Introduction to the theory of quasi-log varieties

Osamu Fujino*

Abstract. This paper is a gentle introduction to the theory of quasi-log varieties by Ambro. We explain the fundamental theorems for the log minimal model program for log canonical pairs. More precisely, we give a proof of the base point free theorem for log canonical pairs in the framework of the theory of quasi-log varieties.

1. Introduction

The aim of this article is to explain the fundamental theorems for the log minimal model program for log canonical pairs. More explicitly, we describe the base point free theorem for log canonical pairs in the framework of the theory of *quasi-log varieties* (see Corollary 4.2). We also treat the cone theorem for log canonical pairs (see Theorem 5.3). This paper is a gentle introduction to Ambro's theory of quasi-log varieties (cf. [A]). It contains no new statements. However, it must be valuable because there are no introductory articles for the theory of quasi-log varieties. The original article [A] seems to be inaccessible even for experts. We basically follow Ambro's arguments (see [A, Section 5]) but we change them slightly to clarify the basic ideas and to remove some ambiguities and mistakes. The book [F7] contains a comprehensive survey of the fundamental theorems of the log minimal model program from the viewpoint of the theory of quasi-log varieties. A new approach to the log minimal model program for log canonical pairs without using quasi-log varieties was found in [F8]. It seems to be more natural and much easier than the theory of quasi-log varieties. The paper [F9] contains all the details of this new approach and is almost self-contained.

Note that we only use $\mathbb{Q}$-divisors for simplicity. Some of the results can be generalized for $\mathbb{R}$-divisors with a little care. We do not treat the relative versions of the fundamental theorems in order to make our arguments transparent. There are no difficulties for the reader to obtain the relative versions once he understands this paper. We hope that this article will make the theory of quasi-log varieties more accessible. Note that the reader does not have to refer to [A] in order to read this article. Our formulation is slightly different from the one in [A]. So, if the reader wants to taste the original flavor of the theory of quasi-log varieties, then he has to see [A].

We summarize the contents of this paper. In Section 2, we quickly review the torsion-freeness and the vanishing theorem in [F7, Chapter 2]. In Section 3, we introduce the notion of *qlc pairs*, which is a special case of Ambro's *quasi-log varieties*, and prove some important and useful lemmas. Theorem 3.6 is a key result in the theory of quasi-log varieties. Section 4 is devoted to the proof of the base point free theorem for qlc pairs. This section is the heart of this paper. In Section 5, we treat the rationality theorem and the cone theorem for log canonical pairs. We note that the rationality theorem directly implies the essential part of the cone theorem and that we do not need the theory of quasi-log varieties for the proof of the rationality theorem. In the final section, Section 6, we explain some related topics.

1.1. Notation and Conventions. We will work over the complex number field $\mathbb{C}$ throughout this paper. But we note that by using the Lefschetz principle, we can extend everything to the case where the base field is an algebraically closed field of characteristic zero. We will use the following notation and the notation in [KM] freely.

Notation. (i) For a $\mathbb{Q}$-Weil divisor $D = \sum_{j=1}^{r} d_j D_j$ such that D_j is a prime divisor for every j and $D_i \neq D_j$ for $i \neq j$, we define the *round-up* $\lceil D \rceil = \sum_{j=1}^{r} \lceil d_j \rceil D_j$ (resp. the *round-down* $\lfloor D \rfloor = \sum_{j=1}^{r} \lfloor d_j \rfloor D_j$), where for every rational number x, $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \leq \lceil x \rceil < x + 1$ (resp. $x - 1 < \lfloor x \rfloor \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \lfloor D \rfloor$. We define

$$D^{=1} = \sum_{d_j = 1} D_j, \quad \text{and} \quad D^{<1} = \sum_{d_j < 1} d_j D_j.$$

We call D a *boundary* (resp. *subboundary*) $\mathbb{Q}$-divisor if $0 \leq d_j \leq 1$ (resp. $d_j \leq 1$) for all j. Note that $\mathbb{Q}$-*linear equivalence* of two $\mathbb{Q}$-divisors B_1 and B_2 is denoted by $B_1 \sim_{\mathbb{Q}} B_2$.

(ii) For a proper birational morphism $f : X \to Y$, the *exceptional locus* $\mathrm{Exc}(f) \subset X$ is the locus where f is not an isomorphism.

(iii) Let X be a normal variety and B an effective $\mathbb{Q}$-divisor on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a resolution such that $\mathrm{Exc}(f) \cup f_*^{-1} B$ has a simple normal crossing support, where $f_*^{-1} B$ is the strict transform of B on Y. We write $K_Y = f^*(K_X + B) + \sum_i a_i E_i$ and $a(E_i, X, B) = a_i$. We say that (X, B) is *lc* if and only if $a_i \geq -1$ for all i. Here, lc is an abbreviation of *log canonical*. Note that the *discrepancy* $a(E, X, B) \in \mathbb{Q}$ can be defined for every prime divisor E *over* X. Let (X, B) be an lc pair. If E is a prime divisor over X such that $a(E, X, B) = -1$, then the center $c_X(E)$ is called an *lc center* of (X, B).

2. Vanishing and torsion-free theorems

In this section, we quickly review Ambro's formulation of torsion-free and vanishing theorems in a simplified form (see [F7, Chapter 2]). First, we fix the notation and the conventions to state theorems.

2.1 (Global embedded simple normal crossing pairs). Let Y be a simple normal crossing divisor on a smooth variety M and D a $\mathbb{Q}$-divisor on M. Assume that $\operatorname{Supp}(D+Y)$ is simple normal crossing and that D and Y have no common irreducible components. We put $B_Y = D|_Y$ and consider the pair (Y, B_Y). We call (Y, B_Y) a *global embedded simple normal crossing pair*. Let $\nu : Y^\nu \to Y$ be the normalization. We put $K_{Y^\nu} + \Theta = \nu^*(K_Y + B_Y)$. A *stratum* of (Y, B_Y) is an irreducible component of Y or the image of some lc center of $(Y^\nu, \Theta^{=1})$. When Y is smooth and B_Y is a $\mathbb{Q}$-divisor on Y such that $\operatorname{Supp} B_Y$ is simple normal crossing, we put $M = Y \times \mathbb{A}^1$ and $D = B_Y \times \mathbb{A}^1$. Then $(Y, B_Y) \simeq (Y \times \{0\}, B_Y \times \{0\})$ satisfies the above conditions, that is, we can consider (Y, B_Y) to be a global embedded simple normal crossing pair.

