TORIC FANO CONTRACTIONS ASSOCIATED TO LONG EXTREMAL RAYS

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Dedicated to Professor Shoetsu Ogata on the occasion of his sixtieth birthday

Abstract. We show that a toric Fano contraction associated to an extremal ray whose length is greater than the dimension of its fiber is a projective space bundle.

Contents

1. Introduction

Let X be a smooth projective variety defined over an algebraically closed field k of arbitrary characteristic. In his epoch-making paper (see [Mo]), Shigefumi Mori established the following famous cone theorem

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum R_j,$$

where $\overline{\mathrm{NE}}(X)$ denotes the Kleiman–Mori cone of X and each R_j is called a K_X-negative extremal ray of $\overline{\mathrm{NE}}(X)$. By the original proof of the above cone theorem, which is based on Mori's bend and break technique to create rational curves, we know that for each K_X-negative extremal ray R there exists a (possibly singular) rational curve C on X such that the numerical equivalence class of C spans R and

$$0 < -K_X \cdot C \leq \dim X + 1$$

holds.

Let X be a $\mathbb{Q}$-Gorenstein projective algebraic variety for which the cone theorem holds. Then for a K_X-negative extremal ray R of $\overline{\mathrm{NE}}(X)$, we put

$$l(R) := \min_{[C] \in R} (-K_X \cdot C)$$

and call it the *length* of R. We have already known that $l(R)$ is an important invariant and that some conditions on $l(R)$ determine the structure of the associated extremal contraction.

In this paper, we are interested in the case where X is a toric variety. We note that $\mathrm{NE}(X) = \overline{\mathrm{NE}}(X)$ holds when X is a projective toric variety. This is because $\mathrm{NE}(X)$ is

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a rational polyhedral cone. We also note that the cone theorem holds for $\mathbb{Q}$-Gorenstein projective toric varieties without any extra assumptions.

From now on, we will only treat $\mathbb{Q}$-factorial projective toric varieties defined over an algebraically closed field k of arbitrary characteristic for simplicity.

For a $\mathbb{Q}$-factorial projective toric n-fold X of Picard number $\rho(X) = 1$, there exists the unique extremal ray of $\mathrm{NE}(X)$. In this case, the following statement holds.

Theorem 1.1 ([F1, Proposition 2.9] and [F2, Proposition 2.1])**.** *Let X be a $\mathbb{Q}$-factorial projective toric n-fold of Picard number $\rho(X) = 1$ with $R = \mathrm{NE}(X)$. Then, the following statements hold.*

(1) *If $l(R) > n$, then $X \simeq \mathbb{P}^n$.*
(2) *If $l(R) \geq n$ and $X \not\simeq \mathbb{P}^n$, then $X \simeq \mathbb{P}(1, 1, 2, \ldots, 2)$.*

For the case where the associated extremal contraction is birational, we have the following estimates which are special cases of [FS2, Theorem 3.2.1].

Theorem 1.2. *Let X be a $\mathbb{Q}$-factorial projective toric n-fold, and let R be a K_X-negative extremal ray of $\mathrm{NE}(X)$. Suppose that the contraction morphism $\varphi_R : X \to W$ associated to R is birational. Then, we obtain*

$$l(R) < d + 1,$$

where

$$d = \max_{w \in W} \dim \varphi_R^{-1}(w) \leq n - 1.$$

When $d = n - 1$, we have a sharper inequality

$$l(R) \leq d = n - 1.$$

In particular, if $l(R) = n - 1$ holds, then $\varphi_R : X \to W$ can be described as follows. There exists a torus invariant smooth point $P \in W$ such that $\varphi_R : X \to W$ is a weighted blow-up at P with the weight $(1, a, \ldots, a)$ for some positive integer a. In this case, the exceptional locus E of φ_R is a torus invariant prime divisor and is isomorphic to $\mathbb{P}^{n-1}$.

This estimate shows that the extremal ray R with $l(R) > n - 1$ must be of fiber type. In this case, we can determine the structure of the associated contraction φ_R as follows.

