Lemma 0.1 ([Kaw, Lemma 4]). *Let $f : X_1 \to X_2$ be a birational morphism of smooth complete varieties and let D_1 and D_2 be simple normal crossing divisors on X_1 and X_2, respectively. We assume that $D_1 = \mathrm{Supp} f^*D_2$. Then we have*

$$Rf_*\mathcal{O}_{X_1}(-D_1) \simeq \mathcal{O}_{X_2}(-D_2).$$

Proof. Note that $D_1 = \lceil \varepsilon f^* D_2 \rceil$ for $0 < \varepsilon \ll 1$. Therefore, by the relative Kawamata–Viehweg vanishing theorem, we have $R^i f_*\mathcal{O}_{X_1}(K_{X_1} + D_1) = 0$ for every $i > 0$. Since (X_2, D_2) is dlt and $K_{X_2} + D_2$ is Cartier, we see that

$$K_{X_1} + D_1 = f^*(K_{X_2} + D_2) + E$$

for some effective f-exceptional Cartier divisor E on X_1. Thus, we obtain $f_*\mathcal{O}_{X_1}(K_{X_1} + D_1) \simeq \mathcal{O}_{X_2}(K_{X_2} + D_2)$. This means that

$$Rf_*\mathcal{O}_{X_1}(K_{X_1} + D_1) \simeq \mathcal{O}_{X_2}(K_{X_2} + D_2).$$

By Grothendieck duality, we have

$$\begin{aligned} Rf_*\mathcal{O}_{X_1}(-D_1) &\simeq R\mathcal{H}om(Rf_*\mathcal{O}_{X_1}(K_{X_1} + D_1), \mathcal{O}_{X_2}(K_{X_2})) \\ &\simeq \mathcal{H}om(\mathcal{O}_{X_2}(K_{X_2} + D_2), \mathcal{O}_{X_2}(K_{X_2})) \\ &\simeq \mathcal{O}_{X_2}(-D_2). \end{aligned}$$

This is the desired quasi-isomorphism. □

References

[Kaw] Y. Kawamata, Addition formula of logarithmic Kodaira dimensions for morphisms of relative dimension one, *Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977)*, 207–217, Kinokuniya Book Store, Tokyo, 1978.

Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan

E-mail address: fujino@math.kyoto-u.ac.jp