceil D \rceil = -\lfloor -D \rfloor$ and $\{D\} = D - \lfloor D \rfloor$. Similarly, we put

$$D^{\geq 1} = \sum_{d_i \geq 1} d_i D_i.$$

Let D be an ℝ-divisor (resp. a ℚ-divisor) as above. We call D a *subboundary* ℝ-divisor (resp. ℚ-divisor) if $D = D^{\leq 1}$ holds. When D is effective and $D = D^{\leq 1}$ holds, we call D a *boundary* ℝ-divisor (resp. ℚ-divisor).

The following definition is standard and is well known.

Definition 3.7 (Singularities of pairs)**.** Let X be a variety and let E be a prime divisor on Y for some proper birational morphism $f \colon Y \to X$ from a normal variety Y. Then E is called a divisor *over* X. A *normal pair* (X, Δ) consists of a normal variety X and an ℝ-divisor Δ on X such that $K_X + \Delta$ is ℝ-Cartier. Let (X, Δ) be a normal pair and let $f \colon Y \to X$ be a proper birational morphism from a normal variety Y. Then we can write

$$K_Y = f^*(K_X + \Delta) + \sum_E a(E, X, \Delta)E$$

with

$$f_*\left(\sum_E a(E, X, \Delta)E\right) = -\Delta,$$

where E runs over prime divisors on Y. We call $a(E, X, \Delta)$ the *discrepancy* of E with respect to (X, Δ). Note that we can define the discrepancy $a(E, X, \Delta)$ for any prime divisor E over X by taking a suitable resolution of singularities of X. If $a(E, X, \Delta) \geq -1$ (resp. > -1) for every prime divisor E over X, then (X, Δ) is called *sub log canonical* (resp. *sub kawamata log terminal*). We further assume that Δ is effective. Then (X, Δ) is called *log canonical* and *kawamata log terminal* if it is sub log canonical and sub kawamata log terminal, respectively.

Let (X, Δ) be a log canonical pair. If there exists a projective birational morphism $f \colon Y \to X$ from a smooth variety Y such that both $\mathrm{Exc}(f)$, the exceptional locus of f, and $\mathrm{Exc}(f) \cup \mathrm{Supp}\, f_*^{-1}\Delta$ are simple normal crossing divisors on Y and that $a(E, X, \Delta) > -1$

holds for every f-exceptional divisor E on Y, then (X, Δ) is called *divisorial log terminal* (*dlt*, for short).

Definition 3.8 (Log canonical centers, non-lc loci, and so on). Let (X, Δ) be a normal pair. If there exist a projective birational morphism $f \colon Y \to X$ from a normal variety Y and a prime divisor E on Y such that (X, Δ) is sub log canonical in a neighborhood of the generic point of $f(E)$ and that $a(E, X, \Delta) = -1$, then $f(E)$ is called a *log canonical center* of (X, Δ).

From now on, we further assume that Δ is effective. Let $f \colon Y \to X$ be a resolution with

$$K_Y + \Delta_Y = f^*(K_X + \Delta)$$

such that $\operatorname{Supp} \Delta_Y$ is a simple normal crossing divisor on Y. We put

$$\mathcal{J}(X, \Delta) = f_* \mathcal{O}_Y(-\lfloor \Delta_Y \rfloor).$$

Then $\mathcal{J}(X, \Delta)$ is an ideal sheaf on X and is known as the *multiplier ideal sheaf* associated to the pair (X, Δ). It is independent of the resolution $f \colon Y \to X$. The closed subscheme $\operatorname{Nklt}(X, \Delta)$ defined by $\mathcal{J}(X, \Delta)$ is called the *non-klt locus* of (X, Δ). It is obvious that (X, Δ) is kawamata log terminal if and only if $\mathcal{J}(X, \Delta) = \mathcal{O}_X$. Similarly, we put

$$\mathcal{J}_{\mathrm{NLC}}(X, \Delta) = f_* \mathcal{O}_X(-\lfloor \Delta_Y \rfloor + \Delta_Y^{=1})$$

and call it the *non-lc ideal sheaf* associated to the pair (X, Δ). We can check that it is independent of the resolution $f \colon Y \to X$. The closed subscheme $\operatorname{Nlc}(X, \Delta)$ defined by $\mathcal{J}_{\mathrm{NLC}}(X, \Delta)$ is called the *non-lc locus* of (X, Δ). It is obvious that (X, Δ) is log canonical if and only if $\mathcal{J}_{\mathrm{NLC}}(X, \Delta) = \mathcal{O}_X$. By definition, the natural inclusion

$$\mathcal{J}(X, \Delta) \subset \mathcal{J}_{\mathrm{NLC}}(X, \Delta)$$

always holds. Therefore, we have

$$\operatorname{Nlc}(X, \Delta) \subset \operatorname{Nklt}(X, \Delta).$$

For the details of $\mathcal{J}_{\mathrm{NLC}}(X, \Delta)$, we recommend the reader to see [F9] and [F14].

4. CLASSICAL BASEPOINT-FREE THEOREMS

In this section, we will reformulate some classical results in order to apply them to the proof of Theorem 1.1. Everything in this section can be proved by the X-method. For the details of the X-method, see [KMM], [KM], and [M]. We note that this section is similar to [F16, Section 2].

Let us start with Shokurov's nonvanishing theorem.

Theorem 4.1 (Shokurov's nonvanishing theorem). *Let X be a smooth variety and let B be an $\mathbb{R}$-divisor on X such that $\operatorname{Supp} B$ is a simple normal crossing divisor on X with $\lfloor B \rfloor \leq 0$. Let $\pi \colon X \to S$ be a proper morphism to a scheme S and let D be a π-nef Cartier divisor on X. Assume that $aD - (K_X + B)$ is nef and big over S for some positive integer a. Then $\pi_* \mathcal{O}_X(mD + \lceil -B \rceil) \neq 0$ for every $m \gg 0$.*

Sketch of Proof. By taking the Stein factorization and considering a sufficiently general fiber of $\pi \colon X \to S$, we may assume that S is a point. By taking a resolution of singularities of X and using Kodaira's lemma, we may further assume that X is projective, $aD - (K_X + B)$ is ample, and B is a $\mathbb{Q}$-divisor. In this case, the statement is well known. For the details, see [KMM], [KM], and [M]. $\qquad\square$

For the proof of Theorem 1.1, the following formulation of the Kawamata–Shokurov basepoint-free theorem is very useful.

Theorem 4.2 (Kawamata–Shokurov basepoint-free theorem)**.** *Let X be a smooth variety and let B be an $\mathbb{R}$-divisor on X such that $\operatorname{Supp} B$ is a simple normal crossing divisor on X with $\lfloor B \rfloor \leq 0$. Let $\pi\colon X \to S$ be a proper morphism to a scheme S and let D be a π-nef Cartier divisor on X. Assume the following conditions:*

    (1) *$aD - (K_X + B)$ is nef and big over S for some positive integer a, and*
    (2) *there exists a positive integer k such that the natural inclusion*

$$\pi_* \mathcal{O}_X(lD) \hookrightarrow \pi_* \mathcal{O}_X(lD + \lceil -B \rceil)$$

    is an isomorphism for every $l \geq k$.