Theorem 2.2 is a special case of the main result in [F7, Chapter 2]. It will play crucial roles in the following sections.

Theorem 2.2 (Torsion-freeness and vanishing theorem). *Let (Y, B_Y) be as above. Assume that B_Y is a boundary $\mathbb{Q}$-divisor. Let $f : Y \to X$ be a proper morphism and L a Cartier divisor on Y.*

(1) *Assume that $L - (K_Y + B_Y)$ is f-semi-ample. Then, for every integer q, every non-zero local section of $R^q f_* \mathcal{O}_Y(L)$ contains in its support the f-image of some stratum of (Y, B_Y).*

(2) *Assume that X is projective and $L - (K_Y + B_Y) \sim_{\mathbb{Q}} f^* H$ for some ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor H on X. Then $H^p(X, R^q f_* \mathcal{O}_Y(L)) = 0$ for every $p > 0$ and $q \geq 0$.*

Remark 2.3. It is obvious that the statement of Theorem 2.2 (1) is equivalent to the following one.

(1′) Assume that $L - (K_Y + B_Y)$ is f-semi-ample. Then, for every integer q, every associated prime of $R^q f_* \mathcal{O}_Y(L)$ is the generic point of the f-image of some stratum of (Y, B_Y).

The above theorem follows from the next theorem.

Theorem 2.4 (Injectivity theorem). *Let (Y, B_Y) be as above. Assume that Y is proper and B_Y is a boundary $\mathbb{Q}$-divisor. Let D be an effective Cartier divisor whose support is contained in $\operatorname{Supp}\{B_Y\}$. Assume that $L \sim_{\mathbb{Q}} K_Y + B_Y$. Then the homomorphism*

$$H^q(Y, \mathcal{O}_Y(L)) \to H^q(Y, \mathcal{O}_Y(L + D)),$$

which is induced by the natural inclusion $\mathcal{O}_Y \to \mathcal{O}_Y(D)$, is injective for every q.

For the proof, which depends on the theory of mixed Hodge structures, we recommend the reader to see [F7, Chapter 2]. It is because [A, Section 3] seems to be inaccessible.

2.1. Idea of the proof. We prove a very special case of Theorem 2.4. This subsection is independent of the other sections. So, the reader can skip it. We adopt Kollár's principle (cf. [KM, Principle 2.46]) here instead of using the arguments by Esnault–Viehweg. We closely follow [KM, 2.4 The Kodaira Vanishing Theorem]. We note that [F6] may help the reader to understand Theorem 2.2. In [F6], we give a short and almost self-contained proof of Theorem 2.2 for the case when Y is smooth.

First, we recall the following Hodge theoretic results. Note that we compute the cohomology groups in the complex analytic setting throughout this subsection.

Theorem 2.5. *Let V be a smooth projective variety and Σ a simple normal crossing divisor on V. Let $\iota : V \setminus \Sigma \to V$ be the natural open immersion. Then the inclusion $\iota_! \mathbb{C}_{V \setminus \Sigma} \subset \mathcal{O}_V(-\Sigma)$ induces surjections*

$$H_c^i(V \setminus \Sigma, \mathbb{C}) = H^i(V, \iota_! \mathbb{C}_{V \setminus \Sigma}) \to H^i(V, \mathcal{O}_V(-\Sigma))$$

for all i.

We note that $\iota_! \mathbb{C}_{V \setminus \Sigma}$ is quasi-isomorphic to the complex $\Omega_V^\bullet(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma)$ and the Hodge to de Rham spectral sequence

$$E_1^{p,q} = H^q(V, \Omega_V^p(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma)) \implies H_c^{p+q}(V \setminus \Sigma, \mathbb{C})$$

degenerates at the E_1-term. See, for example, [E, I.3.], [F7, Section 2.4], or Remark 2.6 below. Theorem 2.5 is a direct consequence of this E_1-degeneration.

Remark 2.6. We put $n = \dim V$. By Poincaré duality, we have

$$H^{2n-(p+q)}(V \setminus \Sigma, \mathbb{C}) \simeq H_c^{p+q}(V \setminus \Sigma, \mathbb{C})^*.$$

On the other hand, by Serre duality, we see that

$$H^{n-q}(V, \Omega_V^{n-p}(\log \Sigma)) \simeq H^q(V, \Omega_V^p(\log \Sigma) \otimes \mathcal{O}_V(-\Sigma))^*.$$

Therefore, the above E_1-degeneration easily follows from the well-known E_1-degeneration of

$${}'E_1^{n-p,n-q} = H^{n-q}(V, \Omega_V^{n-p}(\log \Sigma)) \implies H^{2n-(p+q)}(V \setminus \Sigma, \mathbb{C}).$$

The next theorem is a special case of Theorem 2.4 if we put $Y = X$, $L = K_X + S + M$, and $B_Y = S + \frac{d}{m}D$.

Theorem 2.7. *Let X be a smooth projective variety and S a simple normal crossing divisor on X. Let M be a Cartier divisor on X. Assume that there exists a smooth divisor D on X such that $dD \sim mM$ for some relatively prime positive integers d and m with $d < m$, D and S have no common irreducible components, and $D + S$ is a simple normal crossing divisor on X. Then the homomorphism*

$$H^i(X, \mathcal{O}_X(K_X + S + M)) \to H^i(X, \mathcal{O}_X(K_X + S + M + bD))$$

induced by the natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(bD)$ is injective for every positive integer b and every $i \geq 0$.

