ON ISOLATED LOG CANONICAL SINGULARITIES WITH INDEX ONE

OSAMU FUJINO

Dedicated to Professor Shihoko Ishii on the occasion of her sixtieth birthday

Abstract. We give a method to investigate isolated log canonical singularities with index one which are not log terminal. Our method depends on the minimal model program. One of the main purposes is to show that our invariant coincides with Ishii's Hodge theoretic invariant.

Contents

1. Introduction

Let $P \in X$ be an n-dimensional isolated log canonical singularity with index one which is not log terminal. Let $f : Y \to X$ be a projective resolution such that f is an isomorphism outside P and that $\mathrm{Supp} f^{-1}(P)$ is a simple normal crossing divisor on Y. Then we can write

$$K_Y = f^*K_X + F - E$$

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where E and F are effective Cartier divisors and have no common irreducible components. The divisor E is sometimes called the *essential divisor* for f (see [I2, Definition 7.4.3] and [I4, Definition 2.5]).

In [I1, Propositions 1.4 and 3.7], Shihoko Ishii proves

$$R^{n-1}f_*\mathcal{O}_Y \simeq H^{n-1}(E, \mathcal{O}_E) \simeq \mathbb{C}.$$

For details, see [I2, Propositions 5.3.11, 5.3.12, 7.1.13, 7.4.4, and Theorem 7.1.17]. In this paper, we prove that

$$R^if_*\mathcal{O}_Y \simeq H^i(E, \mathcal{O}_E)$$

for *every* $i > 0$ (cf. Proposition 4.7) and that

$$R^{n-1}f_*\mathcal{O}_Y \simeq \mathbb{C}(P)$$

(cf. Remark 4.8). Our proof depends on the minimal model theory and is different from Ishii's.

By Shihoko Ishii, the singularity $P \in X$ is said to be of type $(0, i)$ if

$$\mathrm{Gr}^W_k H^{n-1}(E, \mathcal{O}_E) = \begin{cases} \mathbb{C} & \text{if} \quad k = i \\ 0 & \text{otherwise} \end{cases}$$

where W is the weight filtration of the mixed Hodge structure on $H^{n-1}(E, \mathbb{C})$. Note that E is a projective connected simple normal crossing variety. Therefore, we have

$$\begin{aligned} &\mathrm{Gr}^W_k H^{n-1}(E, \mathcal{O}_E) \\ &\simeq \mathrm{Gr}^W_k \mathrm{Gr}^0_F H^{n-1}(E, \mathbb{C}) \\ &\simeq \mathrm{Gr}^0_F \mathrm{Gr}^W_k H^{n-1}(E, \mathbb{C}) \end{aligned}$$

where F is the Hodge filtration. We also note that the type of $P \in X$ is independent of the choice of a resolution $f : Y \to X$ by [I1, Proposition 4.2] (see also [I2, Proposition 7.4.6]).

On the other hand, we define $\mu(P \in X)$ by

$$\mu = \mu(P \in X) = \min\{\dim W \mid W \text{ is a stratum of } E\}$$

(see [F2, Definition 4.12]). We prove that $P \in X$ is of type $(0, \mu)$, that is, Ishii's Hodge theoretic invariant coincides with our invariant μ (cf. Theorem 5.5). It was first obtained by Shihoko Ishii in [I3].

By our method based on the minimal model program, we can prove the following properties of E. Let $E = \sum_i E_i$ be the irreducible decomposition. Then $\sum_{i \neq i_0} E_i|_{E_{i_0}}$ has at most two connected components for every irreducible component E_{i_0} of E (cf. Remark 4.10). Let W_1 and W_2 be any two minimal strata of E. Then W_1 is birationally equivalent to W_2 (cf. 4.11 and Remark 4.10). These results seem to be out of reach by the Hodge theoretic method.

Let Γ be the dual complex of E and let $|\Gamma|$ be the topological realization of Γ. Then the dimension of $|\Gamma|$ is $n-1-\mu$ by the definition of μ.

From now on, we assume that $\mu(P \in X) = 0$. In this case, we can prove that

$$H^i(E, \mathcal{O}_E) \simeq H^i(|\Gamma|, \mathbb{C})$$

for every i. Therefore, $P \in X$ is Cohen–Macaulay, equivalently, Gorenstein, if and only if

$$H^i(|\Gamma|, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if} \quad i = 0,\ n-1, \\ 0 & \text{otherwise.} \end{cases}$$

It is Theorem 4.12.

Anyway, by this paper, our approach based on the minimal model program (cf. [F2]) becomes compatible with Ishii's Hodge theoretic method in [I1], [I2], and [I4]. Our approach is more geometric than Ishii's. From our point of view, the main result of [IW] becomes almost obvious. We note that we do not use the notion of *Du Bois singularities*, which is one of the main ingredients of Ishii's Hodge theoretic approach.

We summarize the contents of this paper. Section 2 is a preliminary section. In Section 2.1, we give a criterion of Cohen–Macaulayness. In Section 2.2, we investigate basic properties of dlt pairs. In Section 2.3, we explain the notion of *dlt blow-ups*, which is very useful in the subsequent sections. Section 3 is devoted to the study of dlt pairs with torsion log canonical divisor. In Section 4, we investigate isolated lc singularities with index one which are not log terminal. In Section 5, we prove that our invariant μ coincides with Ishii's Hodge theoretic invariant. The main result (cf. Theorem 5.2) in Section 5 can be applied to special fibers of semi-stable minimal models for varieties with trivial canonical divisor (cf. [F6]).

Notation. Let X be a normal variety and let B be an effective $\mathbb{Q}$-divisor such that $K_X + B$ is $\mathbb{Q}$-Cartier. Then we can define the *discrepancy* $a(E, X, B) \in \mathbb{Q}$ for every prime divisor E *over* X. If $a(E, X, B) \geq -1$ (resp. > -1) for every E, then (X, B) is called *log canonical* (resp. *kawamata log terminal*). We sometimes abbreviate log canonical (resp. kawamata log terminal) to *lc* (resp. *klt*). When $(X, 0)$ is klt, we simply say that X is *log terminal* (*lt*, for short).

Assume that (X, B) is log canonical. If E is a prime divisor over X such that $a(E, X, B) = -1$, then $c_X(E)$ is called a *log canonical center* (*lc center*, for short) of (X, B), where $c_X(E)$ is the closure of the image of E on X.

Let T be a simple normal crossing variety (cf. Definition 2.6) and let $T=\sum_{i\in I} T_i$ be the irreducible decomposition. Then a *stratum* of T is an irreducible component of $T_{i_1}\cap\cdots\cap T_{i_k}$ for some $\{i_1,\cdots,i_k\}\subset I$.

Let r be a rational number. The integral part $\llcorner r\lrcorner$ is the largest integer $\leq r$ and the fractional part $\{r\}$ is defined by $r-\llcorner r\lrcorner$. We put $\ulcorner r\urcorner=-\llcorner -r\lrcorner$ and call it the round-up of r. Let $D=\sum_{i=1}^{r} d_iD_i$ be a $\mathbb{Q}$-divisor where D_i is a prime divisor for every i and $D_i\neq D_j$ for $i\neq j$. We put $\llcorner D\lrcorner=\sum\llcorner d_i\lrcorner D_i$, $\ulcorner D\urcorner=\sum\ulcorner d_i\urcorner D_i$, $\{D\}=\sum\{d_i\}D_i$, and $D^{=1}=\sum_{d_i=1}D_i$.