Then $\mathcal{O}_X(mD)$ is π-generated for every $m \gg 0$.

Sketch of Proof. The proof of the Kawamata–Shokurov basepoint-free theorem, which is now usually called the X-method, works without any changes (see [KMM], [KM], and [M]). For the details, see also the proof of Lemma 4.3 below. □

We need a somewhat artificial lemma for the proof of Theorem 1.1.

Lemma 4.3. *Let $\pi\colon X \to S$ be a proper morphism from a normal variety X to an affine scheme S and let D be a π-nef Cartier divisor on X. Let $p\colon Z \to X$ be a proper birational morphism from a smooth variety Z and let B be an $\mathbb{R}$-divisor on Z such that $\operatorname{Supp} B$ is a simple normal crossing divisor, $B^{<0}$ is p-exceptional, and $B^{\geq 1} \neq 0$. Assume that $ap^*D - (K_Z + B)$ is nef and big over S for some positive real number a and that $\operatorname{Bs}|mD| \cap p(B^{\geq 1}) = \emptyset$ for some positive integer m, where $\operatorname{Bs}|mD|$ denotes the base locus of $|mD|$ with the reduced scheme structure. Then there exists a positive integer s such that*

$$(4.1) \qquad H^0\left(X, \mathcal{O}_X(m^s D) \otimes p_* \mathcal{O}_Z(-\lfloor B \rfloor)\right) \otimes \mathcal{O}_X \to \mathcal{O}_X(m^s D)$$

is surjective on $X \setminus p(B^{\geq 1})$. Note that $p_* \mathcal{O}_Z(-\lfloor B \rfloor)$ is an ideal sheaf on X. In particular, $\mathcal{O}_X(m^s D)$ is π-generated.

Proof. The well-known X-method works with some minor modifications.

Step 1. Let $q\colon Z' \to Z$ be a projective birational morphism from a smooth quasi-projective variety Z' such that $K_{Z'} + B_{Z'} = q^*(K_Z + B)$ and that $\operatorname{Supp} B_{Z'}$ is a simple normal crossing divisor on Z'. It is easy to see that $q_* \mathcal{O}_{Z'}(-\lfloor B_{Z'} \rfloor) = \mathcal{O}_Z(-\lfloor B \rfloor)$. Hence we can replace (Z, B) and $p\colon Z \to X$ with $(Z', B_{Z'})$ and $p \circ q\colon Z' \to X$, respectively. Therefore, we may assume that there exists a simple normal crossing divisor G on Z such that $G = \sum_j F_j$ is the irreducible decomposition with the following properties:

    (1) the support of $G + \operatorname{Supp} B$ is contained in a simple normal crossing divisor on Z,
    (2) $p^* mD = L + \sum_j r_j F_j$ for some nonnegative integers r_j and a $\pi \circ p$-free Cartier divisor L on Z such that

$$H^0(X, \mathcal{O}_X(mD)) = H^0(Z, \mathcal{O}_Z(L))$$

    and that $\operatorname{Bs}|mD| = \bigcup_{r_j > 0} p(F_j)$, and
    (3) $ap^*D - (K_Z + B) - \sum_j p_j F_j$ is ample over S for suitable $0 < p_j \ll 1$.

Step 2. By the usual argument in Kawamata's X-method, we can perturb p_j suitably and choose $c > 0$ such that

$$\left(B + c \sum_j r_j F_j + \sum_j p_j F_j \right)^{>1} = 0$$

on $Z \setminus \operatorname{Supp} B^{\geq 1}$ and

$$\left(B + c \sum_j r_j F_j + \sum_j p_j F_j \right)^{=1}$$

is a prime divisor F on $Z \setminus \operatorname{Supp} B^{\geq 1}$. We note that if $r_j > 0$ then F_j is disjoint from $\operatorname{Supp} B^{\geq 1}$ by the assumption $\operatorname{Bs}|mD| \cap p(B^{\geq 1}) = \emptyset$. We put

$$F + B' = B + c\sum_j r_j F_j + \sum_j p_j F_j.$$

Then, by construction, $\lfloor F + B' \rfloor = F + \lfloor B' \rfloor$, $\operatorname{Supp}(B')^{\geq 1} = \operatorname{Supp} B^{\geq 1}$, and $p_* \mathcal{O}_Z(-\lfloor B' \rfloor) \subset p_* \mathcal{O}_Z(-\lfloor B \rfloor)$. Note that F is disjoint from $\operatorname{Supp}(B')^{\geq 1}$ by construction.

Step 3. Let b be a positive integer such that $b \geq cm + a$. We consider an $\mathbb{R}$-divisor

$$N(b) := bp^* D - (K_Z + F + B')$$

$$= (b - cm - a)p^* D + cL + \left(ap^* D - (K_Z + B) - \sum_j p_j F_j \right).$$

Therefore, we see that $N(b)$ is ample over S. By the Kawamata–Viehweg vanishing theorem,

$$H^1(Z, \mathcal{O}_Z(K_Z + \lceil N(b) \rceil)) = H^1(Z, \mathcal{O}_Z(bp^* D - F - \lfloor B' \rfloor)) = 0.$$

Hence the restriction map

(4.2) $$H^0(Z, \mathcal{O}_Z(bp^* D - \lfloor B' \rfloor)) \to H^0(F, \mathcal{O}_F(bp^* D + \lceil -B'|_F \rceil))$$

is surjective. Note that

$$N(b)|_F = bp^* D|_F - (K_Z + F + B')|_F$$
$$= bp^* D|_F - (K_F + B'|_F)$$

is ample over S. By construction, $\operatorname{Supp} B'|_F$ is a simple normal crossing divisor on F and $\lfloor B'|_F \rfloor \leq 0$ holds. By Theorem 4.1, we obtain that $H^0(F, \mathcal{O}_F(bp^* D + \lceil -B'|_F \rceil)) \neq 0$ for every $b \gg 0$. Therefore, $H^0(Z, \mathcal{O}_Z(bp^* D - \lfloor B' \rfloor))$ has a section which does not vanish on the generic point of F for every $b \gg 0$ by the surjection (4.2). This means that there exists some positive integer t such that

$$H^0\left(X, \mathcal{O}_X(m^t D) \otimes p_* \mathcal{O}_Z(-\lfloor B \rfloor)\right) \otimes \mathcal{O}_X \to \mathcal{O}_X(m^t D)$$

is surjective at the generic point of $p(F)$. It is obvious that $\operatorname{Bs}|m^t D| \cap p(B^{\geq 1}) = \emptyset$ holds. Hence we can apply the same argument as above to $m^t D$. Then, by Noetherian induction, we can find a sufficiently large positive integer s such that the map in (4.1) is surjective.

Step 4. By (4.1), $\operatorname{Bs}|m^s D| \cap (X \setminus p(B^{\geq 1})) = \emptyset$. On the other hand, $\operatorname{Bs}|mD| \cap p(B^{\geq 1}) = \emptyset$ by assumption. Hence $\operatorname{Bs}|m^s D| \cap p(B^{\geq 1}) = \emptyset$. Therefore, we obtain $\operatorname{Bs}|m^s D| = \emptyset$. This means that $\mathcal{O}_X(m^s D)$ is π-generated.