Theorem 1.3. *Let X be a $\mathbb{Q}$-factorial projective toric n-fold with $\rho(X) \geq 2$, and let R be a K_X-negative extremal ray of $\mathrm{NE}(X)$. If $l(R) > n - 1$, then the extremal contraction $\varphi_R : X \to W$ associated to R is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$.*

Remark 1.4. Theorem 1.3 holds for projective $\mathbb{Q}$-Gorenstein toric varieties (without the assumption that X is $\mathbb{Q}$-factorial). For the details, please see [FS2, Proposition 3.2.9].

As a generalization of Theorem 1.3, we prove the following theorem about the structure of extremal contractions of fiber type. More precisely, we will prove a sharper result in Section 3 (see Theorem 3.1). Theorem 1.5 is a direct easy consequence of Theorem 3.1 (see Corollary 3.3).

Theorem 1.5 (Main theorem)**.** *Let X be a $\mathbb{Q}$-factorial projective toric n-fold. Let $\varphi_R : X \to W$ be a Fano contraction associated to a K_X-negative extremal ray $R \subset \mathrm{NE}(X)$ such that the dimension of a fiber of φ_R is d, equivalently, $d = \dim X - \dim W$. If $l(R) > d$, then φ_R is a $\mathbb{P}^d$-bundle over W.*

We show that this result is sharp by Examples 3.2 and 3.5. We note that Theorem 1.5 is nothing but Theorem 1.1 (1) if $\dim W = 0$. Therefore, we can see Theorem 1.5 as a generalization of Theorem 1.1 (1).

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2. Preliminaries

In this section, we introduce some basic results and notation of the toric geometry in order to prove the main theorem. For the details, please see [CLS], [Fu] and [O]. See also [FS1], [Ma, Chapter 14] and [R] for the toric Mori theory.

Let $X = X_\Sigma$ be the toric n-fold associated to a fan Σ in $N = \mathbb{Z}^n$ over an algebraically closed field k of arbitrary characteristic. We will use the notation $\Sigma = \Sigma_X$ to denote the fan associated to a toric variety X. It is well known that there exists a one-to-one correspondence between the r-dimensional cones in Σ and the torus invariant subvarieties of dimension $n - r$ in X. Let $\mathrm{G}(\Sigma)$ be the set of primitive generators for 1-dimensional cones in Σ. Thus, for $v \in \mathrm{G}(\Sigma)$, we have a torus invariant prime divisor corresponding to v.

For an r-dimensional simplicial cone $\sigma \in \Sigma$, let $N_\sigma \subset N$ be the sublattice generated by $\sigma \cap N$ and let $\sigma \cap \mathrm{G}(\Sigma) = \{v_1, \ldots, v_r\}$, that is, $\sigma = \langle v_1, \ldots, v_r\rangle$, where $\langle v_1, \ldots, v_r\rangle$ is the r-dimensional strongly convex cone generated by $\{v_1, \ldots, v_r\}$. Put

$$\mathrm{mult}(\sigma) := [N_\sigma : \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_r]$$

which is the index of the subgroup $\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_r$ in N_σ. The following property is fundamental.

Proposition 2.1. *Let X be a $\mathbb{Q}$-factorial toric n-fold, and let $\tau \in \Sigma$ be an $(n-1)$-dimensional cone and $v \in \mathrm{G}(\Sigma)$. If v and τ generate a maximal cone σ in Σ, then*

$$D \cdot C = \frac{\mathrm{mult}(\tau)}{\mathrm{mult}(\sigma)},$$

where D is the torus invariant prime divisor corresponding to v, while C is the torus invariant curve corresponding to τ.