Proof. We take the usual normalization of the m-fold cyclic cover $\pi : Y \to X$ ramified along the divisor D and defined by $dD \sim mM$. We put $T = \pi^*S$. Then Y is smooth and T is simple normal crossing on Y. Let $\iota : Y \setminus T \to Y$ be the natural open immersion. Then the inclusion $\iota_!\mathbb{C}_{Y \setminus T} \subset \mathcal{O}_Y(-T)$ induces the following surjections

$$H^i(Y, \iota_!\mathbb{C}_{Y \setminus T}) \to H^i(Y, \mathcal{O}_Y(-T))$$

for all i by Theorem 2.5. Since the fibers of π are zero-dimensional, there are no higher direct image sheaves, and

$$H^i(X, \pi_*\iota_!\mathbb{C}_{Y \setminus T}) \to H^i(X, \pi_*\mathcal{O}_Y(-T))$$

is surjective for every $i \geq 0$. The $\mathbb{Z}/m\mathbb{Z}$-action gives eigensheaf decompositions

$$\pi_*\iota_!\mathbb{C}_{Y \setminus T} = \bigoplus_{k=0}^{m-1} G_k$$

and

$$\pi_*\mathcal{O}_Y(-T) = \bigoplus_{k=0}^{m-1} \mathcal{O}_X(-S - kM + \lfloor \tfrac{kd}{m} \rfloor D)$$

such that

$$G_k \subset \mathcal{O}_X(-S - kM + \lfloor \tfrac{kd}{m} \rfloor D)$$

for $0 \leq k \leq m-1$. By taking the $k = 1$ summand, we have the surjections

$$H^i(X, G_1) \to H^i(X, \mathcal{O}_X(-S - M))$$

for all i. It is easy to see that G_1 is a subsheaf of $\mathcal{O}_X(-S - M - bD)$ for every $b \geq 0$. See, for example, [KM, Corollary 2.54, Lemma 2.55]. Therefore,

$$H^i(X, \mathcal{O}_X(-S - M - bD)) \to H^i(X, \mathcal{O}_X(-S - M))$$

is surjective for every i (cf. [KM, Corollary 2.56]). By Serre duality, we have the desired injections. $\square$

By Theorem 2.7, we can easily obtain a very special case of Theorem 2.2 (2). We omit the proof because it is routine work. See, for example, [F1, Section 2.2].

Theorem 2.8. *Let $f : X \to Y$ be a morphism from a smooth projective variety X onto a projective variety Y. Let S be a simple normal crossing divisor on X and L an ample Cartier divisor on Y. Then*

$$H^i(Y, R^jf_*\mathcal{O}_X(K_X + S) \otimes \mathcal{O}_Y(L)) = 0$$

for $i > 0$ and $j \geq 0$.

As a corollary, we obtain a generalization of the Kodaira vanishing theorem (cf. [F6, Theorem 4.4]).

Corollary 2.9 (Kodaira vanishing theorem for log canonical varieties). *Let Y be a projective variety with only log canonical singularities and L an ample Cartier divisor on Y. Then*

$$H^i(Y, \mathcal{O}_Y(K_Y + L)) = 0$$

for $i > 0$.

Proof. Let $f : X \to Y$ be a resolution such that $S = \mathrm{Exc}(f)$ is a simple normal crossing divisor. Then $f_*\mathcal{O}_X(K_X + S) \simeq \mathcal{O}_Y(K_Y)$. Therefore, we have the desired vanishing theorem by Theorem 2.8. $\qquad\square$

We close this subsection with Sommese's example. For the details and other examples, see [F7, Section 2.8].

Example 2.10. We consider $\pi : Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3}) \to \mathbb{P}^1$. Let $\mathcal{M}$ denote the tautological line bundle of $\pi : Y \to \mathbb{P}^1$. We take a general member X of $|(\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1))^{\otimes 4}|$. Then X is a normal Gorenstein projective threefold. Note that X is not lc. We put $\mathcal{O}_Y(L) = \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$. Then L is an ample Cartier divisor on Y. We can check that $H^1(X, \mathcal{O}_X(K_X + L)) = \mathbb{C}$. Thus, the Kodaira vanishing theorem does not necessarily hold for non-lc varieties.

3. Adjunction for qlc varieties

To prove the base point free theorem for log canonical pairs following Ambro's idea, it is better to introduce the notion of *qlc varieties*. For the details, see [F7, Section 3.2].

Definition 3.1 (Qlc varieties). A *qlc variety* is a variety X with a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor ω, and a finite collection $\{C\}$ of reduced and irreducible subvarieties of X such that there is a proper morphism $f : (Y, B_Y) \to X$ from a global embedded simple normal crossing pair as in 2.1 satisfying the following properties:

(1) $f^*\omega \sim_{\mathbb{Q}} K_Y + B_Y$ such that B_Y is a subboundary $\mathbb{Q}$-divisor.

(2) There is an isomorphism

$$\mathcal{O}_X \simeq f_*\mathcal{O}_Y(\ulcorner-(B_Y^{<1})\urcorner).$$

(3) The collection of subvarieties $\{C\}$ coincides with the image of the (Y, B_Y)-strata.

We use the following terminology. The subvarieties C are the *qlc centers* of X, and $f : (Y, B_Y) \to X$ is a *quasi-log resolution* of X. We sometimes simply say that $[X, \omega]$ is a *qlc pair*, or the pair $[X, \omega]$ is *qlc*.

Remark 3.2. By condition (2), we have an isomorphism $\mathcal{O}_X \simeq f_*\mathcal{O}_Y$. In particular, f is a surjective morphism with connected fibers and X is semi-normal.

Proposition 3.3. *Let (X, B) be an lc pair. Then $[X, K_X + B]$ is a qlc pair.*

Proof. Let $f : Y \to X$ be a resolution such that $K_Y + B_Y = f^*(K_X + B)$ and Supp B_Y is a simple normal crossing divisor. Then $\mathcal{O}_X \simeq f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1}) \urcorner)$ because $\ulcorner -(B_Y^{<1}) \urcorner$ is effective and f-exceptional. We note that a qlc center C is X itself or an lc center of (X, B). $\qquad\square$

We start with an easy lemma.