We finish the proof of Lemma 4.3. $\qquad\square$

We close this section with a remark on the X-method.

Remark 4.4. As is well known, the X-method works very well for kawamata log terminal pairs. Precisely speaking, the notion of kawamata log terminal pairs was introduced to make the X-method work well. Unfortunately, however, it is not so powerful for log canonical pairs. Hence we proposed a new approach in order to prove the cone and contraction theorem for log canonical pairs (see [F13] and [F14]). Our approach is based on some vanishing theorems obtained by the theory of mixed Hodge structures on cohomology with compact support (see [F7], [F22], [F23, Chapter 5], and [F27]). In some sense, it is simpler than the X-method.

5. Quasi-log schemes and basic slc-trivial fibrations

In this section, let us quickly review the theory of quasi-log schemes and the framework of basic slc-trivial fibrations.

Let Y be a simple normal crossing divisor on a smooth variety M and let B be an $\mathbb{R}$-divisor on M such that $\operatorname{Supp}(B+Y)$ is a simple normal crossing divisor on M and that B and Y have no common irreducible components. We put $B_Y = B|_Y$ and consider the pair (Y, B_Y). We call (Y, B_Y) a *globally embedded simple normal crossing pair* and M the *ambient space* of (Y, B_Y). A *stratum* of (Y, B_Y) is a log canonical center of $(M, Y + B)$ that is contained in Y.

Let us recall the definition of *quasi-log schemes*.

Definition 5.1 (Quasi-log schemes, see [F23, Definition 6.2.2]). A *quasi-log scheme* is a scheme X with an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) ω on X, a closed subscheme $X_{-\infty} \subsetneq X$, and a finite collection $\{C\}$ of subvarieties of X such that there exists a proper morphism $f \colon (Y, B_Y) \to X$ from a globally embedded simple normal crossing pair satisfying the following properties:

(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$ holds,
(2) the natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil)$ induces an isomorphism

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor),$$

where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$, and
(3) the collection of subvarieties $\{C\}$ coincides with the images of the strata of (Y, B_Y) that are not included in $X_{-\infty}$.

If there is no risk of confusion, then we simply write $[X, \omega]$ to denote the above data

$$(X, \omega, f \colon (Y, B_Y) \to X).$$

The subvarieties C are called the *qlc strata* of $[X, \omega]$ and $X_{-\infty}$ is called the *non-qlc locus* of $[X, \omega]$. We usually use $\operatorname{Nqlc}(X, \omega)$ to denote $X_{-\infty}$. If a qlc stratum C of $[X, \omega]$ is not an irreducible component of X, then it is called a *qlc center* of $[X, \omega]$. The union of $X_{-\infty}$ with all qlc centers of $[X, \omega]$ is denoted by $\operatorname{Nqklt}(X, \omega)$.

We say that $(X, \omega, f \colon (Y, B_Y) \to X)$ or $[X, \omega]$ has a $\mathbb{Q}$-*structure* if B_Y is a $\mathbb{Q}$-divisor, ω is a $\mathbb{Q}$-Cartier divisor (or $\mathbb{Q}$-line bundle), and $f^*\omega \sim_{\mathbb{Q}} K_Y + B_Y$ holds in the above definition.

We need the notion of nef and log big $\mathbb{R}$-divisors in order to formulate the basepoint-free theorem of Reid–Fukuda type.

Definition 5.2 (Nef and log bigness). Let L be an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) on a quasi-log scheme $[X, \omega]$ and let $\pi \colon X \to S$ be a proper morphism between schemes. Then L is said to be *nef and log big over S with respect to $[X, \omega]$* if $L|_W$ is π-nef and $L|_W$ is π-big for every qlc stratum W of $[X, \omega]$.

The following example is very important. By this example, we can apply the results for quasi-log schemes to normal pairs.

Example 5.3. Let X be a normal variety and let B be an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. Let $f \colon Y \to X$ be a proper birational morphism from a smooth variety Y with $K_Y + B_Y = f^*(K_X + B)$ such that $\operatorname{Supp} B_Y$ is a simple normal crossing divisor on Y. Then

$$(X, K_X + B, f \colon (Y, B_Y) \to X)$$

naturally becomes a quasi-log scheme. Note that W is a qlc center of $[X, K_X + B]$ if and only if W is a log canonical center of (X, B). By construction, $\operatorname{Nqlc}(X, K_X + B) = \operatorname{Nlc}(X, B)$. Hence (X, B) is log canonical if and only if $\operatorname{Nqlc}(X, K_X + B) = \emptyset$.

A pair (X, B) which is Zariski locally isomorphic to a globally embedded simple normal crossing pair at any point is called a *simple normal crossing pair*. Let (X, B) be a simple normal crossing pair and let $\nu \colon X^\nu \to X$ be the normalization. We define B^ν by $K_{X^\nu} + B^\nu = \nu^*(K_X + B)$, that is, B^ν is the sum of the inverse images of B and the singular locus of X. We note that X^ν is a disjoint union of smooth varieties and $\operatorname{Supp} B^\nu$ is a simple normal crossing divisor on X^ν. Then we say that W is a *stratum* of (X, B) if and only if W is an irreducible component of X or the ν-image of some log canonical center of (X^ν, B^ν). Let (X, B) be a simple normal crossing pair as above and let $X = \bigcup_{i \in I} X_i$ be the irreducible decomposition of X. Then a *stratum* of X means an irreducible component of $X_{i_1} \cap \cdots \cap X_{i_k}$ for some $\{i_1, \ldots, i_k\} \subset I$. It is not difficult to see that W is a stratum of X if and only if W is a stratum of $(X, 0)$. By definition, it is obvious that a globally embedded simple normal crossing pair is a simple normal crossing pair.

Let us recall the definition of *basic slc-trivial fibrations.*

Definition 5.4 (Basic slc-trivial fibrations). A *basic $\mathbb{Q}$-slc-trivial (resp. $\mathbb{R}$-slc-trivial) fibration* $f \colon (X, B) \to Y$ consists of a projective surjective morphism $f \colon X \to Y$ and a simple normal crossing pair (X, B) satisfying the following properties:

(1) Y is a normal variety,
(2) every stratum of X is dominant onto Y and $f_* \mathcal{O}_X \simeq \mathcal{O}_Y$,
(3) B is a $\mathbb{Q}$-divisor (resp. an $\mathbb{R}$-divisor) such that $B = B^{\leq 1}$ holds over the generic point of Y,
(4) there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor (resp. an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor) D on Y such that $K_X + B \sim_{\mathbb{Q}} f^*D$ (resp. $K_X + B \sim_{\mathbb{R}} f^*D$), and
(5) $\operatorname{rank} f_* \mathcal{O}_X(\lceil -(B^{<1}) \rceil) = 1$.

