LENGTHS OF EXTREMAL RAYS

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Contents

sec16.5

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1.1. **Lengths of extremal rays.** In this subsection, we discuss estimates of lengths of extremal rays. It is indispensable for the log minimal model program with scaling (see, for example, [BCHM]) and the geography of log models (see, for example, [Shokurov] and [SC]). See also the subsection 1.2 below. The results in this subsection were obtained in [Kollar2], [Kollar3], and [Ka2], [Shokurov], [Sh2], and [Birkar] with some extra assumptions.

Let us recall the following easy lemma.

lem145 **Lemma 1.1** (cf. [Sh2, Lemma 1]). *Let (X, B) be a log canonical pair, where B is an $\mathbb{R}$-divisor. Then there are positive real numbers r_i, effective $\mathbb{Q}$-divisors B_i for $1 \le i \le l$, and a positive integer m such that $\sum_{i=1}^{l} r_i = 1$, $K_X + B = \sum_{i=1}^{l} r_i(K_X + B_i)$, (X, B_i) is log canonical for every i, and $m(K_X + B_i)$ is Cartier for every i.*

Proof. Let $\sum_k D_k$ be the irreducible decomposition of $\mathrm{Supp} B$. We consider the finite dimensional real vector space $V = \bigoplus_k \mathbb{R} D_k$. We put

$$\mathcal{Q} = \{D \in V \mid K_X + D \text{ is } \mathbb{R}\text{-Cartier}\}.$$

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Then, it is easy to see that $\mathcal{Q}$ is an affine subspace of V defined over $\mathbb{Q}$. We put

$$\mathcal{L} = \{D \in \mathcal{Q} \mid K_X + D \text{ is log canonical}\}.$$

Thus, by the definition of log canonicity, it is also easy to check that $\mathcal{L}$ is a closed convex rational polytope in V. We note that $\mathcal{L}$ is compact in the classical topology of V. By the assumption, $B \in \mathcal{L}$. Therefore, we can find the desired $\mathbb{Q}$-divisors $B_i \in \mathcal{L}$ and positive real numbers r_i. □

The next result is essentially due to [Ka2] and [Sh2, Proposition 1]. We will prove a more general result in Theorem 1.7 whose proof depends on Theorem 1.2.

prop146 **Theorem 1.2.** *Let X be a normal variety such that (X, B) is lc and let $\pi : X \to S$ be a projective morphism onto a variety S. Let R be a $(K_X + B)$-negative extremal ray. Then we can find a rational curve C on X such that $[C] \in R$ and*

$$0 < -(K_X + B) \cdot C \le 2 \dim X.$$

Proof. By shrinking S, we can assume that S is quasi–projective. By replacing $\pi : X \to S$ with the extremal contraction $\varphi_R : X \to Y$ over S, we can assume that the relative Picard number $\rho(X/S) = 1$. In particular, $-(K_X + B)$ is π-ample. Let $K_X + B = \sum_{i=1}^{l} r_i(K_X + B_i)$ be as in Lemma 1.1. We assume that $-(K_X + B_1)$ is π-ample and $-(K_X + B_i) = -s_i(K_X + B_1)$ in $N^1(X/S)$ with $s_i \le 1$ for every $i \ge 2$. Thus, it is sufficient to find a rational curve C such that $\pi(C)$ is a point and that $-(K_X + B_1) \cdot C \le 2 \dim X$. So, we can assume that $K_X + B$ is $\mathbb{Q}$-Cartier and lc. By Theorem ??,[1] there is a birational morphism $f : (V, B_V) \to (X, B)$ such that $K_V + B_V = f^*(K_X + B)$, V is $\mathbb{Q}$-factorial, and (V, B_V) is dlt. By [Ka2, Theorem 1] and [Matsuki, Theorem 10-2-1], we can find a rational curve C' on V such that $-(K_V + B_V) \cdot C' \le 2 \dim V = 2 \dim X$ and that C' spans a $(K_V + B_V)$-negative extremal ray. By the projection formula, the f-image of C' is a desired rational curve. So, we finish the proof. □

Remark 1.3. It is conjectured that the estimate $\le 2 \dim X$ in Theorem 1.2 should be replaced by $\le \dim X + 1$. When X is smooth projective, it is true by Mori's famous result (cf. [Mori]). See, for example, [KM, Theorem 1.13]. When X is a toric variety, it is also true by [F3] and [F5].

