# A SAMPLE COMPUTATION OF HIGHER COHOMOLOGY GROUPS

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Example 3 is a sample computation of the higher cohomology groups of ample vector bundles on $\mathbb{P}^n$.

**Lemma 1.** *Let $X$ be a projective variety and $E$ an ample vector bundle on $X$. Then $H^0(X, E^*) = 0$.*

*Proof.* We assume that $H^0(X, E^*) \neq 0$. Then there is an injection $0 \to \mathcal{O}_X \to E^*$. By taking the dual, we obtain $E \to \mathcal{O}_X \to 0$. It contradicts the ampleness of $E$. □

**Corollary 2.** *Let $X$ be a projective toric $n$-fold and $E$ an ample vector bundle on $X$. Then $H^n(X, E) = 0$.*

*Proof.* By the Serre duality, it is sufficient to prove $H^0(X, E^* \otimes \mathcal{O}(K_X)) = 0$. On the other hand, $0 \to \mathcal{O}(K_X) \to \mathcal{O}_X$ since $X$ is toric. So, this corollary follows from the above lemma. □

**Example 3.** Let $X = \mathbb{P}^n$ be an $n$-dimensional projective space over $\mathbb{C}$. Let $F : \mathbb{P}^n \to \mathbb{P}^n$ be the $(n+2)$-times multiplication map (see [F]). We put $\mathcal{E} = \bigoplus_{p=1}^{n-1} F^*(\wedge^p T_{\mathbb{P}^n}) \otimes \mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n})$. Then $\mathcal{E}$ is an ample equivariant vector bundle on $\mathbb{P}^n$. We can check that $H^i(\mathbb{P}^n, \mathcal{E}) \neq 0$ for $i < n$ and $H^n(\mathbb{P}^n, \mathcal{E}) = 0$.

*Proof.* We consider the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \to T_{\mathbb{P}^n} \to 0.$$

By taking $F^*$ and $\wedge^p$ for $p = 1, \cdots, n$, we obtain

$$0 \to F^*(\wedge^{p-1} T_{\mathbb{P}^n}) \to \wedge^p(\mathcal{O}_{\mathbb{P}^n}(n+2)^{\oplus n+1}) \to F^*(\wedge^p T_{\mathbb{P}^n}) \to 0.$$

Since $(\wedge^p(\mathcal{O}_{\mathbb{P}^n}(n+2)^{\oplus n+1})) \otimes \mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n}) \simeq \bigoplus \mathcal{O}_{\mathbb{P}^n}(p(n+2) - (n+1))$ is an ample vector bundle on $\mathbb{P}^n$, so is $F^*(\wedge^p T_{\mathbb{P}^n}) \otimes \mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n})$. We know that there is a split injection $\mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n}) \to F_*\mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n})$. Therefore, we have

$$H^i(\mathbb{P}^n, \wedge^p T_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n})) \subset H^i(\mathbb{P}^n, F^*(\wedge^p T_{\mathbb{P}^n}) \otimes \mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n}))$$

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for all $i$. By the Serre duality, the left hand side is not zero if $i = n-p$. So, $H^{n-p}(\mathbb{P}^n, F^*(\wedge^p T_{\mathbb{P}^n})\otimes\mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n})) \neq 0$ by the above inclusion. Therefore, $\mathcal{E}$ is an ample vector bundle on $\mathbb{P}^n$ and has the desired property. □

## References

[F] O. Fujino, Multiplication maps and vanishing theorems for toric varieties, Math. Z. **257** (2007), no. 3, 631–641.

[HMP] M. Hering, M. Mustata, S. Payne, Positivity for toric vector bundles, preprint (2008), arXiv:0805.4035v1.

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