Let X be a projective toric variety. We put

$$\mathrm{Z}_1(X) := \{\text{1-cycles of } X\},$$

and

$$\mathrm{Z}_1(X)_\mathbb{R} := \mathrm{Z}_1(X) \otimes \mathbb{R}.$$

Let

$$\mathrm{Pic}(X) \times \mathrm{Z}_1(X) \to \mathbb{Z}$$

be a pairing defined by $(\mathcal{L}, C) \mapsto \deg_C \mathcal{L}$. By extending it by bilinearity, we have a pairing

$$(\mathrm{Pic}(X) \otimes \mathbb{R}) \times \mathrm{Z}_1(X)_\mathbb{R} \to \mathbb{R}.$$

We define

$$\mathrm{N}^1(X) := (\mathrm{Pic}(X) \otimes \mathbb{R})/\equiv$$

and

$$\mathrm{N}_1(X) := \mathrm{Z}_1(X)_\mathbb{R}/\equiv,$$

where the *numerical equivalence* $\equiv$ is by definition the smallest equivalence relation which makes N^1 and N_1 into dual spaces.

Inside $\mathrm{N}_1(X)$ there is a distinguished cone of effective 1-cycles of X,

$$\mathrm{NE}(X) = \left\{ Z \;\middle|\; Z \equiv \sum a_i C_i \text{ with } a_i \in \mathbb{R}_{\geq 0} \right\} \subset \mathrm{N}_1(X),$$

which is usually called the *Kleiman–Mori cone* of X. It is known that $\mathrm{NE}(X)$ is a rational polyhedral cone. A face $F \subset \mathrm{NE}(X)$ is called an *extremal face* in this case. A one-dimensional extremal face is called an *extremal ray*.

Next, we introduce a combinatorial description of toric Fano contractions which are main objects of this paper. Let $X = X_\Sigma$ be a $\mathbb{Q}$-factorial projective toric n-fold and $\varphi_R : X \to W$ be the extremal contraction associated to an extremal ray $R \subset \mathrm{NE}(X)$ of fiber type. Put

$$d := \dim X - \dim W.$$

Up to automorphisms of N, Σ is constructed as follows:

For the standard basis $\{e_1, \ldots, e_n\} \subset N = \mathbb{Z}^n$, put $N' := \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_d$, while $N'' := \mathbb{Z}e_{d+1} + \cdots + \mathbb{Z}e_n$, that is, $N = N' \oplus N''$. Then, there exist $\{v_1, \ldots, v_{d+1}\} \subset \mathrm{G}(\Sigma) \cap N'$ such that $\{v_1, \ldots, v_{d+1}\} \setminus \{v_i\}$ generates a d-dimensional cone $\sigma_i \in \Sigma$ for any $1 \le i \le d+1$, and $\sigma_1 \cup \cdots \cup \sigma_{d+1} = N' \otimes \mathbb{R}$. Namely, we obtain the complete fan Σ_F in N' whose maximal cones are $\sigma_1, \ldots, \sigma_{d+1}$. Σ_F is associated to a general fiber F of φ_R, and the Picard number $\rho(F)$ is 1. Moreover, for any $\{y_1, \ldots, y_{n-d}\} \subset \mathrm{G}(\Sigma) \setminus \{v_1, \ldots, v_{d+1}\}$ which generates an $(n-d)$-dimensional cone in Σ, $\{v_1, \ldots, v_{d+1}, y_1, \ldots, y_{n-d}\} \setminus \{v_i\}$ generates a maximal cone in Σ for any $1 \le i \le d+1$. Thus, the projection $N = N' \oplus N'' \to N''$ induces φ_R.

Remark 2.2. This description shows that for a toric Fano contraction $\varphi_R : X \to W$, the dimension of any fiber is constant. As we saw above, the general fiber F of φ_R is a projective $\mathbb{Q}$-factorial toric variety of Picard number $\rho(F) = 1$. Moreover, it is known that the fiber $\varphi_R^{-1}(w)_{\mathrm{red}}$ with the reduced structure is isomorphic to F for every closed point $w \in W$ (see [CLS, Proposition 15.4.5] and [Ma, Corollary 14-2-2]).

3. Fano contractions

The following result is the main theorem of this paper.