[1] dlt blow-ups

Remark 1.4. In the proof of Theorem 1.2, we need Kawamata's estimate on the length of an extremal rational curve (cf. [Ka2, Theorem 1] and [Matsuki, Theorem 10-2-1]). It depends on Mori's bend and break technique to create rational curves. So, we need the mod p reduction technique there.

re-03 **Remark 1.5.** Let (X, D) be an lc pair such that D is an $\mathbb{R}$-divisor. Let $\phi : X \to Y$ be a projective morphism and H a Cartier divisor on X. Assume that $H - (K_X + D)$ is f-ample. By Theorem ??,[2] $R^q\phi_*\mathcal{O}_X(H) = 0$ for every $q > 0$ if X and Y are *algebraic* varieties. If this vanishing theorem holds for analytic spaces X and Y, then Kawamata's original argument in [Ka2] works directly for lc pairs. In that case, we do not need the results in [BCHM] in the proof of Theorem 1.2.

We consider the proof of [Matsuki, Theorem 10-2-1] when (X, D) is $\mathbb{Q}$-factorial dlt. We need $R^1\phi_*\mathcal{O}_X(H) = 0$ after shrinking X and Y analytically. In our situation, $(X, D - \varepsilon \llcorner D \lrcorner)$ is klt for $0 < \varepsilon \ll 1$. Therefore, $H - (K_X + D - \varepsilon \llcorner D \lrcorner)$ is ϕ-ample and $(X, D - \varepsilon \llcorner D \lrcorner)$ is klt for $0 < \varepsilon \ll 1$. Thus, we can apply the analytic version of the relative Kawamata–Viehweg vanishing theorem. So, we do not need the analytic version of Theorem ??.[3]

Remark 1.6. We give a remark on [BCHM]. We use the same notation as in [BCHM, 3.8]. In the proof of [BCHM, Corollary 3.8.2], we can assume that $K_X + \Delta$ is klt by [BCHM, Lemma 3.7.4]. By perturbing the coefficients of B slightly, we can further assume that B is a $\mathbb{Q}$-divisor. By applying the usual cone theorem to the klt pair (X, B), we obtain that there are only finitely many $(K_X + \Delta)$-negative extremal rays of $\overline{NE}(X/U)$. We note that [BCHM, Theorem 3.8.1] is only used in the proof of [BCHM, Corollary 3.8.2]. Therefore, we do not need the estimate of lengths of extremal rays in [BCHM]. In particular, we do not need mod p reduction arguments for the proof of the main results in [BCHM].

The final result in this subsection is an estimate of lengths of extremal rays which are relatively ample at non-lc loci (cf. [Kollar2], [Kollar3]).

thm-la **Theorem 1.7.** *Let X be a normal variety, B an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier, and $\pi : X \to S$ a projective morphism onto a variety S. Let R be a $(K_X + B)$-negative extremal*

[2] Kawamata–Viehweg for lc pairs

[3] Kawamata–Viehweg for lc pairs

ray of $\overline{NE}(X/S)$ *which is relatively ample at* $\mathrm{Nlc}(X,B)$*. Then we can find a rational curve* C *on* X *such that* $[C] \in R$ *and*

$$0 < -(K_X + B) \cdot C \leq 2 \dim X.$$

Proof. By shrinking S, we can assume that S is quasi-projective. By replacing $\pi : X \to S$ with the extremal contraction $\varphi_R : X \to Y$ over S (cf. Theorem ?? ??[4]), we can assume that the relative Picard number $\rho(X/S) = 1$ and that π is an isomorphism in a neighborhood of $\mathrm{Nlc}(X,B)$. In particular, $-(K_X + B)$ is π-ample. By Theorem thm91?? ??,[5] there is a projective birational morphism $f : Y \to X$ such that

(i) $K_Y + B_Y = f^*(K_X + B) + \sum_{a(E,X,B)<-1} (a(E,X,B)+1)E$, where $B_Y = f_*^{-1}B + \sum_{E:f\text{-exceptional}} E$,

(ii) (Y, B_Y) is a $\mathbb{Q}$-factorial dlt pair, and

(iii) $D = B_Y + F$, where $F = -\sum_{a(E,X,B)<-1} (a(E,X,B)+1)E \geq 0$.