Lemma 3.4. *Let $f : Z \to Y$ be a proper birational morphism between smooth varieties and B_Y a subboundary $\mathbb{Q}$-divisor on Y such that $\mathrm{Supp}\, B_Y$ is simple normal crossing. Assume that $K_Z + B_Z = f^*(K_Y + B_Y)$ and that $\mathrm{Supp}\, B_Z$ is simple normal crossing. Then we have*

$$f_*\mathcal{O}_Z(\ulcorner -(B_Z^{<1}) \urcorner) \simeq \mathcal{O}_Y(\ulcorner -(B_Y^{<1}) \urcorner).$$

Proof. From $K_Z + B_Z = f^*(K_Y + B_Y)$, we obtain

$$\begin{aligned} K_Z &= f^*(K_Y + B_Y^{=1} + \{B_Y\}) \\ &\quad + f^*(\llcorner B_Y^{<1} \lrcorner) - \llcorner B_Z^{<1} \lrcorner - B_Z^{=1} - \{B_Z\}. \end{aligned}$$

If $a(\nu, Y, B_Y^{=1} + \{B_Y\}) = -1$ for a prime divisor ν over Y, then we can check that $a(\nu, Y, B_Y) = -1$ by using [KM, Lemma 2.45]. Since $f^*(\llcorner B_Y^{<1} \lrcorner) - \llcorner B_Z^{<1} \lrcorner$ is Cartier, we can easily see that $f^*(\llcorner B_Y^{<1} \lrcorner) = \llcorner B_Z^{<1} \lrcorner + E$, where E is an effective f-exceptional divisor. Thus, we obtain

$$f_*\mathcal{O}_Z(\ulcorner -(B_Z^{<1}) \urcorner) \simeq \mathcal{O}_Y(\ulcorner -(B_Y^{<1}) \urcorner).$$

This completes the proof. $\qquad\square$

The following lemma is very important in the study of qlc pairs.

Lemma 3.5. *We use the same notation and assumptions as in* Lemma 3.4. *Let S be a simple normal crossing divisor on Y such that $S \subset \mathrm{Supp}\, B_Y^{=1}$. Let T be the union of the irreducible components of $B_Z^{=1}$ that are mapped into S by f. Assume that $\mathrm{Supp}\, f_*^{-1}B_Y \cup \mathrm{Exc}(f)$ is simple normal crossing on Z. Then we have*

$$f_*\mathcal{O}_T(\ulcorner -(B_T^{<1}) \urcorner) \simeq \mathcal{O}_S(\ulcorner -(B_S^{<1}) \urcorner),$$

where $(K_Z + B_Z)|_T = K_T + B_T$ and $(K_Y + B_Y)|_S = K_S + B_S$.

Proof. We use the same notation as in the proof of Lemma 3.4. We consider the short exact sequence

$$0 \to \mathcal{O}_Z(\ulcorner -(B_Z^{<1}) \urcorner - T) \to \mathcal{O}_Z(\ulcorner -(B_Z^{<1}) \urcorner) \to \mathcal{O}_T(\ulcorner -(B_T^{<1}) \urcorner) \to 0.$$

Since $T = f^*S - F$, where F is an effective f-exceptional divisor, we obtain

$$\ulcorner -(B_Z^{<1}) \urcorner - T = f^*(\ulcorner -(B_Y^{<1}) \urcorner - S) + E + F.$$

Here, we used $f^*(\llcorner B_Y^{\leq 1} \lrcorner) = \llcorner B_Z^{\leq 1} \lrcorner + E$ in the proof of Lemma 3.4. Therefore,

$$f_*\mathcal{O}_Z(\ulcorner -(B_Z^{\leq 1})\urcorner - T) \simeq \mathcal{O}_Y(\ulcorner -(B_Y^{\leq 1})\urcorner - S) \otimes f_*\mathcal{O}_Z(E+F)$$
$$\simeq \mathcal{O}_Y(\ulcorner -(B_Y^{\leq 1})\urcorner - S).$$

It is because E and F are effective and f-exceptional. We note that

$$(\ulcorner -(B_Z^{\leq 1})\urcorner - T) - (K_Z + \{B_Z\} + (B_Z^{=1} - T)) = -f^*(K_Y + B_Y).$$

Therefore, every local section of $R^1 f_*\mathcal{O}_Z(\ulcorner -(B_Z^{\leq 1})\urcorner - T)$ contains in its support the f-image of some stratum of $(Z, \{B_Z\} + B_Z^{=1} - T)$ by Theorem 2.2 (1).

Claim. *No strata of $(Z, \{B_Z\} + B_Z^{=1} - T)$ are mapped into S by f.*

Proof of Claim. Assume that there is a stratum C of $(Z, \{B_Z\} + B_Z^{=1} - T)$ such that $f(C) \subset S$. Note that $\operatorname{Supp} f^*S \subset \operatorname{Supp} f_*^{-1} B_Y \cup \operatorname{Exc}(f)$ and $\operatorname{Supp} B_Z^{=1} \subset \operatorname{Supp} f_*^{-1} B_Y \cup \operatorname{Exc}(f)$. Since C is also a stratum of $(Z, B_Z^{=1})$ and $C \subset \operatorname{Supp} f^*S$, there exists an irreducible component G of $B_Z^{=1}$ such that $C \subset G \subset \operatorname{Supp} f^*S$. Therefore, by the definition of T, G is an irreducible component of T because $f(G) \subset S$ and G is an irreducible component of $B_Z^{=1}$. So, C is not a stratum of $(Z, \{B_Z\} + B_Z^{=1} - T)$. This is a contradiction. $\square$

On the other hand, $f(T) \subset S$. Therefore,

$$f_*\mathcal{O}_T(\ulcorner -(B_T^{\leq 1})\urcorner) \to R^1 f_*\mathcal{O}_Z(\ulcorner -(B_Z^{\leq 1})\urcorner - T)$$

is the zero map by the above claim. Thus, we obtain

$$f_*\mathcal{O}_T(\ulcorner -(B_T^{\leq 1})\urcorner) \simeq \mathcal{O}_S(\ulcorner -(B_S^{\leq 1})\urcorner)$$

by the following commutative diagram.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_Y(\ulcorner -(B_Y^{\leq 1})\urcorner - S) & \longrightarrow & \mathcal{O}_Y(\ulcorner -(B_Y^{\leq 1})\urcorner) & \longrightarrow & \mathcal{O}_S(\ulcorner -(B_S^{\leq 1})\urcorner) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \downarrow & & \\
0 & \longrightarrow & f_*\mathcal{O}_Z(\ulcorner -(B_Z^{\leq 1})\urcorner - T) & \longrightarrow & f_*\mathcal{O}_Z(\ulcorner -(B_Z^{\leq 1})\urcorner) & \longrightarrow & f_*\mathcal{O}_T(\ulcorner -(B_T^{\leq 1})\urcorner) & \longrightarrow & 0
\end{array}
$$

This completes the proof. $\square$

The following theorem (cf. [A, Theorem 4.4]) is one of the key results for the theory of qlc varieties. It is a consequence of Theorem 2.2. See also Theorem 5.2 below.

