NOTES ON TORIC VARIETIES FROM MORI THEORETIC VIEWPOINT, II

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Dedicated to Professor Shigeyuki Kondō on the occasion of his sixtieth birthday

Abstract. We give new estimates of lengths of extremal rays of birational type for toric varieties. We can see that our new estimates are the best by constructing some examples explicitly. As applications, we discuss the nefness and pseudo-effectivity of adjoint bundles of projective toric varieties. We also treat some generalizations of Fujita's freeness and very ampleness for toric varieties.

Contents

1. Introduction

The following theorem is one of the main results of this paper. Our proof of Theorem 1.1 uses the framework of the toric Mori theory developed by [R], [F1], [F2], [F4], [FS1], and so on.

Theorem 1.1 (Theorem 4.2.3 and Corollary 4.2.4)**.** *Let X be a $\mathbb{Q}$-Gorenstein projective toric n-fold and let D be an ample Cartier divisor on X. Then $K_X+(n-1)D$ is pseudo-effective if and only if $K_X+(n-1)D$ is nef. In particular, if X is Gorenstein, then*

$$H^0(X, \mathcal{O}_X(K_X+(n-1)D)) \neq 0$$

if and only if the complete linear system $|K_X+(n-1)D|$ is basepoint-free.

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This theorem was inspired by Lin's paper (see [Li]). Our proof of Theorem 1.1 depends on the following new estimates of lengths of extremal rays of birational type for toric varieties.

Theorem 1.2 (Theorem 3.2.1). *Let $f : X \to Y$ be a projective toric morphism with $\dim X = n$. Assume that K_X is $\mathbb{Q}$-Cartier. Let R be a K_X-negative extremal ray of $\mathrm{NE}(X/Y)$ and let $\varphi_R : X \to W$ be the contraction morphism associated to R. We put*

$$l(R) = \min_{[C] \in R} (-K_X \cdot C).$$

and call it the length of R. Assume that φ_R is birational. Then we obtain

$$l(R) < d + 1,$$

where

$$d = \max_{w \in W} \dim \varphi_R^{-1}(w) \le n - 1.$$

When $d = n - 1$, we have a sharper inequality

$$l(R) \le d = n - 1.$$

In particular, if $l(R) = n - 1$, then $\varphi_R : X \to W$ can be described as follows. There exists a torus invariant smooth point $P \in W$ such that $\varphi_R : X \to W$ is a weighted blow-up at P with the weight $(1, a, \cdots, a)$ for some positive integer a. In this case, the exceptional locus E of φ_R is a torus invariant prime divisor and is isomorphic to $\mathbb{P}^{n-1}$. Moreover, X is $\mathbb{Q}$-factorial in a neighborhood of E.

Theorem 1.2 supplements [F1, Theorem 0.1] (see also [F2, Theorem 3.13]). We will see that the estimates obtained in Theorem 1.2 are the best by constructing some examples explicitly (see Examples 3.3.1 and 3.3.2). For lengths of extremal rays for non-toric varieties, see [K]. As an application of Theorem 1.2, we can prove the following theorem on lengths of extremal rays for $\mathbb{Q}$-Gorenstein toric varieties.

Theorem 1.3 (Theorem 3.2.9). *Let X be a $\mathbb{Q}$-Gorenstein projective toric n-fold with $\rho(X) \ge 2$. Let R be a K_X-negative extremal ray of $\mathrm{NE}(X)$ such that*

$$l(R) = \min_{[C] \in R} (-K_X \cdot C) > n - 1.$$

Then the extremal contraction $\varphi_R : X \to W$ associated to R is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$.

As a direct easy consequence of Theorem 1.3, we obtain the following corollary, which supplements Theorem 1.1.

Corollary 1.4 (Corollary 4.2.5). *Let X be a $\mathbb{Q}$-Gorenstein projective toric n-fold and let D be an ample Cartier divisor on X. If $\rho(X) \ge 3$, then $K_X + (n-1)D$ is always nef. More precisely, if $\rho(X) \ge 2$ and X is not a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$, then $K_X + (n-1)D$ is nef.*

In this paper, we also give some generalizations of Fujita's freeness and very ampleness for toric varieties based on our powerful vanishing theorem (see [F5] and [F6]). As a very special case of our generalization of Fujita's freeness for toric varieties (see Theorem 4.1.1), we can easily recover some parts of Lin's theorem (see [Li, Main Theorem A]).

Theorem 1.5 (Corollary 4.1.2). *Let X be an n-dimensional projective toric variety and let D be an ample Cartier divisor on X. Then the reflexive sheaf $\mathcal{O}_X(K_X + (n+1)D)$ is generated by its global sections.*

By the same way, we can obtain a generalization of Fujita's very ampleness for toric varieties (see Theorem 4.1.8). We note that Sam Payne completely settled Fujita's very ampleness conjecture for singular projective toric varieties by his clever combinatorial approach (see [P]). As was mentioned above, we do not use combinatorial arguments, but apply some vanishing theorems for the proof of Theorem 1.6.

Theorem 1.6 (Theorem 4.1.6)**.** *Let $f : X \to Y$ be a proper surjective toric morphism, let Δ be a reduced torus invariant divisor on X such that $K_X + \Delta$ is Cartier, and let D be an f-ample Cartier divisor on X. Then $\mathcal{O}_X(K_X + \Delta + kD)$ is f-very ample for every $k \geq \max_{y\in Y} \dim f^{-1}(y) + 2$.*

For the precise statements of our generalizations of Fujita's freeness and very ampleness for toric varieties, see Theorems 4.1.1 and 4.1.8. We omit them here since they are technically complicated.

This paper is organized as follows. In Section 2, we collect some basic definitions and results. In subsection 2.1, we explain the basic concepts of the toric geometry. In subsection 2.2, we recall the definitions of *the Kleiman–Mori cone*, *the nef cone*, *the ample cone*, and *the pseudo-effective cone* for toric varieties, and some related results. In subsection 2.3, we explain *subadjunction* for $\mathbb{Q}$-factorial toric varieties. Section 3 is the main part of this paper. After recalling the known estimates of lengths of extremal rays for projective toric varieties in subsection 3.1, we give new estimates of lengths of extremal rays of toric birational contraction morphisms in subsection 3.2. In subsection 3.3, we see that the estimates obtained in subsection 3.2 are the best by constructing some examples explicitly. Section 4 treats Fujita's freeness and very ampleness for toric varieties. The results in subsection 4.1 depend on our powerful vanishing theorem for toric varieties and are independent of our estimates of lengths of extremal rays for toric varieties. Therefore, subsection 4.1 is independent of the other parts of this paper. In subsection 4.2, we discuss Lin's problem (see [Li]) related to Fujita's freeness for toric varieties. We use our new estimates of lengths of extremal rays in this subsection. Subsection 4.3 is a supplement to Fujita's paper: [Fuj]. This paper contains various supplementary results for [Fuj], [Ful], [Li], and so on.

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We will work over an arbitrary algebraically closed field throughout this paper. For the standard notations of the minimal model program, see [F7] and [F8]. For the toric Mori theory, we recommend the reader to see [R], [Ma, Chapter 14], [F1], and [FS1] (see also [CLS]).

2. Preliminaries

This section collects some basic definitions and results.

2.1. **Basics of the toric geometry.**

In this subsection, we recall the basic notion of toric varieties and fix the notation. For the details, see [O], [Ful], [R], or [Ma, Chapter 14] (see also [CLS]).

2.1.1. Let $N \simeq \mathbb{Z}^n$ be a lattice of rank n. A toric variety $X(\Sigma)$ is associated to a *fan* Σ, a correction of convex cones $\sigma \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ satisfying:

(i) Each convex cone σ is a rational polyhedral cone in the sense that there are finitely many $v_1, \cdots, v_s \in N \subset N_{\mathbb{R}}$ such that
$$\sigma = \{r_1 v_1 + \cdots + r_s v_s;\ r_i \geq 0\} =: \langle v_1, \cdots, v_s\rangle,$$
and it is strongly convex in the sense that
$$\sigma \cap -\sigma = \{0\}.$$
(ii) Each face τ of a convex cone $\sigma \in \Sigma$ is again an element in Σ.
(iii) The intersection of two cones in Σ is a face of each.

Definition 2.1.2. The *dimension* $\dim \sigma$ of a cone σ is the dimension of the linear space $\mathbb{R} \cdot \sigma = \sigma + (-\sigma)$ spanned by σ.

We define the sublattice N_σ of N generated (as a subgroup) by $\sigma \cap N$ as follows:
$$N_\sigma := \sigma \cap N + (-\sigma \cap N).$$

If σ is a k-dimensional simplicial cone, and $v_1, \cdots, v_k$ are the first lattice points along the edges of σ, the *multiplicity* of σ is defined to be the *index* of the lattice generated by the $\{v_1, \cdots, v_k\}$ in the lattice N_σ;
$$\mathrm{mult}(\sigma) := [N_\sigma : \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k].$$
We note that the affine toric variety $X(\sigma)$ associated to the cone σ is smooth if and only if $\mathrm{mult}(\sigma) = 1$.

The following is a well-known fact. See, for example, [Ma, Lemma 14-1-1].

Lemma 2.1.3. *A toric variety $X(\Sigma)$ is $\mathbb{Q}$-factorial if and only if each cone $\sigma \in \Sigma$ is simplicial.*

2.1.4. The *star* of a cone τ can be defined abstractly as the set of cones σ in Σ that contain τ as a face. Such cones σ are determined by their images in $N(\tau) := N/N_\tau$, that is, by
$$\overline{\sigma} = \sigma + (N_\tau)_{\mathbb{R}}/(N_\tau)_{\mathbb{R}} \subset N(\tau)_{\mathbb{R}}.$$
These cones $\{\overline{\sigma}; \tau \prec \sigma\}$ form a fan in $N(\tau)$, and we denote this fan by $\mathrm{Star}(\tau)$. We set $V(\tau) = X(\mathrm{Star}(\tau))$, that is, the toric variety associated to the fan $\mathrm{Star}(\tau)$. It is well known that $V(\tau)$ is an $(n-k)$-dimensional closed toric subvariety of $X(\Sigma)$, where $\dim \tau = k$. If $\dim V(\tau) = 1$ (resp. $n-1$), then we call $V(\tau)$ a *torus invariant curve* (resp. *torus invariant divisor*). For the details about the correspondence between τ and $V(\tau)$, see [Ful, 3.1 Orbits].

