Injectivity theorems

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Dedicated to Professor Yujiro Kawamata on the occasion of his sixtieth birthday

Abstract.

We prove some injectivity theorems. Our proof depends on the theory of mixed Hodge structures on cohomology with compact support. Our injectivity theorems would play crucial roles in the minimal model theory for higher-dimensional algebraic varieties. We also treat some applications.

Contents

§1. Introduction

The following theorem is the main theorem of this paper, which is a slight generalization of [Fuj3, Proposition 2.23] (see also [Fuj2] and [Fuj8, Theorem 3.1]) and is inspired by the main theorem of [Amb2]. We note that there are many contributors to this kind of injectivity theorem, for example, Tankeev, Kollár, Esnault–Viehweg, Ambro, Fujino, and others.

Theorem 1.1 (Main theorem)**.** *Let X be a proper simple normal crossing algebraic variety and let Δ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $\mathrm{Supp}\Delta$ is a simple normal crossing divisor on X and that Δ is a boundary $\mathbb{R}$-divisor on X. Let L be a Cartier divisor on X and let D be an effective Weil divisor on X whose support is contained in $\mathrm{Supp}\Delta$. Assume that $L \sim_{\mathbb{R}} K_X + \Delta$. Then the natural homomorphism*

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D))$$

induced by the inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$ is injective for every q.

Remark 1.2. In [Fuj3, Proposition 2.23], the support of D is assumed to be contained in $\mathrm{Supp}\{\Delta\}$, where $\{\Delta\}$ is the fractional part of Δ.

Remark 1.3. We will prove the relative version of Theorem 1.1 in Theorem 6.1. The proof of Theorem 6.1 uses [BiVP]. Therefore, Theorem 6.1 is a nontrivial generalization of Theorem 1.1.

We note that Theorem 1.1 contains Theorem 1.4, which is equivalent to the main theorem of [Amb2] (see [Amb2, Theorem 2.3]). Theorem 1.4 shows that the notion of *maximal non-lc ideal sheaves* introduced in [FST] is useful and has some nontrivial applications. For the details, see Section 5.

Theorem 1.4. *Let X be a proper smooth algebraic variety and let Δ be a boundary $\mathbb{R}$-divisor on X such that $\mathrm{Supp}\Delta$ is a simple normal crossing divisor on X. Let L be a Cartier divisor on X and let D be an effective Cartier divisor on X whose support is contained in $\mathrm{Supp}\Delta$. Assume that $L \sim_{\mathbb{R}} K_X + \Delta$. Then the natural homomorphism*

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D))$$

induced by the inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$ is injective for every q.

A special case of Theorem 1.1 implies a very powerful vanishing and torsion-free theorem for simple normal crossing pairs (see [Fuj8, Theorem 1.1]). See also [Fuj2] and [Fuj3, Theorem 2.38 and Theorem 2.39]. It plays crucial roles for the study of *semi-log canonical pairs* and *quasi-log varieties* (see, [Fuj3], [Fuj5], [Fuj9], and [FF]).

More precisely, we obtain the following injectivity theorem for simple normal crossing pairs by using a special case of Theorem 1.1.

Theorem 1.5 (see [Fuj8, Theorem 3.4])**.** *Let (X, Δ) be a simple normal crossing pair such that X is a proper algebraic variety and that Δ is a boundary $\mathbb{R}$-divisor on X. Let L be a Cartier divisor on X and*

let D be an effective Cartier divisor that is permissible with respect to (X, Δ). Assume the following conditions:

(i) $L \sim_{\mathbb{R}} K_X + \Delta + H$,

(ii) H *is a semi-ample $\mathbb{R}$-divisor, and*

(iii) $tH \sim_{\mathbb{R}} D + D'$ *for some positive real number t, where D' is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor that is permissible with respect to (X, Δ).*

Then the homomorphism

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D)),$$

which is induced by the natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$, is injective for every q.

As an application of Theorem 1.5, we obtain Theorem 1.6, which is very important for the study of higher-dimensional algebraic varieties.

Theorem 1.6 (see [Fuj8, Theorem 1.1]). *Let (Y, Δ) be a simple normal crossing pair such that Δ is a boundary $\mathbb{R}$-divisor on Y. Let $f : Y \to X$ be a proper morphism between algebraic varieties and let L be a Cartier divisor on Y such that $L - (K_Y + \Delta)$ is f-semi-ample. Let q be an arbitrary non-negative integer. Then we have the following properties.*

(i)   *Every associated prime of $R^q f_* \mathcal{O}_Y(L)$ is the generic point of the f-image of some stratum of (Y, Δ).*

(ii)   *Let $\pi : X \to V$ be a projective morphism to an algebraic variety V such that*

$$L - (K_Y + \Delta) \sim_{\mathbb{R}} f^* H$$

for some π-ample $\mathbb{R}$-divisor H on X. Then $R^q f_* \mathcal{O}_Y(L)$ is π_*-acyclic, that is,

$$R^p \pi_* R^q f_* \mathcal{O}_Y(L) = 0$$

for every $p > 0$.

In this paper, we do not prove Theorem 1.5 and Theorem 1.6. We only treat Theorem 1.1 and Theorem 1.4. For the details of Theorem 1.5 and Theorem 1.6, we recommend the reader to see [Fuj8].

Here, we quickly explain the main idea of the proof.

1.7 (Idea of the proof). We give a proof of Theorem 1.4 under the assumption that Δ is reduced and that $L \sim K_X + \Delta$.

It is well-known that

$$E_1^{p,q} = H^q(X, \Omega_X^p(\log \Delta) \otimes \mathcal{O}_X(-\Delta)) \Rightarrow H_c^{p+q}(X - \Delta, \mathbb{C})$$

degenerates at E_1 by Deligne's theory of mixed Hodge structures. This implies that the natural inclusion

$$\iota_! \mathbb{C}_{X-\Delta} \subset \mathcal{O}_X(-\Delta),$$

where $\iota : X - \Delta \to X$, induces surjections

$$\varphi^i : H^i(X, \iota_! \mathbb{C}_{X-\Delta}) \to H^i(X, \mathcal{O}_X(-\Delta))$$

for all i. On the other hand, we can easily see that

$$\iota_! \mathbb{C}_{X-\Delta} \subset \mathcal{O}_X(-\Delta - D) \subset \mathcal{O}_X(-\Delta)$$

because $\mathrm{Supp} D \subset \mathrm{Supp} \Delta$. Thus φ^i factors as

$$H^i(X, \iota_! \mathbb{C}_{X-\Delta}) \to H^i(X, \mathcal{O}_X(-\Delta - D)) \to H^i(X, \mathcal{O}_X(-\Delta))$$

for every i. Hence

$$H^i(X, \mathcal{O}_X(-\Delta - D)) \to H^i(X, \mathcal{O}_X(-\Delta))$$

is surjective for every i. By Serre duality, we obtain that

$$H^q(X, \mathcal{O}_X(K_X + \Delta)) \to H^q(X, \mathcal{O}_X(K_X + \Delta + D))$$

is injective for every q.

In this paper, we use the notion of Du Bois complexes and Du Bois singularities for the proof of Theorem 1.1 and Theorem 1.4. More precisely, we use the notion of Du Bois complexes for pairs, which is related to the mixed Hodge structures on cohomology with compact support. Consequently, the proof of Theorem 1.1 is simpler than the arguments in [Fuj3, Section 2.3 and Section 2.4] (see also Section 3 and Section 4 in [Fuj2]). Note that we just need the E_1-degeneration of Hodge to de Rham type spectral sequences associated to the mixed Hodge structures on cohomology with compact support. We do not need the explicit descriptions of the weight filtrations.

We strongly recommend the reader to see [Fuj8]. This paper and [Fuj8] simplify and generalize the main part of [Fuj3, Chapter 2] (see also [Fuj2, Sections 3, 4, and 5]). We note that the foundation of the theory of semi-log canonical pairs discussed in [Fuj9] is composed of the results established in this paper and [Fuj8] (see [Fuj2] and [Fuj3]).