Theorem 3.6 (Adjunction and vanishing theorem)**.** *Let $[X, \omega]$ be a qlc pair and X' a union of some qlc centers of $[X, \omega]$.*

(i) *Then $[X', \omega']$ is a qlc pair, where $\omega' = \omega|_{X'}$. Moreover, the qlc centers of $[X', \omega']$ are exactly the qlc centers of $[X, \omega]$ that are included in X'.*

(ii) *Assume that X is projective. Let L be a Cartier divisor on X such that $L - \omega$ is ample. Then $H^q(X, \mathcal{O}_X(L)) = 0$ and $H^q(X, \mathcal{I}_{X'} \otimes \mathcal{O}_X(L)) = 0$ for $q > 0$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X' on X. Note that $H^q(X', \mathcal{O}_{X'}(L)) = 0$ for every $q > 0$ because $[X', \omega']$ is a qlc pair by (i) and $L|_{X'} - \omega'$ is ample.*

Proof. (i) Let $f : (Y, B_Y) \to X$ be a quasi-log resolution. Let M be the ambient space of Y and D a subboundary $\mathbb{Q}$-divisor on M such that $B_Y = D|_Y$. By taking blow-ups of M, we can assume that the union of all strata of (Y, B_Y) mapped into X', which is denoted by Y', is a union of irreducible components of Y (cf. Lemma 3.5). We put $Y'' = Y - Y'$. We define $(K_Y + B_Y)|_{Y'} = K_{Y'} + B_{Y'}$ and consider $f : (Y', B_{Y'}) \to X'$. We claim that $[X', \omega']$ is a qlc pair, where $\omega' = \omega|_{X'}$, and $f : (Y', B_{Y'}) \to X'$ is a quasi-log resolution. From the definition, $B_{Y'}$ is a subboundary and $f^*\omega' \sim_{\mathbb{Q}} K_{Y'} + B_{Y'}$ on Y'. We consider the following short exact sequence

$$0 \to \mathcal{O}_{Y''}(-Y') \to \mathcal{O}_Y \to \mathcal{O}_{Y'} \to 0.$$

We put $A = \ulcorner -(B_Y^{<1}) \urcorner$. Then we have

$$0 \to \mathcal{O}_{Y''}(A - Y') \to \mathcal{O}_Y(A) \to \mathcal{O}_{Y'}(A) \to 0.$$

Applying f_*, we obtain

$$0 \to f_*\mathcal{O}_{Y''}(A - Y') \to \mathcal{O}_X \to f_*\mathcal{O}_{Y'}(A) \to R^1 f_*\mathcal{O}_{Y''}(A - Y') \to \cdots.$$

The support of every non-zero local section of $R^1 f_*\mathcal{O}_{Y''}(A - Y')$ can not be contained in $f(Y') = X'$ by Theorem 2.2 (1). We note that

$$-f^*\omega \sim_{\mathbb{Q}} (A - Y')|_{Y''} - (K_{Y''} + \{B_{Y''}\} + B_{Y''}^{=1} - Y'|_{Y''})$$

on Y'', where $(K_Y + B_Y)|_{Y''} = K_{Y''} + B_{Y''}$, and that $Y'|_{Y''}$ is contained in $B_{Y''}^{=1}$. Therefore, $f_*\mathcal{O}_{Y'}(A) \to R^1 f_*\mathcal{O}_{Y''}(A - Y')$ is the zero map. We note that the surjection $\mathcal{O}_X \to f_*\mathcal{O}_{Y'}(A)$ decomposes as

$$\mathcal{O}_X \to \mathcal{O}_{X'} \to f_*\mathcal{O}_{Y'} \to f_*\mathcal{O}_{Y'}(A)$$

since $f(Y') = X'$. Therefore, we obtain

$$\mathcal{O}_{X'} \simeq f_*\mathcal{O}_{Y'}(A) = f_*\mathcal{O}_{Y'}(\ulcorner -(B_{Y'}^{<1}) \urcorner).$$

Thus, we see that $f_*\mathcal{O}_{Y''}(A - Y') \simeq \mathcal{I}_{X'}$, the defining ideal sheaf of X' on X. The statement for qlc centers is obvious by the construction of the quasi-log resolution. So, we obtain (i).

(ii) Let $f : (Y, B_Y) \to X$ be a quasi-log resolution as in the proof of (i). Apply Theorem 2.2 (2). Then we obtain $H^q(X, \mathcal{O}_X(L)) = 0$ for every $q > 0$ because

$$f^*(L - \omega) \sim_{\mathbb{Q}} f^*L - (K_Y + B_Y)$$
$$= f^*L + \ulcorner -(B_Y^{<1}) \urcorner - (K_Y + \{B_Y\} + B_Y^{=1})$$

and $f_*\mathcal{O}_Y(f^*L + \ulcorner -(B_Y^{\leq 1}) \urcorner) \simeq \mathcal{O}_X(L)$. We consider $f : Y'' \to X$. We put $(K_Y + B_Y)|_{Y''} = K_{Y''} + B_{Y''}$. Then

$$f^*(L - \omega) \sim_\mathbb{Q} (f^*L - (K_Y + B_Y))|_{Y''}$$
$$= (f^*L + A - Y')|_{Y''} - (K_{Y''} + \{B_{Y''}\} + B_{Y''}^{=1} - Y'|_{Y''})$$

on Y''. Note that $Y'|_{Y''}$ is contained in $B_{Y''}^{=1}$. Therefore, we obtain

$$H^q(X, f_*\mathcal{O}_{Y''}(A - Y') \otimes \mathcal{O}_X(L)) = 0$$

for every $q > 0$ by Theorem 2.2 (2). Thus this completes the proof by the isomorphism $f_*\mathcal{O}_{Y''}(A - Y') \simeq \mathcal{I}_{X'}$ obtained in the proof of (i). $\qquad\square$

Corollary 3.7. *Let $[X, \omega]$ be a qlc pair and X' an irreducible component of X. Then $[X', \omega']$, where $\omega' = \omega|_{X'}$, is a qlc pair.*

Proof. By Definition 3.1 and Remark 3.2, X' is a qlc center of $[X, \omega]$. Therefore, by Theorem 3.6 (i), $[X', \omega']$ is a qlc pair. $\qquad\square$

We use the next definition in Section 4.