Theorem 3.1. *Let $X = X_\Sigma$ be a $\mathbb{Q}$-factorial projective toric n-fold. Let $\varphi_R : X \to W$ be a Fano contraction associated to a K_X-negative extremal ray $R \subset \mathrm{NE}(X)$, and $d = n - \dim W$ be the dimension of a fiber of φ_R. If a general fiber of φ_R is isomorphic to $\mathbb{P}^d$ and*

$$-K_X \cdot C > \frac{d+1}{2}$$

holds for any curve C on X contracted by φ_R, then φ_R is a $\mathbb{P}^d$-bundle over W.

Proof. We may assume that $\varphi_R : X \to W$ is induced by the following projection:

$$\begin{array}{ccc} N = \mathbb{Z}^n & \xrightarrow{p} & \mathbb{Z}^{n-d} \\ \cup & & \cup \\ (x_1, \ldots, x_n) & \longmapsto & (x_{d+1}, \ldots, x_n). \end{array}$$

Let $\{e_1, \ldots, e_n\}$ be the standard basis for $N = \mathbb{Z}^n$. We put

$$v_1 := e_1, \quad \ldots, \quad v_d := e_d, \quad \text{and} \quad v_{d+1} := -(e_1 + \cdots + e_d).$$

Then Σ contains the d-dimensional subfan Σ_F corresponding to a general fiber $F \simeq \mathbb{P}^d$ whose maximal cones are

$$\langle \{v_1, \ldots, v_{d+1}\} \setminus \{v_i\} \rangle \quad (1 \le i \le d+1).$$

Let $V_\sigma \subset N \otimes_{\mathbb{Z}} \mathbb{R}$ be the linear subspace spanned by σ for any $(n-d)$-dimensional cone σ in Σ such that $(\sigma \cap \mathrm{G}(\Sigma)) \cap \{v_1, \ldots, v_{d+1}\} = \emptyset$. Then it is sufficient to show that

$$V_\sigma \cap \mathbb{Z}^n \xrightarrow{p} \mathbb{Z}^{n-d} \qquad (3.1)$$

is bijective. This is because the restriction of $\varphi_R : X \to W$ to the affine toric open subset U corresponding to an $(n-d)$-dimensional cone $p(\sigma)$ is the second projection $\mathbb{P}^d \times U \to U$ if p in (3.1) is bijective. The injectivity of (3.1) is trivial. Therefore, we will show the surjectivity of (3.1).

Let $y_1, \ldots, y_{n-d} \in \mathrm{G}(\Sigma) \setminus \{v_1, \ldots, v_{d+1}\}$ be the primitive generators for any $(n-d)$-dimensional cone in Σ such that $p(\langle y_1, \ldots, y_{n-d}\rangle)$ is also $(n-d)$-dimensional. Put

$$y_1 = (b_{1,1}, \ldots, b_{d,1}, a_{1,1}, \ldots, a_{n-d,1}),$$
$$\vdots$$
$$y_{n-d} = (b_{1,n-d}, \ldots, b_{d,n-d}, a_{1,n-d}, \ldots, a_{n-d,n-d}).$$

For any $(z_1, \ldots, z_{n-d}) \in \mathbb{Z}^{n-d}$, we can take $(c_1, \ldots, c_{n-d}) \in \mathbb{R}^{n-d}$ satisfying

$$p(c_1y_1 + \cdots + c_{n-d}y_{n-d}) = c_1p(y_1) + \cdots + c_{n-d}p(y_{n-d}) = (z_1, \ldots, z_{n-d}).$$

We note that the matrix

$$A := \begin{pmatrix} a_{1,1} & \cdots & a_{1,n-d} \\ \vdots & \ddots & \vdots \\ a_{n-d,1} & \cdots & a_{n-d,n-d} \end{pmatrix}$$

is regular as a real matrix because $p(y_1), \ldots, p(y_{n-d})$ generates an $(n-d)$-dimensional cone. Therefore, $(c_1, \ldots, c_{n-d})$ is uniquely determined by

$$\begin{pmatrix} c_1 \\ \vdots \\ c_{n-d} \end{pmatrix} = A^{-1} \begin{pmatrix} z_1 \\ \vdots \\ z_{n-d} \end{pmatrix} \in \mathbb{Q}^{n-d}.$$

Thus, all we have to do is to show that

$$c_1b_{r,1} + \cdots + c_{n-d}b_{r,n-d} \in \mathbb{Z}$$

for any $1 \leq r \leq d$.