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In this paper, we will work over $\mathbb{C}$, the complex number field. We will freely make use of the standard notation and definition in [KM].

2. Preliminaries

In this section, we prove some preliminary results.

2.1. **A criterion of Cohen–Macaulayness.** The main purpose of this subsection is to prove Corollary 2.3, which seems to be well known to experts. Here, we give a global proof based on the Kawamata–Viehweg vanishing theorem for the reader's convenience. See also the arguments in [F5, 4.3.1].

Lemma 2.1. *Let X be a normal variety with an isolated singularity $P\in X$. Let $f:Y\to X$ be any resolution. If X is Cohen–Macaulay, then $R^if_*\mathcal{O}_Y=0$ for $0<i<n-1$, where $n=\dim X$.*

Proof. Without loss of generality, we may assume that X is projective. We consider the following spectral sequence

$$E_2^{p,q}=H^p(X,R^qf_*\mathcal{O}_Y\otimes L^{-1})\Rightarrow H^{p+q}(Y,f^*L^{-1})$$

for a sufficiently ample line bundle L on X. By the Kawamata–Viehweg vanishing theorem, $H^{p+q}(Y,f^*L^{-1})=0$ for $p+q<n$. On the other

hand, $E_2^{p,0} = H^p(X, L^{-1}) = 0$ for $p < n$ since X is Cohen–Macaulay. By using the exact sequence

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to E_2^{2,0} \to E^2 \to \cdots ,$$

we obtain $E_2^{0,1} \simeq E_2^{2,0} = 0$ when $n \geq 3$. This implies $R^1 f_* \mathcal{O}_Y = 0$. We note that $\mathrm{Supp} R^i f_* \mathcal{O}_Y \subset \{P\}$ for every $i > 0$. Inductively, we obtain $R^i f_* \mathcal{O}_Y \simeq H^0(X, R^i f_* \mathcal{O}_Y \otimes L^{-1}) = E_2^{0,i} \simeq E_\infty^{0,i} = 0$ for $0 < i < n-1$. □

Lemma 2.2. *Let X be a normal projective n-fold and let $f : Y \to X$ be a resolution. Assume that $R^i f_* \mathcal{O}_Y = 0$ for $0 < i < n-1$. Then X is Cohen–Macaulay.*

Proof. It is sufficient to prove $H^i(X, L^{-1}) = 0$ for any ample line bundle L on X for all $i < n$ (see [KM, Corollary 5.72]). We consider the spectral sequence

$$E_2^{p,q} = H^p(X, R^q f_* \mathcal{O}_Y \otimes L^{-1}) \Rightarrow H^{p+q}(Y, f^* L^{-1}).$$

As before, $H^{p+q}(Y, f^*L^{-1}) = 0$ for $p+q < n$ by the Kawamata–Viehweg vanishing theorem. By the exact sequence

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to E_2^{2,0} \to E^2 \to \cdots ,$$

we obtain $H^1(X, L^{-1}) = 0$ and $H^2(X, L^{-1}) = 0$ if $n \geq 3$. Inductively, we can check that $H^i(X, L^{-1}) = E_2^{i,0} \simeq E_\infty^{i,0} = 0$ for $i < n$. We finish the proof. □

Combining the above two lemmas, we obtain the next corollary.

Corollary 2.3. *Let $P \in X$ be a normal isolated singularity and let $f : Y \to X$ be a resolution. Then X is Cohen–Macaulay if and only if $R^i f_* \mathcal{O}_Y = 0$ for $0 < i < n-1$, where $n = \dim X$.*

Proof. We shrink X and assume that X is affine. Then we compactify X and may assume that X is projective. Therefore, we can apply Lemmas 2.1 and 2.2. □

2.2. **Basic properties of dlt pairs.** In this subsection, we prove supplementary results on dlt pairs. For the definition of dlt pairs, see [KM, Definition 2.37, Theorem 2.44]. See also [F4] for details of singularities of pairs.

The following proposition generalizes [FA, 17.5 Corollary], where it was only proved that S is semi-normal and S_2. In the subsequent sections, we will use the arguments in the proof of Proposition 2.4.

Proposition 2.4 (cf. [F5, Theorem 4.4]). *Let (X, Δ) be a dlt pair and let $\llcorner \Delta \lrcorner =: S = S_1 + \cdots + S_k$ be the irreducible decomposition. We put $T = S_1 + \cdots + S_l$ for $1 \leq l \leq k$. Then T is semi-normal, Cohen–Macaulay, and has only Du Bois singularities.*

Proof. We put $B = \{\Delta\}$. Let $f : Y \to X$ be a resolution such that $K_Y + S' + B' = f^*(K_X + S + B) + E$ with the following properties: (i) S' (resp. B') is the strict transform of S (resp. B), (ii) $\mathrm{Supp}(S' + B') \cup \mathrm{Exc}(f)$ and $\mathrm{Exc}(f)$ are simple normal crossing divisors on Y, (iii) f is an isomorphism over the generic point of every lc center of $(X, S + B)$, and (iv) $\ulcorner E \urcorner \geq 0$. We write $S = T + U$. Let T' (resp. U') be the strict transform of T (resp. U) on Y. We consider the following short exact sequence

$$0 \to \mathcal{O}_Y(-T' + \ulcorner E \urcorner) \to \mathcal{O}_Y(\ulcorner E \urcorner) \to \mathcal{O}_{T'}(\ulcorner E|_{T'} \urcorner) \to 0.$$

Since $-T' + E \sim_{\mathbb{Q},f} K_Y + U' + B'$ and $E \sim_{\mathbb{Q},f} K_Y + S' + B'$, we have $-T' + \ulcorner E \urcorner \sim_{\mathbb{Q},f} K_Y + U' + B' + \{-E\}$ and $\ulcorner E \urcorner \sim_{\mathbb{Q},f} K_Y + S' + B' + \{-E\}$. By the vanishing theorem of Reid–Fukuda type (see, for example, [F5, Lemma 4.10]),

$$R^i f_* \mathcal{O}_Y(-T' + \ulcorner E \urcorner) = R^i f_* \mathcal{O}_Y(\ulcorner E \urcorner) = 0$$

for every $i > 0$. Note that we used the assumption that f is an isomorphism over the generic point of every lc center of $(X, S + B)$. Therefore, we have

$$0 \to f_* \mathcal{O}_Y(-T' + \ulcorner E \urcorner) \to \mathcal{O}_X \to f_* \mathcal{O}_{T'}(\ulcorner E|_{T'} \urcorner) \to 0$$