We note that $K_Y + D = f^*(K_X + B)$. Therefore, we have

$$f_*(\overline{NE}(Y/S)_{K_Y+D\geq 0}) \subseteq \overline{NE}(X/S)_{K_X+B\geq 0} = \{0\}.$$

We also note that

$$f_*(\overline{NE}(Y/S)_{\mathrm{Nlc}(Y,D)}) = \{0\}.$$

Thus, there is a $(K_Y + D)$-negative extremal ray R' of $\overline{NE}(Y/S)$ which is relatively ample at $\mathrm{Nlc}(Y,D)$. By Theorem thm144?? ??,[6] R' is spanned by a curve $C^\dagger$. Since $-(K_Y + D) \cdot C^\dagger > 0$, we see that $f(C^\dagger)$ is a curve. If $C^\dagger \subset \mathrm{Supp}F$, then $f(C^\dagger) \subset \mathrm{Nlc}(X,B)$. It is a contradiction because $\pi \circ f(C^\dagger)$ is a point. Thus, $C^\dagger \not\subset \mathrm{Supp}F$. Since $-(K_Y + B_Y) = -(K_Y + D) + F$, we can see that R' is a $(K_Y + B_Y)$-negative extremal ray of $\overline{NE}(Y/S)$. Therefore, we can find a rational curve C' on Y such that C' spans R' and that

$$0 < -(K_Y + B_Y) \cdot C' \leq 2 \dim X$$

by Theorem prop146 1.2. By the above argument, we can easily see that $C' \not\subset \mathrm{Supp}F$. Therefore, we obtain

$$\begin{aligned} 0 < -(K_Y + D) \cdot C' &= -(K_Y + B_Y) \cdot C' - F \cdot C' \\ &\leq -(K_Y + B_Y) \cdot C' \leq 2 \dim X. \end{aligned}$$

Since $K_Y + D = f^*(K_X + B)$, $C = f(C')$ is a rational curve on X such that $\pi(C)$ is a point and $0 < -(K_X + B) \cdot C \leq 2 \dim X$. □

[4]cone and contraction theorem

[5]dlt blow-ups

[6]cone theorem

Remark 1.8. In Theorem 1.7, we can prove $0 \le -(K_X + B) \cdot C \le \dim X + 1$ when $\dim X \le 2$. For details, see [F16, Proposition 3.7].

sub-a1

1.2. **Shokurov's polytopes.** In this subsection, we discuss a very important result obtained by Shokurov (cf. [Shokurov, 6.2. First Main Theorem]), which is an application of Theorem 1.2. We closely follow Birkar's treatment in [Birkar2, Section 3].

say-a01

1.9. Let $\pi : X \to S$ be a projective morphism from a normal variety X to a variety S. A curve Γ on X is called *extremal* over S if the following properties hold.

(1) Γ generates an extremal ray R of $\overline{NE}(X/S)$.
(2) There is a π-ample Cartier divisor H on X such that

$$H \cdot \Gamma = \min\{H \cdot C\},$$

where C ranges over curves generating R.

We note that every $(K_X + \Delta)$-negative extremal ray R of $\overline{NE}(X/S)$ is spanned by a curve if Δ is an effective $\mathbb{R}$-divisor on X such that (X, Δ) is log canonical. It is a consequence of the cone and contraction theorem (cf. Theorem ??[7]).

Let B be an effective $\mathbb{R}$-divisor on X such that (X, B) is log canonical and let R be a $(K_X + B)$-negative extremal ray of $\overline{NE}(X/S)$. Then we can take a rational curve C such that C spans R and that $0 < -(K_X + B) \cdot C \le 2 \dim X$ by Theorem 1.2. Let Γ be an extremal curve generating R. Then we have

$$\frac{-(K_X + B) \cdot \Gamma}{H \cdot \Gamma} = \frac{-(K_X + B) \cdot C}{H \cdot C}.$$

Therefore,

$$-(K_X + B) \cdot \Gamma = (-(K_X + B) \cdot C) \cdot \frac{H \cdot \Gamma}{H \cdot C} \le 2 \dim X.$$

Let F be a reduced divisor on X. We consider the finite dimensional real vector space $V = \bigoplus_k \mathbb{R} F_k$ where $F = \sum_k F_k$ is the irreducible decomposition. We have already seen that

$$\mathcal{L} = \{\Delta \in V \,|\, (X, \Delta) \text{ is log canonical}\}$$

is a rational polytope in V, that is, it is the convex hull of finitely many rational points in V (see Lemma 1.1).