By considering the principal Cartier divisors of the dual basis of $\{e_1, \ldots, e_n\}$, we obtain the relations

$$(3.2) \qquad \left\{ \begin{array}{rcl} D_1 - D_{d+1} + b_{1,1}E_1 + \cdots + b_{1,n-d}E_{n-d} + H_1 & = & 0, \\ & \vdots & \\ D_d - D_{d+1} + b_{d,1}E_1 + \cdots + b_{d,n-d}E_{n-d} + H_d & = & 0, \\ a_{1,1}E_1 + \cdots + a_{1,n-d}E_{n-d} + H_{d+1} & = & 0, \\ & \vdots & \\ a_{n-d,1}E_1 + \cdots + a_{n-d,n-d}E_{n-d} + H_n & = & 0 \end{array} \right.$$

in $\mathrm{N}^1(X)$, where $D_1, \ldots, D_{d+1}, E_1, \ldots, E_{n-d}$ are the torus invariant prime divisors corresponding to $v_1, \ldots, v_{d+1}, y_1, \ldots, y_{n-d}$, respectively, and $H_1, \ldots, H_n$ are some linear combinations of torus invariant prime divisors other than $D_1, \ldots, D_{d+1}, E_1, \ldots, E_{n-d}$. Let $C = C_r$ $(1 \leq r \leq d)$ be the torus invariant curve corresponding to the $(n-1)$-dimensional cone

$$\langle \{v_1, \ldots, v_d, y_1, \ldots, y_{n-d}\} \setminus \{v_r\} \rangle .$$

Since $H_i \cdot C = 0$ for any $1 \leq i \leq n$, we may ignore $H_1, \ldots, H_n$ in the following calculation. Since the matrix A is regular, we have

$$E_1 \cdot C = \cdots = E_{n-d} \cdot C = 0,$$

and

$$D_1 \cdot C = D_2 \cdot C = \cdots = D_{d+1} \cdot C$$

by the above equalities (3.2) in $N^1(X)$. Thus, we obtain

$$-K_X \cdot C = (d+1)D_i \cdot C$$

for any $1 \leq i \leq d+1$.

Put

$$\alpha := \operatorname{mult}\left(\langle \{v_1, \ldots, v_d, y_1, \ldots, y_{n-d}\} \setminus \{v_r\}\rangle\right)$$

and

$$\beta := \operatorname{mult}\left(\langle \{v_1, \ldots, v_d, y_1, \ldots, y_{n-d}\}\rangle\right).$$

Then we get

$$D_r \cdot C = \frac{\alpha}{\beta}$$

by Proposition 2.1. We note that $\alpha \mid \beta$ always holds. Obviously, $\beta = |\det A|$. On the other hand, α is the product of the elementary divisors of the $n \times (n-1)$ matrix

$$\left({}^{\mathrm t}v_1, \ldots, \overset{\vee}{{}^{\mathrm t}v_r}, \ldots, {}^{\mathrm t}v_d, {}^{\mathrm t}y_1, \ldots, {}^{\mathrm t}y_{n-d}\right) = \begin{pmatrix} 1 & & & & & & b_{1,1} & \ldots & b_{1,n-d} \\ & \ddots & & & \text{\Large 0} & & \vdots & \ddots & \vdots \\ & & 1 & & & & b_{r-1,1} & \ldots & b_{r-1,n-d} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & b_{r,1} & \ldots & b_{r,n-d} \\ & & & 1 & & & b_{r+1,1} & \ldots & b_{r+1,n-d} \\ & & & & \ddots & & \vdots & \ddots & \vdots \\ & & & & & 1 & b_{d,1} & \ldots & b_{d,n-d} \\ & & & & & & a_{1,1} & \ldots & a_{1,n-d} \\ & \text{\Large 0} & & & & & \vdots & \ddots & \vdots \\ & & & & & & a_{n-d,1} & \ldots & a_{n-d,n-d} \end{pmatrix},$$