We summarize the contents of this paper. In Section 2, we collect some basic definitions and notations. In Section 3, we briefly review Du Bois complexes and Du Bois singularities. Section 4 is devoted to the proof of Theorem 1.1 and Theorem 1.4. In Section 5, we collect some miscellaneous comments on related topics, for example, Ambro's proof of the injectivity theorem in [Amb2], the extension theorem from log canonical centers, etc. We also explain some interesting applications of Theorem 1.4 due to Ambro ([Amb2]) in order to show how to use Theorem 1.4. In Section 6, we discuss the relative version of the main theorem: Theorem 6.1. We also discuss some applications.

We will work over $\mathbb{C}$, the field of complex numbers, throughout this paper. In this paper, a variety means a (not necessarily equidimensional) reduced separated scheme of finite type over $\mathbb{C}$. We will make use of the standard notation of the minimal model program as in [Fuj7].

§2. Preliminaries

First, we briefly recall basic definitions of divisors. We note that we have to deal with reducible varieties in this paper. For the details, see, for example, [Har, Section 2] and [Liu, Section 7.1].

2.1. Let X be a noetherian scheme with structure sheaf $\mathcal{O}_X$ and let $\mathcal{K}_X$ be the sheaf of total quotient rings of $\mathcal{O}_X$. Let $\mathcal{K}_X^*$ denote the (multiplicative) sheaf of invertible elements in $\mathcal{K}_X$, and $\mathcal{O}_X^*$ the sheaf of invertible elements in $\mathcal{O}_X$. We note that $\mathcal{O}_X \subset \mathcal{K}_X$ and $\mathcal{O}_X^* \subset \mathcal{K}_X^*$.

2.2 (Cartier, $\mathbb{Q}$-Cartier, and $\mathbb{R}$-Cartier divisors)**.** A *Cartier divisor* D on X is a global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$, that is, D is an element of $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. A $\mathbb{Q}$-*Cartier divisor* (resp. $\mathbb{R}$-*Cartier divisor*) is an element of $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \otimes_{\mathbb{Z}} \mathbb{R}$).

2.3 (Linear, $\mathbb{Q}$-linear, and $\mathbb{R}$-linear equivalence)**.** Let D_1 and D_2 be two $\mathbb{R}$-Cartier divisors on X. Then D_1 is *linearly* (resp. $\mathbb{Q}$-*linearly*, or $\mathbb{R}$-*linearly*) *equivalent* to D_2, denoted by $D_1 \sim D_2$ (resp. $D_1 \sim_{\mathbb{Q}} D_2$, or

$D_1 \sim_{\mathbb{R}} D_2$) if

$$D_1 = D_2 + \sum_{i=1}^{k} r_i(f_i)$$

such that $f_i \in \Gamma(X, \mathcal{K}_X^*)$ and $r_i \in \mathbb{Z}$ (resp. $r_i \in \mathbb{Q}$, or $r_i \in \mathbb{R}$) for every i. We note that (f_i) is a *principal Cartier divisor* associated to f_i, that is, the image of f_i by $\Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$.

2.4 (Supports). Let D be a Cartier divisor on X. The *support* of D, denoted by $\mathrm{Supp}\,D$, is the subset of X consisting of points x such that a local equation for D is not in $\mathcal{O}_{X,x}^*$. The support of D is a closed subset of X.

2.5 (Weil divisors, $\mathbb{Q}$-divisors, and $\mathbb{R}$-divisors). Let X be an equidimensional reduced separated scheme of finite type over $\mathbb{C}$. We note that X is not necessarily regular in codimension one. A (*Weil*) *divisor* D on X is a finite formal sum

$$\sum_{i=1}^{n} d_i D_i$$

where D_i is an irreducible reduced closed subscheme of X of pure codimension one and d_i is an integer for every i such that $D_i \neq D_j$ for $i \neq j$.

If $d_i \in \mathbb{Q}$ (resp. $d_i \in \mathbb{R}$) for every i, then D is called a $\mathbb{Q}$-*divisor* (resp. $\mathbb{R}$-*divisor*). We define the *round-up* $\lceil D \rceil = \sum_{i=1}^{r} \lceil d_i \rceil D_i$ (resp. the *round-down* $\lfloor D \rfloor = \sum_{i=1}^{r} \lfloor d_i \rfloor D_i$), where for every real number x, $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \leq \lceil x \rceil < x + 1$ (resp. $x - 1 < \lfloor x \rfloor \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \lfloor D \rfloor$. We call D a *boundary* $\mathbb{R}$-divisor if $0 \leq d_i \leq 1$ for every i.

We put

$$D^{\leq k} = \sum_{d_i \leq k} d_i D_i, \quad D^{\geq k} = \sum_{d_i \geq k} d_i D_i, \quad D^{=k} = \sum_{d_i = k} d_i D_i = k \sum_{d_i = k} D_i$$

and

$$^k D = \sum_{d_i = k} D_i$$

for every $k \in \mathbb{R}$. We note that $D^{=1} = {}^1 D$.

Next, we recall the definition of *simple normal crossing pairs*.

Definition 2.6 (Simple normal crossing pairs). We say that the pair (X, D) is *simple normal crossing* at a point $a \in X$ if X has a Zariski open neighborhood U of a that can be embedded in a smooth variety

Y, where Y has regular system of parameters $(x_1, \cdots, x_p, y_1, \cdots, y_r)$ at $a = 0$ in which U is defined by a monomial equation

$$x_1 \cdots x_p = 0$$

and

$$D = \sum_{i=1}^{r} \alpha_i(y_i = 0)|_U, \quad \alpha_i \in \mathbb{R}.$$

We say that (X, D) is a *simple normal crossing pair* if it is simple normal crossing at every point of X. If $(X, 0)$ is a simple normal crossing pair, then X is called a *simple normal crossing variety*. If X is a simple normal crossing variety, then X has only Gorenstein singularities. Thus, it has an invertible dualizing sheaf ω_X. Therefore, we can define the *canonical divisor K_X* such that $\omega_X \simeq \mathcal{O}_X(K_X)$ (cf. [Liu, Section 7.1 Corollary 1.19]). It is a Cartier divisor on X and is well-defined up to linear equivalence.

We note that a simple normal crossing pair is called a *semi-snc pair* in [Kol, Definition 1.10].

Definition 2.7 (Strata and permissibility). Let X be a simple normal crossing variety and let $X = \bigcup_{i \in I} X_i$ be the irreducible decomposition of X. A *stratum* of X is an irreducible component of $X_{i_1} \cap \cdots \cap X_{i_k}$ for some $\{i_1, \cdots, i_k\} \subset I$. A Cartier divisor D on X is *permissible* if D contains no strata of X in its support. A finite $\mathbb{Q}$-linear (resp. $\mathbb{R}$-linear) combination of permissible Cartier divisors is called a *permissible $\mathbb{Q}$-divisor* (resp. $\mathbb{R}$-*divisor*) on X.

2.8. Let X be a simple normal crossing variety. Let $\mathrm{PerDiv}(X)$ be the abelian group generated by permissible Cartier divisors on X and let $\mathrm{Weil}(X)$ be the abelian group generated by Weil divisors on X. Then we can define natural injective homomorphisms of abelian groups

$$\psi : \mathrm{PerDiv}(X) \otimes_{\mathbb{Z}} \mathbb{K} \to \mathrm{Weil}(X) \otimes_{\mathbb{Z}} \mathbb{K}$$

for $\mathbb{K} = \mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. Let $\nu : \widetilde{X} \to X$ be the normalization. Then we have the following commutative diagram.

$$
\begin{array}{ccc}
\mathrm{Div}(\widetilde{X}) \otimes_{\mathbb{Z}} \mathbb{K} & \xrightarrow[\widetilde{\psi}]{\sim} & \mathrm{Weil}(\widetilde{X}) \otimes_{\mathbb{Z}} \mathbb{K} \\
{\scriptstyle \nu^*} \uparrow & & \downarrow {\scriptstyle \nu_*} \\
\mathrm{PerDiv}(X) \otimes_{\mathbb{Z}} \mathbb{K} & \xrightarrow[\psi]{} & \mathrm{Weil}(X) \otimes_{\mathbb{Z}} \mathbb{K}
\end{array}
$$

Note that $\mathrm{Div}(\widetilde{X})$ is the abelian group generated by Cartier divisors on $\widetilde{X}$ and that $\widetilde{\psi}$ is an isomorphism since $\widetilde{X}$ is smooth.