Let $B_1, \cdots, B_r$ be the vertices of $\mathcal{L}$ and let m be a positive integer such that $m(K_X + B_j)$ is Cartier for every j. We take an $\mathbb{R}$-divisor $B \in \mathcal{L}$. Then we can find non-negative real numbers $a_1, \cdots, a_r$ such that $B = \sum_j a_j B_j$, $\sum_j a_j = 1$, and (X, B_j) is log canonical for every

[7] cone theorem

j (see Lemma lem145 1.1). For every curve C on X, the intersection number $-(K_X+B)\cdot C$ can be written as

$$\sum_j a_j \frac{n_j}{m}$$

such that $n_j \in \mathbb{Z}$ for every j. If C is an extremal curve, then we can see that $n_j \leq 2m \dim X$ for every j by the above arguments.

On the real vector space V, we consider the following norm

$$\|B\| = \max_j \{|b_j|\},$$

where $B = \sum_j b_j F_j$.

We explain Shokurov's important results (cf. shokurov-models [Shokurov]) following birkar2 [Birkar2, Proposition 3.2].

theorem-a01

Theorem 1.10. *We use the same notation as in* say-a01 *1.9. We fix an $\mathbb{R}$-divisor $B \in \mathcal{L}$. Then we can find positive real numbers α and δ, which depend on (X, B) and F, with the following properties.*

(1) *If Γ is any extremal curve over S and $(K_X+B)\cdot\Gamma > 0$, then $(K_X+B)\cdot\Gamma > \alpha$.*
(2) *If $\Delta \in \mathcal{L}$, $\|\Delta - B\| < \delta$, and $(K_X+\Delta)\cdot R \leq 0$ for an extremal curve Γ, then $(K_X+B)\cdot\Gamma \leq 0$.*
(3) *Let $\{R_t\}_{t\in T}$ be any set of extremal rays of $\overline{NE}(X/S)$. Then*
$$\mathcal{N}_T = \{\Delta \in \mathcal{L} \,|\, (K_X+\Delta)\cdot R_t \geq 0 \text{ for every } t \in T\}$$
is a rational polytope in V.

Proof. (1) If B is a $\mathbb{Q}$-divisor, then the claim is obvious even if Γ is not extremal. We assume that B is not a $\mathbb{Q}$-divisor. Then we can write $K_X + B = \sum_j a_j(K_X+B_j)$ as in say-a01 1.9. Then $(K_X+B)\cdot\Gamma = \sum_j a_j(K_X+B_j)\cdot\Gamma$. If $(K_X+B)\cdot\Gamma < 1$, then

$$\begin{aligned}-2\dim X \leq (K_X+B_{j_0})\cdot\Gamma &< \frac{1}{a_{j_0}}\{-\sum_{j\neq j_0} a_j(K_X+B_j)\cdot\Gamma + 1\}\\ &\leq \frac{2\dim X + 1}{a_{j_0}}\end{aligned}$$

for $a_{j_0} \neq 0$. It is because $(K_X+B_j)\cdot\Gamma \geq -2\dim X$ for every j. Thus there are only finitely many possibilities of the intersection numbers $(K_X+B_j)\cdot\Gamma$ for $a_j \neq 0$ when $(K_X+B)\cdot\Gamma < 1$. Therefore, the existence of α is obvious.