2.1.5 (Intersection theory for $\mathbb{Q}$-factorial toric varieties)**.** Assume that Σ is simplicial. If $\sigma, \tau \in \Sigma$ span $\gamma \in \Sigma$ with $\sigma \cap \tau = \{0\}$, then
$$V(\sigma) \cdot V(\tau) = \frac{\mathrm{mult}(\sigma) \cdot \mathrm{mult}(\tau)}{\mathrm{mult}(\gamma)} V(\gamma)$$
in the *Chow group* $A^*(X)_{\mathbb{Q}}$. For the details, see [Ful, 5.1 Chow groups]. If σ and τ are contained in no cone of Σ, then $V(\sigma) \cdot V(\tau) = 0$.

2.2. **Cones of divisors.** In this subsection, we explain various cones of divisors and some related topics.

2.2.1. Let $f : X \to Y$ be a proper toric morphism; a 1-cycle of X/Y is a formal sum $\sum a_i C_i$ with complete curves C_i in the fibers of f, and $a_i \in \mathbb{Z}$. We put
$$Z_1(X/Y) := \{\text{1-cycles of } X/Y\},$$
and
$$Z_1(X/Y)_{\mathbb{R}} := Z_1(X/Y) \otimes \mathbb{R}.$$

There is a pairing

$$\mathrm{Pic}(X) \times Z_1(X/Y)_{\mathbb{R}} \to \mathbb{R}$$

defined by $(\mathcal{L}, C) \mapsto \deg_C \mathcal{L}$, extended by bilinearity. We define

$$N^1(X/Y) := (\mathrm{Pic}(X) \otimes \mathbb{R})/\equiv$$

and

$$N_1(X/Y) := Z_1(X/Y)_{\mathbb{R}}/\equiv,$$

where the *numerical equivalence* $\equiv$ is by definition the smallest equivalence relation which makes N^1 and N_1 into dual spaces.

Inside $N_1(X/Y)$ there is a distinguished cone of effective 1-cycles of X/Y,

$$\mathrm{NE}(X/Y) = \{\, Z \mid Z \equiv \sum a_i C_i \text{ with } a_i \in \mathbb{R}_{\geq 0}\} \subset N_1(X/Y),$$

which is usually called the *Kleiman–Mori cone* of $f : X \to Y$. It is known that $\mathrm{NE}(X/Y)$ is a rational polyhedral cone. A face $F \prec \mathrm{NE}(X/Y)$ is called an *extremal face* in this case. A one-dimensional extremal face is called an *extremal ray*.

We define the *relative Picard number* $\rho(X/Y)$ by

$$\rho(X/Y) := \dim_{\mathbb{Q}} N^1(X/Y) < \infty.$$

An element $D \in N^1(X/Y)$ is called *f-nef* if $D \geq 0$ on $\mathrm{NE}(X/Y)$.

If X is complete and Y is a point, then we write $\mathrm{NE}(X)$ and $\rho(X)$ for $\mathrm{NE}(X/Y)$ and $\rho(X/Y)$, respectively. We note that $N_1(X/Y) \subset N_1(X)$, and $N^1(X/Y)$ is the corresponding quotient of $N^1(X)$.

From now on, we assume that X is complete. We define the *nef cone* $\mathrm{Nef}(X)$, the *ample cone* $\mathrm{Amp}(X)$, and the *pseudo-effective cone* $\mathrm{PE}(X)$ in $N^1(X)$ as follows.

$$\mathrm{Nef}(X) = \{D \,|\, D \text{ is nef}\},$$

$$\mathrm{Amp}(X) = \{D \,|\, D \text{ is ample}\}$$

and

$$\mathrm{PE}(X) = \left\{ D \;\middle|\; \begin{array}{l} D \equiv \sum a_i D_i \text{ such that } D_i \text{ is an effective} \\ \text{Cartier divisor and } a_i \in \mathbb{R}_{\geq 0} \text{ for every } i \end{array} \right\}.$$

It is easy to see that

$$\mathrm{Amp}(X) \subset \mathrm{Nef}(X) \subset \mathrm{PE}(X).$$

The reader can find various examples of cones of divisors and curves in [F3], [FP], and [FS2].

Lemma 2.2.2. *Let X be a complete toric variety and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then D is pseudo-effective if and only if $\kappa(X, D) \geq 0$, that is, there exists a positive integer m such that mD is Cartier and that*

$$H^0(X, \mathcal{O}_X(mD)) \neq 0.$$

More generally, g^*D is pseudo-effective for some projective birational toric morphism $g : Z \to X$ from a smooth projective toric variety Z if and only if $\kappa(X, D) \geq 0$.

Proof. It is sufficient to prove that $\kappa(X, D) \geq 0$ when g^*D is pseudo-effective. By replacing X and D with Z and g^*D, we may assume that X is a smooth projective toric variety. In this case, it is easy to see that $\mathrm{PE}(X)$ is spanned by the numerical equivalence classes of torus invariant prime divisors (see, for example, [CLS, Lemma 15.1.8]). Therefore, we can write $D \equiv \sum_i a_i D_i$ where D_i is a torus invariant prime divisor and $a_i \in \mathbb{Q}_{>0}$ for every i since D is a $\mathbb{Q}$-divisor. Thus, we obtain $D \sim_{\mathbb{Q}} \sum_i a_i D_i \geq 0$. This implies $\kappa(X, D) \geq 0$. $\square$

2.2.3. Let X be a complete toric variety and let $g : Z \to X$ be a projective birational toric morphism from a smooth projective toric variety Z. Then

$$g^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Z)$$

induces a natural inclusion

$$N^1(X) \hookrightarrow N^1(Z).$$

By this inclusion, we can see $N^1(X)$ as a linear subspace of $N^1(Z)$. It is well known that $\mathrm{PE}(Z)$ is a rational polyhedral cone in $N^1(Z)$ (see, for example, [CLS, Lemma 15.1.8]). Note that the inclusion $\mathrm{PE}(X) \subset N^1(X) \cap \mathrm{PE}(Z)$ is obvious. The opposite inclusion $\mathrm{PE}(X) \supset N^1(X) \cap \mathrm{PE}(Z)$ follows from Lemma 2.2.2. Anyway, the equality

$$\mathrm{PE}(X) = N^1(X) \cap \mathrm{PE}(Z)$$

holds. In particular, we have the following statement.

Proposition 2.2.4. *Let X be a complete toric variety. Then $\mathrm{PE}(X)$ is a rational polyhedral cone in $N^1(X)$.*

The following lemma is well known and is very important. We will use it in the subsequent sections repeatedly.

Lemma 2.2.5. *Let $f : X \to Y$ be a proper toric morphism and let D be an f-nef Cartier divisor on X. Then D is f-free, that is,*

$$f^*f_*\mathcal{O}_X(D) \to \mathcal{O}_X(D)$$

is surjective.

Proof. See, for example, [N, Chapter VI. 1.13. Lemma]. □

We close this subsection with an easy example. It is well known that $\mathrm{NE}(X)$ is spanned by the numerical equivalence classes of torus invariant irreducible curves. However, the dual cone $\mathrm{Nef}(X)$ of $\mathrm{NE}(X)$ is not always spanned by the numerical equivalence classes of torus invariant prime divisors.

Example 2.2.6. We consider $\mathbb{P}^1 \times \mathbb{P}^1$. Let $p_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the i-th projection for $i = 1, 2$. Let D_1, D_2 (resp. D_3, D_4) be the torus invariant curves in the fibers of p_1 (resp. p_2). Let $X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the blow-up at the point $P = D_1 \cap D_3$ and let E be the exceptional curve on X. Let D'_i denote the strict transform of D_i on X for all i. Then $\mathrm{NE}(X)$ is spanned by the numerical equivalence classes of E, D'_1, and D'_3. On the other hand, $\mathrm{Nef}(X) \subset N^1(X)$ is spanned by D'_2, D'_4, and $D'_1 + D'_3 + E$. Therefore, the extremal ray of $\mathrm{Nef}(X)$ is not necessarily spanned by a torus invariant prime divisor.

2.3. **Subadjunction.** In this subsection, we quickly explain *subadjunction* for $\mathbb{Q}$-factorial toric varieties for the reader's convenience. We note that subadjunction plays an important role in the theory of minimal models (see, for example, [F7, §14. Shokurov's differents]).

Lemma 2.3.1 (Subadjunction)**.** *Let X be a $\mathbb{Q}$-factorial toric variety and let $\{D_i\}_{i\in I}$ be the set of all torus invariant prime divisors on X. We consider $D = \sum_{i\in I} d_i D_i$, where $d_i \in \mathbb{Q}$ and $0 \le d_i \le 1$ for every i. Since X is a toric variety, we can put $K_X = -\sum_{i\in I} D_i$. We assume $d_{i_0} = 1$ for some $i_0 \in I$. We put $S = D_{i_0}$. Let $\{B_j\}_{j\in J}$ be the set of all torus invariant prime divisors on S. Then the following formula*

$$(K_X + D)|_S = K_S + \sum_{j\in J} b_j B_i \tag{2.1}$$

holds, where $K_S = -\sum_{j\in J} B_j$, $b_j \in \mathbb{Q}$ and $0 \le b_j \le 1$ for every j. Moreover, $b_j = 1$ holds in (2.1) *if and only if there exists $i(j) \in I$ such that $d_{i(j)} = 1$ and that $B_j = D_{i(j)} \cap S$. We note that $\sum_{j\in J} b_j B_j$ in* (2.1) *is usually called a* different.

Proof. We note that

$$K_X + D = K_X + \sum_{i\in I} D_i - \sum_{i\in I}(1-d_i)D_i = -\sum_{i\in I}(1-d_i)D_i.$$

Therefore, we have

$$(K_X + D)|_S = -\sum_{i\in I}(1-d_i)D_i\cdot S = K_S + \sum_{j\in J} B_j - \sum_{i\in I}(1-d_i)D_i\cdot S.$$

We put

$$\sum_{j\in J} b_j B_j = \sum_{j\in J} B_j - \sum_{i\in I}(1-d_i)D_i\cdot S. \tag{2.2}$$

Then we obtain $b_j \in \mathbb{Q}$ and $0 \le b_j \le 1$ for every j by 2.1.5. By (2.2), it is easy to see that $b_j = 1$ holds if and only if there exists $i(j) \in I$ such that $d_{i(j)} = 1$ and $D_{i(j)} \cap S = B_j$. □

3. Lengths of extremal rays

In this section, we discuss some estimates of lengths of extremal rays of projective toric morphisms.

3.1. **Quick review of the known estimates.** In this subsection, we recall the known estimates of lengths of extremal rays for toric varieties. The first result is [F1, Theorem 0.1] (see also [F2, Theorem 3.13]).