where ${}^{\mathrm t}v$ stands for the transpose of v. By interchanging rows of this matrix, one can easily check that α is also the product of the elementary divisors of the $(n-d+1) \times (n-d)$ matrix

$$\overline{A} = \begin{pmatrix} b_{r,1} & \ldots & b_{r,n-d} \\ a_{1,1} & \ldots & a_{1,n-d} \\ \vdots & \ddots & \vdots \\ a_{n-d,1} & \ldots & a_{n-d,n-d} \end{pmatrix}.$$

Suppose that $D_r \cdot C < 1$ holds. Then, more strongly, we obtain the inequality $D_r \cdot C \leq \frac{1}{2}$ by the relation $\alpha \mid \beta$. Thus, the following inequality

$$-K_X \cdot C = (d+1)D_r \cdot C \leq \frac{d+1}{2}$$

holds. However, this contradicts the assumption that $\frac{d+1}{2} < -K_X \cdot C$. Therefore, the equality

$$\frac{\alpha}{\beta} = D_r \cdot C = 1$$

must always hold. Since the general theory of elementary divisors says that α is the greatest common divisor of the $(n-d) \times (n-d)$ minor determinants of $\overline{A}$, the $(n-d) \times (n-d)$ determinant

$$\begin{vmatrix} b_{r,1} & \ldots & b_{r,n-d} \\ a_{1,1} & \ldots & a_{1,n-d} \\ \vdots & \vdots & \vdots \\ a_{i-1,1} & \ldots & a_{i-1,n-d} \\ a_{i+1,1} & \ldots & a_{i+1,n-d} \\ \vdots & \vdots & \vdots \\ a_{n-d,1} & \ldots & a_{n-d,n-d} \end{vmatrix}$$

is divisible by $\det A$ for any $1 \le i \le n-d$. Let

$$\widetilde{A} := \begin{pmatrix} \widetilde{a}_{1,1} & \dots & \widetilde{a}_{1,n-d} \\ \vdots & \ddots & \vdots \\ \widetilde{a}_{n-d,1} & \dots & \widetilde{a}_{n-d,n-d} \end{pmatrix}$$

be the cofactor matrix of A. Then,

$$\begin{aligned} & c_1 b_{r,1} + \cdots + c_{n-d} b_{r,n-d} \\ = & \frac{1}{\det A}\left(\widetilde{a}_{1,1}z_1 + \cdots + \widetilde{a}_{1,n-d}z_{n-d}\right) b_{r,1} + \cdots + \frac{1}{\det A}\left(\widetilde{a}_{n-d,1}z_1 + \cdots + \widetilde{a}_{n-d,n-d}z_{n-d}\right) b_{r,n-d} \\ = & \frac{\widetilde{a}_{1,1}b_{r,1} + \cdots + \widetilde{a}_{n-d,1}b_{r,n-d}}{\det A} \times z_1 + \cdots + \frac{\widetilde{a}_{1,n-d}b_{r,1} + \cdots + \widetilde{a}_{n-d,n-d}b_{r,n-d}}{\det A} \times z_{n-d} \end{aligned}$$

is an integer. This completes the proof. □

The following example shows that Theorem 3.1 is sharp.