(2) If we take δ sufficiently small, then, for every $\Delta \in \mathcal{L}$ with $\|\Delta - B\| < \delta$, we can always find $\Delta' \in \mathcal{L}$ such that

$$K_X + \Delta = (1-s)(K_X+B) + s(K_X+\Delta')$$

with
$$0 \le s \le \frac{\alpha}{\alpha + 2\dim X}.$$
Since Γ is extremal, we have $(K_X + \Delta') \cdot \Gamma \ge -2\dim X$ for every $\Delta' \in \mathcal{L}$. We assume that $(K_X + B) \cdot \Gamma > 0$ Then $(K_X + B) \cdot \Gamma > \alpha$ by (1). Therefore,
$$\begin{aligned}(K_X + \Delta) \cdot \Gamma &= (1 - s)(K_X + B) \cdot \Gamma + s(K_X + \Delta') \cdot \Gamma \\ &> (1 - s)\alpha + s(-2\dim X) \ge 0.\end{aligned}$$
It is a contradiction. Therefore, we obtain $(K_X + B) \cdot \Gamma \le 0$. We complete the proof of (2).

(3) For every $t \in T$, we can assume that there is some $\Delta_t \in \mathcal{L}$ such that $(K_X + \Delta) \cdot R_t < 0$. We note that $(K_X + \Delta) \cdot R_t < 0$ for some $\Delta \in \mathcal{L}$ implies $(K_X + B_j) \cdot R_t < 0$ for some j. Therefore, we can assume that T is contained in $\mathbb{N}$. It is because there are only countably many (K_X+B_j)-negative extremal rays for every j by the cone theorem (cf. Theorem ??[8]). We note that $\mathcal{N}_T$ is a closed convex subset of $\mathcal{L}$ by definition. If T is a finite set, then the claim is obvious. Thus, we can assume that $T = \mathbb{N}$. By (2) and by the compactness of $\mathcal{N}_T$, we can take $\Delta_1, \cdots, \Delta_n \in \mathcal{N}_T$ and $\delta_1, \cdots, \delta_n > 0$ such that $\mathcal{N}_T$ is covered by
$$\mathcal{B}_i = \{\Delta \in \mathcal{L} \,|\, \|\Delta - \Delta_i\| < \delta_i\}$$
and that if $\Delta \in \mathcal{B}_i$ with $(K_X + \Delta) \cdot R_t < 0$ for some t, then $(K_X + \Delta_i) \cdot R_t = 0$. If we put
$$T_i = \{t \in T \,|\, (K_X + \Delta) \cdot R_t < 0 \text{ for some } \Delta \in \mathcal{B}_i\},$$
then $(K_X + \Delta_i) \cdot R_t = 0$ for every $t \in T_i$ by the above construction. Since $\{\mathcal{B}_i\}_{i=1}^n$ gives an open covering of $\mathcal{N}_T$, we have $\mathcal{N}_T = \bigcap_{1\le i\le n} \mathcal{N}_{T_i}$.

claim-a0 **Claim.** $\mathcal{N}_T = \bigcap_{1\le i\le n} \mathcal{N}_{T_i}$.

Proof of Claim. We note that $\mathcal{N}_T \subset \bigcap_{1\le i\le n} \mathcal{N}_{T_i}$ is obvious. We assume that $\mathcal{N}_T \subsetneq \bigcap_{1\le i\le n} \mathcal{N}_{T_i}$. We take $\Delta \in \bigcap_{1\le i\le n} \mathcal{N}_{T_i} \setminus \mathcal{N}_T$ which is very close to $\mathcal{N}_T$. Since $\mathcal{N}_T$ is covered by $\{\mathcal{B}_i\}_{i=1}^n$, there is some i_0 such that $\Delta \in \mathcal{B}_{i_0}$. Since $\Delta \notin \mathcal{N}_T$, there is some $t_0 \in T$ such that $(K_X+\Delta)\cdot R_{t_0} < 0$. Thus, $t_0 \in T_{i_0}$. It is a contradiction because $\Delta \in \mathcal{N}_{T_{i_0}}$. Therefore, $\mathcal{N}_T = \bigcap_{1\le i\le n} \mathcal{N}_{T_i}$. □

So, it is sufficient to see that each $\mathcal{N}_{T_i}$ is a rational polytope in V. By replacing T with T_i, we can assume that there is some $\Delta \in \mathcal{N}_T$ such that $(K_X + \Delta) \cdot R_t = 0$ for every $t \in T$.