Theorem 3.1.1. *Let $f : X \to Y$ be a projective toric morphism with $\dim X = n$ and let $\Delta = \sum \delta_i \Delta_i$ be an $\mathbb{R}$-divisor on X such that Δ_i is a torus invariant prime divisor and $0 \le \delta_i \le 1$ for every i. Assume that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let R be an extremal ray of $\mathrm{NE}(X/Y)$. Then there exists a curve C on X such that $[C] \in R$ and*

$$-(K_X + \Delta)\cdot C \le n + 1.$$

More precisely, we can choose C such that

$$-(K_X + \Delta)\cdot C \le n$$

unless $X \simeq \mathbb{P}^n$ and $\sum \delta_i < 1$. We note that if X is complete then we can make C a torus invariant curve on X.

Our proof of Theorems 3.1.1 and 3.2.1 below heavily depends on Reid's description of toric extremal contraction morphisms (see [R] and [Ma, Chapter 14]).

3.1.2 (Reid's description of toric extremal contraction morphisms)**.** Let $f : X \to Y$ be a projective surjective toric morphism from a complete $\mathbb{Q}$-factorial toric n-fold and let R be an extremal ray of $\mathrm{NE}(X/Y)$. Let $\varphi_R : X \to W$ be the extremal contraction associated to R. We write

$$\begin{array}{rccc} & A & \longrightarrow & B \\ & \cap & & \cap \\ \varphi_R : & X & \longrightarrow & W, \end{array}$$

where A is the exceptional locus of φ_R and B is the image of A by φ_R. Then there exist torus invariant prime divisors $E_1, \cdots, E_\alpha$ on X with $0 \le \alpha \le n-1$ such that E_i is negative on R for $1 \le i \le \alpha$ and that A is $E_1 \cap \cdots \cap E_\alpha$. In particular, A is an irreducible torus invariant subvariety of X with $\dim A = n - \alpha$. Note that $\alpha = 0$ if and only if $A = X$, that is, φ_R is a Fano contraction. There are torus invariant prime divisors $E_{\beta+1}, \cdots, E_{n+1}$ on X with $\alpha \le \beta \le n-1$ such that E_i is positive on R for $\beta + 1 \le i \le n+1$. Let F be a general fiber of $A \to B$. Then F is a $\mathbb{Q}$-factorial toric Fano variety with $\rho(F) = 1$ and $\dim F = n - \beta$. The divisors $E_{\beta+1}|_F, \cdots, E_{n+1}|_F$ define all the torus invariant prime divisors on F. In particular, B is an irreducible torus invariant subvariety of W with

$\dim B = \beta - \alpha$. When X is not complete, we can reduce it to the case where X is complete by the equivariant completion theorem in [F2]. For the details, see [S].

3.1.3. We quickly review the idea of the proof of Theorem 3.1.1 in [F1]. We will use the same idea in the proof of Theorem 3.2.1 below. By replacing X with its projective $\mathbb{Q}$-factorialization, we may assume that X is $\mathbb{Q}$-factorial. Let R be an extremal ray of $\mathrm{NE}(X/Y)$. Then we consider the extremal contraction $\varphi_R : X \to W$ associated to R. If X is not projective, then we can reduce it to the case where X is projective by the equivariant completion theorem (see [F2]). By Reid's combinatorial description of φ_R, any fiber F of φ_R is a $\mathbb{Q}$-factorial projective toric variety with $\rho(F) = 1$. By subadjunction (see Lemma 2.3.1), we can compare $-(K_X + \Delta) \cdot C$ with $-K_F \cdot C$, where C is a curve on F. So, the key ingredient of the proof of Theorem 3.1.1 is the following proposition.

Proposition 3.1.4. *Let X be a $\mathbb{Q}$-factorial projective toric n-fold with $\rho(X) = 1$. Assume that $-K_X \cdot C > n$ for every integral curve C on X. Then $X \simeq \mathbb{P}^n$.*

For the proof, see [F1, Proposition 2.9]. Our proof heavily depends on the calculation described in 3.1.8 below.

3.1.5 (Supplements to [F4])**.** By the same arguments as in the proof of Proposition 3.1.4, we can obtain the next proposition, which is nothing but [F4, Proposition 2.1].

Proposition 3.1.6. *Let X be a $\mathbb{Q}$-factorial projective toric n-fold with $\rho(X) = 1$ such that $X \not\simeq \mathbb{P}^n$. Assume that $-K_X \cdot C \geq n$ for every integral curve C on X. Then X is isomorphic to the weighted projective space $\mathbb{P}(1, 1, 2, \cdots, 2)$.*

The following proposition, which is missing in [F4], is a characterization of hyperquadrics for toric varieties (see Corollary of [KO, Theorem 2.1]). This proposition says that the results in [F4] are compatible with [Fuj, Theorem 2 (a)].

Proposition 3.1.7. *Let X be a projective toric n-fold with $n \geq 2$. We assume that $-K_X \equiv nD$ for some Cartier divisor D on X and $\rho(X) = 1$. Then D is very ample and $\Phi_{|D|} : X \hookrightarrow \mathbb{P}^{n+1}$ embeds X into $\mathbb{P}^{n+1}$ as a hyperquadric.*

Proof. By [F4, Theorem 3.2, Remark 3.3, and Theorem 3.4], there exists a crepant toric resolution $\varphi : Y \to X$, where $Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ or $Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. We note that $X = \mathbb{P}(1, 1, 2, \cdots, 2)$ when $Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$. We also note that X is not $\mathbb{Q}$-factorial if $Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. Let $\mathcal{O}_Y(1)$ be the tautological line bundle of the $\mathbb{P}^{n-1}$-bundle Y over $\mathbb{P}^1$. Then we have $\mathcal{O}_Y(-K_Y) \simeq \mathcal{O}_Y(n)$. We can directly check that $\dim H^0(Y, \mathcal{O}_Y(1)) = n + 2$. We consider $\Phi_{|\mathcal{O}_Y(1)|} : Y \to \mathbb{P}^{n+1}$. By construction,

$$\Phi_{|\mathcal{O}_Y(1)|} : Y \xrightarrow{\varphi} X \xrightarrow{\pi} \mathbb{P}^{n+1}$$

is the Stein factorization of $\Phi_{|\mathcal{O}_Y(1)|} : Y \to \mathbb{P}^{n+1}$. By construction again, we have $\mathcal{O}_Y(1) \simeq \varphi^*\mathcal{O}_X(D)$. Since we can directly check that

$$\mathrm{Sym}^k H^0(Y, \mathcal{O}_Y(1)) \to H^0(Y, \mathcal{O}_Y(k))$$

is surjective for every $k \in \mathbb{Z}_{>0}$, we see that

$$\mathrm{Sym}^k H^0(X, \mathcal{O}_X(D)) \to H^0(X, \mathcal{O}_X(kD))$$

is also surjective for every $k \in \mathbb{Z}_{>0}$. This means that $\mathcal{O}_X(D)$ is very ample. In particular, $\pi : X \to \mathbb{P}^{n+1}$ is nothing but the embedding $\Phi_{|D|} : X \hookrightarrow \mathbb{P}^{n+1}$. Since $D^n = (\mathcal{O}_Y(1))^n = 2$, X is a hyperquadric in $\mathbb{P}^{n+1}$. $\square$

As was mentioned above, the following calculation plays an important role in the proof of Proposition 3.1.4.

3.1.8 (Fake weighted projective spaces)**.** Now we fix $N \simeq \mathbb{Z}^n$. Let $\{v_1, \cdots, v_{n+1}\}$ be a set of primitive vectors of N such that $N_{\mathbb{R}} = \sum_i \mathbb{R}_{\geq 0} v_i$. We define n-dimensional cones

$$\sigma_i := \langle v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_{n+1} \rangle$$

for $1 \leq i \leq n+1$. Let Σ be the complete fan generated by n-dimensional cones σ_i and their faces for all i. Then we obtain a complete toric variety $X = X(\Sigma)$ with Picard number $\rho(X) = 1$. We call it a *$\mathbb{Q}$-factorial toric Fano variety with Picard number one.* It is sometimes called a *fake weighted projective space.* We define $(n-1)$-dimensional cones $\mu_{i,j} = \sigma_i \cap \sigma_j$ for $i \neq j$. We can write $\sum_i a_i v_i = 0$, where $a_i \in \mathbb{Z}_{>0}$, $\gcd(a_1, \cdots, a_{n+1}) = 1$, and $a_1 \leq a_2 \leq \cdots \leq a_{n+1}$ by changing the order. Then we obtain

$$0 < V(v_{n+1}) \cdot V(\mu_{n,n+1}) = \frac{\mathrm{mult}(\mu_{n,n+1})}{\mathrm{mult}(\sigma_n)} \leq 1,$$

$$V(v_i) \cdot V(\mu_{n,n+1}) = \frac{a_i}{a_{n+1}} \cdot \frac{\mathrm{mult}(\mu_{n,n+1})}{\mathrm{mult}(\sigma_n)},$$

and

$$\begin{aligned} -K_X \cdot V(\mu_{n,n+1}) &= \sum_{i=1}^{n+1} V(v_i) \cdot V(\mu_{n,n+1}) \\ &= \frac{1}{a_{n+1}} \left(\sum_{i=1}^{n+1} a_i \right) \frac{\mathrm{mult}(\mu_{n,n+1})}{\mathrm{mult}(\sigma_n)} \leq n+1. \end{aligned}$$

We note that

$$\frac{\mathrm{mult}(\sigma_n)}{\mathrm{mult}(\mu_{n,n+1})} \in \mathbb{Z}_{>0}.$$

For the procedure to compute intersection numbers, see 2.1.5 or [Ful, p.100]. If $-K_X \cdot V(\mu_{n,n+1}) = n+1$, then $a_i = 1$ for every i and $\mathrm{mult}(\mu_{n,n+1}) = \mathrm{mult}(\sigma_n)$.

We note that the above calculation plays crucial roles in [F1], [F4], [FI], and this paper (see the proof of Theorem 3.2.1, and so on).

Lemma 3.1.9. *We use the same notation as in* 3.1.8. *We consider the sublattice N' of N spanned by $\{v_1, \cdots, v_{n+1}\}$. Then the natural inclusion $N' \to N$ induces a finite toric morphism $f : X' \to X$ from a weighted projective space X' such that f is étale in codimension one. In particular, $X(\Sigma)$ is a weighted projective space if and only if $\{v_1, \cdots, v_{n+1}\}$ generates N.*

We note the above elementary lemma. Example 3.1.10 shows that there are many fake weighted projective spaces which are not weighted projective spaces.