By ψ, every permissible divisor (resp. $\mathbb{Q}$-divisor or $\mathbb{R}$-divisor) can be considered as a Weil divisor (resp. $\mathbb{Q}$-divisor or $\mathbb{R}$-divisor). Therefore, various operations, for example, $\lfloor D \rfloor$, $\{D\}$, and so on, make sense for a permissible $\mathbb{R}$-divisor D on X.

Definition 2.9 (Simple normal crossing divisors)**.** Let X be a simple normal crossing variety and let D be a Cartier divisor on X. If (X, D) is a simple normal crossing pair and D is reduced, then D is called a *simple normal crossing divisor* on X.

Remark 2.10. Let X be a simple normal crossing variety and let D be a $\mathbb{K}$-divisor on X where $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$. If $\mathrm{Supp}\,D$ is a simple normal crossing divisor on X and D is $\mathbb{K}$-Cartier, then $\lfloor D \rfloor$ and $\lceil D \rceil$ (resp. $\{D\}$, $D^{<1}$, and so on) are Cartier (resp. $\mathbb{K}$-Cartier) divisors on X (cf. [BiVP, Section 8]).

The following lemma is easy but important.

Lemma 2.11. *Let X be a simple normal crossing variety and let B be a permissible $\mathbb{R}$-divisor on X such that $\lfloor B \rfloor = 0$. Let A be a Cartier divisor on X. Assume that $A \sim_{\mathbb{R}} B$. Then there exists a permissible $\mathbb{Q}$-divisor C on X such that $A \sim_{\mathbb{Q}} C$, $\lfloor C \rfloor = 0$, and $\mathrm{Supp}\,C = \mathrm{Supp}\,B$.*

Proof. We can write $B = A + \sum_{i=1}^{k} r_i(f_i)$, where $f_i \in \Gamma(X, \mathcal{K}_X^*)$ and $r_i \in \mathbb{R}$ for every i. Let $P \in X$ be a scheme theoretic point corresponding to some stratum of X. We consider the following affine map

$$\mathbb{K}^k \to H^0(X_P, \mathcal{K}_{X_P}^*/\mathcal{O}_{X_P}^*) \otimes_{\mathbb{Z}} \mathbb{K}$$

induced by $(a_1, \cdots, a_k) \mapsto A + \sum_{i=1}^{k} a_i(f_i)$, where $X_P = \mathrm{Spec}\,\mathcal{O}_{X,P}$ and $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$. Then we can check that

$$\mathcal{P} = \{(a_1, \cdots, a_k) \in \mathbb{R}^k \,|\, A + \sum_i a_i(f_i) \text{ is permissible}\} \subset \mathbb{R}^k$$

is an affine subspace of $\mathbb{R}^k$ defined over $\mathbb{Q}$. Therefore, we see that

$$\mathcal{S} = \{(a_1, \cdots, a_k) \in \mathcal{P} \,|\, \mathrm{Supp}(A + \sum_i a_i(f_i)) \subset \mathrm{Supp}\,B\} \subset \mathcal{P}$$

is an affine subspace of $\mathbb{R}^k$ defined over $\mathbb{Q}$. Since $(r_1, \cdots, r_k) \in \mathcal{S}$, we know that $\mathcal{S} \neq \emptyset$. We take a point $(s_1, \cdots, s_k) \in \mathcal{S} \cap \mathbb{Q}^k$ which is general in $\mathcal{S}$ and sufficiently close to $(r_1, \cdots, r_k)$ and put $C = A + \sum_{i=1}^{k} s_i(f_i)$. By construction, C is a permissible $\mathbb{Q}$-divisor such that $C \sim_{\mathbb{Q}} A$, $\lfloor C \rfloor = 0$, and $\mathrm{Supp}\,C = \mathrm{Supp}\,B$.   Q.E.D.

§3. A quick review of Du Bois complexes

In this section, we briefly review Du Bois complexes and Du Bois singularities. For the details, see, for example, [DuB], [Ste], [GNPP, Exposé V], [Sai], [PS], [Kov2], and [Kol, Chapter 6].
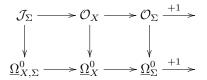
3.1 (Du Bois complexes). Let X be an algebraic variety. Then we can associate a filtered complex $(\underline{\Omega}_X^\bullet, F)$ called the *Du Bois complex* of X in a suitable derived category $D_{\mathrm{diff,coh}}^b(X)$ (see [DuB, 1. Complexes filtrés d'opérateurs différentiels d'ordre ≤ 1]). We put

$$\underline{\Omega}_X^0 = \mathrm{Gr}_F^0 \underline{\Omega}_X^\bullet.$$

There is a natural map $(\Omega_X^\bullet, \sigma) \to (\underline{\Omega}_X^\bullet, F)$. It induces $\mathcal{O}_X \to \underline{\Omega}_X^0$. If $\mathcal{O}_X \to \underline{\Omega}_X^0$ is a quasi-isomorphism, then X is said to have *Du Bois singularities*. We sometimes simply say that X is *Du Bois*. Let Σ be a reduced closed subvariety of X. Then there is a natural map $\rho : (\underline{\Omega}_X^\bullet, F) \to (\underline{\Omega}_\Sigma^\bullet, F)$ in $D_{\mathrm{diff,coh}}^b(X)$. By taking the cone of ρ with a shift by one, we obtain a filtered complex $(\underline{\Omega}_{X,\Sigma}^\bullet, F)$ in $D_{\mathrm{diff,coh}}^b(X)$. Note that $(\underline{\Omega}_{X,\Sigma}^\bullet, F)$ was essentially introduced by Steenbrink in [Ste, Section 3]. We put

$$\underline{\Omega}_{X,\Sigma}^0 = \mathrm{Gr}_F^0 \underline{\Omega}_{X,\Sigma}^\bullet.$$

Then there are a map $\mathcal{J}_\Sigma \to \underline{\Omega}_{X,\Sigma}^0$, where $\mathcal{J}_\Sigma$ is the defining ideal sheaf of Σ on X, and the following commutative diagram

$$
\begin{array}{cccccc}
\mathcal{J}_\Sigma & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_\Sigma & \xrightarrow{+1} \\
\downarrow & & \downarrow & & \downarrow & \\
\underline{\Omega}_{X,\Sigma}^0 & \longrightarrow & \underline{\Omega}_X^0 & \longrightarrow & \underline{\Omega}_\Sigma^0 & \xrightarrow{+1}
\end{array}
$$

in the derived category $D_{\mathrm{coh}}^b(X)$ (see also Remark 3.3 below).

By using the theory of mixed Hodge structures on cohomology with compact support, we have the following theorem.

Theorem 3.2. *Let X be a variety and let Σ be a reduced closed subvariety of X. We put $j : X - \Sigma \hookrightarrow X$. Then we have the following properties.*

   (1)   *The complex $(\underline{\Omega}_{X,\Sigma}^\bullet)^{\mathrm{an}}$ is a resolution of $j_! \mathbb{C}_{X^{\mathrm{an}} - \Sigma^{\mathrm{an}}}$.*
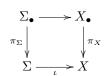
   (2)   *If in addition X is proper, then the spectral sequence*

$$E_1^{p,q} = \mathbb{H}^q(X, \underline{\Omega}_{X,\Sigma}^p) \Rightarrow H^{p+q}(X^{\mathrm{an}}, j_! \mathbb{C}_{X^{\mathrm{an}} - \Sigma^{\mathrm{an}}})$$

degenerates at E_1, where $\underline{\Omega}_{X,\Sigma}^p = \mathrm{Gr}_F^p \underline{\Omega}_{X,\Sigma}^\bullet[p]$.