If there is no danger of confusion, we simply use basic slc-trivial fibrations to denote basic $\mathbb{Q}$-slc-trivial fibrations or basic $\mathbb{R}$-slc-trivial fibrations. Let $f \colon (X, B) \to Y$ be a basic slc-trivial fibration as in Definition 5.4 and let $\nu \colon X^\nu \to X$ be the normalization with $K_{X^\nu} + B^\nu = \nu^*(K_X + B)$ as before. Let P be a prime divisor on Y. By shrinking Y around the generic point of P, we assume that P is a Cartier divisor. We set

$$b_P = \max \left\{ t \in \mathbb{R} \,\middle|\, \begin{array}{l} (X^\nu, B^\nu + t\nu^* f^* P) \text{ is sub log canonical} \\ \text{over the generic point of } P \end{array} \right\}$$

and put

$$B_Y = \sum_P (1 - b_P) P,$$

where P runs over prime divisors on Y. Then B_Y is a well-defined $\mathbb{R}$-divisor on Y and is called the *discriminant $\mathbb{R}$-divisor* of $f \colon (X, B) \to Y$. We set

$$M_Y = D - K_Y - B_Y$$

and call M_Y the *moduli $\mathbb{R}$-divisor* of $f \colon (X, B) \to Y$. By definition, we have

$$K_X + B \sim_{\mathbb{R}} f^*(K_Y + B_Y + M_Y).$$

Let $\sigma \colon Y' \to Y$ be a proper birational morphism from a normal variety Y'. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
(X^\sharp, B^\sharp) & \xrightarrow{\ \mu\ } & (X^\nu, B^\nu) \\
\downarrow & & \downarrow{\scriptstyle f \circ \nu} \\
Y' & \xrightarrow[\ \sigma\ ]{} & Y,
\end{array}
$$

where $X^\sharp$ is the normalization of the main components of $X \times_Y Y'$ and $B^\sharp$ is defined by $K_{X^\sharp} + B^\sharp = \mu^*(K_{X^\nu} + B^\nu)$. As above, we can define $\mathbb{R}$-divisors $B_{Y'}$, $K_{Y'}$ and $M_{Y'}$

for $(X^\sharp, B^\sharp) \to Y'$ such that $\sigma^* D = K_{Y'} + B_{Y'} + M_{Y'}$, $\sigma_* B_{Y'} = B_Y$, $\sigma_* K_{Y'} = K_Y$ and $\sigma_* M_{Y'} = M_Y$ hold. Hence there exist a unique $\mathbb{R}$-b-divisor $\mathbf{B}$ such that $\mathbf{B}_{Y'} = B_{Y'}$ for every $\sigma \colon Y' \to Y$ and a unique $\mathbb{R}$-b-divisor $\mathbf{M}$ such that $\mathbf{M}_{Y'} = M_{Y'}$ for every $\sigma \colon Y' \to Y$. We call $\mathbf{B}$ the *discriminant $\mathbb{R}$-b-divisor* associated to $f \colon (X, B) \to Y$. The $\mathbb{R}$-b-divisor $\mathbf{M}$ is called the *moduli $\mathbb{R}$-b-divisor* associated to $f \colon (X, B) \to Y$. When $f \colon (X, B) \to Y$ is a basic $\mathbb{Q}$-slc-trivial fibration, it is easy to see that $\mathbf{B}$ and $\mathbf{M}$ are $\mathbb{Q}$-b-divisors by construction.

By using the theory of mixed Hodge structures on cohomology with compact support (see [F23, Chapter 5]), we have:

Theorem 5.5 ([F23, Theorem 6.3.5]). *Let $[X, \omega]$ be a quasi-log scheme and let X' be the union of $X_{-\infty}$ with a (possibly empty) union of some qlc strata of $[X, \omega]$. Then we have the following properties.*

    (i) *(Adjunction for quasi-log schemes). Assume that $X' \neq X_{-\infty}$. Then X' naturally becomes a quasi-log scheme with $\omega' = \omega|_{X'}$ and $X'_{-\infty} = X_{-\infty}$. Moreover, the qlc strata of $[X', \omega']$ are exactly the qlc strata of $[X, \omega]$ that are included in X'.*

    (ii) *(Vanishing theorem for quasi-log schemes). Assume that $\pi \colon X \to S$ is a proper morphism between schemes. Let L be a Cartier divisor on X such that $L - \omega$ is nef and log big over S with respect to $[X, \omega]$. Then $R^i \pi_* (\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)) = 0$ for every $i > 0$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X' on X.*

Theorem 5.5 is a key result in the theory of quasi-log schemes.

Remark 5.6. If $\mathrm{Nqklt}(X, \omega) \neq \mathrm{Nqlc}(X, \omega)$, then $[\mathrm{Nqklt}(X, \omega), \omega|_{\mathrm{Nqklt}(X, \omega)}]$ naturally becomes a quasi-log scheme by adjunction (see Theorem 5.5 (i)).

By using the theory of variations of mixed Hodge structure on cohomology with compact support (see [FF] and [FFS]), we have:

Theorem 5.7 ([FH2, Theorem 5.1]). *Let $f \colon (X, B) \to Y$ be a basic $\mathbb{R}$-slc-trivial fibration such that Y is a smooth quasi-projective variety. We write $K_X + B \sim_{\mathbb{R}} f^* D$. Assume that there exists a simple normal crossing divisor Σ on Y such that $\mathrm{Supp}\, D \subset \Sigma$ and that every stratum of $(X, \mathrm{Supp}\, B)$ is smooth over $Y \setminus \Sigma$. Let $\mathbf{B}$ and $\mathbf{M}$ be the discriminant and moduli $\mathbb{R}$-b-divisors associated to $f \colon (X, B) \to Y$, respectively. Then*

    (i) $\mathbf{K} + \mathbf{B} = \overline{K_Y + B_Y}$ *holds, where $\mathbf{K}$ is the canonical b-divisor of Y, and*

    (ii) $\mathbf{M}_Y$ *is a potentially nef $\mathbb{R}$-divisor on Y with $\mathbf{M} = \overline{\mathbf{M}_Y}$.*

Theorem 5.7 is the most fundamental property of basic slc-trivial fibrations.

Remark 5.8. It is conjectured that $\mathbf{M}_Y$ is semi-ample in Theorem 5.7. Unfortunately, however, it is still widely open. We note that $\mathbf{M}_Y$ is known to be semi-ample when Y is a curve (see [FFL] and [FH2]).

Very roughly speaking, in the author's opinion, the theory of quasi-log schemes is a powerful framework to use mixed Hodge structures on cohomology with compact support for the study of higher-dimensional algebraic varieties and the theory of basic slc-trivial fibrations was constructed in order to make the theory of variations of mixed Hodge structure on cohomology with compact support applicable for some geometric problems.

Finally, we note that Theorems 5.5 and 5.7 hold true over any algebraically closed field k of characteristic zero (see Section 2).

6. ON NORMAL IRREDUCIBLE QUASI-LOG SCHEMES

In this section, we will prove Theorem 1.3. For the proof of Theorem 1.3, we prepare an elementary lemma.

Lemma 6.1. *Let m be any positive integer and let a be any nonnegative real number. If $t \leq -\frac{a}{m}$, then the following inequality*

(6.1)
$$m\left(1 + \lfloor -t \rfloor\right) \geq 1 + \lfloor a \rfloor$$

holds.