Example 3.1.10. We put $N = \mathbb{Z}^3$. Let $\{e_1, e_2, e_3\}$ be the standard basis of N. We put $v_1 = e_1$, $v_2 = e_2$, $v_3 = e_3$, and $v_4 = -e_1 - e_2 - e_3$. The fan Σ is the subdivision of $N_{\mathbb{R}}$ by $\{v_1, v_2, v_3, v_4\}$. Then $Y = X(\Sigma) \simeq \mathbb{P}^3$. We consider a new lattice

$$N^\dagger = N + \left(\frac{1}{2}, \frac{1}{2}, 0 \right) \mathbb{Z}.$$

The natural inclusion $N \to N^\dagger$ induces a finite toric morphism $Y \to X$, which is étale in codimension one. It is easy to see that K_X is Cartier and $-K_X \sim 4D_4$, where $D_4 = V(v_4)$ is not Cartier but $2D_4$ is Cartier. Since $\{v_1, v_2, v_3, v_4\}$ does not span the lattice $N^\dagger$, X is not a weighted projective space. Of course, X is a fake weighted projective space.

3.2. **New estimate of lengths of extremal rays.** The following theorem is one of the main theorems of this paper, in which we prove new estimates of K_X-negative extremal rays of birational type. We will see that they are the best by Examples 3.3.1 and 3.3.2.

Theorem 3.2.1 (Lengths of extremal rays of birational type, Theorem 1.2)**.** *Let $f : X \to Y$ be a projective toric morphism with $\dim X = n$. Assume that K_X is $\mathbb{Q}$-Cartier. Let R be a K_X-negative extremal ray of $\mathrm{NE}(X/Y)$ and let $\varphi_R : X \to W$ be the contraction morphism associated to R. We put*

$$l(R) = \min_{[C]\in R}(-K_X \cdot C).$$

and call it the length of R. Assume that φ_R is birational. Then we obtain

$$l(R) < d + 1,$$

where

$$d = \max_{w\in W} \dim \varphi_R^{-1}(w) \le n - 1.$$

When $d = n - 1$, we have a sharper inequality

$$l(R) \le d = n - 1.$$

In particular, if $l(R) = n - 1$, then $\varphi_R : X \to W$ can be described as follows. There exists a torus invariant smooth point $P \in W$ such that $\varphi_R : X \to W$ is a weighted blow-up at P with the weight $(1, a, \cdots, a)$ for some positive integer a. In this case, the exceptional locus E of φ_R is a torus invariant prime divisor and is isomorphic to $\mathbb{P}^{n-1}$. Moreover, X is $\mathbb{Q}$-factorial in a neighborhood of E.

The idea of the proof of Theorem 3.2.1 is the same as that of Theorem 3.1.1.

Proof of Theorem 3.2.1. In Step 1, we will explain how to reduce problems to the case where X is $\mathbb{Q}$-factorial. Then we will prove the inequality $l(R) < d + 1$ in Step 2. In Step 3, we will treat the case where X is $\mathbb{Q}$-factorial and $l(R) \ge n - 1$. Finally, in Step 4, we will treat the case where $l(R) \ge n - 1$ under the assumption that X is not necessarily $\mathbb{Q}$-factorial.

Step 1. In this step, we will explain how to reduce problems to the case where X is $\mathbb{Q}$-factorial.

Without loss of generality, we may assume that $W = Y$. Let $\pi : \widetilde{X} \to X$ be a small projective $\mathbb{Q}$-factorialization (see, for example, [F1, Corollary 5.9]). Then we can take an extremal ray $\widetilde{R}$ of $\mathrm{NE}(\widetilde{X}/W)$ and construct the following commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\varphi_{\widetilde{R}}"] \arrow[d, "\pi"'] & \widetilde{W} \arrow[d] \\ X \arrow[r, "\varphi_R"'] & W \end{tikzcd}$$

where $\varphi_{\widetilde{R}}$ is the contraction morphism associated to $\widetilde{R}$. We note that $\varphi_{\widetilde{R}}$ must be small when φ_R is small, because the composition of small morphisms π and φ_R is also a small morphism. We write

$$\begin{array}{rccc} & \widetilde{A} & \longrightarrow & \widetilde{B} \\ & \cap & & \cap \\ \varphi_{\widetilde{R}}: & \widetilde{X} & \longrightarrow & \widetilde{W}, \end{array}$$

where $\widetilde{A}$ is the exceptional locus of $\varphi_{\widetilde{R}}$ and $\widetilde{B}$ is the image of $\widetilde{A}$. Let $\widetilde{F}$ be a general fiber of $\widetilde{A} \to \widetilde{B}$. Then $\widetilde{F}$ is a fake weighted projective space as in 3.1.2, that is, $\widetilde{F}$ is a $\mathbb{Q}$-factorial toric Fano variety with Picard number one. Since $\rho(\widetilde{F}) = 1$, $\pi : \widetilde{F} \to F := \pi(\widetilde{F})$ is finite.

Therefore, by definition, $\dim \widetilde{F} = \dim F \leq d$ since $\varphi_R(F)$ is a point. Let $\widetilde{C}$ be a curve in $\widetilde{F}$ and let C be the image of $\widetilde{C}$ by π with the reduced scheme structure. Then we obtain

$$-K_{\widetilde{X}} \cdot \widetilde{C} = -\pi^* K_X \cdot \widetilde{C} = -mK_X \cdot C,$$

where m is the mapping degree of $\widetilde{C} \to C$. Thus, if $-K_{\widetilde{X}} \cdot \widetilde{C}$ satisfies the desired inequality, then $-K_X \cdot C$ also satisfies the same inequality. Therefore, for the proof of $l(R) < d+1$, we may assume that X is $\mathbb{Q}$-factorial and $W = Y$ by replacing X and Y with $\widetilde{X}$ and $\widetilde{W}$, respectively.

Step 2. In this step, we will prove the desired inequality $l(R) < d+1$ under the assumption that X is $\mathbb{Q}$-factorial.

We write

$$\begin{array}{rccc} & A & \longrightarrow & B \\ & \cap & & \cap \\ \varphi_R : & X & \longrightarrow & W, \end{array}$$

where A is the exceptional locus of φ_R and B is the image of A. We note that A is irreducible (see 3.1.2). We put $\dim A = n - \alpha$ and $\dim B = \beta - \alpha$ as in 3.1.2. Let F be a general fiber of $A \to B$. Then we know that F is a $\mathbb{Q}$-factorial toric Fano variety with Picard number one and that there exist torus invariant prime divisors $E_1, \cdots, E_\alpha$ on X such that E_i is negative on R for every i and A is $E_1 \cap \cdots \cap E_\alpha$ (see 3.1.2). By using subadjunction (see Lemma 2.3.1) repeatedly, we have

$$(K_X + E_1 + \cdots + E_\alpha)|_A = K_A + D$$

for some effective $\mathbb{Q}$-divisor D on A. Let C be a curve in F. Then

$$\begin{aligned} -K_X \cdot C &= -(K_A + D) \cdot C + E_1 \cdot C + \cdots + E_\alpha \cdot C \\ &< -(K_A + D) \cdot C = -(K_F + D|_F) \cdot C \leq -K_F \cdot C. \end{aligned} \tag{3.1}$$

We note that $D|_F$ is effective and $K_A|_F = K_F$ holds since F is a general fiber of $A \to B$. We also note that $D|_F \cdot C \geq 0$ since $\rho(F) = 1$. By [F1, Proposition 2.9] (see also 3.1.8), there exists a torus invariant curve C on F such that $-K_F \cdot C \leq \dim F + 1 = n - \beta + 1$. Therefore, we obtain

$$-K_X \cdot C < n - \beta + 1 = d + 1 \leq n$$

since $\beta \geq \alpha \geq 1$. This means that $l(R) < d+1$. By combining it with Step 1, we have $l(R) < d+1$ without assuming that X is $\mathbb{Q}$-factorial.

We close this step with easy useful remarks.

Remark 3.2.2. We note that if $F \not\simeq \mathbb{P}^{n-\beta}$ in the above argument, then we can choose C such that $-K_F \cdot C \leq \dim F = n - \beta$ (see Theorem 3.1.1).

Remark 3.2.3. If X is Gorenstein, then $-K_X \cdot C < n$ implies $-K_X \cdot C \leq n-1$. Therefore, by combining it with Step 1, we can easily see that the estimate $l(R) \leq n-1$ always holds for Gorenstein (not necessarily $\mathbb{Q}$-factorial) toric varieties.

If φ_R is small, then we can find C such that $-K_X \cdot C < n-1$ and $[C] \in R$ since we know $\beta \geq \alpha \geq 2$. Therefore, by combining it with Step 1, the estimate $l(R) < n-1$ always holds for (not necessarily $\mathbb{Q}$-factorial) toric varieties, when φ_R is small.

Step 3. In this step, we will investigate the case where $l(R) \geq n-1$ under the assumption that X is $\mathbb{Q}$-factorial.

We will use the same notation as in Step 2. In this case, we see that $-K_X \cdot C \geq n-1$ for every curve C on F. Then, we see that $\dim A = \dim F = n-1$, $F \simeq \mathbb{P}^{n-1}$ and $\dim B = 0$ (see Remark 3.2.2). Equivalently, φ_R contracts $F \simeq \mathbb{P}^{n-1}$ to a torus invariant point $P \in W$. Let $\langle e_1, \cdots, e_n \rangle$ be the n-dimensional cone corresponding to $P \in W$. Then X is obtained by the star subdivision of $\langle e_1, \cdots, e_n \rangle$ by e_{n+1}, where $be_{n+1} = a_1e_1 +$

$\cdots + a_n e_n$, $b \in \mathbb{Z}_{>0}$ and $a_i \in \mathbb{Z}_{>0}$ for all i. We may assume that $\gcd(b, a_1, \cdots, a_n) = 1$, $\gcd(b, a_1, \cdots, a_{i-1}, a_{i+1}, \cdots a_n) = 1$ for all i, and $\gcd(a_1, \cdots, a_n) = 1$. Without loss of generality, we may assume that $a_1 \leq \cdots \leq a_n$ by changing the order. We write $\sigma_i = \langle e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_{n+1}\rangle$ for all i and $\mu_{k,l} = \sigma_k \cap \sigma_l$ for $k \neq l$. Then

$$-K_X \cdot V(\mu_{k,n}) = \frac{1}{a_n}\left(\sum_{i=1}^{n} a_i - b\right)\frac{\mathrm{mult}(\mu_{k,n})}{\mathrm{mult}(\sigma_k)} \geq n-1 \tag{3.2}$$

for $1 \leq k \leq n-1$. Then $\mathrm{mult}(\mu_{k,n}) = \mathrm{mult}(\sigma_k)$ for $1 \leq k \leq n-1$. Thus, a_k divides a_n for $1 \leq k \leq n-1$.