and $R^i f_* \mathcal{O}_{T'}(\ulcorner E|_{T'} \urcorner) = 0$ for all $i > 0$. Note that $\ulcorner E \urcorner$ is effective and f-exceptional. Thus, $\mathcal{O}_T \simeq f_* \mathcal{O}_{T'} \simeq f_* \mathcal{O}_{T'}(\ulcorner E'|_{T'} \urcorner)$. Since T' is a simple normal crossing divisor, T is semi-normal. By the above vanishing result, we obtain $Rf_* \mathcal{O}_{T'}(\ulcorner E|_{T'} \urcorner) \simeq \mathcal{O}_T$ in the derived category. Therefore, the composition $\mathcal{O}_T \to Rf_* \mathcal{O}_{T'} \to Rf_* \mathcal{O}_{T'}(\ulcorner E|_{T'} \urcorner) \simeq \mathcal{O}_T$ is a quasi-isomorphism. Apply $R\mathcal{H}om_T(\underline{}, \omega_T^\bullet)$ to the quasi-isomorphism $\mathcal{O}_T \to Rf_* \mathcal{O}_{T'} \to \mathcal{O}_T$. Then the composition $\omega_T^\bullet \to Rf_* \omega_{T'}^\bullet \to \omega_T^\bullet$ is a quasi-isomorphism by the Grothendieck duality. By the vanishing theorem (see, for example, [F5, Lemma 2.33]), $R^i f_* \omega_{T'} = 0$ for $i > 0$. Hence, $h^i(\omega_T^\bullet) \subseteq R^i f_* \omega_{T'}^\bullet \simeq R^{i+d} f_* \omega_{T'}$, where $d = \dim T = \dim T'$. Therefore, $h^i(\omega_T^\bullet) = 0$ for $i > -d$. Thus, T is Cohen–Macaulay. This argument is the same as the proof of Theorem 1 in [K2]. Since T' is a simple normal crossing divisor, T' has only Du Bois singularities. The quasi-isomorphism $\mathcal{O}_T \to Rf_* \mathcal{O}_{T'} \to \mathcal{O}_T$ implies that T has only Du Bois singularities (cf. [K1, Corollary 2.4]). Since T' is a simple normal crossing divisor on Y and $\omega_{T'}$ is an invertible sheaf on T', every associated prime of $\omega_{T'}$ is the generic point of some irreducible component of T'. By f, every irreducible component of T' is mapped birationally onto an irreducible component of T. Therefore, $f_*\omega_{T'}$ is torsion-free on T. Since the composition $\omega_T \to f_*\omega_{T'} \to \omega_T$ is an isomorphism, we obtain $f_*\omega_{T'} \simeq \omega_T$. It is because $f_*\omega_{T'}$ is torsion-free and $f_*\omega_{T'}$ is generically isomorphic to ω_T. By the Grothendieck duality,

$$Rf_*\mathcal{O}_{T'} \simeq R\mathcal{H}om_T(Rf_*\omega^\bullet_{T'}, \omega^\bullet_T) \simeq R\mathcal{H}om_T(\omega^\bullet_T, \omega^\bullet_T) \simeq \mathcal{O}_T.$$

So, $R^if_*\mathcal{O}_{T'} = 0$ for all $i > 0$. □

We obtain the following vanishing theorem in the proof of Proposition 2.4.

Corollary 2.5. *Under the notation in the proof of* Proposition 2.4*, $R^if_*\mathcal{O}_{T'} = 0$ for every $i > 0$ and $f_*\mathcal{O}_{T'} \simeq \mathcal{O}_T$.*

We close this subsection with a useful lemma for simple normal crossing varieties.

Definition 2.6 (Normal crossing and simple normal crossing varieties)**.** A variety X has *normal crossing singularities* if, for every closed point $x \in X$,

$$\widehat{\mathcal{O}}_{X,x} \simeq \frac{\mathbb{C}[[x_0, \cdots, x_N]]}{(x_0 \cdots x_k)}$$

for some $0 \leq k \leq N$, where $N = \dim X$. Furthermore, if each irreducible component of X is smooth, X is called a *simple normal crossing* variety.

Lemma 2.7. *Let $f : V_1 \to V_2$ be a birational morphism between projective simple normal crossing varieties. Assume that there is a Zariski open subset U_1 (resp. U_2) of V_1 (resp. V_2) such that U_1 (resp. U_2) contains the generic point of any stratum of V_1 (resp. V_2) and that f induces an isomorphism between U_1 and U_2. Then $R^if_*\mathcal{O}_{V_1} = 0$ for every $i > 0$ and $f_*\mathcal{O}_{V_1} \simeq \mathcal{O}_{V_2}$.*

Proof. We can write

$$K_{V_1} = f^*K_{V_2} + E$$

such that E is f-exceptional. We consider the following commutative diagram

$$\begin{array}{ccc} V_1^\nu & \xrightarrow{f^\nu} & V_2^\nu \\ {\scriptstyle \nu_1}\downarrow & & \downarrow{\scriptstyle \nu_2} \\ V_1 & \xrightarrow[f]{} & V_2 \end{array}$$

where $\nu_1 : V_1^\nu \to V_1$ and $\nu_2 : V_2^\nu \to V_2$ are the normalizations. We can write $K_{V_1^\nu} + \Theta_1 = \nu_1^*K_{V_1}$ and $K_{V_2^\nu} + \Theta_2 = \nu_2^*K_{V_2}$, where Θ_1 and Θ_2 are the *conductor* divisors. By pulling back $K_{V_1}=f^*K_{V_2}+E$ to V_1^ν by ν_1, we have

$$K_{V_1^\nu}+\Theta_1=(f^\nu)^*(K_{V_2^\nu}+\Theta_2)+\nu_1^*E.$$

Note that V_2^ν is smooth and Θ_2 is a reduced simple normal crossing divisor on V_2^ν. By the assumption, f^ν is an isomorphism over the generic point of any lc center of the pair (V_2^ν,Θ_2). Therefore, ν_1^*E is effective since $K_{V_2^\nu}+\Theta_2$ is Cartier. Thus, we obtain that E is effective. We can easily check that f has connected fibers by the assumptions. Since V_2 is semi-normal and satisfies Serre's S_2 condition, we have $\mathcal{O}_{V_2}\simeq f_*\mathcal{O}_{V_1}$ and $f_*\mathcal{O}_{V_1}(K_{V_1})\simeq\mathcal{O}_{V_2}(K_{V_2})$. On the other hand, we obtain $R^if_*\mathcal{O}_{V_1}(K_{V_1})=0$ for every $i>0$ by [F5, Lemma 2.33]. Therefore, $Rf_*\mathcal{O}_{V_1}(K_{V_1})\simeq\mathcal{O}_{V_2}(K_{V_2})$ in the derived category. Since V_1 and V_2 are Gorenstein, we have $Rf_*\mathcal{O}_{V_1}\simeq\mathcal{O}_{V_2}$ in the derived category by the Grothendieck duality (cf. the proof of Proposition 2.4). □

2.3. **Dlt blow-ups.** Let us recall the notion of *dlt blow-ups*. Theorem 2.8 was first obtained by Christopher Hacon (cf. [F7, Section 10]). For a simplified proof, see [F6, Section 4].

Theorem 2.8 (Dlt blow-up)**.** *Let (X,Δ) be a quasi-projective lc pair. Then we can construct a projective birational morphism $f:Y\to X$ such that $K_Y+\Delta_Y=f^*(K_X+\Delta)$ with the following properties.*

(a) *(Y,Δ_Y) is a $\mathbb{Q}$-factorial dlt pair.*
(b) *$a(E,X,\Delta)=-1$ for every f-exceptional divisor E.*

When (X,Δ) is dlt, we can make f small and an isomorphism over the generic point of every lc center of (X,Δ).

Note that Theorem 2.8 was proved by the minimal model program with scaling (cf. [BCHM]).

As a corollary of Theorem 2.8, we obtain the following useful lemma.

Lemma 2.9. *Let $P\in X$ be an isolated lc singularity with index one, where X is quasi-projective. Then there exists a projective birational morphism $g:Z\to X$ such that $K_Z+D=g^*K_X$, (Z,D) is a $\mathbb{Q}$-factorial dlt pair, g is an isomorphism outside P, and D is a reduced divisor on Z.*

Remark 2.10. If $P\in X$ is $\mathbb{Q}$-factorial, then $f^{-1}(P)$ is a divisor. So, we have $\mathrm{Supp}D=f^{-1}(P)$. In general, we have only $\mathrm{Supp}D\subset f^{-1}(P)$.