Proof. We can uniquely write

$$a = mk + l$$

for some nonnegative integer k with $0 \leq l < m$. Then

$$
\begin{aligned}
m(1 + \lfloor -t \rfloor) - (1 + \lfloor a \rfloor) &\geq m\left(1 + \left\lfloor \frac{a}{m} \right\rfloor\right) - (1 + \lfloor a \rfloor) \\
&\geq m(1 + k) - (1 + mk + m - 1) \\
&= 0
\end{aligned}
$$

holds. This implies the desired inequality. $\qquad\square$

Let us prove Theorem 1.3.

Proof of Theorem 1.3. We divide the proof into several small steps. Steps 1 and 2 are more or less well known. Step 3 is new.

Step 1. By definition, $\mathcal{I}_{X_{-\infty}} = f_* \mathcal{O}_Y(\lceil -(B_Y^{\leq 1}) \rceil - \lfloor B_Y^{>1} \rfloor)$ is an ideal sheaf on X and $\operatorname{Supp} \lfloor B_Y^{>1} \rfloor$ is not dominant onto X by f. Therefore, $\operatorname{rank} f_* \mathcal{O}_Y(\lceil -(B_Y^{\leq 1}) \rceil) = 1$ holds. In particular, we have $\operatorname{rank} f_* \mathcal{O}_X = 1$. Let $f \colon Y \to Z \to X$ be the Stein factorization of $f \colon Y \to X$. Since every irreducible component of Y is dominant onto X and $\operatorname{rank} f_* \mathcal{O}_Y = 1$, $Z \to X$ is a finite birational morphism from a variety Z onto a normal variety X. Then, by Zariski's main theorem, $Z \to X$ is an isomorphism. This means that the natural map $\mathcal{O}_X \to f_* \mathcal{O}_Y$ is an isomorphism. Hence $f \colon (Y, B_Y) \to X$ is a basic $\mathbb{R}$-slc-trivial fibration. By [FH2, Corollary 5.2], we can take a birational morphism $p \colon X' \to X$ from a smooth quasi-projective variety X' satisfying (i), (ii), and (iii). By the argument in Step 3 in the proof of [F30, Theorem 7.1], we can make $p \colon X' \to X$ satisfy (iv). We note that we can directly apply the argument in Step 3 in the proof of [F30, Theorem 7.1] to the basic $\mathbb{R}$-slc-trivial fibration $f \colon (Y, B_Y) \to X$ by [FH2, Corollary 5.2]. Moreover, by the same argument, we can check (v).

Step 2. In this step, we will check that $\mathcal{J}_{\mathrm{Ngklt}}$ and $\mathcal{J}_{\mathrm{Nglc}}$ are well-defined ideal sheaves on X, that is, they are independent of $p \colon X' \to X$.

Let $q \colon X'' \to X'$ be a projective birational morphism from a smooth quasi-projective variety X'' such that $\mathbf{K}_{X''} + \mathbf{B}_{X''} = q^*(\mathbf{K}_{X'} + \mathbf{B}_{X'})$ and that $\operatorname{Supp} \mathbf{B}_{X''}$ is a simple normal crossing divisor on X''. Since $(X', \{\mathbf{B}_{X'}\})$ is kawamata log terminal, $q^* \lfloor \mathbf{B}_{X'} \rfloor - \lfloor \mathbf{B}_{X''} \rfloor$ is an effective q-exceptional divisor on X''. Hence we have

$$q_* \mathcal{O}_{X''}(-\lfloor \mathbf{B}_{X''} \rfloor) = \mathcal{O}_{X'}(-\lfloor \mathbf{B}_{X'} \rfloor)$$

by projection formula. By this fact and Hironaka's resolution of singularities, we can easily see that $\mathcal{J}_{\mathrm{Ngklt}}$ is independent of $p \colon X' \to X$. Since $\mathbf{B}_X$ is effective by construction (see [F30, Lemma 11.2]), $\operatorname{Supp} \mathbf{B}_{X'}^{<0}$ is p-exceptional. Hence $\mathcal{J}_{\mathrm{Ngklt}}$ is an ideal sheaf on X and $\mathcal{J}_{\mathrm{Ngklt}} = p_* \mathcal{O}_{X'}\left(-\lfloor \mathbf{B}_{X'}^{\geq 1} \rfloor\right)$ holds. Of course, $\mathcal{J}_{\mathrm{Ngklt}}$ is a generalization of well-known multiplier ideal sheaves (see [La]). Similarly, since $(X', \{\mathbf{B}_{X'}\} + \mathbf{B}_{X'}^{=1})$ is divisorial log terminal, $q^*\left(\lfloor \mathbf{B}_{X'} \rfloor - \mathbf{B}_{X'}^{=1}\right) - \left(\lfloor \mathbf{B}_{X''} \rfloor - \mathbf{B}_{X''}^{=1}\right)$ is an effective q-exceptional divisor on X''. Hence
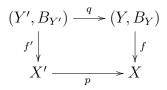
$$q_* \mathcal{O}_{X''}\left(-\lfloor \mathbf{B}_{X''} \rfloor + \mathbf{B}_{X''}^{=1}\right) = \mathcal{O}_{X'}\left(-\lfloor \mathbf{B}_{X'} \rfloor + \mathbf{B}_{X'}^{=1}\right).$$

This means that $\mathcal{J}_{\mathrm{Nglc}}$ is independent of $p \colon X' \to X$ by Hironaka's resolution of singularities. Since $\lceil -(\mathbf{B}_{X'}^{<1}) \rceil$ is effective and p-exceptional, $\mathcal{J}_{\mathrm{Nglc}}$ is an ideal sheaf on X and

$\mathcal{J}_{\mathrm{Nglc}} = p_*\mathcal{O}_{X'}\left(-\lfloor\mathbf{B}_{X'}^{\geq 1}\rfloor\right)$ holds. By definition, $\mathcal{J}_{\mathrm{Nglc}}$ is a generalization of non-lc ideal sheaves.

Step 3. In this step, we will check the inclusions $\mathcal{J}_{\mathrm{Ngklt}} \subset \mathcal{I}_{\mathrm{Nqklt}(X,\omega)}$ and $\mathcal{J}_{\mathrm{Nglc}} \subset \mathcal{I}_{\mathrm{Nqlc}(X,\omega)}$.