Case 1. If $a_1 = a_n$, then $a_1 = \cdots = a_n = 1$. In this case $-K_X \cdot V(\mu_{k,n}) \geq n-1$ implies $b = 1$. And we have $\mathrm{mult}(\mu_{k,l}) = \mathrm{mult}(\sigma_k)$ for $1 \leq k \leq n$, $1 \leq l \leq n$, and $k \neq l$. In particular, $\mathrm{mult}(\sigma_1) = \mathrm{mult}(\mu_{1,l})$ for $2 \leq l \leq n$. This implies $\mathrm{mult}(\sigma_1) = 1$. Since $e_{n+1} = e_1 + \cdots + e_n$, $\langle e_1, \cdots, e_n\rangle$ is a nonsingular cone. Therefore, $\varphi_R : X \to W$ is a blow-up at a smooth point P. Of course, $l(R) = n-1$.

Case 2. Assume that $a_1 \neq a_n$. If $a_2 \neq a_n$, then $\frac{a_1}{a_n} \leq \frac{1}{2}$ and $\frac{a_2}{a_n} \leq \frac{1}{2}$. This contradicts $-K_X \cdot V(\mu_{k,l}) \geq n-1$. Therefore, $a_1 = 1$ and $a_2 = \cdots = a_n = a$ for some positive integer $a \geq 2$. The condition $-K_X \cdot V(\mu_{k,n}) \geq n-1$ implies $b = 1$. Thus, $\mathrm{mult}(\mu_{k,l}) = \mathrm{mult}(\sigma_k)$ for $1 \leq k \leq n$, $2 \leq l \leq n$, and $k \neq l$. In particular, $\mathrm{mult}(\sigma_1) = \mathrm{mult}(\mu_{1,l})$ for $2 \leq l \leq n$. Thus, $\mathrm{mult}(\sigma_1) = 1$. Since

$$e_{n+1} = e_1 + ae_2 + \cdots + ae_n,$$

$\langle e_1, \cdots, e_n\rangle$ is a nonsingular cone. Therefore, $\varphi_R : X \to W$ is a weighted blow-up at a smooth point $P \in W$ with the weight $(1, a, \cdots, a)$. In this case, $K_X = \varphi_R^* K_W + (n-1)aE$, where $E \simeq \mathbb{P}^{n-1}$ is the exceptional divisor and $l(R) = n-1$ (see Proposition 3.2.6 below).

Anyway, when X is $\mathbb{Q}$-factorial, we obtain that $l(R) \geq n-1$ implies $l(R) = n-1$. Therefore, the estimate $l(R) \leq n-1$ always holds when X is $\mathbb{Q}$-factorial and φ_R is birational.

Step 4. In this final step, we will treat the case where $l(R) \geq n-1$ under the assumption that X is not necessarily $\mathbb{Q}$-factorial.

Let $\pi : \widetilde{X} \to X$ be a small projective $\mathbb{Q}$-factorialization as in Step 1. By the argument in Step 1, we can find a $K_{\widetilde{X}}$-negative extremal ray $\widetilde{R}$ of $\mathrm{NE}(\widetilde{X}/W)$ such that $l(\widetilde{R}) \geq n-1$. Therefore, by Step 3, the associated contraction morphism $\varphi_{\widetilde{R}} : \widetilde{X} \to \widetilde{W}$ is a weighted blow-up at a smooth point $\widetilde{P} \in \widetilde{W}$ with the weight $(1, a, \cdots, a)$ for some positive integer a. Let $\widetilde{E}$ ($\simeq \mathbb{P}^{n-1}$) be the $\varphi_{\widetilde{R}}$-exceptional divisor on $\widetilde{X}$. We put $E = \pi(\widetilde{E})$. Then it is easy to see that $E \simeq \mathbb{P}^{n-1}$ and that $\pi : \widetilde{E} \to E$ is an isomorphism.

Lemma 3.2.4. *$\pi : \widetilde{X} \to X$ is an isomorphism over some open neighborhood of E.*

Proof of Lemma 3.2.4. We will get a contradiction by assuming that $\pi : \widetilde{X} \to X$ is not an isomorphism over any open neighborhood of E. Since $\varphi_{\widetilde{R}}$ is a weighted blow-up as described in the case where X is $\mathbb{Q}$-factorial (see Step 3) and π is a crepant small toric morphism by construction, the fan of $\widetilde{X}$ contains n-dimensional cones

$$\sigma_i := \langle \{e_1, \ldots, e_{n+1}\} \setminus \{e_i\}\rangle,$$

for $1 \leq i \leq n$, where $\{e_1, \ldots, e_n\}$ is the standard basis of $N = \mathbb{Z}^n$ and $e_{n+1} := e_1 + ae_2 + \cdots + ae_n$ with $a \in \mathbb{Z}_{>0}$. Since we assume that $\pi : \widetilde{X} \to X$ is not an isomorphism over any open neighborhood of E, there exists at least one non-simplicial n-dimensional cone σ in the fan of X such that σ contains one of the above n-dimensional cones. By symmetry, it is sufficient to consider the two cases where σ contains σ_n or σ_1.

First, we suppose $\sigma_n \subset \sigma$. Let $x = x_1e_1 + \cdots + x_ne_n \in N$ be the primitive generator for some one-dimensional face of σ which is not contained in σ_n. Then, by considering the facets of σ_n, we have the inequalities $ax_1 - x_n \geq 0$, $x_i - x_n \geq 0$ for $2 \leq i \leq n-1$, and $x_n < 0$. If $x_1 - x_n < 0$, then $x_1 < x_n < 0$. This means that $ax_1 - x_n \leq x_1 - x_n < 0$. This is a contradiction. Therefore, the inequality $x_1 - x_n \geq 0$ also holds.

Claim. *$x_i \leq 0$ for every $i \neq n$.*

Proof of Claim. Suppose $x_i > 0$ for some $i \neq n$. We note that x must be contained in the hyperplane passing through the points $e_1, \ldots, e_{n-1}, e_{n+1}$ since π is crepant, that is, $K_{\widetilde{X}} = \pi^*K_X$. So the equality

$$\begin{aligned} 1 &= x_1 + \cdots + x_{n-1} - (n-2)x_n \\ &= (x_1 - x_n) + \cdots + (x_{i-1} - x_n) + x_i + (x_{i+1} - x_n) + \cdots + (x_{n-1} - x_n) \end{aligned}$$

holds. Therefore, $x_j - x_n = 0$ must hold for every $j \neq i$, and $x_i = 1$. If $i \neq 1$, then we have $a = 1$ since $ax_1 - x_n = (a-1)x_n \geq 0$ and $x_n < 0$. However, the linear relation

$$x + (-x_n)e_{n+1} = (1 - x_n)e_i$$

means that π contracts a divisor $V(e_i)$. This is a contradiction because π is small by construction. If $i = 1$, then we have $ax_1 - x_n = a - x_n > 0$ since $a > 0$ and $-x_n > 0$. However, the linear relation

$$ax + (-x_n)e_{n+1} = (a - x_n)e_1$$

means that π contracts a divisor $V(e_1)$. This is a contradiction because π is small by construction. In any case, we obtain that $x_i \leq 0$ holds for $1 \leq i \leq n-1$. □

Therefore, the linear relation

$$(-x_1)e_1 + \cdots + (-x_n)e_n + x = 0$$

says that the cone $\langle e_1, \ldots, e_n, x\rangle$ contains a positive dimensional linear subspace of $N_{\mathbb{R}}$ because $-x_i \geq 0$ for $1 \leq i \leq n-1$ and $-x_n > 0$. Since $\langle e_1, \ldots, e_n, x\rangle$ must be contained in a strongly convex cone in the fan of W, this is a contradiction.

Next, we suppose $\sigma_1 \subset \sigma$. We can apply the same argument as above. Let $x = x_1e_1 + \cdots + x_ne_n \in N$ be the primitive generator for some one-dimensional face of σ which is not contained in σ_1. In this case, we can obtain the inequalities $x_i - ax_1 \geq 0$ for $2 \leq i \leq n$, and $x_1 < 0$ by considering the facets of σ_1, and the equality $(1-(n-1)a)x_1 + x_2 + \cdots + x_n = 1$ by the fact that π is crepant. If $x_i > 0$ for some $2 \leq i \leq n$, then the equality

$$\begin{aligned} 1 &= (1-(n-1)a)x_1 + x_2 + \cdots + x_n \\ &= (1-a)x_1 + (x_2 - ax_1) + \cdots + (x_{i-1} - ax_1) + x_i + (x_{i+1} - ax_1) + \cdots + (x_n - ax_1) \end{aligned}$$

tells us that $a = 1$ because $x_1 < 0$, and that $x_j - x_1 = 0$ for every $j \neq i$ and $x_i = 1$. Therefore, as in the proof of Claim, we get a contradiction by the linear relation

$$x + (-x_1)e_{n+1} = (1 - x_1)e_i.$$

So we obtain that $x_i \leq 0$ holds for $2 \leq i \leq n$. Thus we get a linear relation

$$(-x_1)e_1 + \cdots + (-x_n)e_n + x = 0$$

as above, where $-x_i \geq 0$ for $2 \leq i \leq n$ and $-x_1 > 0$. This means that the cone $\langle e_1, \cdots, e_n, x\rangle$ contains a positive dimensional linear subspace of $N_{\mathbb{R}}$. This is a contradiction as explained above.

In any case, we get a contradiction. Therefore, $\pi : \widetilde{X} \to X$ is an isomorphism over some open neighborhood of E. □

Since $\pi : \widetilde{X} \to X$ is an isomorphism over some open neighborhood of E by Lemma 3.2.4, we see that E is $\mathbb{Q}$-Cartier. Therefore, the exceptional locus of φ_R coincides with $E \simeq \mathbb{P}^{n-1}$. Thus $\varphi_R : X \to W$ is a weighted blow-up at a torus invariant smooth point $P \in W$ with the weight $(1, a, \cdots, a)$ for some positive integer a.

So, we complete the proof of Theorem 3.2.1. □

Remark 3.2.5. If B is complete, then we can make C a torus invariant curve on X in Theorem 3.2.1. For the details, see the proof of [F1, Theorem 0.1].