Definition 3.8. *Let $[X, \omega]$ be a qlc pair. Let X' be the union of qlc centers of X that are not any irreducible components of X. Then X' with $\omega' = \omega|_{X'}$ is a qlc variety by Theorem 3.6 (i). We denote it by* $\mathrm{Nqklt}(X, \omega)$.

We close this section with the following very useful lemma, which seems to be indispensable for the proof of the base point free theorem in Section 4.

Lemma 3.9. *Let $f : (Y, B_Y) \to X$ be a quasi-log resolution of a qlc pair $[X, \omega]$. Let E be a Cartier divisor on X such that $\mathrm{Supp}\, E$ contains no qlc centers of $[X, \omega]$. By blowing up M, the ambient space of Y, inside $\mathrm{Supp}\, f^*E$, we can assume that $(Y, B_Y + f^*E)$ is a global embedded simple normal crossing pair.*

Proof. First, we take a blow-up of M along f^*E and apply Hironaka's resolution theorem to M. Then we can assume that there exists a Cartier divisor F on M such that $\mathrm{Supp}\,(F \cap Y) = \mathrm{Supp}\, f^*E$. Next, we apply Szabó's resolution lemma to $\mathrm{Supp}\,(D + Y + F)$ on M. Thus, we obtain the desired properties by Lemma 3.5. $\qquad\square$

4. Base point free theorem

The next theorem is the main theorem of this section. It is a special case of [A, Theorem 5.1]. This formulation is indispensable for the inductive treatment of log canonical pairs in the framework of the theory of quasi-log varieties. For the details, see [F7, Section 3.2.2].

Theorem 4.1. *Let $[X, \omega]$ be a projective qlc pair and L a nef Cartier divisor on X. Assume that $qL - \omega$ is ample for some $q > 0$. Then $\mathcal{O}_X(mL)$ is generated by global sections for every $m \gg 0$, that is, there exists a positive number m_0 such that $\mathcal{O}_X(mL)$ is generated by global sections for every $m \geq m_0$.*

Proof. First, we note that the statement is obvious when $\dim X = 0$.

Claim 1. *We can assume that X is irreducible.*

Let X' be an irreducible component of X. Then X' with $\omega' = \omega|_{X'}$ has a natural qlc structure induced by $[X, \omega]$ by adjunction (see Corollary 3.7). By the vanishing theorem (see Theorem 3.6 (ii)), we have $H^1(X, \mathcal{I}_{X'} \otimes \mathcal{O}_X(mL)) = 0$ for all $m \geq q$. We consider the following commutative diagram.

$$
\begin{array}{ccccc}
H^0(X, \mathcal{O}_X(mL)) \otimes \mathcal{O}_X & \xrightarrow{\alpha} & H^0(X', \mathcal{O}_{X'}(mL)) \otimes \mathcal{O}_{X'} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\mathcal{O}_X(mL) & \longrightarrow & \mathcal{O}_{X'}(mL) & \longrightarrow & 0
\end{array}
$$

Since α is surjective for $m \geq q$, we can assume that X is irreducible when we prove this theorem.

Claim 2. *For every $m \gg 0$, $\mathcal{O}_X(mL)$ is generated by global sections on an open neighborhood of $\mathrm{Nqklt}(X, \omega)$.*

We put $X' = \mathrm{Nqklt}(X, \omega)$. Then $[X', \omega']$, where $\omega' = \omega|_{X'}$, is a qlc pair by adjunction (see Definition 3.8 and Theorem 3.6 (i)). By induction on the dimension, $\mathcal{O}_{X'}(mL)$ is generated by global sections for every $m \gg 0$. By the following commutative diagram:

$$
\begin{array}{ccccc}
H^0(X, \mathcal{O}_X(mL)) \otimes \mathcal{O}_X & \xrightarrow{\alpha} & H^0(X', \mathcal{O}_{X'}(mL)) \otimes \mathcal{O}_{X'} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\mathcal{O}_X(mL) & \longrightarrow & \mathcal{O}_{X'}(mL) & \longrightarrow & 0,
\end{array}
$$

we know that, for every $m \gg 0$, $\mathcal{O}_X(mL)$ is generated by global sections on an open neighborhood of X'.

Claim 3. *For every $m \gg 0$, $\mathcal{O}_X(mL)$ is generated by global sections on a non-empty Zariski open set.*

By Claim 2, we can assume that $\mathrm{Nqklt}(X, \omega)$ is empty. If L is numerically trivial, then $H^0(X, \mathcal{O}_X(L)) = H^0(X, \mathcal{O}_X(-L)) = \mathbb{C}$. It is because $h^0(X, \mathcal{O}_X(\pm L)) = \chi(X, \mathcal{O}_X(\pm L)) = \chi(X, \mathcal{O}_X) = 1$ by Theorem 3.6 (ii) and [Kl, Chapter II §2 Theorem 1]. Therefore, $\mathcal{O}_X(L)$ is trivial. So, we can assume that L is not numerically trivial. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution. Let $x \in X$ be a general smooth point. Then we can take a $\mathbb{Q}$-divisor D such that $\mathrm{mult}_x D > \dim X$ and $D \sim_{\mathbb{Q}} (q + r)L - \omega$ for some $r > 0$ (see [KM, 3.5 Step 2]). By blowing up M, we can assume that $(Y, B_Y + f^*D)$ is a global embedded simple normal crossing pair by Lemma 3.9. We note that every stratum of (Y, B_Y) is mapped onto X by the assumption. By the construction of D, we can find a positive rational number $c < 1$ such that $B_Y + cf^*D$ is a subboundary and some stratum of $(Y, B_Y + cf^*D)$ does not dominate X. Note that $f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil) \simeq \mathcal{O}_X$. Then