We note that $\mathcal{I}_{\mathrm{Nqklt}(X,\omega)} = f_*\mathcal{O}_Y(-\lfloor B_Y\rfloor)$ holds (see [F23, Propositions 6.3.1 and 6.3.2] and the proof of [F23, Theorem 6.3.5 (i)]). Let P be an irreducible component of $\mathrm{Supp}\, B_Y^{\geq 1}$. We take a suitable birational modification $p\colon X' \to X$ satisfying (i)–(v) and consider the induced basic $\mathbb{R}$-slc-trivial fibration $f'\colon (Y', B_{Y'}) \to X'$ (see [FH2, Definition 3.4]). We have the following commutative diagram.

$$
\begin{array}{ccc}
(Y', B_{Y'}) & \xrightarrow{\ q\ } & (Y, B_Y) \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
X' & \xrightarrow[\ p\ ]{} & X
\end{array}
$$

Note that $f'\colon (Y', B_{Y'}) \to X'$ is a basic $\mathbb{R}$-slc-trivial fibration with $K_{Y'}+B_{Y'} = q^*(K_Y+B_Y)$ such that $f'\colon (Y', B_{Y'}) \to X'$ coincides with the base change of $f\colon (Y, B_Y) \to X$ by $p\colon X' \to X$ over some nonempty Zariski open subset of X'. Without loss of generality, by using the flattening theorem (see [RG, Théorème (5.2.2)]), we may assume that the image of $P' := q_*^{-1}P$ by f' is a prime divisor Q on X'. We put $\mathrm{coeff}_P\, B_Y = 1+a$ with $a \geq 0$ and $\mathrm{coeff}_{P'}\, f'^*Q = m > 0$. Then $\mathrm{coeff}_Q\, \mathbf{B}_{X'} = 1-t$ with $t \leq -\frac{a}{m}$. By Lemma 6.1,

$$m\left(1 + \lfloor -t\rfloor\right) \geq 1 + \lfloor a\rfloor.$$

This means that if

$$h \in \Gamma(U, \mathcal{J}_{\mathrm{Ngklt}}) = \Gamma\left(U, p_*\mathcal{O}_{X'}\left(-\lfloor\mathbf{B}_{X'}^{\geq 1}\rfloor\right)\right)$$

then

$$f^*h \in \Gamma\left(f^{-1}(U), \mathcal{O}_Y\left(-\lfloor B_Y^{\geq 1}\rfloor\right)\right)$$

for any Zariski open subset U of X. Therefore, we obtain the desired inclusion

$$\mathcal{J}_{\mathrm{Ngklt}} \subset f_*\mathcal{O}_Y(-\lfloor B_Y\rfloor) = \mathcal{I}_{\mathrm{Nqklt}(X,\omega)}.$$

This is what we wanted. The same argument as above works for $\mathcal{J}_{\mathrm{Nglc}}$ and $\mathcal{I}_{\mathrm{Nqlc}(X,\omega)} = f_*\mathcal{O}_Y(-\lfloor B_Y\rfloor + B_Y^{=1})$. Hence we obtain the desired inclusion $\mathcal{J}_{\mathrm{Nglc}} \subset \mathcal{I}_{\mathrm{Nqlc}(X,\omega)}$.

We finish the proof of Theorem 1.3. $\qquad\square$

7. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 and Corollary 1.2. Our proof of Theorem 1.1 here, which is a combination of Kawamata's X-method with Theorem 1.3 in the framework of quasi-log schemes, is completely different from the proof given in [F20]. A key idea of the proof of Theorem 1.1 below is due to the argument in [FLh2]. Let us prove Theorem 1.1.

Proof of Theorem 1.1. We divide the proof into several small steps.

Step 1. If $\dim X \setminus X_{-\infty} = 0$, then Theorem 1.1 obviously holds true. From now on, we assume that Theorem 1.1 holds for any quasi-log scheme Z with $\dim Z\setminus Z_{-\infty} < \dim X\setminus X_{-\infty}$ by induction on $\dim X \setminus X_{-\infty}$.

Step 2. We take a qlc stratum W of $[X,\omega]$. We put $X' = W \cup X_{-\infty}$. Then, by adjunction (see Theorem 5.5 (i)), X' has a natural quasi-log scheme structure induced by $[X,\omega]$. By the vanishing theorem (see Theorem 5.5 (ii)), we have

$$R^1\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(mL)) = 0$$

for every $m \geq q$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X' on X. Therefore, we obtain that the restriction map

$$\pi_* \mathcal{O}_X(mL) \to \pi_* \mathcal{O}_{X'}(mL)$$

is surjective for every $m \geq q$. Thus, we may assume that $\overline{X \setminus X_{-\infty}}$ is irreducible for the proof of Theorem 1.1 by the following commutative diagram.

$$
\begin{array}{ccccc}
\pi^* \pi_* \mathcal{O}_X(mL) & \longrightarrow & \pi^* \pi_* \mathcal{O}_{X'}(mL) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\mathcal{O}_X(mL) & \longrightarrow & \mathcal{O}_{X'}(mL) & \longrightarrow & 0
\end{array}
$$

Step 3. We put $W = \overline{X \setminus X_{-\infty}}$. By Step 2, W is irreducible. In this step, we further assume that $W \cap \mathrm{Nqklt}(X, \omega) = \emptyset$. In this case, W is a normal variety (see [F23, Lemma 6.3.9]) and $[W, \omega|_W]$ is a quasi-log canonical pair (see [F23, Lemma 6.3.12]). By [F30, Theorem 1.7] (see also Theorem 1.3), there exists a projective birational morphism $p: W' \to W$ from a smooth quasi-projective variety W' such that

$$p^* \omega|_W = K_{W'} + B_{W'} + M_{W'},$$

Supp $B_{W'}$ is a simple normal crossing divisor, $B_{W'} = B_{W'}^{<1}$, Supp $B_{W'}^{<0}$ is p-exceptional, and $M_{W'}$ is a potentially nef $\mathbb{R}$-divisor on W'. Hence $q(p^* L|_W) - (K_{W'} + B_{W'})$ is nef and big over S and $\lceil -B_{W'} \rceil$ is effective and $\pi \circ p$-exceptional. Then, by Theorem 4.2, $\mathcal{O}_{W'}(mp^* L|_W)$ is $\pi \circ p$-generated for every $m \gg 0$. Hence $\mathcal{O}_W(mL|_W)$ is π-generated for every $m \gg 0$. Since $W \cap X_{-\infty} = \emptyset$ by assumption, $\mathcal{O}_X(mL)$ is π-generated for every $m \gg 0$. Therefore, from now on, we may assume that $W \cap \mathrm{Nqklt}(X, \omega) \neq \emptyset$.

Step 4. By [F35, Lemma 4.19], $[W, \omega|_W]$ has a natural quasi-log scheme structure induced by $[X, \omega]$ such that $\mathcal{I}_{\mathrm{Nqklt}(W, \omega|_W)} = \mathcal{I}_{\mathrm{Nqklt}(X, \omega)}$ holds. Let $\nu: Z \to W$ be the normalization. Then, by [F35, Theorem 1.9] (see also [FLh1]), there exists a proper surjective morphism $f': Y' \to Z$ from a quasi-projective globally embedded simple normal crossing pair $(Y', B_{Y'})$ such that every stratum of Y' is dominant onto Z and that

$$(Z, \nu^*(\omega|_W)), f': (Y', B_{Y'}) \to Z$$

naturally becomes a quasi-log scheme with

$$\nu_* \mathcal{I}_{\mathrm{Nqklt}(Z, \nu^*(\omega|_W))} = \mathcal{I}_{\mathrm{Nqklt}(W, \omega|_W)} = \mathcal{I}_{\mathrm{Nqklt}(X, \omega)}.$$

By Theorem 1.3, we can take a projective birational morphism $p: Z' \to Z$ from a smooth quasi-projective variety Z' with

$$\mathbf{K}_{Z'} + \mathbf{B}_{Z'} + \mathbf{M}_{Z'} = p^* \nu^*(\omega|_W)$$

satisfying (i)–(v) in Theorem 1.3 such that the following inclusion

$$\mathcal{J}_{\mathrm{Ngklt}} = p_* \mathcal{O}_{Z'}(-\lfloor \mathbf{B}_{Z'} \rfloor) \subset \mathcal{I}_{\mathrm{Nqklt}(Z, \nu^*(\omega|_W))}$$

holds.