We explicitly state the basic properties of the weighted blow-up in Theorem 3.2.1 for the reader's convenience.

Proposition 3.2.6. *Let $\varphi : X \to \mathbb{A}^n$ be the weighted blow-up at $0 \in \mathbb{A}^n$ with the weight $(1, a, \cdots, a)$ for some positive integer a. If $a = 1$, then φ is the standard blow-up at 0. In particular, X is smooth. If $a \geq 2$, then X has only canonical Gorenstein singularities which are not terminal singularities. Furthermore, the exceptional locus E of φ is isomorphic to $\mathbb{P}(1, a, \cdots, a) \simeq \mathbb{P}^{n-1}$ and*

$$K_X = \varphi^* K_{\mathbb{A}^n} + (n-1)aE.$$

We note that E is not Cartier on X if $a \neq 1$. However, aE is a Cartier divisor on X.

Proof. We can check the statements by direct calculation. □

Let us see an important related example.

Example 3.2.7. We fix $N = \mathbb{Z}^n$ and let $\{e_1, \cdots, e_n\}$ be the standard basis of N. We consider the cone $\sigma = \langle e_1, \cdots, e_n \rangle$ in $N' = N + \mathbb{Z}e_{n+1}$, where $e_{n+1} = \frac{1}{b}(1, a, \cdots, a)$. Here, a and b are positive integers such that $\gcd(a, b) = 1$. We put $Y = X(\sigma)$ is the associated affine toric variety which has only one singular point P. We take a weighted blow-up of Y at P with the weight $\frac{1}{b}(1, a, \cdots, a)$. This means that we divide σ by e_{n+1} and obtain a fan Σ of $N'_{\mathbb{R}}$. We define $X = X(\Sigma)$. Then the induced projective birational toric morphism $f : X \to Y$ is the desired weighted blow-up. It is obvious that X is $\mathbb{Q}$-factorial and $\rho(X/Y) = 1$. We can easily obtain

$$K_X = f^* K_Y + \left(\frac{1 + (n-1)a}{b} - 1 \right) E,$$

where $E = V(e_{n+1}) \simeq \mathbb{P}^{n-1}$ is the exceptional divisor of f, and

$$-K_X \cdot C = (n-1) - \frac{b-1}{a},$$

where $C = V(\langle e_2, \cdots, e_{n-1}, e_{n+1} \rangle)$ is a torus invariant irreducible curve on E. We note that

$$-(K_X + \delta E) \cdot C > n - 1$$

if and only if

$$\delta > \frac{b-1}{b}$$

since $E \cdot C = -\frac{b}{a}$.

In subsection 3.3, we will see more sophisticated examples (see Examples 3.3.1 and 3.3.2), which show the estimates obtained in Theorem 3.2.1 are the best.

By the proof of Theorem 3.2.1, we can prove the following theorem.

Theorem 3.2.8. *Let $f : X \to Y$ be a projective toric morphism with $\dim X = n$ and let $\Delta = \sum \delta_i \Delta_i$ be an effective $\mathbb{R}$-divisor on X, where Δ_i is a torus invariant prime divisor and $0 \leq \delta_i \leq 1$ for every i. Let R be an extremal ray of $\mathrm{NE}(X/Y)$ and let $\varphi_R : X \to W$*

be the extremal contraction morphism associated to R. Assume that X is $\mathbb{Q}$-factorial and φ_R is birational. If

$$\min_{[C]\in R}(-(K_X+\Delta)\cdot C) > n-1,$$

then $\varphi_R : X \to W$ is the weighted blow-up described in Example 3.2.7 and $\mathrm{Supp}\Delta \supset E$*, where $E \simeq \mathbb{P}^{n-1}$ is the exceptional divisor of φ_R.*

Proof. We use the same notation as in Step 2 in the proof of Theorem 3.2.1. Since

$$(E_1+\cdots+E_\alpha-\Delta)\cdot C \le 0,$$

we obtain

$$\begin{aligned}-(K_X+\Delta)\cdot C &= -(K_A+D)\cdot C + (E_1+\cdots+E_\alpha-\Delta)\cdot C\\ &\le -(K_A+D)\cdot C\\ &\le -K_F\cdot C\end{aligned}$$

(see (3.1)). By assumption, $-(K_X+\Delta)\cdot C > n-1$. This implies that $n-1 < -K_F\cdot C$. Therefore, we obtain $\dim A = \dim F = n-1$, $F \simeq \mathbb{P}^{n-1}$ and $\dim B = 0$. In this situation,

$$-(K_X+\Delta)\cdot C \le -(K_X+A)\cdot C$$

always holds. Thus we have

$$-(K_X+A)\cdot V(\mu_{k,n}) = \frac{1}{a_n}\left(\sum_{i=1}^{n} a_i\right)\frac{\mathrm{mult}(\mu_{k,n})}{\mathrm{mult}(\sigma_k)} > n-1$$

for $1 \le k \le n-1$ (see (3.2)). We note that $A = V(e_{n+1})$. Thus, by the same arguments as in the proof of Theorem 3.2.1, we see that φ_R is the weighted blow-up described in Example 3.2.7. More precisely, we obtain that $(a_1,\cdots,a_n) = (1,\cdots,1)$ or $(1,a,\cdots,a)$ and that σ_1 is a nonsingular cone. However, b is not necessarily 1 in the proof of Theorem 3.2.1. By direct calculation, we have $\mathrm{Supp}\Delta \supset E$, where $E(=A=F)$ is the exceptional divisor of φ_R. □

Finally, we prove the following theorem.

Theorem 3.2.9 (Theorem 1.3)**.** *Let X be a $\mathbb{Q}$-Gorenstein projective toric n-fold with $\rho(X)\ge 2$. Let R be a K_X-negative extremal ray of* $\mathrm{NE}(X)$ *such that*

$$l(R) = \min_{[C]\in R}(-K_X\cdot C) > n-1.$$

Then the extremal contraction $\varphi_R : X \to W$ associated to R is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$.

Proof. We divide the proof into several steps. From Step 1 to Step 4, we will prove this theorem under the extra assumption that X is $\mathbb{Q}$-factorial. In Step 5, we will prove that X is always $\mathbb{Q}$-factorial if there exists an extremal ray R with $l(R) > n-1$.

Step 1. We consider the contraction morphism $\varphi_R : X \to W$ associated to R. By Theorem 3.2.1, φ_R is a Fano contraction, that is, $\dim W < \dim X$. Let F be a general fiber of φ_R and let C be a curve on F. Then, by adjunction, we have

$$-K_X\cdot C = -K_F\cdot C.$$

We note that F is a fake weighted projective space. By Theorem 3.1.1, $F\simeq\mathbb{P}^{n-1}$, $W=\mathbb{P}^1$, and $\rho(X)=2$.

Step 2. Without loss of generality, $\varphi_R : X \to W$ is induced by the projection $\pi : N = \mathbb{Z}^n \to \mathbb{Z}, (x_1, \cdots, x_n) \mapsto x_n$. We put

$$\begin{aligned} v_1 &= (1, 0, \cdots, 0), & v_2 &= (0, 1, 0, \cdots, 0), & &\cdots, \\ v_{n-1} &= (0, \cdots, 0, 1, 0), & v_n &= (-1, \cdots, -1, 0), & v_+ &= (b_1, \cdots, b_{n-1}, a_+), \\ v_- &= (c_1, \cdots, c_{n-1}, -a_-), \end{aligned}$$

where a_+ and a_- are positive integers. More precisely, v_i denotes the vector with a 1 in the ith coordinate and 0's elsewhere for $1 \le i \le n-1$. We may assume that the fan Σ corresponding to the toric variety X is the subdivision of $N_{\mathbb{R}}$ by $v_1, \cdots, v_n, v_+$, and v_-. We note that the following equalities

$$\begin{cases} D_1 - D_n + b_1 D_+ + c_1 D_- = 0 \\ D_2 - D_n + b_2 D_+ + c_2 D_- = 0 \\ \qquad\qquad \vdots \\ D_{n-1} - D_n + b_{n-1} D_+ + c_{n-1} D_- = 0 \\ a_+ D_+ - a_- D_- = 0 \end{cases} \tag{3.3}$$

hold, where $D_i = V(v_i)$ for every i and $D_\pm = V(v_\pm)$. We note that it is sufficient to prove that $a_+ = a_- = 1$.

Step 3. In this step, we will prove that $a_+ = 1$ holds.

By taking a suitable coordinate change, we may assume that

$$0 \le b_1, \cdots, b_{n-1} < a_+$$

holds. If $b_i = 0$ for every i, then $a_+ = 1$ since v_+ is a primitive vector of N. From now on, we assume that $b_{i_0} \ne 0$ for some i_0. Without loss of generality, we may assume that $b_1 \ne 0$. We put

$$C = V(\langle v_2, \cdots, v_{n-1}, v_+\rangle).$$

Then C is a torus invariant curve contained in a fiber of $\varphi_R : X \to W$. We have

$$D_1 \cdot C = \frac{\mathrm{mult}(\langle v_2, \cdots, v_{n-1}, v_+\rangle)}{\mathrm{mult}(\langle v_1, \cdots, v_{n-1}, v_+\rangle)} = \frac{\gcd(a_+, b_1)}{a_+}$$

(see 2.1.5) and $D_+ \cdot C = D_- \cdot C = 0$. We note that $D_i \cdot C = D_1 \cdot C$ for every i by (3.3). Therefore, we obtain

$$-K_X \cdot C = \frac{n \gcd(a_+, b_1)}{a_+}.$$

Since $0 < b_1 < a_+$, we see $\gcd(a_+, b_1) < a_+$. Thus, the following inequality

$$\frac{\gcd(a_+, b_1)}{a_+} \le \frac{1}{2}$$

holds. This means that

$$-K_X \cdot C \le \frac{n}{2} \le n - 1.$$

This is a contradiction. Therefore, $b_i = 0$ for every i and $a_+ = 1$.

Step 4. By the same argument, we get $a_- = 1$. Thus, we see that $\varphi_R : X \to W$ is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$.

Step 5. In this step, we will prove that X is $\mathbb{Q}$-factorial.