the pair $[X, \omega + cD]$ is qlc and $f : (Y, B_Y + cf^*D) \to X$ is a quasi-log resolution. We note that $q'L - (\omega + cD)$ is ample since $c < 1$, where $q' = q + cr$. By construction, $\mathrm{Nqklt}(X, \omega + cD)$ is non-empty. Therefore, by applying Claim 2 to $[X, \omega + cD]$, for every $m \gg 0$, $\mathcal{O}_X(mL)$ is generated by global sections on an open neighborhood of $\mathrm{Nqklt}(X, \omega + cD)$. So, we obtain Claim 3.

Let p be a prime number and l a large integer. Then $|p^l L| \neq \emptyset$ by Claim 3 and $|p^l L|$ is free on an open neighborhood of $\mathrm{Nqklt}(X, \omega)$ by Claim 2.

Claim 4. *If the base locus* $\mathrm{Bs}|p^l L|$ *(with reduced scheme structure) is not empty, then* $\mathrm{Bs}|p^{l'} L|$ *is strictly smaller than* $\mathrm{Bs}|p^l L|$ *for some* $l' > l$.

Let $f : (Y, B_Y) \to X$ be a quasi-log resolution. We take a general member $D \in |p^l L|$. We note that $|p^l L|$ is free on an open neighborhood of $\mathrm{Nqklt}(X, \omega)$. Thus, f^*D intersects all strata of $(Y, \mathrm{Supp}\, B_Y)$ transversally over $X \setminus \mathrm{Bs}|p^l L|$ by Bertini and f^*D contains no strata of (Y, B_Y). By taking blow-ups of M suitably, we can assume that $(Y, B_Y + f^*D)$ is a global embedded simple normal crossing pair (cf. Lemmas 3.9 and 3.5). We take the maximal positive rational number c such that $B_Y + cf^*D$ is a subboundary. We note that $c \leq 1$. Here, we used $\mathcal{O}_X \simeq f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1}) \urcorner)$. Then $f : (Y, B_Y + cf^*D) \to X$ is a quasi-log resolution of $[X, \omega' = \omega + cD]$. Note that $[X, \omega']$ has a qlc center C that intersects $\mathrm{Bs}|p^l L|$ by the construction. By induction on the dimension, $\mathcal{O}_C(mL)$ is generated by global sections for all $m \gg 0$. We can lift the sections of $\mathcal{O}_C(mL)$ to X for $m \geq q + cp^l$ by Theorem 3.6 (ii). Then we obtain that, for every $m \gg 0$, $\mathcal{O}_X(mL)$ is generated by global sections on an open neighborhood of C. Therefore, $\mathrm{Bs}|p^{l'} L|$ is strictly smaller than $\mathrm{Bs}|p^l L|$ for some $l' > l$.

Claim 5. $\mathcal{O}_X(mL)$ *is generated by global sections for every* $m \gg 0$.

By Claim 4 and noetherian induction, $\mathcal{O}_X(p^l L)$ and $\mathcal{O}_X(p'^{l'} L)$ are generated by global sections for large l and l', where p and p' are prime numbers and $p \neq p'$. So, there exists a positive number m_0 such that $\mathcal{O}_X(mL)$ is generated by global sections for every $m \geq m_0$. $\square$

The next corollary is obvious from Theorem 4.1 and Proposition 3.3.

Corollary 4.2 (Base point free theorem for lc pairs). *Let* (X, B) *be a projective lc pair and* L *a nef Cartier divisor on* X. *Assume that* $qL - (K_X + B)$ *is ample for some* $q > 0$. *Then* $\mathcal{O}_X(mL)$ *is generated by global sections for every* $m \gg 0$.

The reader can find another proof of Corollary 4.2 in [F8, Section 4]. It does not need the notion of qlc pairs.

5. Cone theorem

In this section, we will state the cone theorem for lc pairs (cf. Theorem 5.3). The essential part of the cone theorem follows from the rationality theorem, Theorem 5.1. The rationality theorem is in turn implied by the vanishing theorem for

lc centers (cf. Theorem 5.2) by the standard argument (for the details, see [F8, Section 5]). Note that Theorem 5.2 is a special case of Theorem 3.6 (ii), but it can be proved much more easily (see, for example, [F6, Theorem 4.1] or [F8, Theorem 2.2]). Note that we do not need the theory of quasi-log varieties in this section. So, we omit the details.

5.1. Rationality theorem. Here, we explain the rationality theorem for log canonical pairs. It implies the essential part of the cone theorem for log canonical pairs.

Theorem 5.1 (Rationality theorem). *Let (X, B) be a projective lc pair such that $a(K_X + B)$ is Cartier for a positive integer a. Let H be an ample Cartier divisor on X. Assume that $K_X + B$ is not nef. We put*

$$r = \max\{t \in \mathbb{R} : H + t(K_X + B) \text{ is nef}\}.$$

Then r is a rational number of the form u/v $(u, v \in \mathbb{Z})$ where $0 < v \leq a(\dim X + 1)$.

As we explained above, Theorem 5.1 can be proved easily by using the following very special case of Theorem 3.6 (ii).

Theorem 5.2 (Vanishing theorem for lc centers). *Let X be a projective variety and B a boundary $\mathbb{Q}$-divisor on X such that (X, B) is log canonical. Let D be a Cartier divisor on X. Assume that $D - (K_X + B)$ is ample. Let C be an lc center of the pair (X, B) with the reduced scheme structure. Then we have*

$$H^i(X, \mathcal{I}_C \otimes \mathcal{O}_X(D)) = 0, \quad H^i(C, \mathcal{O}_C(D)) = 0$$

for all $i > 0$, where $\mathcal{I}_C$ is the defining ideal sheaf of C on X. In particular, the restriction map

$$H^0(X, \mathcal{O}_X(D)) \to H^0(C, \mathcal{O}_C(D))$$

is surjective.