From now on, we will simply write X (resp. $\mathcal{O}_X$ and so on) to express X^{an} (resp. $\mathcal{O}_{X^{\mathrm{an}}}$ and so on) if there is no risk of confusion.

Proof. Here, we use the formulation of [PS, §3.3 and §3.4]. We assume that X is proper. We take cubical hyperresolutions $\pi_X : X_\bullet \to X$ and $\pi_\Sigma : \Sigma_\bullet \to \Sigma$ fitting in a commutative diagram.

$$
\begin{array}{ccc}
\Sigma_\bullet & \longrightarrow & X_\bullet \\
{\scriptstyle \pi_\Sigma} \downarrow & & \downarrow {\scriptstyle \pi_X} \\
\Sigma & \xrightarrow{\ \iota\ } & X
\end{array}
$$

Let $\mathcal{H}dg(X) := R\pi_{X*}\mathcal{H}dg^\bullet(X_\bullet)$ be a mixed Hodge complex of sheaves on X giving the natural mixed Hodge structure on $H^\bullet(X, \mathbb{Z})$ (see [PS, Definition 5.32 and Theorem 5.33]). We can obtain a mixed Hodge complex of sheaves $\mathcal{H}dg(\Sigma) := R\pi_{\Sigma*}\mathcal{H}dg^\bullet(\Sigma_\bullet)$ on Σ analogously. Roughly speaking, by forgetting the weight filtration and the $\mathbb{Q}$-structure of $\mathcal{H}dg(X)$ and considering it in $D^b_{\mathrm{diff,coh}}(X)$, we obtain the Du Bois complex $(\underline{\Omega}^\bullet_X, F)$ of X (see [GNPP, Exposé V (3.3) Théoréme]). We can also obtain the Du Bois complex $(\underline{\Omega}^\bullet_\Sigma, F)$ of Σ analogously. By taking the mixed cone of $\mathcal{H}dg(X) \to \iota_*\mathcal{H}dg(\Sigma)$ with a shift by one, we obtain a mixed Hodge complex of sheaves on X giving the natural mixed Hodge structure on $H^\bullet_c(X-\Sigma, \mathbb{Z})$ (see [PS, 5.5 Relative Cohomology]). Roughly speaking, by forgetting the weight filtration and the $\mathbb{Q}$-structure, we obtain the desired filtered complex $(\underline{\Omega}^\bullet_{X,\Sigma}, F)$ in $D^b_{\mathrm{diff,coh}}(X)$. When X is not proper, we take completions of $\overline{X}$ and $\overline{\Sigma}$ of X and Σ and apply the above arguments to $\overline{X}$ and $\overline{\Sigma}$. Then we restrict everything to X. The properties (1) and (2) obviously hold by the above description of $(\underline{\Omega}^\bullet_{X,\Sigma}, F)$. By the above construction and description of $(\underline{\Omega}^\bullet_{X,\Sigma}, F)$, we know that the map $\mathcal{J}_\Sigma \to \underline{\Omega}^0_{X,\Sigma}$ in $D^b_{\mathrm{coh}}(X)$ is induced by natural maps of complexes.   Q.E.D.

Remark 3.3. Note that the Du Bois complex $\underline{\Omega}^\bullet_X$ is nothing but the filtered complex $R\pi_{X*}(\Omega^\bullet_{X_\bullet}, F)$. For the details, see [GNPP, Exposé V (3.3) Théoréme and (3.5) Définition]. Therefore, the Du Bois complex of the pair (X, Σ) is given by

$$
\mathrm{Cone}^\bullet(R\pi_{X*}(\Omega^\bullet_{X_\bullet}, F) \to \iota_* R\pi_{\Sigma*}(\Omega^\bullet_{\Sigma_\bullet}, F))[-1].
$$

By the construction of $\underline{\Omega}^\bullet_X$, there is a natural map $a_X : \mathcal{O}_X \to \underline{\Omega}^\bullet_X$ which induces $\mathcal{O}_X \to \underline{\Omega}^0_X$ in $D^b_{\mathrm{coh}}(X)$. Moreover, the composition of $a_X^{\mathrm{an}} : \mathcal{O}_{X^{\mathrm{an}}} \to (\underline{\Omega}^\bullet_X)^{\mathrm{an}}$ with the natural inclusion $\mathbb{C}_{X^{\mathrm{an}}} \subset \mathcal{O}_{X^{\mathrm{an}}}$ induces a quasi-isomorphism $\mathbb{C}_{X^{\mathrm{an}}} \xrightarrow{\simeq} (\underline{\Omega}^\bullet_X)^{\mathrm{an}}$. We have a natural map $a_\Sigma :$

$\mathcal{O}_\Sigma \to \underline{\Omega}^\bullet_\Sigma$ with the same properties as a_X and the following commutative diagram.

$$
\begin{array}{ccc}
\mathcal{O}_X & \longrightarrow & \mathcal{O}_\Sigma \\
{\scriptstyle a_X}\downarrow & & \downarrow{\scriptstyle a_\Sigma} \\
\underline{\Omega}^\bullet_X & \longrightarrow & \underline{\Omega}^\bullet_\Sigma
\end{array}
$$

Therefore, we have a natural map $b : \mathcal{J}_\Sigma \to \underline{\Omega}^\bullet_{X,\Sigma}$ such that b induces $\mathcal{J}_\Sigma \to \underline{\Omega}^0_{X,\Sigma}$ in $D^b_{\mathrm{coh}}(X)$ and that the composition of $b^{\mathrm{an}} : (\mathcal{J}_\Sigma)^{\mathrm{an}} \to (\underline{\Omega}^\bullet_{X,\Sigma})^{\mathrm{an}}$ with the natural inclusion $j_!\mathbb{C}_{X^{\mathrm{an}}-\Sigma^{\mathrm{an}}} \subset (\mathcal{J}_\Sigma)^{\mathrm{an}}$ induces a quasi-isomorphism $j_!\mathbb{C}_{X^{\mathrm{an}}-\Sigma^{\mathrm{an}}} \xrightarrow{\simeq} (\underline{\Omega}^\bullet_{X,\Sigma})^{\mathrm{an}}$. We need the weight filtration and the $\mathbb{Q}$-structure in order to prove the E_1-degeneration of Hodge to de Rham type spectral sequence. We used the framework of [PS, §3.3 and §3.4] because we had to check that various diagrams related to comparison morphisms are commutative (see [PS, Remark 3.23]) for the proof of Theorem 3.2 (2) and so on.

Let us recall the definition of *Du Bois pairs* by [Kov2, Definition 3.13].

Definition 3.4 (Du Bois pairs)**.** With the notation of 3.1 and Theorem 3.2, if the map $\mathcal{J}_\Sigma \to \underline{\Omega}^0_{X,\Sigma}$ is a quasi-isomorphism, then the pair (X, Σ) is called a *Du Bois pair*.

By the definitions, we can easily check the following useful proposition.

Proposition 3.5. *With the notation of 3.1 and Theorem 3.2, we assume that both X and Σ are Du Bois. Then the pair (X, Σ) is a Du Bois pair, that is, $\mathcal{J}_\Sigma \to \underline{\Omega}^0_{X,\Sigma}$ is a quasi-isomorphism.*

Let us recall the following well-known results on Du Bois singularities.

Theorem 3.6. *Let X be a normal algebraic variety with only quotient singularities. Then X has only rational singularities. In particular, X is Du Bois.*

Theorem 3.6 follows from, for example, [DuB, 5.2. Théorème], and [Kov1]. Lemma 3.7 will play an important role in the proof of Theorem 1.4.