We assume that X is not $\mathbb{Q}$-factorial. Let $\pi : \widetilde{X} \to X$ be a small projective $\mathbb{Q}$-factorialization. We note that $\rho(\widetilde{X}) > \rho(X) \ge 2$ since X is not $\mathbb{Q}$-factorial. By the argument in Step 1 in the proof of Theorem 3.2.1, there exists an extremal ray $\widetilde{R}$ of $\mathrm{NE}(\widetilde{X}/W)$ with $l(\widetilde{R}) > n-1$. Let $\varphi_{\widetilde{R}} : \widetilde{X} \to \widetilde{W}$ be the contraction morphism associated to $\widetilde{R}$. Then, by the argument in Step 1, we see that $\rho(\widetilde{X}) = 2$ and that $\varphi_{\widetilde{R}}$ is nothing but $\varphi_R \circ \pi$. This is a contradiction because $\rho(\widetilde{X}) > 2$. This means that X is always $\mathbb{Q}$-factorial.

Therefore, we get the desired statement. □

We close this subsection with an easy example, which shows that Theorem 3.2.9 is sharp.

Example 3.2.10. We consider $N = \mathbb{Z}^2$, $v_1 = (0,1), v_2 = (0,-1), v_+ = (2,1), v_- = (-1,0)$ and the projection $\pi : N = \mathbb{Z}^2 \to \mathbb{Z}, (x_1, x_2) \mapsto x_1$. Let Σ be the fan obtained by subdividing $N_{\mathbb{R}}$ by $\{v_1, v_2, v_+, v_-\}$. Then $X = X(\Sigma)$ is a projective toric surface with $\rho(X) = 2$. The map $\pi : N \to \mathbb{Z}$ induces a Fano contraction morphism $\varphi : X \to \mathbb{P}^1$. Let R be the corresponding extremal ray of $\mathrm{NE}(X)$. Then $l(R) = 1 = 2 - 1$. Note that X is not a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

The reader can find a generalization of Theorem 3.2.9, that is, a characterization of toric projective bundles for $\mathbb{Q}$-factorial projective toric varieties, in [FS3].

3.3. **Examples.** In this subsection, we will see that the estimates in Theorem 3.2.1 are the best by the following examples.

Example 3.3.1. We use the same notation as in Example 3.2.7. In Example 3.2.7, we put $a = k^2$ and $b = mk + 1$ for any positive integers k and m. Then it is obvious that $\gcd(a, b) = 1$. So, we can apply the construction in Example 3.2.7. Then we obtain a toric projective birational morphism $f : X \to Y$ such that X is $\mathbb{Q}$-factorial and $\rho(X/Y) = 1$. We can easily check that

$$K_X = f^*K_Y + \left(\frac{1 + k^2(n-1)}{mk+1} - 1\right) E$$

and

$$-K_X \cdot C = n - 1 - \frac{m}{k}.$$

Therefore, we see that the minimal lengths of extremal rays do not satisfy the ascending chain condition in this birational setting. More precisely, the minimal lengths of extremal rays can take any values in $\mathbb{Q} \cap (0, n-1)$. For a related topic, see [FI].

Let us construct small contraction morphisms with a long extremal ray.

Example 3.3.2. We fix $N = \mathbb{Z}^n$ with $n \geq 3$. Let $\{v_1, \cdots, v_n\}$ be the standard basis of N. We put

$$v_{n+1} = (\underbrace{a, \cdots, a}_{n-k+1}, \underbrace{-1, \cdots, -1}_{k-1})$$

with $2 \leq k \leq n-1$, where a is any positive integer. Let Σ^+ be the fan in $\mathbb{R}^n$ such that the set of maximal cones of Σ^+ is

$$\{\langle \{v_1, \cdots, v_{n+1}\} \setminus \{v_i\} \rangle \,|\, n-k+2 \leq i \leq n+1\}.$$

Let us consider the smooth toric variety $X^+ = X(\Sigma^+)$ associated to the fan Σ^+. We note that the equality

$$v_{n-k+2} + \cdots + v_n + v_{n+1} = av_1 + \cdots + av_{n-k+1}$$

holds. We can get an antiflipping contraction $\varphi^+ : X^+ \to W$, that is, a K_{X^+}-positive small contraction morphism, when

$$a > \frac{k}{n-k+1}.$$

In this case, we have the following flipping diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "\varphi"'] & & X^+ \arrow[dl, "\varphi^+"] \\ & W & \end{tikzcd}$$

By construction, $\varphi : X \to W$ is a flipping contraction whose exceptional locus is isomorphic to $\mathbb{P}^{n-k}$. The exceptional locus of φ^+ is isomorphic to $\mathbb{P}^{k-1}$. Of course, φ (resp. φ^+) contracts $\mathbb{P}^{n-k}$ (resp. $\mathbb{P}^{k-1}$) to a point in W. We can directly check that

$$-K_X \cdot C = n - k + 1 - \frac{k}{a}$$

for every torus invariant curve C in the φ-exceptional locus $\mathbb{P}^{n-k}$.

Example 3.3.2 shows that the estimate for small contractions in Theorem 3.2.1 is sharp.

Remark 3.3.3. If $(n, k) = (3, 2)$ and $a \geq 2$ in Example 3.3.2, then $\varphi : X \to W$ is a threefold toric flipping contraction whose length of the extremal ray is $\geq 3 - 2 = 1$. We note that the lengths of extremal rays of three-dimensional terminal (not necessarily toric) flipping contractions are less than one.

4. Basepoint-free theorems

This section is a supplement to Fujita's freeness conjecture for toric varieties (see [F1], [Fuj], [Lat], [Li], [Mu], and [P]) and Fulton's book: [Ful].

4.1. **Variants of Fujita's conjectures for toric varieties.** One of the most general formulations of Fujita's freeness conjecture for toric varieties is [F1, Corollary 0.2]. However, it does not cover the first part of [Li, Main theorem A]. So, we give a generalization here with a very simple proof. It is an easy application of the vanishing theorem (see [F5] and [F6]).

Theorem 4.1.1 (Basepoint-freeness). *Let $g : Z \to X$ be a proper toric morphism and let A and B be reduced torus invariant Weil divisors on Z without common irreducible components. Let $f : X \to Y$ be a proper surjective toric morphism and let D be an f-ample Cartier divisor on X. Then*

$$R^q g_*(\widetilde{\Omega}^a_Z(\log(A + B))(-A)) \otimes \mathcal{O}_X(kD)$$

is f-free, that is,

$$f^* f_*(R^q g_*(\widetilde{\Omega}^a_Z(\log(A + B))(-A)) \otimes \mathcal{O}_X(kD)) \to R^q g_*(\widetilde{\Omega}^a_Z(\log(A + B))(-A)) \otimes \mathcal{O}_X(kD)$$

is surjective for every $a \geq 0$, $q \geq 0$, and $k \geq \max_{y \in Y} \dim f^{-1}(y) + 1$.

As a very special case, we can recover the following result.

Corollary 4.1.2 (cf. [Li, Main theorem A]). *Let X be an n-dimensional projective toric variety and let D be an ample Cartier divisor on X. Then the reflexive sheaf $\mathcal{O}_X(K_X + (n + 1)D)$ is generated by its global sections.*

Proof. In Theorem 4.1.1, we assume that $g : Z \to X$ is the identity, $A = B = 0$, $a = \dim X$, $q = 0$, and Y is a point. Then we obtain the desired statement. □

Example 4.1.3. Let us consider $X = \mathbb{P}(1, 1, 1, 2)$. Let P be the unique $\frac{1}{2}(1, 1, 1)$-singular point of X and let D be an ample Cartier divisor on X. We can find a torus invariant curve C on X such that

$$K_X \cdot C \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}.$$

Therefore, for every effective Weil divisor E on X such that $E \sim K_X + 4D$, we have $P \in \mathrm{Supp}E$. On the other hand, by Corollary 4.1.2, the reflexive sheaf $\mathcal{O}_X(K_X + 4D)$ is generated by its global sections.

Before proving Theorem 4.1.1, let us recall the definition of the reflexive sheaf $\widetilde{\Omega}^a_X(\log(A+B))(-A)$ and the vanishing theorem in [F6].

Definition 4.1.4. Let W be any Zariski open set of Z such that W is smooth and $\mathrm{codim}_Z(Z \setminus W) \geq 2$. In this case, $A+B$ is a simple normal crossing divisor on W. On this assumption, $\Omega^a_W(\log(A+B))$ is a well-defined locally free sheaf on W. Let $\iota : W \hookrightarrow Z$ be the natural open immersion. Then we put

$$\widetilde{\Omega}^a_Z(\log(A+B))(-A) = \iota_*(\Omega^a_W(\log(A+B)) \otimes \mathcal{O}_W(-A))$$

for every $a \geq 0$. It is easy to see that the reflexive sheaf

$$\widetilde{\Omega}^a_Z(\log(A+B))(-A)$$

on Z does not depend on the choice of W.

The next theorem is one of the vanishing theorems obtained in [F6]. For the proof and other vanishing theorems, see [F5] and [F6].

Theorem 4.1.5 ([F6, Theorem 4.3])**.** *Let $g : Z \to X$ be a proper toric morphism and let A and B be reduced torus invariant Weil divisors on Z without common irreducible components. Let $f : X \to Y$ be a proper surjective toric morphism and let L be an f-ample line bundle on X. Then*

$$R^pf_*(R^qg_*(\widetilde{\Omega}^a_Z(\log(A+B))(-A)) \otimes L) = 0$$

for every $p > 0$, $q \geq 0$, and $a \geq 0$.

Let us prove Theorem 4.1.1.

Proof of Theorem 4.1.1. By the vanishing theorem: Theorem 4.1.5, we have

$$R^pf_*(R^qg_*(\widetilde{\Omega}^a_Z(\log(A+B))(-A)) \otimes \mathcal{O}_X((k-p)D)) = 0$$

for every $p > 0$, $q \geq 0$, $a \geq 0$, and $k \geq \max_{y\in Y} \dim f^{-1}(y) + 1$. Since $\mathcal{O}_X(D)$ is f-free, we obtain that

$$R^qg_*(\widetilde{\Omega}^a_Z(\log(A+B))(-A)) \otimes \mathcal{O}_X(kD)$$

is f-free by the Castelnuovo–Mumford regularity (see, for example, [Laz, Example 1.8.24]). □

Here, we treat some generalizations of Fujita's very ampleness for toric varieties as applications of Theorem 4.1.5. For the details of Fujita's very ampleness for toric varieties, see [P].

Theorem 4.1.6. *Let $f : X \to Y$ be a proper surjective toric morphism, let Δ be a reduced torus invariant divisor on X such that $K_X+\Delta$ is Cartier, and let D be an f-ample Cartier divisor on X. Then $\mathcal{O}_X(K_X+\Delta+kD)$ is f-very ample for every $k \geq \max_{y\in Y} \dim f^{-1}(y)+2$.*

Proof. It follows from the Castelnuovo–Mumford regularity by the vanishing theorem: Theorem 4.1.5. For the details, see [Laz, Example 1.8.22]. □

The following corollary is a special case of the above theorem. This result was first proved by Hui-Wen Lin (see [Li]) for the case where X is $\mathbb{Q}$-factorial and $\dim X \leq 6$, and then completed by Sam Payne (see [P]).

