A MEMO ON "SPECIAL TERMINATION AND REDUCTION TO PL FLIPS"BY O. FUJINO

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1. Y. Takano and H. Uehara pointed out that Example 4.4.2 in [F] is incorrect. The vector e_1 is contained in the cone $\langle e_2, e_4, e_5\rangle$. I overlooked this mistake for a long time. I thank Y. Takano and H. Uehara.

Let $\varphi : X \to Y$ be a 3-dimensional toric flipping contraction such that X has only terminal singularities and that Y is affine. Then we can prove that X is $\mathbb{Q}$-factorial and the unique rational curve that is contracted by φ passes through only one singular point of X. Therefore, $\varphi : X \to Y$ is the flip described in [M, Example-Claim 14-2-5].

2. Here, we give one example of 3-dimensional non-$\mathbb{Q}$-factorial toric flips. Please replace [F, Example 4.4.2] with the following example. **Note that there are no 3-dimensional non-$\mathbb{Q}$-factorial *terminal* flips!**

Example 3 (3-dimensional non-$\mathbb{Q}$-factorial flip)**.** We fix a lattice $N = \mathbb{Z}^3$. Pick lattice points $v_1 = (1, 0, 1), v_2 = (-1, 1, 1), v_3 = (-1, 0, 1), v_4 = (0, -1, 1)$, and $v_5 = (1, 2, 0)$. We consider the following fans.

$$\begin{aligned}
\Delta_X &= \{\langle v_1, v_2, v_3, v_4\rangle, \langle v_1, v_2, v_5\rangle, \text{and their faces}\},\\
\Delta_W &= \{\langle v_1, v_2, v_3, v_4, v_5\rangle, \text{and its faces}\}, \text{and}\\
\Delta_{X^+} &= \{\langle v_1, v_4, v_5\rangle, \langle v_2, v_3, v_5\rangle, \langle v_3, v_4, v_5\rangle, \text{and their faces}\}.
\end{aligned}$$

We put $X := X(\Delta_X), X^+ := X(\Delta_{X^+})$, and $W := X(\Delta_W)$. Then we have a commutative diagram of toric varieties:

$$\begin{array}{ccc}
X & \dashrightarrow & X^+ \\
& \searrow \quad \swarrow & \\
& W &
\end{array}$$

such that

(i) $\varphi\colon X \to W$ and $\varphi^+\colon X^+ \to W$ are small projective toric morphisms,
(ii) $\rho(X/W) = 1$ and $\rho(X^+/W) = 2$,
(iii) X has two isolated singular points and X^+ has only one terminal quotient singularity,

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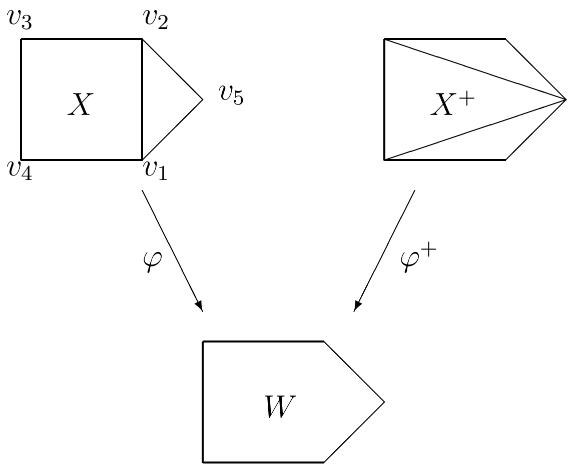
(iv) $-K_X$ is φ-ample and K_{X^+} is φ^+-ample, and
(v) X is not $\mathbb{Q}$-factorial, but X^+ is $\mathbb{Q}$-factorial.

Thus, this diagram is a toric flip. Note that the ampleness of $-K_X$ (resp. K_{X^+}) follows from the convexity (resp. concavity) of the roofs of the maximal cones in Δ_X (resp. Δ_{X^+}). The figure below should help to understand this example.



One can check the following properties:

(1) X has one isolated non-quotient canonical Gorenstein singularity and one terminal quotient singularity,
(2) the flipping locus is $\mathbb{P}^1$ and it passes through the singular points of X,
(3) X^+ has only one terminal quotient singularity, and
(4) the flipped locus is $\mathbb{P}^1 \cup \mathbb{P}^1$ and these two $\mathbb{P}^1$s intersect each other at the singular point of X^+.

This example implies that the relative Picard number may increase after a flip when X is not $\mathbb{Q}$-factorial. So, we do not use the Picard number directly to prove the termination of the log MMP.

4. We can construct a 3-dimensional toric flipping diagram

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow \quad \swarrow & \\ & W & \end{array}$$

with the following properties,

(i) X has only canonical Gorenstein singularities,
(ii) $\rho(X/W) = 1$ and $\rho(X^+/W) = n$ for any $n \geq 2$, and

(iii) X is smooth.

I will discuss this example elsewhere.

References

[F] O. Fujino, Special termination and reduction to pl flips, in *Flips for* 3*-folds and* 4*-folds* (Alessio Corti, ed.), 63–75, Oxford University Press, 2007.

[M] K. Matsuki, *Introduction to the Mori program*, Universitext. Springer-Verlag, New York, 2002.

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