For non-degenerate isolated hypersurface log canonical singularities, we can use the toric geometry to construct dlt blow-ups as in Lemma 2.9 (see [FS, Section 6]).

3. Dlt pairs with torsion log canonical divisor

This section is a supplement to [F1, Section 2] and [F2, Section 2]. We introduce a new invariant for dlt pairs with torsion log canonical divisor.

Definition 3.1. Let (X, D) be a projective dlt pair such that $K_X + D \sim_{\mathbb{Q}} 0$. We put

$$\widetilde{\mu} = \widetilde{\mu}(X, D) = \min\{\dim W \mid W \text{ is an lc center of } (X, D)\}.$$

It is related to the invariant μ, which is defined in [F2] and will play important roles in the subsequent sections. See 4.11 below.

Remark 3.2. By [CKP, Theorem 1] or [G, Theorem 1.2], $K_X + D \equiv 0$ if and only if $K_X + D \sim_{\mathbb{Q}} 0$.

As we pointed out in [FG], [F1, Section 2] works in any dimension by using the minimal model program with scaling (cf. [BCHM]). Therefore, we obtain the following proposition (cf. [F2, Proposition 2.4]).

Proposition 3.3. *Let (X, D) be a projective dlt pair such that $K_X + D \sim_{\mathbb{Q}} 0$. Let W be any minimal lc center of (X, D). Then $\dim W = \widetilde{\mu}(X, D)$. Moreover, all the minimal lc centers of (X, D) are birational each other and $\llcorner D \lrcorner$ has at most two connected components.*

Sketch of the proof. By Theorem 2.8, we may assume that X is $\mathbb{Q}$-factorial. The induction on dimension and [F1, Proposition 2.1] implies the desired properties. More precisely, all the minimal lc centers are B-birational each other (cf. [F1, Definition 1.5]). Note that Proof of Claims in the proof of [F1, Lemma 4.9] may help us understand this proposition. □

The next lemma is new. We will use it in Section 4.

Lemma 3.4. *Let (X, D) be an n-dimensional projective dlt pair such that $K_X + D \sim_{\mathbb{Q}} 0$. Assume that $\llcorner D \lrcorner \neq 0$. Then there exists an irreducible component D_0 of $\llcorner D \lrcorner$ such that $h^i(X, \mathcal{O}_X) \leq h^i(D_0, \mathcal{O}_{D_0})$ for every i.*

Proof. By using the dlt blow-up (cf. Theorem 2.8), we can construct a small projective $\mathbb{Q}$-factorialization of X. So, by replacing X with its $\mathbb{Q}$-factorialization, we may assume that X is $\mathbb{Q}$-factorial. By the assumption, $K_X + D - \varepsilon \llcorner D \lrcorner$ is not pseudo-effective for $0 < \varepsilon \ll 1$. Let H be an effective ample $\mathbb{Q}$-divisor on X such that $K_X + D - \varepsilon \llcorner D \lrcorner + H$ is nef and klt. Apply the minimal model program on $K_X + D - \varepsilon \llcorner D \lrcorner$

with scaling of H. Then we obtain a sequence of divisorial contractions and flips:

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_k,$$

and an extremal Fano contraction $\varphi : X_k \to Z$ (cf. [F6, Section 2]). By the construction, there is an irreducible component D_0 of $\llcorner D \lrcorner$ such that the strict transform D'_0 of D_0 on X_k dominates Z. Since X and X_k have only rational singularities, we have $h^i(X, \mathcal{O}_X) = h^i(X_k, \mathcal{O}_{X_k})$ for every i. Since $R^i\varphi_*\mathcal{O}_{X_k} = 0$ for every $i > 0$, we have $h^i(X_k, \mathcal{O}_{X_k}) = h^i(Z, \mathcal{O}_Z)$ for every i. Since D_0 and Z have only rational singularities (cf. [F3, Corollary 1.5]), $h^i(Z, \mathcal{O}_Z) \le h^i(D_0, \mathcal{O}_{D_0})$ for every i (see, for example, [PS, Theorem 2.29]). Therefore, we have the desired inequality $h^i(X, \mathcal{O}_X) \le h^i(D_0, \mathcal{O}_{D_0})$ for every i. □

Example 3.5. Let $X = \mathbb{P}^2$ and let D be an elliptic curve on $X = \mathbb{P}^2$. Then (X, D) is a projective dlt pair such that $K_X + D \sim 0$. In this case, $h^1(X, \mathcal{O}_X) = 0 < h^1(D, \mathcal{O}_D) = 1$.

By combining the above results, we obtain the next proposition.

Proposition 3.6. *Let (X, D) be a projective dlt pair such that $K_X + D \sim_{\mathbb{Q}} 0$. We assume that $\widetilde{\mu}(X, D) = 0$. Then $h^i(X, \mathcal{O}_X) = 0$ for every $i > 0$. Moreover, X is rationally connected.*

Proof. If $\dim X = 1$, then the statement is trivial since $X \simeq \mathbb{P}^1$. From now on, we assume that $\dim X \ge 2$. Since $\widetilde{\mu}(X, D) = 0$, we obtain that (X, D) is not klt. Thus we know $\llcorner D \lrcorner \ne 0$. Let D_0 be any irreducible component of $\llcorner D \lrcorner$. By adjunction, we obtain $(K_X + D)|_{D_0} = K_{D_0} + B$ such that (D_0, B) is dlt, $K_{D_0} + B \sim_{\mathbb{Q}} 0$, and $\widetilde{\mu}(D_0, B) = 0$ by Proposition 3.3. By the induction on dimension, we know that every irreducible component D_0 of $\llcorner D \lrcorner$ is rationally connected and $h^i(D_0, \mathcal{O}_{D_0}) = 0$ for every $i > 0$. Thus, by Lemma 3.4, we have that $h^i(X, \mathcal{O}_X) = 0$ for every $i > 0$. In the proof of Lemma 3.4, Z has only log terminal singularities by [F3, Corollary 4.5]. Since D_0 is rationally connected, so is Z by [HM, Corollary 1.5]. On the other hand, the general fiber of $\varphi : X_k \to Z$ is rationally connected (cf. [Z, Theorem 1] and [HM, Corollaries 1.3 and 1.5]). By [GHS, Corollary 1.3], X_k is rationally connected. Thus, X is rationally connected by [HM, Corollary 1.5]. □

By Proposition 3.6, we obtain a corollary: Corollary 3.7.