If $\dim_{\mathbb{R}} \mathcal{L} = 1$, then this already implies the claim. We assume $\dim_{\mathbb{R}} \mathcal{L} > 1$. Let $\mathcal{L}^1, \cdots, \mathcal{L}^p$ be the proper faces of $\mathcal{L}$. Then $\mathcal{N}^i_T =$

[8] cone theorem

$\mathcal{N}_T \cap \mathcal{L}^i$ is a rational polytope by induction on dimension. Moreover, for each $\Delta'' \in \mathcal{N}_T$ which is not Δ, there is Δ' on some proper face of $\mathcal{L}$ such that Δ'' is on the line segment determined by Δ and Δ'. Since $(K_X+\Delta)\cdot R_t = 0$ for every $t \in T$, if $\Delta' \in \mathcal{L}^i$, then $\Delta' \in \mathcal{N}^i_T$. Therefore, $\mathcal{N}_T$ is the convex hull of Δ and all the $\mathcal{N}^i_T$. Thus, there is a finite subset $T' \subset T$ such that

$$\bigcup_i \mathcal{N}^i_T = \mathcal{N}_{T'} \cap (\bigcup_i \mathcal{L}^i).$$

Therefore, the convex hull of Δ and $\bigcup_i \mathcal{N}^i_T$ is just $\mathcal{N}_{T'}$. We complete the proof of (3). □

By Theorem 1.10 (3), Lemma 2.6 in [Birkar] holds for lc pairs. It may be useful for the log minimal model program with scaling.

bir-prop **Theorem 1.11** (cf. [Birkar, Lemma 2.6])**.** *Let (X, B) be an lc pair, B an $\mathbb{R}$-divisor, and $\pi : X \to S$ a projective morphism between algebraic varieties. Let H be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $K_X + B + H$ is π-nef and $(X, B + H)$ is lc. Then, either $K_X + B$ is also π-nef or there is a $(K_X + B)$-negative extremal ray R such that $(K_X + B + \lambda H) \cdot R = 0$, where*

$$\lambda := \inf\{t \geq 0 \,|\, K_X + B + tH \text{ is } \pi\text{-nef}\}.$$

Of course, $K_X + B + \lambda H$ is π-nef.

Proof. Assume that $K_X + B$ is not π-nef. Let $\{R_j\}$ be the set of $(K_X + B)$-negative extremal rays over S. Let C_j be an extremal curve spanning R_j for every j. We put $\mu = \sup_j\{\mu_j\}$, where

$$\mu_j = \frac{-(K_X + B)\cdot C_j}{H \cdot C_j}.$$

Obviously, $\lambda = \mu$ and $0 < \mu \leq 1$. So, it is sufficient to prove that $\mu = \mu_l$ for some l. There are positive real numbers $r_1, \cdots, r_l$ such that $\sum_i r_i = 1$ and a positive integer m, which are independent of j, such that

$$-(K_X + B)\cdot C_j = \sum_{i=1}^{l} \frac{r_i n_{ij}}{m} > 0$$

(see Lemma 1.1, Theorem 1.2, and 1.9). Since C_j is extremal, n_{ij} is an integer with $n_{ij} \leq 2m \dim X$ for every i and j. If $(K_X+B+H)\cdot R_l = 0$ for some l, then there are nothing to prove since $\lambda = 1$ and $(K_X + B + H) \cdot R = 0$ with $R = R_l$. Thus, we assume that $(K_X + B + H) \cdot R_j > 0$

for every j. We put $F=\mathrm{Supp}(B+H)$, $F=\sum_k F_k$ is the irreducible decomposition, $V=\bigoplus_k \mathbb{R}F_k$,

$$\mathcal{L}=\{\Delta\in V\,|\,(X,\Delta)\text{ is log canonical}\},$$

and

$$\mathcal{N}=\{\Delta\in\mathcal{L}\,|\,(K_X+\Delta)\cdot R_j\geq 0\text{ for every }j\}.$$