Corollary 4.1.7 ([Li, Main Theorem B] and [P, Theorem 1])**.** *Let X be an n-dimensional projective Gorenstein toric variety and let D be an ample Cartier divisor on X. Then $\mathcal{O}_X(K_X + (n+2)D)$ is very ample.*

We think that the following theorem has not been stated explicitly in the literature.

Theorem 4.1.8 (Very ampleness)**.** *Let $g : Z \to X$ be a proper toric morphism and let A and B be reduced torus invariant Weil divisors on Z without common irreducible components. Let $f : X \to Y$ be a proper surjective toric morphism and let D be an f-ample Cartier divisor on X. Assume that $R^q g_*(\widetilde{\Omega}^a_Z(\log(A+B))(-A))$ is locally free. Then*

$$R^q g_*(\widetilde{\Omega}^a_Z(\log(A+B))(-A)) \otimes \mathcal{O}_X(kD)$$

is f-very ample for every $k \geq \max_{y\in Y} \dim f^{-1}(y) + 2$.

Proof. The proof of Theorem 4.1.6 works for this theorem since

$$R^q g_*(\widetilde{\Omega}^a_Z(\log(A+B))(-A))$$

is locally free by assumption (see [Laz, Example 1.8.22]). □

4.2. **Lin's problem.** In this subsection, we treat Lin's problem raised in [Li]. In [Li, Lemma 4.3], she claimed the following lemma, which is an exercise in [Ful, p.90], without proof.

Lemma 4.2.1. *Let X be a complete Gorenstein toric variety and let D be an ample (Cartier) divisor. If $\Gamma(X, K+D) \neq 0$ then $K+D$ is generated by its global sections. In fact, P_{K+D} is the convex hull of $\mathrm{Int}P_D \cap M$.*

Sam Payne pointed out that Lemma 4.2.1 does not seem to have a known valid proof (see [Li, p.500 Added in proof]). Unfortunately, the following elementary example is a counterexample to Lemma 4.2.1. So, Lemma 4.2.1 is NOT true. Therefore, the alternative proof of Theorem A in [Li] does not work.

Example 4.2.2. Let $Y = \mathbb{P}^n$ and let $P \in Y$ be a torus invariant closed point. Let $f : X \to Y$ be the blow-up at P. We put $B = \sum_{i=1}^{n+1} B_i$, where B_i is a torus invariant prime divisor on Y for every i. Then it is well known that $\mathcal{O}_Y(K_Y) \simeq \mathcal{O}_Y(-B)$. We define $D = f^*B - E$, where E is the exceptional divisor of f. In this case, we have

$$K_X = f^*K_Y + (n-1)E$$

and it is not difficult to see that D is ample. Therefore,

$$K_X + (n-1)D = f^*(K_Y + (n-1)B)$$

is nef, that is, $\mathcal{O}_X(K_X + (n-1)D)$ is generated by its global sections. We note that $H^0(X, \mathcal{O}_X(K_X + aD)) \neq 0$ for every positive integer a. However, $K_X + aD$ is not nef for any real number $a < n-1$. In particular, $H^0(X, \mathcal{O}_X(K_X + D)) \neq 0$ but $\mathcal{O}_X(K_X + D)$ is not generated by its global sections.

The following theorem is the main theorem of this section. It follows from Theorem 3.2.1.

Theorem 4.2.3 (see Theorem 1.1)**.** *Let X be a $\mathbb{Q}$-Gorenstein projective toric n-fold and let D be an ample Cartier divisor on X. Then $K_X + (n-1)D$ is pseudo-effective if and only if $K_X + (n-1)D$ is nef.*

Proof. If $K_X + (n-1)D$ is nef, then $K_X + (n-1)D$ is obviously pseudo-effective. So, all we have to do is to see that $K_X + (n-1)D$ is nef when it is pseudo-effective. From now on, we assume that $K_X + (n-1)D$ is pseudo-effective. We take a positive rational number τ such that $K_X + \tau D$ is nef but not ample. In some literature, $1/\tau$ is called the *nef threshold of D with respect to X*. It is not difficult to see that τ is rational since the Kleiman–Mori cone is a rational polyhedral cone in our case. If $\tau \le n-1$, then the theorem is obvious since

$$K_X + (n-1)D = K_X + \tau D + (n-1-\tau)D$$

and D is ample. Therefore, we assume that $\tau > n-1$. We take a sufficiently large positive integer m such that $m(K_X+\tau D)$ is Cartier. We consider the toric morphism $f := \Phi_{|m(K_X+\tau D)|} : X \to Y$. By the definition of τ, f is not an isomorphism. Let R be an extremal ray of $\mathrm{NE}(X/Y)$. Let C be any integral curve on X such that $[C] \in R$. Since $(K_X+\tau D)\cdot C = 0$, we obtain $-K_X \cdot C = \tau D \cdot C > n-1$. Therefore, f is not birational by Theorem 3.2.1. Equivalently, $K_X + \tau D$ is not big. Thus, the numerical equivalence class of $K_X+\tau D$ is on the boundary of the pseudo-effective cone $\mathrm{PE}(X)$ of X. So,

$$K_X + (n-1)D = K_X + \tau D - (\tau - (n-1))D$$

is outside $\mathrm{PE}(X)$. This is a contradiction. Therefore, $K_X + (n-1)D$ is nef when $K_X + (n-1)D$ is pseudo-effective. □

As a corollary, we obtain the following result, which is a correction of Lemma 4.2.1. It is a variant of Fujita's freeness conjecture for toric varieties. Example 4.2.2 shows that the constant $n-1$ in Corollary 4.2.4 is the best.

Corollary 4.2.4 (see Theorem 1.1)**.** *Let X be a Gorenstein projective toric n-fold and let D be an ample Cartier divisor on X. If $H^0(X, \mathcal{O}_X(K_X+(n-1)D)) \neq 0$, then $\mathcal{O}_X(K_X + (n-1)D)$ is generated by its global sections.*

Proof. If $H^0(X, \mathcal{O}_X(K_X+(n-1)D)) \neq 0$, then $K_X+(n-1)D$ is obviously pseudo-effective. Then, by Theorem 4.2.3, $K_X+(n-1)D$ is nef. If $K_X+(n-1)D$ is a nef Cartier divisor, then the complete linear system $|K_X+(n-1)D|$ is basepoint-free by Lemma 2.2.5. □

By Theorem 3.2.9, we can check the following result.

Corollary 4.2.5 (Corollary 1.4)**.** *Let X be a $\mathbb{Q}$-Gorenstein projective toric n-fold and let D be an ample Cartier divisor on X. If $\rho(X) \ge 3$, then $K_X+(n-1)D$ is always nef. More precisely, if $\rho(X) \ge 2$ and X is not a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$, then $K_X+(n-1)D$ is nef.*

Proof. By Theorem 3.2.9, $K_X+(n-1)D$ is nef since $\rho(X) \ge 2$ and X is not a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$. □

4.3. **Supplements to Fujita's paper.** This subsection supplements Fujita's paper: [Fuj].

We have never seen Corollary 4.2.4 in the literature. However, we believe that Fujita could prove Corollary 4.2.4 without any difficulties (see Theorem 4.3.2 below). We think that he was not interested in the toric geometry when he wrote [Fuj]. If he was familiar with the toric geometry, then he would have adopted Example 4.3.1 in [Fuj, (3.5) Remark]. This example supplements Fujita's remark: [Fuj, (3.5) Remark]. We think that our example is much simpler.

Example 4.3.1. We fix $N = \mathbb{Z}^2$. We put $e_1 = (1,0)$, $e_2 = (0,1)$, $e_3 = (-1,-1)$, and $e_4 = (1,2)$. We consider the fan Σ obtained by subdividing $N_{\mathbb{R}}$ with e_1, e_2, e_3, and e_4. We write $X = X(\Sigma)$, the associated toric variety. Then X is Gorenstein and $-K_X$ is ample. We put $D = -K_X$. It is obvious that $K_X + D \sim 0$. It is easy to see that the Kleiman–Mori cone $\mathrm{NE}(X)$ is spanned by the two torus invariant curves $E = V(e_4)$ and $E' = V(e_2)$. So, we have two extremal contractions. By removing e_4 from Σ, we obtain a contraction morphism $f : X \to \mathbb{P}^2$. In this case, E is not Cartier although $2E$ is Cartier. We note that $-K_X \cdot E = 1$. The morphism f is the weighted blow-up with the weight $(1,2)$ described in Proposition 3.2.6. Another contraction is obtained by removing e_2. It is a contraction morphism from X to $\mathbb{P}(1,1,2)$. Note that E' is a Cartier divisor on X.

We close this subsection with the following theorem. In Theorem 4.3.2, we treat normal Gorenstein projective varieties defined over $\mathbb{C}$ with only rational singularities, which are not necessarily toric. So, the readers who are interested only in the toric geometry can skip this final theorem.

Theorem 4.3.2 (see [Fuj]). *Let X be a normal projective variety defined over $\mathbb{C}$ with only rational Gorenstein singularities. Let D be an ample Cartier divisor on X. If $K_X+(n-1)D$ is pseudo-effective with $n=\dim X$, then $K_X+(n-1)D$ is nef.*

Proof. We take a positive rational number τ such that $K_X+\tau D$ is nef but not ample. It is well known that $\tau \leq n+1$ (see [Fuj, Theorem 1]). If $\tau \leq n-1$, then the theorem is obvious. Therefore, we assume that $n-1<\tau\leq n+1$. If $\tau=n+1$, then $X\simeq \mathbb{P}^n$ and $\mathcal{O}_X(D)\simeq \mathcal{O}_{\mathbb{P}^n}(1)$. In this case, $K_X+(n-1)D$ is not pseudo-effective. Thus, we have $n-1<\tau\leq n$ by [Fuj, Theorem 1]. By [Fuj, Theorem 2] and its proof, it can be checked easily that $\tau=n$ and $K_X+\tau D=K_X+nD$ is nef but is not big. Therefore, K_X+nD is on the boundary of the pseudo-effective cone of X. So, $K_X+(n-1)D=K_X+nD-D$ is not pseudo-effective. This is a contradiction. Anyway, we obtain that $K_X+(n-1)D$ is nef if $K_X+(n-1)D$ is pseudo-effective. □

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