Corollary 3.7. *Let (X, D) be a projective dlt pair such that $K_X + D \sim_{\mathbb{Q}} 0$. Let $f : Y \to X$ be any resolution such that $K_Y + D_Y = f^*(K_X + D)$ and that* $\mathrm{Supp} D_Y$ *is a simple normal crossing divisor on Y. Assume that $\widetilde{\mu}(X, D) = 0$. Then every stratum of $D_Y^{=1}$ is rationally connected.*

Moreover, $h^i(W, \mathcal{O}_W) = 0$ *for every* $i > 0$ *where* W *is a stratum of* $D_Y^{=1}$*.*

Proof. Let W be a stratum of $D_Y^{=1}$. Let $\pi : Y' \to Y$ be a blow-up at W and let E_W be the exceptional divisor of π. Then it is sufficient to prove that E_W is rationally connected and $h^i(E_W, \mathcal{O}_{E_W}) = 0$ for every $i > 0$. Therefore, by replacing Y with Y', we may assume that W is an irreducible component of $D_Y^{=1}$. We can construct a dlt blow-up $f' : Y' \to X$ such that $K_{Y'} + D_{Y'} = f'^*(K_X + D)$ and that $f'^{-1} \circ f : Y \dashrightarrow Y'$ is an isomorphism at the generic point of W (cf. [F6, Section 6]). Since $K_{Y'} + D_{Y'} \sim_{\mathbb{Q}} 0$ and we can easily check that $\widetilde{\mu}(Y', D_{Y'}) = 0$ (cf. [F1, Claim (A_n)]), we see that W', the strict transform of W, is rationally connected and $h^i(W', \mathcal{O}_{W'}) = 0$ for every $i > 0$ by Proposition 3.6. Thus, W is rationally connected (cf. [HM, Corollary 1.5]) and $h^i(W, \mathcal{O}_W) = 0$ for every $i > 0$. □

4. Isolated log canonical singularities with index one

In this section, we consider when an isolated log canonical singularity with index one is Cohen–Macaulay or not.

4.1. Let $P \in X$ be an n-dimensional isolated lc singularity with index one. By the algebraization theorem (cf. [HR], [A1, Corollary 1.6], and [A2, Theorem 3.8]), we always assume that X is an algebraic variety in this paper (see also [I2, Theorems 3.2.3 and 3.2.4]). Assume that $P \in X$ is not lt. We consider a resolution $f : Y \to X$ such that (i) f is an isomorphism outside $P \in X$, and (ii) $f^{-1}(P)$ is a simple normal crossing divisor on Y. In this setting, we can write

$$K_Y = f^*K_X + F - E,$$

where F and E are both effective Cartier divisors without common irreducible components. In particular, E is a reduced simple normal crossing divisor on Y.

Lemma 4.2. *The cohomology group* $H^i(E, \mathcal{O}_E)$ *is independent of* f *for every* i*.*

Proof. Let $f' : Y' \to X$ be another resolution with $K_{Y'} = f'^*K_X + F' - E'$ as in 4.1. By the weak factorization theorem (see [M, Theorem 5-4-1] or [AKMW, Theorem 0.3.1(6)]), we may assume that $\varphi : Y' \to Y$ is a blow-up whose center $C \subset \mathrm{Supp} f^{-1}(P)$ is smooth, irreducible, and has simple normal crossing with $\mathrm{Supp} f^{-1}(P)$. It means that at each point $p \in \mathrm{Supp} f^{-1}(P)$ there exists a regular coordinate system $\{x_1, \cdots, x_n\}$

in a neighborhood $p \in U_p$ such that

$$\mathrm{Supp} f^{-1}(P) \cap U_p = \left\{ \prod_{j \in J} x_j = 0 \right\}$$

and $C \cap U_p = \{x_i = 0 \text{ for } i \in I\}$ for some subsets $I, J \subset \{1, \cdots, n\}$. Thus, we can directly check that $H^i(E, \mathcal{O}_E) \simeq H^i(E', \mathcal{O}_{E'})$ for every i. $\square$

4.3. Let Γ be the dual complex of E and let $|\Gamma|$ be the topological realization of Γ. Note that the vertices of Γ correspond to the components E_i, the edges correspond to $E_i \cap E_j$, and so on, where $E = \sum_i E_i$ is the irreducible decomposition of E. More precisely, E defines a conical polyhedral complex Δ (see [KKMS, Chapter II, Definition 5]). By [KKMS, p.70 Remark], we get a compact polyhedral complex Δ_0 from Δ. The dual complex Γ of E is essentially the same as this compact polyhedral complex Δ_0 and $|\Gamma| = |\Delta_0|$ as topological spaces. See the construction of the *dual complex* in [S] and [P, Section 2] for details. Therefore, we obtain the following lemma.

Lemma 4.4. *The dual complex Γ is well defined and $|\Gamma|$ is independent of f.*

Proof. As we explained above, the well-definedness of Γ is in [KKMS, Chapter II]. By the weak factorization theorem (see [M, Theorem 5-4-1] or [AKMW, Theorem 0.3.1(6)]), we can easily check that the topological realization $|\Gamma|$ does not depend on f. $\square$

Remark 4.5. The paper [S] discusses the dual complex of $\mathrm{Supp} f^{-1}(P)$ by the same method. Case 1) in the proof of [S, Lemma] is sufficient for our purposes. Note that we treat the dual complex Γ of E. In general, $\mathrm{Supp} E \subsetneq \mathrm{Supp} f^{-1}(P)$.

4.6. Let $g : Z \to X$ be a projective birational morphism as in Lemma 2.9. Then we have $0 \to \mathcal{O}_Z(-D) \to \mathcal{O}_Z \to \mathcal{O}_D \to 0$. By the vanishing theorem, we obtain $R^i g_* \mathcal{O}_Z(K_Z) = 0$ for every $i > 0$. Therefore, we have

$$R^i g_* \mathcal{O}_Z \simeq R^i g_* \mathcal{O}_D \simeq H^i(D, \mathcal{O}_D)$$

for every $i > 0$. We note that D is connected since $\mathcal{O}_X \simeq g_* \mathcal{O}_Z \to g_* \mathcal{O}_D$ is surjective. By applying Corollary 2.5, we can construct a resolution $h : Y \to Z$ such that

$$K_Y + E - F = h^*(K_Z + D) = f^* K_X,$$

where F and E are both effective Cartier divisors without common irreducible components, $\mathrm{Supp} E$ is a simple normal crossing divisor,

$f = g \circ h$, h is an isomorphism outside $g^{-1}(P)$, h is an isomorphism over the generic point of any lc center of (Z, D), $R^i h_* \mathcal{O}_E = 0$ for every $i > 0$, and $h_* \mathcal{O}_E \simeq \mathcal{O}_D$. Therefore, $H^i(D, \mathcal{O}_D) \simeq H^i(E, \mathcal{O}_E)$ for every i. Apply the principalization to the defining ideal sheaf $\mathcal{I}$ of $f^{-1}(P)$. Then we obtain a sequence of blow-ups whose centers have simple normal crossing with E (cf. [K1, Theorem 3.35]). In this process, $H^i(E, \mathcal{O}_E)$ does not change for every i (cf. the proof of Lemma 4.2). Therefore, we may assume that $f^{-1}(P)$ is a divisor on Y. We further take a sequence of blow-ups whose centers have simple normal crossing with E. Then we can make $\mathrm{Supp} f^{-1}(P)$ a simple normal crossing divisor on Y (cf. [BEV, Corollary 7.9] or [K2, Proposition 6]). We note that we may assume that f is an isomorphism outside $P \in X$. We also note that $R^i g_* \mathcal{O}_Z \simeq R^i f_* \mathcal{O}_Y$ for every i because Z has only rational singularities. So, we obtain the next proposition.