Lemma 3.7. *Let X be a variety with closed subvarieties X_1 and X_2 such that $X = X_1 \cup X_2$. Assume that X_1, X_2, and $X_1 \cap X_2$ are Du Bois. Note that, in particular, we assume that $X_1 \cap X_2$ is reduced. Then X is Du Bois.*

For the proof of Lemma 3.7, see, for example, [Sch, Lemma 3.4]. We close this section with a remark on Du Bois singularities.

Remark 3.8 (Du Bois singularities and log canonical singularities)**.** Kollár and Kovács established that log canonical singularities are Du Bois in [KolK]. Moreover, semi-log canonical singularities are Du Bois (see [Kol, Corollary 6.32]). We note that the arguments in [KolK] heavily depend on the recent developments of the minimal model program by Birkar–Cascini–Hacon–McKernan and the results by Ambro and Fujino (see, for example, [Amb1], [Fuj3], [Fuj6], and [Fuj7]). We need a special case of Theorem 1.6 for the arguments in [KolK]. In this paper, we will just use Du Bois complexes for cyclic covers of simple normal crossing pairs. Our proof in Section 4 is independent of the deep result in [KolK].

The fact that (semi-)log canonical singularities are Du Bois does not seem to be so useful when we consider various Kodaira-type vanishing theorems for (semi-)log canonical pairs. This is because (semi-)log canonical singularities are not necessarily Cohen–Macaulay. The approach to various Kodaira-type vanishing theorems for semi-log canonical pairs in [Fuj9] is based on the vanishing theorem in [Fuj8] (see Theorem 1.6, [Fuj2], and [Fuj3]) and the theory of partial resolution of singularities for reducible varieties (see [BiVP]).

§4.   Proof of theorems

In this section, we prove Theorem 1.1 and Theorem 1.4.

Proof of Theorem 1.4.   Without loss of generality, we may assume that X is connected. We set $S = \lfloor \Delta \rfloor$ and $B = \{\Delta\}$. By perturbing B, we may assume that B is a $\mathbb{Q}$-divisor (cf. Lemma 2.11). We set $\mathcal{M} = \mathcal{O}_X(L - K_X - S)$. Let N be the smallest positive integer such that $NL \sim N(K_X + S + B)$. In particular, NB is an integral Weil divisor. We take the N-fold cyclic cover

$$\pi' : Y' = \mathrm{Spec}_X \bigoplus_{i=0}^{N-1} \mathcal{M}^{-i} \to X$$

associated to the section $NB \in |\mathcal{M}^N|$. More precisely, let $s \in H^0(X, \mathcal{M}^N)$ be a section whose zero divisor is NB. Then the dual of $s : \mathcal{O}_X \to \mathcal{M}^N$ defines an $\mathcal{O}_X$-algebra structure on $\bigoplus_{i=0}^{N-1} \mathcal{M}^{-i}$. Let $Y \to Y'$ be the normalization and let $\pi : Y \to X$ be the composition morphism. It is well-known that

$$Y = \mathrm{Spec}_X \bigoplus_{i=0}^{N-1} \mathcal{M}^{-i}(\lfloor iB \rfloor).$$

For the details, see [EV, 3.5. Cyclic covers]. Note that Y has only quotient singularities. We set $T = \pi^* S$. Let $T = \sum_{i \in I} T_i$ be the irreducible decomposition. Then every irreducible component of $T_{i_1} \cap \cdots \cap T_{i_k}$ has only quotient singularities for every $\{i_1, \cdots, i_k\} \subset I$. Hence it is easy to see that both Y and T have only Du Bois singularities by Theorem 3.6 and Lemma 3.7 (see also [Ish]). Therefore, the pair (Y, T) is a Du Bois pair by Proposition 3.5. This means that $\mathcal{O}_Y(-T) \to \underline{\Omega}^0_{Y,T}$ is a quasi-isomorphism. See also [FFS, 3.4]. We note that T is Cartier. Hence $\mathcal{O}_Y(-T)$ is the defining ideal sheaf of T on Y. The E_1-degeneration of

$$E_1^{p,q} = \mathbb{H}^q(Y, \underline{\Omega}^p_{Y,T}) \Rightarrow H^{p+q}(Y, j_! \mathbb{C}_{Y-T})$$

implies that the homomorphism

$$H^q(Y, j_! \mathbb{C}_{Y-T}) \to H^q(Y, \mathcal{O}_Y(-T))$$

induced by the natural inclusion

$$j_! \mathbb{C}_{Y-T} \subset \mathcal{O}_Y(-T)$$

is surjective for every q (see Remark 3.3). By taking a suitable direct summand

$$\mathcal{C} \subset \mathcal{M}^{-1}(-S)$$

of

$$\pi_*(j_! \mathbb{C}_{Y-T}) \subset \pi_* \mathcal{O}_Y(-T),$$

we obtain a surjection

$$H^q(X, \mathcal{C}) \to H^q(X, \mathcal{M}^{-1}(-S))$$

induced by the natural inclusion $\mathcal{C} \subset \mathcal{M}^{-1}(-S)$ for every q. We can check the following simple property by examining the monodromy action of the Galois group $\mathbb{Z}/N\mathbb{Z}$ of $\pi : Y \to X$ on $\mathcal{C}$ around $\mathrm{Supp}B$.

Lemma 4.1 (cf. [KolM, Corollary 2.54]). *Let $U \subset X$ be a connected open set such that $U \cap \mathrm{Supp}\Delta \neq \emptyset$. Then $H^0(U, \mathcal{C}|_U) = 0$.*

Proof. If $U \cap \mathrm{Supp}B \neq \emptyset$, then $H^0(U, \mathcal{C}|_U) = 0$ since the monodromy action on $\mathcal{C}|_{U \setminus \mathrm{Supp}B}$ around $\mathrm{Supp}B$ is nontrivial. If $U \cap \mathrm{Supp}S \neq \emptyset$, then $H^0(U, \mathcal{C}|_U) = 0$ since $\mathcal{C}$ is a direct summand of $\pi_*(j_! \mathbb{C}_{Y-T})$ and $T = \pi^* S$. Q.E.D.

This property is utilized via the following fact. The proof is obvious.

Lemma 4.2 (cf. [KolM, Lemma 2.55]). *Let F be a sheaf of Abelian groups on a topological space X and $F_1, F_2 \subset F$ subsheaves. Let $Z \subset X$ be a closed subset. Assume that*

(1) $F_2|_{X-Z} = F|_{X-Z}$, and

(2) *if U is connected, open and $U \cap Z \neq \emptyset$, then $H^0(U, F_1|U) = 0$.*

Then F_1 *is a subsheaf of* F_2.

As a corollary, we obtain:

Corollary 4.3 (cf. [KolM, Corollary 2.56]). *Let $M \subset \mathcal{M}^{-1}(-S)$ be a subsheaf such that $M|_{X-\mathrm{Supp}\Delta} = \mathcal{M}^{-1}(-S)|_{X-\mathrm{Supp}\Delta}$. Then the injection*

$$\mathcal{C} \to \mathcal{M}^{-1}(-S)$$

factors as

$$\mathcal{C} \to M \to \mathcal{M}^{-1}(-S).$$

Therefore,

$$H^q(X, M) \to H^q(X, \mathcal{M}^{-1}(-S))$$

is surjective for every q.

Proof.   The first part is clear from Lemma 4.1 and Lemma 4.2. This implies that we have maps

$$H^q(X, \mathcal{C}) \to H^q(X, M) \to H^q(X, \mathcal{M}^{-1}(-S)).$$

As we saw above, the composition is surjective. Hence so is the map on the right.   Q.E.D.

Therefore, $H^q(X, \mathcal{M}^{-1}(-S-D)) \to H^q(X, \mathcal{M}^{-1}(-S))$ is surjective for every q. By Serre duality, we obtain that

$$H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{M}(S)) \to H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{M}(S+D))$$

is injective for every q. This means that

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D))$$

is injective for every q.   Q.E.D.