Then $\mathcal{N}$ is a rational polytope in V by Theorem 1.10 (3) and $B+H$ is in the relative interior of $\mathcal{N}$ by the above assumption. Therefore, we can write

$$K_X+B+H=\sum_{p=1}^{q} r'_p(K_X+\Delta_p),$$

where $r'_1,\cdots,r'_q$ are positive real numbers such that $\sum_p r'_p=1$, (X,Δ_p) is lc for every p, $m'(K_X+\Delta_p)$ is Cartier for some positive integer m' and every p, and $(K_X+\Delta_p)\cdot C_j>0$ for every p and j. So, we obtain

$$(K_X+B+H)\cdot C_j=\sum_{p=1}^{q}\frac{r'_p n'_{pj}}{m'}$$

with $0<n'_{pj}=m'(K_X+\Delta_p)\cdot C_j\in\mathbb{Z}$. Note that m' and r'_p are independent of j for every p. We also note that

$$\begin{aligned}\frac{1}{\mu_j}=\frac{H\cdot C_j}{-(K_X+B)\cdot C_j}&=\frac{(K_X+B+H)\cdot C_j}{-(K_X+B)\cdot C_j}+1\\&=\frac{m\sum_{p=1}^{q} r'_p n'_{pj}}{m'\sum_{i=1}^{l} r_j n_{ij}}+1.\end{aligned}$$

Since

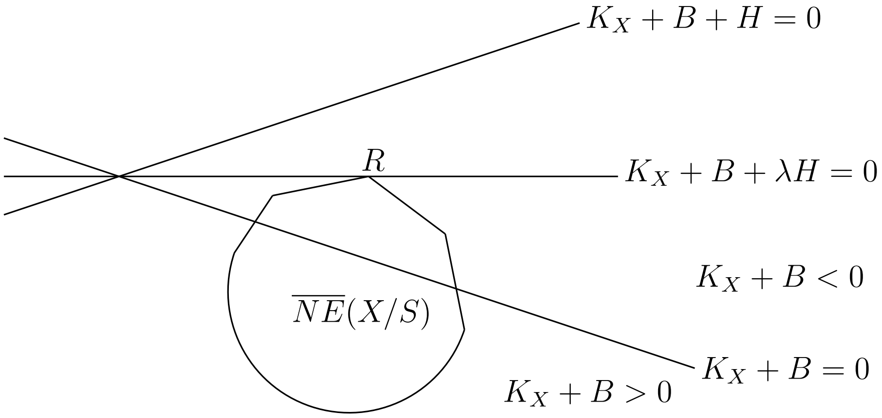
$$\sum_{i=1}^{l}\frac{r_i n_{ij}}{m}>0$$

for every j and $n_{ij}\leq 2m\dim X$ with $n_{ij}\in\mathbb{Z}$ for every i and j, the number of the set $\{n_{ij}\}_{i,j}$ is finite. Thus,

$$\inf_j\left\{\frac{1}{\mu_j}\right\}=\frac{1}{\mu_l}$$

for some l. Therefore, we obtain $\mu=\mu_l$. We finish the proof. □

The following picture helps the reader to understand Theorem 1.11.



1.12 (Abundance conjectures)**.** We close this subsection with applications of Theorem 1.10 (3) to abundance conjectures for $\mathbb{R}$-divisors (cf. [Shokurov, 2.7. Theorem on log semi-ampleness for 3-folds]).

The following proposition is a useful application of Theorem 1.10 (cf. [Shokurov, 2.7]).

proposition-a02 **Proposition 1.13.** *Let $f : X \to Y$ be a projective morphism between algebraic varieties. Let B be an effective $\mathbb{R}$-divisor on X such that (X, B) is log canonical and that $K_X + B$ is f-nef. Assume that the abundance conjecture holds for $\mathbb{Q}$-divisors. More precisely, we assume that $K_X + \Delta$ is f-semi-ample if $\Delta \in \mathcal{L}$, Δ is a $\mathbb{Q}$-divisor, and $K_X + \Delta$ is f-nef, where*

$$\mathcal{L} = \{\Delta \in V \mid (X, \Delta) \text{ is log canonical}\},$$

$V = \bigoplus_k \mathbb{R}F_k$, and $\sum_k F_k$ is the irreducible decomposition of Supp$B$. *Then $K_X + B$ is f-semi-ample.*

Proof. Let $\{R_t\}_{t \in T}$ be the set of all extremal rays of $\overline{NE}(X/Y)$. We consider $\mathcal{N}_T$ as in 1.9. Then $\mathcal{N}_T$ is a rational polytope in $\mathcal{L}$ by Theorem 1.10 (3). We can easily see that

$$\mathcal{N}_T = \{\Delta \in \mathcal{L} \mid K_X + \Delta \text{ is } f\text{-nef}\}.$$