Step 5. By induction on $\dim W$ (see Step 1) or the assumption that $\mathcal{O}_{X_{-\infty}}(mL)$ is π-generated for every $m \gg 0$, $\mathcal{O}_{\mathrm{Nqklt}(X, \omega)}(mL|_{\mathrm{Nqklt}(X, \omega)})$ is π-generated for every $m \gg 0$. As in Step 2, the restriction map

$$\pi_* \mathcal{O}_X(mL) \to \pi_* \mathcal{O}_{\mathrm{Nqklt}(X, \omega)}(mL|_{\mathrm{Nqklt}(X, \omega)})$$

is surjective for $m \geq q$ since $R^1 \pi_* (\mathcal{I}_{\mathrm{Nqklt}(X, \omega)} \otimes \mathcal{O}_X(mL)) = 0$ for $m \geq q$ by the vanishing theorem (see Theorem 5.5 (ii)). Therefore, the relative base locus of $\mathcal{O}_X(mL)$ is disjoint from $\mathrm{Nqklt}(X, \omega)$ for every $m \gg 0$.

Step 6. Without loss of generality, we may further assume that S is affine. Let ℓ be a sufficiently large prime integer. Then $\mathrm{Bs}\,|\ell L| \cap \mathrm{Nqklt}(Z, \nu^*(\omega|_W)) = \mathrm{Bs}\,|\ell L| \cap p(\mathbf{B}_{Z'}^{\geq 1}) = \emptyset$ by Step 5. We note that $\mathbf{B}_{Z'}^{\leq 0}$ is p-exceptional and that $\mathbf{M}_{Z'}$ is nef over S. By Lemma 4.3,

$$H^0\left(Z, \mathcal{O}_Z(\ell^s \nu^*(L|_W)) \otimes p_* \mathcal{O}_{Z'}(-\lfloor \mathbf{B}_{Z'} \rfloor)\right) \otimes \mathcal{O}_Z \to \mathcal{O}_Z(\ell^s \nu^*(L|_W))$$

is surjective on $Z \setminus p(\mathbf{B}_{Z'}^{\geq 1})$ for some positive integer s. We note that

$$
\begin{aligned}
H^0 & \left(Z, \mathcal{O}_Z(m\nu^*(L|_W)) \otimes p_* \mathcal{O}_{Z'}(-\lfloor \mathbf{B}_{Z'} \rfloor)\right) \\
& \subset H^0\left(Z, \mathcal{O}_Z(m\nu^*(L|_W)) \otimes \mathcal{I}_{\mathrm{Nqklt}(Z, \nu^*(\omega|_W))}\right) \\
& = H^0\left(W, \mathcal{O}_W(mL|_W) \otimes \mathcal{I}_{\mathrm{Nqklt}(W, \omega|_W)}\right) \\
& = H^0\left(X, \mathcal{O}_X(mL) \otimes \mathcal{I}_{\mathrm{Nqklt}(X, \omega)}\right)
\end{aligned}
$$

holds for every integer m. Therefore, $\mathrm{Bs}\,|\ell^s L| \cap (X \setminus \mathrm{Nqklt}(X, \omega)) = \emptyset$. This implies that $\mathrm{Bs}\,|\ell^s L| = \emptyset$. We take a sufficiently large prime integer ℓ' with $\ell' \neq \ell$. By the same argument as above, we can find a positive integer s' such that $\mathrm{Bs}\,|\ell'^{s'} L| = \emptyset$. Hence $\mathrm{Bs}\,|mL| = \emptyset$ for every $m \gg 0$ by Lemma 7.1 below.

We finish the proof of Theorem 1.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We have already used the following easy lemma. We give a proof for the sake of completeness.

Lemma 7.1. *Let a and b be positive integers with $1 < a < b$ such that $\gcd(a, b) = 1$. Then, for any positive integer m with $m \geq a\left(b - \lceil \frac{b}{a} \rceil\right)$, there exist nonnegative integers u and v such that $m = ua + vb$.*

Proof. We can uniquely write $m = qa + r$ such that q and r are integers with $q \geq b - \lceil \frac{b}{a} \rceil$ and $0 \leq r \leq a - 1$. If $r = 0$, then it is sufficient to put $u = q$ and $v = 0$. From now on, we assume $r \neq 0$. Then there exists a positive integer c such that $cb = \lfloor \frac{cb}{a} \rfloor a + r$ with $1 \leq c \leq a - 1$. Hence $m = \left(q - \lfloor \frac{cb}{a} \rfloor\right) a + cb$. Note that

$$q - \left\lfloor \frac{cb}{a} \right\rfloor \geq b - \left\lceil \frac{b}{a} \right\rceil - \left\lfloor \frac{(a-1)b}{a} \right\rfloor = 0.$$

Thus it is sufficient to put $u = q - \lfloor \frac{cb}{a} \rfloor$ and $v = c$. $\qquad\qquad\qquad\square$

We close this section with the proof of Corollary 1.2.

Proof of Corollary 1.2. Let (X, Δ) be a log canonical pair. We put $\omega = K_X + \Delta$. Then $[X, \omega]$ naturally becomes a quasi-log scheme with $\mathrm{Nqlc}(X, \omega) = \emptyset$ (see Example 5.3). By assumption, $qL - \omega$ is nef and log big over S with respect to $[X, \omega]$. Hence $\mathcal{O}_X(mL)$ is π-generated for every $m \gg 0$ by Theorem 1.1. $\qquad\qquad\qquad\square$

8. Comments on Ambro's paper: Quasi-log varieties

In this section, we make many comments on [A] to help the reader understand differences between Ambro's original approach in [A] and our framework of quasi-log schemes mainly discussed in [F23, Chapter 6]. Note that [F11] is a gentle introduction to the theory of quasi-log varieties. We also note that a *quasi-log variety* in [A] and [F11] is called a *quasi-log scheme* in the author's recent papers because it may have non-reduced components.

8.0. **Introduction.** By [A, Definition 1], a *generalized log variety* (X, B) is a pair consisting of a normal variety X and an effective $\mathbb{R}$-divisor B on X such that $K_X + B$ is $\mathbb{R}$-Cartier. Note that we sometimes call (X, B) a *normal pair* in some literature. When (X, B) is log canonical, it is called a *log variety*. The main result of [A] is the cone and contraction theorem for generalized log varieties (see [A, Theorem 2]). In order to establish the cone and contraction theorem, Ambro introduced the notion of *quasi-log varieties*, which was motivated by Kawamata's X-method. He also said that the motivation behind [A] is Shokurov's idea that log varieties and non-kawamata log terminal loci should be treated on an equal footing. In [F14], we recovered [A, Theorem 2] without using the framework of quasi-log schemes. The approach in [F14] was influenced by not only Kawamata's X-method but also the theory of (algebraic) multiplier ideal sheaves (see [La]). If the reader is interested only in the cone and contraction theorem for generalized log varieties, then we recommend him or her to see [F14], which seems to be more accessible than [A]. We note that [F7] and [F13] may help the reader understand [F14]. The cone and contraction theorem for generalized log varieties (see [A, Theorem 2] and [F14, Theorem 1.1]) plays a crucial role in the minimal model theory for log surfaces (see [F15], [F19], [F32], [F33], and [FT]).