Proposition 4.7. *Let $f : Y \to X$ be a resolution as in* 4.1. *Then $R^i f_* \mathcal{O}_Y \simeq H^i(E, \mathcal{O}_E)$ for every $i > 0$. Therefore, $P \in X$ is Cohen–Macaulay, equivalently, $P \in X$ is Gorenstein, if and only if $H^i(E, \mathcal{O}_E) = 0$ for $0 < i < n - 1$.*

Proof. It is a direct consequence of Lemma 4.2 and Corollary 2.3 by 4.6. ☐

Remark 4.8. In 4.6, $(K_Z + D)|_D = K_D \sim 0$. Therefore, $H^{n-1}(D, \mathcal{O}_D)$ is dual to $H^0(D, \mathcal{O}_D)$, where $n = \dim X$. So, $R^{n-1} g_* \mathcal{O}_Z \simeq \mathbb{C}(P)$. Thus, $P \in X$ is not a rational singularity.

Remark 4.9. Shihoko Ishii proves

$$R^i f_* \mathcal{O}_Y \simeq H^i(f^{-1}(P)_{\mathrm{red}}, \mathcal{O}_{f^{-1}(P)_{\mathrm{red}}})$$

for every $i > 0$ by the theory of Du Bois singularities (cf. [I1, Corollary 1.5, Theorem 2.3] and [I2, Proposition 7.1.13, Theorem 7.1.17]). For details, see [I1] and [I2].

By using the minimal model program with scaling, we can prove Proposition 4.7 without appealing to Lemma 4.2.

Remark 4.10. Let $f : Y \to X$ with $K_Y + E = f^* K_X + F$ be as in 4.1. Let H be an effective f-ample $\mathbb{Q}$-divisor on Y such that $(Y, E + H)$ is dlt and that $K_Y + E + H$ is nef over X. We can run the minimal model program on $K_Y + E$ over X with scaling of H. Then we obtain a dlt blow-up $f' : Y' \to X$ such that (Y', E') is a $\mathbb{Q}$-factorial dlt pair and that $K_{Y'} + E' = f'^* K_X$ where E' is the pushforward of E on Y' (cf. [F6, Section 4]). We note that each step of the minimal model program

$$Y \dashrightarrow Y_1 \dashrightarrow Y_2 \dashrightarrow \cdots \dashrightarrow Y'$$

is an isomorphism at the generic point of any lc center of (Y, E). By 4.6, $R^if_*\mathcal{O}_Y \simeq R^if'_*\mathcal{O}_{Y'} \simeq R^if'_*\mathcal{O}_{E'} \simeq H^i(E', \mathcal{O}_{E'})$ for every $i > 0$. By taking a common resolution

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ Y \arrow[rr, dashed] & & Y' \end{tikzcd}$$

such that α (resp. β) is an isomorphism over the generic point of any lc center of (Y, E) (resp. (Y', E')) and that $\mathrm{Exc}(\alpha)$, $\mathrm{Exc}(\beta)$, and $\mathrm{Exc}(\alpha) \cup \mathrm{Exc}(\beta) \cup \mathrm{Supp}\alpha_*^{-1}E$ are simple normal crossing divisors on W, we can easily check that

$$H^i(E, \mathcal{O}_E) \simeq H^i(E', \mathcal{O}_{E'})$$

for every i because $R\alpha_*\mathcal{O}_T \simeq \mathcal{O}_E$ and $R\beta_*\mathcal{O}_T \simeq \mathcal{O}_{E'}$ (cf. Corollary 2.5). Note that $K_W + \Delta_1 = \alpha^*(K_Y + E)$ and $K_W + \Delta_2 = \beta^*(K_{Y'} + E')$ with $\Delta_1^{=1} = T = \Delta_2^{=1}$ such that T is a reduced simple normal crossing divisor on W. Therefore,

$$H^i(E, \mathcal{O}_E) \simeq H^i(E', \mathcal{O}_{E'}) \simeq R^if_*\mathcal{O}_Y$$

for $i > 0$.

Let $E = \sum_i E_i$ be the irreducible decomposition and let $E' = \sum_i E'_i$ be the corresponding irreducible decomposition. Let E_{i_0} be an irreducible component of E and let T_{i_0} be the strict transform of E_{i_0} on W. By applying the connectedness lemma (cf. [KM, Theorem 5.48]) to $\alpha : T_{i_0} \to E_{i_0}$ and $\beta : T_{i_0} \to E'_{i_0}$, we know that the number of the connected components of $\sum_{i \neq i_0} E_i|_{E_{i_0}}$ coincides with that of $\sum_{i \neq i_0} E'_i|_{E'_{i_0}}$. Therefore, $\sum_{i \neq i_0} E_i|_{E_{i_0}}$ has at most two connected components by applying Proposition 3.3 to $(E'_{i_0}, \sum_{i \neq i_0} E'_i|_{E'_{i_0}})$. Note that $(E'_{i_0}, \sum_{i \neq i_0} E'_i|_{E'_{i_0}})$ is dlt and $K_{E'_{i_0}} + \sum_{i \neq i_0} E'_i|_{E'_{i_0}} \sim 0$.

4.11 (Invariant μ)**.** Let $P \in X$ be an isolated lc singularity with index one which is not lt. Let $g : Z \to X$ be a projective birational morphism such that $K_Z + D = g^*K_X$ and that (Z, D) is a $\mathbb{Q}$-factorial dlt pair. We define

$$\mu = \mu(P \in X) = \min\{\dim W \,|\, W \text{ is an lc center of } (Z, D)\}.$$

This invariant μ was first introduced in [F2, Definition 4.12]. Let $D = \sum_i D_i$ be the irreducible decomposition. Then $K_{D_i} + \Delta_i := (K_Z + D)|_{D_i} \sim 0$ and (D_i, Δ_i) is dlt. By applying Proposition 3.3 to each (D_i, Δ_i), every minimal lc center of (Z, D) is μ-dimensional and all the minimal lc centers are birational each other. Note that D is connected.

Let $g' : Z' \to X$ be another projective birational morphism such that $K_{Z'} + D' = g'^*K_X$ and that (Z', D') is a $\mathbb{Q}$-factorial dlt pair. Then it is easy to see that $(Z, D) \dashrightarrow (Z', D')$ is B-birational. This means that there is a common resolution

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ Z \arrow[rr, dashed] & & Z' \end{tikzcd}$$

such that $\alpha^*(K_Z + D) = \beta^*(K_{Z'} + D')$. Then we can easily check that

$$\begin{aligned} &\min\{\dim W \mid W \text{ is an lc center of } (Z, D)\} \\ &= \min\{\dim W' \mid W' \text{ is an lc center of } (Z', D')\}. \end{aligned}$$

See, for example, the proof of [F1, Lemma 4.9]. Therefore, $\mu(P \in X)$ is well-defined. Let $f : Y \to X$ with $K_Y = f^*K_X + F - E$ be as in 4.1. Then it is easy to see that

$$\mu = \mu(P \in X) = \min\{\dim W \mid W \text{ is a stratum of } E\}$$

by Remark 4.10.

Now, the following theorem is not difficult to prove.