The reader can find the details of the rationality theorem in [F8, Section 5].

5.2. Cone theorem. Let us state the main theorem of this section.

Theorem 5.3 (Cone theorem). *Let (X, B) be a projective lc pair. Then we have*

(i) *There are (countably many) rational curves $C_j \subset X$ such that $0 < -(K_X + B) \cdot C_j \leq 2 \dim X$, and*

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X + B) \geq 0} + \sum \mathbb{R}_{\geq 0}[C_j].$$

(ii) *For any $\varepsilon > 0$ and ample $\mathbb{Q}$-divisor H,*

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X + B + \varepsilon H) \geq 0} + \sum_{finite} \mathbb{R}_{\geq 0}[C_j].$$

(iii) *Let $F \subset \overline{NE}(X)$ be a $(K_X + B)$-negative extremal face. Then there is a unique morphism $\varphi_F : X \to Z$ such that $(\varphi_F)_* \mathcal{O}_X \simeq \mathcal{O}_Z$, Z is projective, and an irreducible curve $C \subset X$ is mapped to a point by φ_F if and only if $[C] \in F$. The map φ_F is called the* contraction *of F.*

(iv) *Let F and φ_F be as in* (iii). *Let L be a line bundle on X such that $(L \cdot C) = 0$ for every curve C with $[C] \in F$. Then there is a line bundle L_Z on Z such that $L \simeq \varphi_F^* L_Z$.*

Proof. The upper bound $2 \dim X$ and the fact that C_j is a rational curve in (i) can be proved by Kawamata's argument in [Ka] with the aid of [BCHM]. For the details, see [F7, Section 3.1.3] or [F9, Section 18]. The other statements in (i) and (ii) are formal consequences of the rationality theorem (cf. Theorem 5.1). For the proof, see [KM, Theorem 3.15]. The statements (iii) and (iv) are obvious by Corollary 4.2 and the statements (i) and (ii). See Steps 7 and 9 in [KM, 3.3 The Cone Theorem]. □

6. Related topics

In this paper, we did not prove Theorem 2.2, which is a key result for the theory of quasi-log varieties. For the proof, see [F7, Chapter 2]. The paper [F6] is a gentle introduction to the vanishing and torsion-free theorems. In [F7, Chapters 3, 4], we give a proof of the existence of fourfold lc flips and prove the base point free theorem of Reid–Fukuda type for lc pairs. The base point free theorem for lc pairs is generalized in [F2], where we obtain Kollár's effective base point free theorem for lc pairs. In [F3], we prove the effective base point free theorem of Angehrn–Siu type for lc pairs. We introduce the notion of non-lc ideal sheaves and prove the restriction theorem in [F4]. It is a generalization of Kawakita's inversion of adjunction on log canonicity for normal divisors. See also [FST]. In [F5], we prove that the log canonical ring is finitely generated in dimension four. In [F8], we obtain the fundamental theorems of the log minimal model program for log canonical pairs without using the theory of quasi-log varieties. Our new approach in [F8] seems to be more natural and simpler than Ambro's theory of quasi-log varieties. In [F9], we go ahead with this new approach. We strongly recommend the reader to see [F8] and [F9]. Finally, in [F10], the minimal model theory for log surfaces is discussed under much weaker assumptions than everybody expected.

7. References

[A]   F. Ambro, Quasi-log varieties, Tr. Mat. Inst. Steklova **240** (2003), Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 220–239; translation in Proc. Steklov Inst. Math. 2003, no. 1 (240), 214–233.

[BCHM] C. Birkar, P. Cascini, C. Hacon, and J. McKernan, Existence of minimal models for varieties of log general type, J. Amer. Math. Soc. **23**, no. 2, 405–468.

[E]   F. Elzein, Mixed Hodge structures, Trans. Amer. Math. Soc. **275** (1983), no. 1, 71–106.

[F1]   O. Fujino, Higher direct images of log canonical divisors, J. Differential Geom. **66** (2004), no. 3, 453–479.

[F2]   O. Fujino, Effective base point free theorem for log canonical pairs—Kollár type theorem, Tohoku Math. J. **61** (2009), 475–481.

[F3]   O. Fujino, Effective base point free theorem for log canonical pairs II—Angehrn–Siu type theorems—, to appear in Michigan Math. J.

[F4]   O. Fujino, Theory of non-lc ideal sheaves: basic properties, Kyoto Journal of Mathematics, Vol. **50**, No. 2 (2010), 225–245.

[F5]   O. Fujino, Finite generation of the log canonical ring in dimension four, to appear in Nagata memorial issue of Kyoto Journal of Mathematics.

[F6]   O. Fujino, On injectivity, vanishing and torsion-free theorems for algebraic varieties, Proc. Japan Acad. Ser. A Math. Sci. **85** (2009), no. 8, 95–100.

[F7]   O. Fujino, Introduction to the log minimal model program for log canonical pairs, preprint 2009.

[F8]   O. Fujino, Non-vanishing theorem for log canonical pairs, to appear in Journal of Algebraic Geometry.

[F9]   O. Fujino, Fundamental theorems for the log minimal model program, preprint 2009.

[F10]   O. Fujino, Minimal model theory for log surfaces, preprint 2010.

[FST]   O. Fujino, K. Schwede, and S. Takagi, Supplements to non-lc ideal sheaves, preprint 2010.

[Ka]   Y. Kawamata, On the length of an extremal rational curve, Invent. Math. **105** (1991), no. 3, 609–611.

[Kl]   S. L. Kleiman, Toward a numerical theory of ampleness, Ann. of Math. (2) **84** (1966), 293–344.

[KM]   J. Kollár, S. Mori, *Birational geometry of algebraic varieties. With the collaboration of C. H. Clemens and A. Corti*, Translated from the 1998 Japanese original. Cambridge Tracts in Mathematics, **134**. Cambridge University Press, Cambridge, 1998.

Osamu Fujino, Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502 Japan

E-mail: fujino@math.kyoto-u.ac.jp