Let us prove Theorem 1.1, the main theorem of this paper. The proof of Theorem 1.4 works for Theorem 1.1 with some minor modifications.

Proof of Theorem 1.1.   Without loss of generality, we may assume that X is connected. We can take an effective Cartier divisor D' on X such that $D' - D$ is effective and $\mathrm{Supp}D' \subset \mathrm{Supp}\Delta$. Therefore, by replacing D with D', we may assume that D is a Cartier divisor. We set $S = \lfloor \Delta \rfloor$ and $B = \{\Delta\}$. By Lemma 2.11, we may assume that B is a $\mathbb{Q}$-divisor. We set $\mathcal{M} = \mathcal{O}_X(L - K_X - S)$. Let N be the smallest positive integer such that $NL \sim N(K_X + S + B)$. We define an $\mathcal{O}_X$-algebra

structure of $\bigoplus_{i=0}^{N-1} \mathcal{M}^{-i}(\lfloor iB \rfloor)$ by $s \in H^0(X, \mathcal{M}^N)$ with $(s = 0) = NB$. We set

$$\pi : Y = \mathrm{Spec}_X \bigoplus_{i=0}^{N-1} \mathcal{M}^{-i}(\lfloor iB \rfloor) \to X$$

and $T = \pi^* S$. Let $Y = \sum_{j \in J} Y_j$ be the irreducible decomposition. Then every irreducible component of $Y_{j_1} \cap \cdots \cap Y_{j_l}$ has only quotient singularities for every $\{j_1, \cdots, j_l\} \subset J$. Let $T = \sum_{i \in I} T_i$ be the irreducible decomposition. Then every irreducible component of $T_{i_1} \cap \cdots \cap T_{i_k}$ has only quotient singularities for every $\{i_1, \cdots, i_k\} \subset I$. Hence it is easy to see that both Y and T are Du Bois by Theorem 3.6 and Lemma 3.7 (see also [Ish]). Therefore, the pair (Y, T) is a Du Bois pair by Proposition 3.5. This means that $\mathcal{O}_Y(-T) \to \underline{\Omega}^0_{Y,T}$ is a quasi-isomorphism. See also [FFS, 3.4]. We note that T is Cartier. Hence $\mathcal{O}_Y(-T)$ is the defining ideal sheaf of T on Y. The E_1-degeneration of

$$E_1^{p,q} = \mathbb{H}^q(Y, \underline{\Omega}^p_{Y,T}) \Rightarrow H^{p+q}(Y, j_! \mathbb{C}_{Y-T})$$

implies that the homomorphism

$$H^q(Y, j_! \mathbb{C}_{Y-T}) \to H^q(Y, \mathcal{O}_Y(-T))$$

induced by the natural inclusion

$$j_! \mathbb{C}_{Y-T} \subset \mathcal{O}_Y(-T)$$

is surjective for every q (see Remark 3.3). By taking a suitable direct summand

$$\mathcal{C} \subset \mathcal{M}^{-1}(-S)$$

of

$$\pi_*(j_! \mathbb{C}_{Y-T}) \subset \pi_* \mathcal{O}_Y(-T),$$

we obtain a surjection

$$H^q(X, \mathcal{C}) \to H^q(X, \mathcal{M}^{-1}(-S))$$

induced by the natural inclusion $\mathcal{C} \subset \mathcal{M}^{-1}(-S)$ for every q. It is easy to see that Lemma 4.1 holds for this new setting. Hence Corollary 4.3 also holds without any modifications. Therefore,

$$H^q(X, \mathcal{M}^{-1}(-S - D)) \to H^q(X, \mathcal{M}^{-1}(-S))$$

is surjective for every q. By Serre duality, we obtain that

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D))$$

is injective for every q. $\hspace{2cm}$ Q.E.D.

§5. Miscellaneous comments

In this section, we collect some miscellaneous comments on related topics.

5.1. Ambro's injectivity theorems

Let X be a smooth variety and let Σ be a simple normal crossing divisor on X. In order to prove the main theorem of [Amb2] (see Theorem 1.4), Ambro used the complex $(\Omega_X^\bullet(*\Sigma), F)$ and the natural inclusion

$$(\Omega_X^\bullet(\log \Sigma), F) \subset (\Omega_X^\bullet(*\Sigma), F).$$

Hence the arguments in [Amb2] are different from the proof of Theorem 1.4 given in Section 4. We do not know how to generalize his approach to the case when X is a simple normal crossing variety and Σ is a simple normal crossing divisor on X.

5.2. Extension theorem from log canonical centers

The following result is a slight generalization of [Amb2, Theorem 6.4]. Note that [FG, Proposition 5.12], which is closely related to the abundance conjecture, is a special case of Theorem 5.2.1.

Theorem 5.2.1 (Extension theorem). *Let (X, Δ) be a proper log canonical pair. Let L be a Cartier divisor on X such that $H = L - (K_X + \Delta)$ is a semi-ample $\mathbb{R}$-divisor on X. Let D be an effective $\mathbb{R}$-divisor on X such that $D \sim_{\mathbb{R}} tH$ for some positive real number t and let Z be the union of the log canonical centers of (X, Δ) contained in $\mathrm{Supp}D$. Then the natural restriction map*

$$H^0(X, \mathcal{O}_X(L)) \to H^0(Z, \mathcal{O}_Z(L))$$

is surjective.

Proof. Let $f : Y \to X$ be a birational morphism from a smooth projective variety Y such that $\mathrm{Exc}(f) \cup \mathrm{Supp}f_*^{-1}\Delta$ is a simple normal crossing divisor on Y. Then we can write

$$K_Y + \Delta_Y = f^*(K_X + \Delta) + E$$

where E is an effective f-exceptional Cartier divisor and Δ_Y is a boundary $\mathbb{R}$-divisor. Without loss of generality, we may further assume that $f^{-1}(Z)$ is a divisor on Y. Let W be the union of all the log canonical centers of (Y, Δ_Y) whose images by f are contained in Z. Note that W is a divisor on Y such that $W \leq \lfloor \Delta_Y \rfloor$. We consider the short exact sequence

$$0 \to \mathcal{O}_Y(E - W) \to \mathcal{O}_Y(E) \to \mathcal{O}_W(E) \to 0.$$

Since
$$E - W = K_Y + (\Delta_Y - W) - f^*(K_X + \Delta),$$
there are no associated primes of $R^1 f_* \mathcal{O}_Y(E - W)$ in $Z = f(W)$ by [Fuj7, Theorem 6.3 (i)]. Therefore, the connecting homomorphism
$$\delta : f_* \mathcal{O}_W(E) \to R^1 f_* \mathcal{O}_Y(E - W)$$
is zero. Hence we obtain
$$\mathcal{O}_X \simeq f_* \mathcal{O}_Y(E) \to f_* \mathcal{O}_W(E)$$
is surjective. This implies that $f_* \mathcal{O}_W(E) \simeq \mathcal{O}_Z$. Since $H^0(Y, \mathcal{O}_Y(f^*L + E)) \simeq H^0(X, \mathcal{O}_X(L))$ and $H^0(W, \mathcal{O}_W(f^*L + E)) \simeq H^0(Z, \mathcal{O}_Z(L))$, it is sufficient to prove that the natural restriction map
$$H^0(Y, \mathcal{O}_Y(f^*L + E)) \to H^0(W, \mathcal{O}_W(f^*L + E))$$
is surjective. By assumption, there is a morphism $g : X \to V$ such that V is a normal projective variety, $g_* \mathcal{O}_X \simeq \mathcal{O}_V$, and $H \sim_{\mathbb{R}} g^*A$, where A is an ample $\mathbb{R}$-divisor on V. We note that
$$(f^*L + E - W) - (K_Y + \Delta_Y - W) = f^*(L - (K_X + \Delta))$$
$$\sim_{\mathbb{R}} f^*g^*A.$$
By the assumption on D and the construction of
$$Y \xrightarrow{f} X \xrightarrow{g} V,$$
we can find an effective ample Cartier divisor D_1 and an effective ample $\mathbb{R}$-divisor D_2 on V such that $D_1 + D_2 \sim_{\mathbb{R}} sA$ for some positive real number s, $W \leq f^*g^*D_1$, and that $\mathrm{Supp} f^*g^*(D_1 + D_2)$ contains no log canonical centers of $(Y, \Delta_Y - W)$. Hence
$$H^i(Y, \mathcal{O}_Y(f^*L + E - W)) \to H^i(Y, \mathcal{O}_Y(f^*L + E))$$
is injective for every i (see [Fuj7, Theorem 6.1]). See also Theorem 1.5. In particular,
$$H^1(Y, \mathcal{O}_Y(f^*L + E - W)) \to H^1(Y, \mathcal{O}_Y(f^*L + E))$$
is injective. Thus we obtain that
$$H^0(Y, \mathcal{O}_Y(f^*L + E)) \to H^0(W, \mathcal{O}_W(f^*L + E))$$
is surjective. Therefore, we obtain the desired surjection.   Q.E.D.