Theorem 4.12. *We use the notation in* 4.1. *We assume* $\mu(P \in X) = 0$. *Then* $H^i(E, \mathcal{O}_E) \simeq H^i(|\Gamma|, \mathbb{C})$. *Therefore,* $P \in X$ *is Cohen–Macaulay, equivalently,* $P \in X$ *is Gorenstein, if and only if*

$$H^i(|\Gamma|, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{for } i = 0, n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We use the spectral sequence in 4.13 to calculate $H^i(E, \mathcal{O}_E)$. By Corollary 3.7, $H^q(E^{[p]}, \mathcal{O}_{E^{[p]}}) = 0$ for every $q > 0$. Therefore, we obtain $E_2^{i,0} \simeq H^i(|\Gamma|, \mathbb{C})$ for every i and the spectral sequence degenerates at E_2. Thus we have $H^i(E, \mathcal{O}_E) \simeq H^i(|\Gamma|, \mathbb{C})$ for every i. $\square$

4.13. Let E be a simple normal crossing variety and let $E = \sum_i E_i$ be the irreducible decomposition. We put $E^{[0]} = \coprod_i E_i$, $E^{[1]} = \coprod_{i,j}(E_i \cap E_j)$, $\cdots$, $E^{[p]} = \coprod_{i_0, \cdots, i_p}(E_{i_0} \cap \cdots \cap E_{i_p})$, $\cdots$. Let $a_p : E^{[p]} \to E$ be the obvious map. Then it is well known that

$$(a_0)_*\mathcal{O}_{E^{[0]}} \to (a_1)_*\mathcal{O}_{E^{[1]}} \to \cdots \to (a_p)_*\mathcal{O}_{E^{[p]}} \to \cdots$$

is a resolution of $\mathcal{O}_E$. By taking the associated hypercohomology, we obtain a spectral sequence

$$E_1^{p,q} = H^q(E^{[p]}, \mathcal{O}_{E^{[p]}}) \Rightarrow H^{p+q}(E, \mathcal{O}_E).$$

We close this section with the following obvious two propositions.

Proposition 4.14. *We assume that the dimension of X is ≥ 3. By the above spectral sequence, if $P \in X$ is Cohen–Macaulay, then $H^1(|\Gamma|, \mathbb{C}) = 0$.*

Proof. By the spectral sequence in 4.13, it is easy to see that $H^1(|\Gamma|, \mathbb{C}) \neq 0$ implies $H^1(E, \mathcal{O}_E) \neq 0$. □

Proposition 4.15. *Let $P \in X$ be an n-dimensional isolated lc singularity with index one which is not lt. If $P \in X$ is Cohen–Macaulay, then*

$$\begin{aligned}\chi(\mathcal{O}_E) := \sum_i (-1)^i h^i(E, \mathcal{O}_E) &= 1 + (-1)^{n-1} \\ &= \sum_{p,q} (-1)^{p+q} \dim H^q(E^{[p]}, \mathcal{O}_{E^{[p]}}).\end{aligned}$$

Remark 4.16. Tsuchihashi's cusp singularities (cf. [T1] and [T2]) give us many examples of three dimensional index one isolated lc singularities with $\mu = 0$ which are not Cohen–Macaulay.

5. Ishii's Hodge theoretic invariant

In this section, we give a Hodge theoretic characterization of our invariant μ. It shows that our invariant μ coincides with Ishii's Hodge theoretic invariant.

Let us quickly recall Ishii's definition of singularities of type $(0, i)$. For the details, see [I2, Section 7] and [I4, 2.6 and Definition 2.7].

5.1 (Type $(0, i)$ singularities due to Shihoko Ishii)**.** Let $P \in X$ be an n-dimensional isolated lc singularity with index one which is not lt. Let $f : Y \to X$ be a resolution such that

$$K_Y = f^*K_X + F - E$$

as in 4.1. Shihoko Ishii proves that $H^{n-1}(E, \mathcal{O}_E) = \mathbb{C}$ (cf. Proposition 4.7 and Remark 4.8). In [I2, Definition 7.4.5] and [I4, Definition 2.7], she defines that the singularity $P \in X$ is of type $(0, i)$ if

$$\mathrm{Gr}_i^W H^{n-1}(E, \mathcal{O}_E) \neq 0.$$

Note that E is a projective simple normal crossing variety, W is the weight filtration of the natural mixed Hodge structure on $H^{n-1}(E, \mathbb{C})$, and that $H^{n-1}(E, \mathcal{O}_E) \simeq \mathrm{Gr}_F^0 H^{n-1}(E, \mathbb{C})$ where F is the natural Hodge filtration. Therefore, we have

$$\begin{aligned}&\mathrm{Gr}_k^W H^{n-1}(E, \mathcal{O}_E) \\ &\simeq \mathrm{Gr}_k^W \mathrm{Gr}_F^0 H^{n-1}(E, \mathbb{C}) \\ &\simeq \mathrm{Gr}_F^0 \mathrm{Gr}_k^W H^{n-1}(E, \mathbb{C})\end{aligned}$$

By Deligne's theory of mixed Hodge structures, we know that $0 \le i \le n-1$.

The main purpose of this section is to show that $\mu(P \in X) = i$ where $P \in X$ is of type $(0, i)$.

The following theorem corresponds to [I1, Theorem 4.3] in our framework. For the definition of *sdlt pairs*, see [F1, Definition 1.1]. Let (X, Δ) be an sdlt pair. Then X is S_2, normal crossing in codimension one, and every irreducible component of X is normal. Let V be sdlt. Then there is the smallest Zariski closed subset Z of V such that $V \setminus Z$ is a simple normal crossing variety and the codimension of Z in V is ≥ 2. We define a *stratum* of V as the closure of a stratum of $V \setminus Z$.

Theorem 5.2. *Let V be an m-dimensional connected projective sdlt variety such that $K_V \sim 0$. Let $f : V' \to V$ be a projective birational morphism from a simple normal crossing variety V'. Assume that there is a Zariski open subset U' (resp. U) of V' (resp. V) such that U' (resp. U) contains the generic point of any stratum of V' (resp. V) and that f induces an isomorphism between U' and U. We further assume that the exceptional locus* Exc(f) *is a simple normal crossing divisor on V' (cf.* [F5, Definition 2.11]) *and that*

$$K_{V'} = f^*K_V + E$$

where E is effective. Then $H^m(V', \mathcal{O}_{V'}) = \mathbb{C}$. Moreover, we obtain that

$$\begin{aligned}
&\mathrm{Gr}_F^0 \mathrm{Gr}_k^W H^m(V', \mathbb{C}) \\
&\simeq \mathrm{Gr}_k^W \mathrm{Gr}_F^0 H^m(V', \mathbb{C}) \\
&\simeq \mathrm{Gr}_k^W H^m(V', \mathcal{O}_{V'}) \\
&= \begin{cases} \mathbb{C} & \text{if} \quad k = \mu \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

where μ is the dimension of the minimal stratum of V'. Note that F is the Hodge filtration and W is the weight filtration of the natural mixed Hodge structure on $H^m(V', \mathbb{C})$.

Proof. First we prove that $H^m(V', \mathcal{O}_{V'}) = \mathbb{C}$.

Step 1. Since V is simple normal crossing in codimension one and S_2, V is semi-normal. We can easily check that f has connected fibers by the assumptions. Therefore, we obtain $f_*\mathcal{O}_{V'} \simeq \mathcal{O}_V$. We note that E is f-exceptional by the assumptions. Since E is effective, f-exceptional, and V satisfies Serre's S_2 condition, we see that $f_*\mathcal{O}_{V'}(E) \simeq \mathcal{O}_V$. On the other hand, we obtain $R^if_*\mathcal{O}_{V'}(E) \simeq R^if_*\mathcal{O}_{V'}(K_{V'}) = 0$ for every

$i > 0$ (cf. [F5, Lemma 2.33]). Therefore, we have $Rf_*\mathcal{O}_{V'}(E) \simeq \mathcal{O}_V$ in the derived category. By the same arguments as in the proof of Proposition 2.4, we obtain that V is Cohen–Macaulay. Moreover, $R^i f_*\mathcal{O}_{V'} = 0$ for every $i > 0$ (see the proof of Proposition 2.4) and $f_*\mathcal{O}_{V'} \simeq \mathcal{O}_V$. Thus, $H^m(V', \mathcal{O}_{V'}) \simeq H^m(V, \mathcal{O}_V) = \mathbb{C}$. We note that $K_V \sim 0$ and V is Cohen–Macaulay.