By assumption, $B \in \mathcal{N}_T$. Let $\mathcal{F}$ be the minimal face of $\mathcal{N}_T$ containing B. Then we can find $\mathbb{Q}$-divisors $D_1, \cdots, D_l$ on X such that D_i is in the relative interior of $\mathcal{F}$, $K_X + B = \sum_i d_i(K_X + D_i)$, where d_i is a positive real number for every i and $\sum_i d_i = 1$. By assumption, $K_X + D_i$ is f-semi-ample for every i. Therefore, $K_X + B$ is f-semi-ample. $\square$

Remark 1.14 (Stability of Iitaka fibrations)**.** In the proof of Proposition 1.13, we note the following property. If C is a curve on X such that

$f(C)$ is a point and $(K_X+D_{i_0})\cdot C=0$ for some i_0, then $(K_X+D_i)\cdot C=0$ for every i. It is because we can find $\Delta'\in\mathcal{F}$ such that $(K_X+\Delta')\cdot C<0$ if $(K_X+D_i)\cdot C>0$ for some $i\neq i_0$. It is a contradiction. Therefore, there exists a contraction morphism $g:X\to Z$ over Y and h-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors $A_1,\cdots,A_l$ on Z, where $h:Z\to Y$, such that $K_X+D_i\sim_{\mathbb{Q}} g^*A_i$ for every i. In particular,

$$K_X+B\sim_{\mathbb{R}} g^*(\sum_i d_iA_i).$$

Note that $\sum_i d_iA_i$ is h-ample. Roughly speaking, the Iitaka fibration of K_X+B is the same as that of K_X+D_i for every i.

corollary-a04 **Corollary 1.15.** *Let $f:X\to Y$ be a projective morphism between algebraic varieties. Assume that (X,B) is lc and that K_X+B is f-nef. We further assume one of the following conditions.*

(i) $\dim X\leq 3$.
(ii) $\dim X=4$ *and* $\dim Y\geq 1$.

Then K_X+B is f-semi-ample.

Proof. It is obvious by Proposition 1.13 and the log abundance theorems for threefolds and fourfolds (see, for example, [KeMM, 1.1. Theorem] and [F18, Theorem 3.10]). □

Corollary 1.16. *Let $f:X\to Y$ be a projective morphism between algebraic varieties. Assume that (X,B) is klt and K_X+B is f-nef. We further assume that $\dim X-\dim Y\leq 3$. Then K_X+B is f-semi-ample.*

Proof. If B is a $\mathbb{Q}$-divisor, then it is well known that $K_{X_\eta}+B_\eta$ is semi-ample, where X_η is the generic fiber of f and $B_\eta=B|_{X_\eta}$ (see, for example, [KeMM, 1.1. Theorem]). Therefore, K_X+B is f-semi-ample by [F17, Theorem 1.1]. When B is an $\mathbb{R}$-divisor, we can take $\mathbb{Q}$-divisors $D_1,\cdots,D_l\in\mathcal{F}$ as in the proof of Proposition 1.13 such that (X,D_i) is klt for every i. Since K_X+D_i is f-semi-ample by the above argument, we obtain that K_X+B is f-semi-ample. □

Remark 1.17 (Log surfaces)**.** In [F16, Sections 6, 7, and 8], we discuss the log abundance theorem for log surfaces. The results in [F16] are much stronger than everybody expected.

References

birkar [Birkar] Existence

birkar2 [Birkar2] Existence II.

bchm [BCHM]

fuji-tor [F3] Notes on toric

fuji-to2 [F5] Equivariant
fujino16 [F16] Log surfaces
fuji-ka [F17] On Kawamata
fuji-finite [F18] Finite
kawamata [Ka2] On
kemm [KeMM] Log abundance
kollar2 [Kollar2]
kollar3 [Kollar3]
km [KM] Kollár–Mori
matsuki [Matsuki]
mori-th [Mori] Threefolds
shokurov-models [Shokurov] 3-flod log models
sho-7 [Sh2]
shokurov-choi [SC] Shokurov, Choi

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