The proof of Theorem 5.2.1 is essentially the same as that of [FG, Proposition 5.12] and is different from the arguments in [Amb2, Section 6]. The framework discussed in [Fuj7] is sufficient for Theorem 5.2.1. We recommend the reader to compare the above proof with the proof of [Amb2, Theorem 6.4], which is shorter than our proof and is based on [Amb2, Theorem 6.2]. We will present the original proof of Theorem 5.2.1 as an application of Theorem 5.3.3 below for the reader's convenience. For the relative version of Theorem 5.2.1, see Theorem 6.4 below.

5.3.   The maximal non-lc ideal sheaves

By combining Theorem 1.4 with the notion of *maximal* non-lc ideal sheaves, we have some interesting results due to Ambro ([Amb2]). Note that the ideal sheaf defined in [Amb2, Definition 4.3] is nothing but the *maximal non-lc ideal sheaf* introduced in [FST, Definition 7.1] (see also [Fuj7, Remark 7.6]).

Let us recall the definition of maximal non-lc ideal sheaves.

Definition 5.3.1 (Maximal non-lc ideal sheaves)**.** Let X be a normal variety and let Δ be an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a resolution with

$$K_Y + \Delta_Y = f^*(K_X + \Delta)$$

such that $\mathrm{Supp}\,\Delta_Y$ is a simple normal crossing divisor. Then we put

$$\mathcal{J}'(X, \Delta) = f_*\mathcal{O}_Y(\lceil K_Y - f^*(K_X + \Delta) + \varepsilon F \rceil)$$

for $0 < \varepsilon \ll 1$, where $F = \mathrm{Supp}\,\Delta_Y^{\geq 1}$. We call $\mathcal{J}'(X, \Delta)$ the *maximal non-lc ideal sheaf* associated to (X, Δ). It is easy to see that

$$\mathcal{J}'(X, \Delta) = f_*\mathcal{O}_Y(-\lfloor \Delta_Y \rfloor + \sum_{k=1}^{\infty} {}^k\Delta_Y).$$

Note that there is a positive integer k_0 such that ${}^k\Delta_Y = 0$ for every $k > k_0$. Therefore,

$$\sum_{k=1}^{\infty} {}^k\Delta_Y = {}^1\Delta_Y + {}^2\Delta_Y + \cdots + {}^{k_0}\Delta_Y.$$

We also note that

$$\mathcal{J}_{\mathrm{NLC}}(X, \Delta) = f_*\mathcal{O}_Y(-\lfloor \Delta_Y \rfloor + \Delta_Y^{=1})$$

is the (*minimal*) *non-lc ideal sheaf* associated to (X, Δ) and that

$$\mathcal{J}(X, \Delta) = f_* \mathcal{O}_Y(-\lfloor \Delta_Y \rfloor)$$

is the *multiplier ideal sheaf* associated to (X, Δ). It is obvious that

$$\mathcal{J}(X, \Delta) \subset \mathcal{J}_{\mathrm{NLC}}(X, \Delta) \subset \mathcal{J}'(X, \Delta).$$

For the details of $\mathcal{J}'(X, \Delta)$, see [FST] (see also [Fuj4]).

Remark 5.3.2 (Non-F-pure ideals)**.** A positive characteristic analog of $\mathcal{J}'(X, \Delta)$, which we call the *non-F-pure ideal* associated to (X, Δ) and is denoted by $\sigma(X, \Delta)$, introduced in [FST] is now becoming a very important tool for higher-dimensional algebraic geometry in positive characteristic.

Theorem 5.3.3 is a nontrivial application of Theorem 1.4. For the relative version of Theorem 5.3.3, see Theorem 6.2 below.

Theorem 5.3.3 ([Amb2, Theorem 6.2])**.** *Let X be a proper normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let L be a Cartier divisor on X such that $L - (K_X + \Delta)$ is semi-ample. Let $\mathcal{J}'(X, \Delta)$ be the maximal non-lc ideal sheaf associated to (X, Δ) and let Y be the closed subscheme defined by $\mathcal{J}'(X, \Delta)$. Then we have a short exact sequence*

$$0 \to H^0(X, \mathcal{J}'(X, \Delta) \otimes \mathcal{O}_X(L))$$
$$\to H^0(X, \mathcal{O}_X(L)) \to H^0(Y, \mathcal{O}_Y(L)) \to 0.$$

We describe the proof of Theorem 5.3.3 for the reader's convenience (see also [Amb2]).

Proof. We take an effective general $\mathbb{R}$-divisor D with small coefficients such that $L - (K_X + \Delta) \sim_{\mathbb{R}} D$. By replacing Δ with $\Delta + D$, we may assume that $L \sim_{\mathbb{R}} K_X + \Delta$. Let $Z \to X$ be a resolution such that $K_Z + \Delta_Z = f^*(K_X + \Delta)$. We may assume that $\mathrm{Supp}\,\Delta_Z$ is a simple normal crossing divisor. We note that

$$-\lfloor \Delta_Z \rfloor + \sum_{k=1}^{\infty} {}^k\Delta_Z = (K_Z + \{\Delta_Z\} + \sum_{k=1}^{\infty} {}^k\Delta_Z) - f^*(K_X + \Delta).$$

We write

$$-\lfloor \Delta_Z \rfloor + \sum_{k=1}^{\infty} {}^k\Delta_Z = P - N$$

where P and N are effective and have no common irreducible components. Note that P is f-exceptional since Δ is effective. Therefore,

$$f^*L + P - N \sim_{\mathbb{R}} K_Z + \{\Delta_Z\} + \sum_{k=1}^{\infty} {}^k\Delta_Z.$$

Thus
$$H^i(Z, \mathcal{O}_Z(f^*L + P - N)) \to H^i(Z, \mathcal{O}_Z(f^*L + P))$$

is injective for every i by Theorem 1.4. This is because

$$\operatorname{Supp} N \subset \operatorname{Supp}(\{\Delta_Z\} + \sum_{k=1}^{\infty} {}^k\Delta_Z).$$

We note that

$$f_*\mathcal{O}_Z(f^*L + P - N) \simeq \mathcal{J}'(X, \Delta) \otimes \mathcal{O}_X(L)$$

and
$$f_*\mathcal{O}_Z(f^*L + P) \simeq \mathcal{O}_X(L).$$

By the following commutative diagram:

$$
\begin{array}{ccc}
H^1(Z, \mathcal{O}_Z(f^*L + P - N)) & \xrightarrow{\ b\ } & H^1(Z, \mathcal{O}_Z(f^*L + P)) \\
\big\uparrow{\scriptstyle a} & & \big\uparrow{\scriptstyle c} \\
H^1(X, \mathcal{J}'(X, \Delta) \otimes \mathcal{O}_X(L)) & \xrightarrow[\ d\ ]{} & H^1(X, \mathcal{O}_X(L)),
\end{array}
$$

we obtain that

$$H^1(X, \mathcal{J}'(X, \Delta) \otimes \mathcal{O}_X(L)) \to H^1(X, \mathcal{O}_X(L))$$

is injective. Note that a and c are injective by the Leray spectral sequences and that b is injective by the above argument. Hence the natural restriction map

$$H^0(X, \mathcal{O}_X(L)) \to H^0(Y, \mathcal{O}_Y(L))$$

is surjective. We obtain the desired short exact sequence.   Q.E.D.