We use the induction on dimension for the latter statement. The statement is obvious for a 0-dimensional variety.

Step 2. When V is irreducible, the statement is obvious. It is because V' is a smooth connected projective variety. So, $H^m(V', \mathbb{C})$ has the natural pure Hodge structure of weight m.

Step 3. From now on, we assume that V is reducible. Let V'_1 be an irreducible component of V' and let V_1 be the corresponding irreducible component of V. We write $V' = V'_1 \cup V'_2$ and $V = V_1 \cup V_2$. Consider the Mayer-Vietoris exact sequence:

$$(\spadesuit) \qquad H^{m-1}(V'_1 \cap V'_2, \mathcal{O}_{V'_1 \cap V'_2}) \xrightarrow{\delta} H^m(V', \mathcal{O}_{V'}) \\ \to H^m(V'_1, \mathcal{O}_{V'_1}) \oplus H^m(V'_2, \mathcal{O}_{V'_2}).$$

By the Serre duality, $H^m(V'_i, \mathcal{O}_{V'_i})$ is dual to $H^0(V'_i, \mathcal{O}_{V'_i}(K_{V'_i}))$. We put $f_i = f|_{V'_i}$ for $i = 1, 2$. We can write

$$K_{V'_i} + V'_j|_{V'_i} = f_i^*(K_{V_i} + V_j|_{V_i}) + E|_{V'_i} \sim E|_{V'_i} =: F_i$$

for $\{i, j\} = \{1, 2\}$ where F_i is an effective f_i-exceptional divisor. We note that $K_{V_i} + V_j|_{V_i} = K_V|_{V_i} \sim 0$. Let H be an ample Cartier divisor on V. Then $(f_i^*H)^{m-1} \cdot K_{V'_i} < 0$ because $V'_j|_{V'_i} \neq 0$ for $i = 1, 2$. Thus $H^0(V'_i, \mathcal{O}_{V'_i}(K_{V'_i})) = 0$ for $i = 1, 2$. This means that $H^m(V'_i, \mathcal{O}_{V'_i}) = 0$ for $i = 1, 2$. So the last term in ($\spadesuit$) is zero. Therefore, we obtain that

$$\mathrm{Gr}_k^W H^{m-1}(V'_1 \cap V'_2, \mathcal{O}_{V'_1 \cap V'_2}) \to \mathrm{Gr}_k^W H^m(V', \mathcal{O}_{V'})$$

is surjective for every k. We note that $V'_1 \cap V'_2$ is an $(m-1)$-dimensional projective simple normal crossing variety and that $V'_1 \cap V'_2$ has at most two connected components by Proposition 3.3 and [KM, Theorem 5.48]. Note that $(V_1, V_2|_{V_1})$ is dlt and $K_{V_1} + V_2|_{V_1} \sim 0$. Moreover, each connected component of $V'_1 \cap V'_2$ satisfies the assumptions of this theorem and the dimension of the minimal stratum of each connected component of $V'_1 \cap V'_2$ is also μ. Therefore, by the induction on dimension, we obtain that $\mathrm{Gr}_k^W H^m(V', \mathcal{O}_{V'}) \neq 0$ if and only if $k = \mu$.

We obtain all the desired results. $\square$

Remark 5.3. By Step 3 in the proof of Theorem 5.2, we obtain the following description. Let C be any minimal stratum of V'. Then we obtain an isomorphism

$$\mathbb{C} = H^{\mu}(C, \mathcal{O}_C) \underset{\delta_\mu}{\simeq} \cdots \underset{\delta_k}{\simeq} \cdots \underset{\delta_{m-1}}{\simeq} H^m(V', \mathcal{O}_{V'}) = \mathbb{C}$$

where each δ_k is the connecting homomorphism of a suitable Mayer–Vietoris exact sequence for $\mu \leq k \leq m-1$. Note that C has only canonical singularities with $K_C \sim 0$.

Remark 5.4 (Semi-stable minimal models for varieties with trivial canonical divisor)**.** Let $f : X \to Y$ be a projective surjective morphism from a smooth quasi-projective variety X to a smooth quasi-projective curve Y. Assume that f is smooth over $Y \setminus P$, $K_{f^{-1}(Q)} \sim 0$ for every $Q \in Y \setminus P$, and f^*P is a reduced simple normal crossing divisor on X. Then we obtain a relative good minimal model $f' : X' \to Y$ of $f : X \to Y$ by [F6, Theorem 1.1]. Then the special fiber $S = f'^*P$ is an sdlt variety with $K_S \sim 0$. So, we can apply Theorem 5.2 to S.

As an application of Theorem 5.2, we obtain the following theorem.

Theorem 5.5. *Let $P \in X$ be an isolated lc singularity with index one which is not lt. Then $P \in X$ is of type $(0, i)$ if and only if $\mu(P \in X) = i$.*

Proof. We use the notations in Remark 4.10. Let $f : Y \to X$ be as in 4.1. First, we apply Theorem 5.2 to $\beta : T \to E'$. Then we obtain

$$\mathrm{Gr}^W_\mu H^{n-1}(T, \mathcal{O}_T) \neq 0$$

where $\mu = \mu(P \in X)$. Next, we consider $\alpha : T \to E$. Let C be a minimal stratum of E and let C' be the corresponding stratum of T. By Step 3 in the proof of Theorem 5.2, Remark 5.3, and Lemma 2.7, we can construct the following commutative diagram.

$$\begin{array}{ccc}
\mathbb{C} = H^{\mu}(C', \mathcal{O}_{C'}) & \xrightarrow{\delta_1} & H^{n-1}(T, \mathcal{O}_T) = \mathbb{C} \\
{\scriptstyle \alpha|^*_{C'}}\uparrow & & \uparrow{\scriptstyle \alpha|^*_T} \\
\mathbb{C} = H^{\mu}(C, \mathcal{O}_C) & \xrightarrow{\delta_2} & H^{n-1}(E, \mathcal{O}_E) = \mathbb{C}
\end{array}$$

Note that δ_1 and δ_2 are isomorphisms, which are the compositions of the connecting homomorphisms of suitable Mayer–Vietoris exact sequences (cf. Remark 5.3), and that $\alpha|^*_{C'}$ and $\alpha|^*_T$ are isomorphisms (cf. Lemma 2.7). By taking Gr^W, we obtain that

$$\mathrm{Gr}^W_\mu H^{n-1}(E, \mathcal{O}_E) \neq 0.$$

This means that $P \in X$ is of type $(0, \mu)$. We note that

$$\mathrm{Gr}^W_\mu H^\mu(C, \mathcal{O}_C) = H^\mu(C, \mathcal{O}_C)$$

since C is smooth and projective. $\square$

We note that Theorem 5.5 also follows from [I2, Proposition 7.4.8] and [I3] (see [F2, Remark 4.13]).

Anyway, by Theorem 5.5, our approach in [F2] and this paper is compatible with Ishii's theory developed in [I1], [I2], and [I4].

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Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan

E-mail address: fujino@math.kyoto-u.ac.jp