Example 3.2. Let $\{e_1, \ldots, e_n\}$ be the standard basis for $N = \mathbb{Z}^n$ and $p : N \to \mathbb{Z}^{n-d}$ be the projection

$$(x_1, \ldots, x_d, x_{d+1}, \ldots, x_n) \mapsto (x_{d+1}, \ldots, x_n)$$

for $1 \le d < n$. Put

$$v_1 := e_1, \ \ldots, \ v_d := e_d, \ v_{d+1} := -(e_1 + \cdots + e_d),$$

$$y_1 := e_{d+1}, \ \ldots, \ y_{n-d-1} := e_{n-1}, \ y_{n-d} := e_1 + e_{d+1} + \cdots + e_{n-1} + 2e_n.$$

Let Σ be the fan in N whose maximal cones are generated by $\{v_1, \ldots, v_{d+1}, y_1, \ldots, y_{n-d}\} \setminus \{v_i\}$ for $1 \le i \le d+1$. In this case, $X = X_\Sigma$ has a Fano contraction whose general fiber is isomorphic to $\mathbb{P}^d$. Moreover, every fiber with the reduced structure is isomorphic to $\mathbb{P}^d$ (see Remark 2.2). However, X does not decompose into $\mathbb{P}^d$ and a toric affine $(n-d)$-fold, because

$$\frac{p(y_1) + \cdots + p(y_{n-d})}{2} = e_{d+1} + \cdots + e_n \in \mathbb{Z}^{n-d},$$

while

$$\frac{y_1 + \cdots + y_{n-d}}{2} = \frac{1}{2}e_1 + e_{d+1} + \cdots + e_n \notin N.$$

From this noncomplete variety, one can easily construct a projective toric n-fold which has a Fano contraction associated to an extremal ray of length $\frac{d+1}{2}$ (for example, add the generator $y_{n-d+1} := -(e_{d+1} + \cdots + e_n)$ and compactify Σ).

If we make the inequality in Theorem 3.1 stronger, then the assumption that a general fiber of a Fano contraction is isomorphic to the projective space automatically holds as follows.

Corollary 3.3. *Let $X = X_\Sigma$ be a $\mathbb{Q}$-factorial projective toric n-fold. Let $\varphi_R : X \to W$ be a Fano contraction associated to a K_X-negative extremal ray $R \subset \mathrm{NE}(X)$, and $d = n - \dim W$ be the dimension of a fiber of φ_R. If $-K_X \cdot C > d$ holds for any curve C on X contracted by φ_R, then φ_R is a $\mathbb{P}^d$-bundle over W.*

Proof. Let F be a general fiber of φ_R and let C be any curve on F. Then, by adjunction, we have

$$d < -K_X \cdot C = -K_F \cdot C.$$

Therefore, by Theorem 1.1 (1), $F \simeq \mathbb{P}^d$ holds. Since $\frac{d+1}{2} \le d$, we can apply Theorem 3.1. □

As an easy consequence of Corollary 3.3, we obtain:

Corollary 3.4. *Let $X = X_\Sigma$ be a $\mathbb{Q}$-factorial projective toric n-fold and let Δ be any effective (not necessarily torus invariant) $\mathbb{R}$-divisor on X. Let $\varphi_R : X \to W$ be a Fano contraction associated to a $(K_X + \Delta)$-negative extremal ray $R \subset \mathrm{NE}(X)$ with $d = n - \dim W$. If $-(K_X + \Delta) \cdot C > d$ for any curve C on X contracted by φ_R, then φ_R is a $\mathbb{P}^d$-bundle over W.*

Proof. We can easily see that $D \cdot C \geq 0$ for any effective Weil divisor D on X and any curve C on X contracted by φ_R since $\varphi_R : X \to W$ is a toric Fano contraction of a $\mathbb{Q}$-factorial projective toric variety X. Therefore, we get

$$d < -(K_X + \Delta) \cdot C \leq -K_X \cdot C$$

for any curve C on X contracted by φ_R. Thus, we see that $\varphi_R : X \to W$ is a $\mathbb{P}^d$-bundle over W by Corollary 3.3. □

The following example shows that Corollary 3.3 is sharp.

Example 3.5. Let $F := \mathbb{P}(1, 1, 2, \ldots, 2)$ be the d-dimensional weighted projective space and W a $\mathbb{Q}$-factorial projective toric $(n - d)$-fold. Then, the length of the extremal ray corresponding to the first projection $\varphi : X = W \times F \to W$ is d (see [F2, Proposition 2.1] and [FS2, Proposition 3.1.6]).

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