8.1. **Section 1. Preliminary.** In Section 1 in [A], some standard definitions are collected. In the theory of quasi-log schemes, we usually treat highly singular reducible schemes. Moreover, we have to treat $\mathbb{R}$-Cartier divisors and $\mathbb{R}$-line bundles. We note that Kleiman's famous ampleness criterion does not necessarily hold true for singular complete non-projective schemes (see [F5] and [F32]). On the other hand, the Nakai–Moishezon ampleness criterion for $\mathbb{R}$-line bundles holds for any complete schemes (see [FMi2]). It sometimes may be very useful when we treat $\mathbb{R}$-line bundles on highly singular schemes. Moreover, the Nakai–Moishezon ampleness criterion on complete algebraic spaces is crucial for the proof of the projectivity of some moduli spaces (see [F28]).

8.2. **Section 2. Normal crossing pairs.** Ambro defined *multicrossing singularities* and considered their associated hypercoverings (see [A, Definition 2.1 and Lemma 2.2]). Moreover, he defined *multicrossing divisors*. Then he finally introduced the notion of (*embedded*) *normal crossing pairs* (see [A, Definitions 2.3 and 2.7]). He used embedded normal crossing pairs to define quasi-log varieties. On the other hand, our framework of the theory of quasi-log schemes in [F23, Chapter 6] uses the notion of *globally embedded simple normal crossing pairs*. Note that [F23, Propositions 6.3.1, 6.3.2, and 6.3.3] is much more flexible than [A, Proposition 2.8]. We think that our approach is more accessible than Ambro's because *globally embedded simple normal crossing pairs* are much easier to treat than *embedded normal crossing pairs*. We can use the standard techniques in the theory of minimal models for higher-dimensional algebraic varieties to treat globally embedded simple normal crossing pairs. In general, simple normal crossing divisors behave much better than normal crossing divisors (see [F6]).

8.3. **Section 3. Vanishing theorems.** Section 3 in [A] is a short section on injectivity, vanishing, and torsion-free theorems. The proof of [A, Theorems 3.1 and 3.2] is hard to follow. In [F23, Chapter 5] (see also [F7], [F14], [F22], [F27], and so on), we give a rigorous proof of [A, Theorems 3.1 and 3.2] and treat some more general results. Our approach is slightly different from Ambro's and is based on the theory of mixed Hodge structures on cohomology with compact support. A survey article [F26] may help the reader understand our approach to vanishing theorems. The reader can find some related vanishing theorems in [F17], [F18], [F29], and so on.

8.4. **Section 4. Quasi-log varieties.** As we mentioned above, a quasi-log variety is called a quasi-log scheme in [F23]. Section 4 is the main part of [A]. In Section 4, Ambro defined quasi-log varieties in [A, Definition 4.1]. Ambro's definition is slightly different from ours

in [F23, Definition 6.2.2]. Note that (Y, B_Y) in [A, Definition 4.1] is an embedded normal crossing pair and (Y, B_Y) in [F23, Definition 6.2.2] is a globally embedded simple normal crossing pair. For the details of this difference, see [F24]. The most important result in this section is [A, Theorem 4.4], which is *adjunction* and *vanishing* for quasi-log varieties. In [F23, Section 6.3], we prepare some useful propositions (see [F23, Propositions 6.3.1, 6.3.2, and 6.3.3]) and prove adjunction and vanishing for quasi-log schemes in [F23, Theorem 6.3.5]. We also discuss some other basic properties of quasi-log schemes in [F23, Sections 6.3 and 6.4]. Note that a *qlc center* in [A] is called a *qlc stratum* in [F23, Chapter 6]. We also note that a *qlc center which is not maximal with respect to the inclusion* in [A] is called a *qlc center* in [F23, Chapter 6]. Hence $\mathrm{LCS}(X)$ in [A, Definition 4.6] is nothing but $\mathrm{Nqklt}(X, \omega)$ in [F23, Chapter 6]. We can recover [A, Proposition 4.7] by [F23, Lemma 6.3.9]. Our proof seems to be simpler. Note that [A, Proposition 4.8] is [F23, Theorem 6.3.11]. In [FH2] (see also [FH3]), we completely generalize [A, Theorem 4.9]. Our treatment depends on the theory of variations of mixed Hodge structure on cohomology with compact support (see [FF], [FFS], [F30], and [FFL]). On the other hand, [A, Theorem 4.11] only uses the theory of variations of pure Hodge structure.

8.5. **Section 5. The cone theorem.** In [A, Section 5], the cone and contraction theorem for quasi-log schemes was established in full generality. The results in [A, Section 5] are recovered in [F23, Sections 6.5, 6.6, and 6.7]. Although we slightly changed and improved some arguments, the treatment in [F23, Sections 6.5, 6.6, and 6.7] is essentially the same as that in [A, Section 5]. The basepoint-free theorem for quasi-log schemes (see [A, Theorem 5.1] and [F23, Theorem 6.5.1]) is generalized in various directions (see [F8], [F10], [F21], [F25], [F34], [FLh2], [FMi1], and [Li]). The theory of quasi-log schemes gives a very powerful framework for basepoint-freeness (see also the proof of Theorem 1.1 in Section 7 in this paper).

8.6. **Section 6. Quasi-log Fano contractions.** In [A, Section 6], Ambro specialized some results in [A, Section 5] for quasi-log Fano contraction morphisms. In [F23, Section 6.8], we treat (*relative*) *quasi-log Fano schemes*. Moreover, in [FLw], [F31], [F35], and [FH1], we discuss simple connectedness, rationally chain connectedness, lengths of rational curves for (relative) quasi-log Fano schemes. In [F31] and [F35], we use not only quasi-log schemes but also some results obtained by the theory of basic slc-trivial fibrations. Moreover, in [FH1], we also use the minimal model program for log canonical pairs. Hence the results in [F31], [F35], and [FH1] are much more general than those in [A, Section 6].

8.7. **Section 7. The log big case.** In Section 7, Ambro replaced the ampleness in some theorems with the *nef and log bigness*. Note that [A, Theorem 7.2] is the basepoint-free theorem of Reid–Fukuda type for quasi-log schemes, which is Theorem 1.1 in this paper. In [A], there is no detail of the proof of [A, Theorem 7.2]. Now we have a rigorous proof of [A, Theorem 7.2]. The reader can find vanishing theorems for nef and log big divisors in [F23, Theorem 5.8.2 and Theorem 6.3.5 (ii)]. We note that [F23, Theorem 6.3.8] is a slight generalization of [A, Theorem 7.3]. We know that everything in [A, Section 7] holds true. We note that the proof of Theorem 1.1 in this paper depends on some deep results in the theory of variations of mixed Hodge structure on cohomology with compact support (see [FF], [FFS], [F30], [F35], [FH2], and so on).

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