Theorem 5.3.3 shows that $\mathcal{J}'(X, \Delta)$ is useful for some applications. We give the original proof of Theorem 5.2.1 as an application of Theorem 5.3.3.

Proof of Theorem 5.2.1. Let ε be a small positive number. Then it is easy to see that $\mathcal{J}'(X, \Delta + \varepsilon D) = \mathcal{I}_Z$, where $\mathcal{I}_Z$ is the defining ideal sheaf of Z. Since $L - (K_X + \Delta + \varepsilon D) \sim_{\mathbb{R}} (1 - \varepsilon t)H$ is semi-ample, we have the following short exact sequence

$$0 \to H^0(X, \mathcal{J}'(X, \Delta + \varepsilon D) \otimes \mathcal{O}_X(L)) \to H^0(X, \mathcal{O}_X(L))$$
$$\to H^0(Z, \mathcal{O}_Z(L)) \to 0$$

by Theorem 5.3.3. In particular, the natural restriction map

$$H^0(X, \mathcal{O}_X(L)) \to H^0(Z, \mathcal{O}_Z(L))$$

is surjective. Q.E.D.

The following theorem is Ambro's inversion of adjunction. For the relative version of Theorem 5.3.4, see Theorem 6.3 below.

Theorem 5.3.4 ([Amb2, Theorem 6.3])**.** *Let X be a proper normal irreducible variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $-(K_X + \Delta)$ is semi-ample. Suppose that the non-lc locus $\mathrm{Nlc}(X, \Delta)$ of (X, Δ) is not empty, that is, (X, Δ) is not log canonical. Then $\mathrm{Nlc}(X, \Delta)$ is connected and intersects every log canonical center of (X, Δ).*

We describe Ambro's proof of Theorem 5.3.4 based on Theorem 1.4 in order to show how to use Theorem 1.4.

Proof. We take an effective general $\mathbb{R}$-divisor D with small coefficients such that $D \sim_{\mathbb{R}} -(K_X + \Delta)$. By replacing Δ with $\Delta + D$, we may assume that $K_X + \Delta \sim_{\mathbb{R}} 0$. We set $Y = \mathrm{Nlc}(X, \Delta)$. By Theorem 5.3.3, we have the following short exact sequence:

$$0 \to H^0(X, \mathcal{J}'(X, \Delta)) \to H^0(X, \mathcal{O}_X) \to H^0(Y, \mathcal{O}_Y) \to 0.$$

This implies that $H^0(Y, \mathcal{O}_Y) \simeq \mathbb{C}$. Hence Y is connected. Let C be a log canonical center of (X, Δ). Let $f : Z \to X$ be a resolution such that $\mathrm{Exc}(f) \cup \mathrm{Supp} f_*^{-1}\Delta$ is a simple normal crossing divisor and that $f^{-1}(C)$ is a divisor. We set $K_Z + \Delta_Z = f^*(K_X + \Delta)$. Let W be the union of all the irreducible components of $\Delta_Z^{=1}$ whose images by f are contained in C. It is obvious that $f(W) = C$. By construction, we have

$$-\lfloor \Delta_Z \rfloor + \sum_{k=1}^{\infty} {}^k\Delta_Z - W \sim_{\mathbb{R}} K_Z + \{\Delta_Z\} + \sum_{k=1}^{\infty} {}^k\Delta_Z - W$$

since $K_Z + \Delta_Z \sim_{\mathbb{R}} 0$. We set

$$-\lfloor \Delta_Z \rfloor + \sum_{k=1}^{\infty} {}^k\Delta_Z = P - N$$

where P and N are effective and have no common irreducible components. Note that P is f-exceptional. By Theorem 1.4,

$$H^i(Z, \mathcal{O}_Z(P - N - W)) \to H^i(Z, \mathcal{O}_Z(P - W))$$

is injective for every i because $\mathrm{Supp} N \subset \mathrm{Supp}(\{\Delta_Z\} + \sum_{k=1}^{\infty} {}^k\Delta_Z - W)$. Thus the natural restriction map

$$H^0(Z, \mathcal{O}_Z(P - W)) \to H^0(N, \mathcal{O}_N(P - W))$$

is surjective. Since $H^0(Z, \mathcal{O}_Z(P - W)) = 0$, we obtain $H^0(N, \mathcal{O}_N(P - W)) = 0$. On the other hand,

$$H^0(N, \mathcal{O}_N(P - W)) \subset H^0(N, \mathcal{O}_N(P)) \neq 0$$

implies $N \cap W \neq \emptyset$. Thus we obtain $C \cap Y \neq \emptyset$.   Q.E.D.

Remark 5.3.5. If X is *projective* in Theorem 5.3.4, then we can prove Theorem 5.3.4 without using Theorem 5.3.3. We give a sketch of the proof. We may assume that $K_X + \Delta \sim_{\mathbb{R}} 0$. Let $f : Y \to X$ be a dlt blow-up with $K_Y + \Delta_Y = f^*(K_X + \Delta)$. We may assume that $a(E, X, \Delta) \leq -1$ for every f-exceptional divisor and that $(Y, \Delta_Y^{\leq 1} + S)$ is a dlt pair where $S = \mathrm{Supp}\Delta_Y^{>1}$. We run a minimal model program with respect to $K_Y + \Delta_Y^{\leq 1} + S$. Note that $K_Y + \Delta_Y^{\leq 1} + S \sim_{\mathbb{R}} S - \Delta_Y^{>1} \neq 0$ is not pseudo-effective. By the similar argument to the proof of [Fuj1, Proposition 2.1] (cf. [Fuj3, Theorem 3.47]), we can recover Theorem 5.3.4 when X is projective. We leave the details as exercises for the interested reader.

§6. Relative version

In this section, we discuss the relative version of Theorem 1.1 and some related results.

Theorem 6.1 (Relative injectivity theorem)**.** *Let X be a simple normal crossing variety and let Δ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $\mathrm{Supp}\Delta$ is a simple normal crossing divisor on X and that Δ is a boundary $\mathbb{R}$-divisor on X. Let $\pi : X \to V$ be a proper morphism between algebraic varieties and let L be a Cartier divisor on X and let D be an effective Weil divisor on X whose support is contained in $\mathrm{Supp}\Delta$. Assume that $L \sim_{\mathbb{R},\pi} K_X + \Delta$, that is, there is an $\mathbb{R}$-Cartier divisor B on V such that $L \sim_{\mathbb{R}} K_X + \Delta + \pi^*B$. Then the natural homomorphism*

$$R^q\pi_*\mathcal{O}_X(L) \to R^q\pi_*\mathcal{O}_X(L + D)$$

induced by the inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$ is injective for every q.

By using [BiVP] (see [Fuj8, Lemma 3.6]), we can reduce Theorem 6.1 to Theorem 1.1.

Proof. By shrinking V, we may assume that V is affine and $L \sim_{\mathbb{R}} K_X + \Delta$. Without loss of generality, we may assume that X is connected. Let $\overline{V}$ be a projective compactification of V. By [Fuj8, Lemma 3.6], we can compactify $\pi : X \to V$ to $\overline{\pi} : \overline{X} \to \overline{V}$. By the same argument as in Step 2 in the proof of [Fuj8, Theorem 3.7 (i)], we may assume that there is a Cartier divisor $\overline{L}$ on $\overline{X}$ such that $\overline{L}|_X = L$. We can write

$$L - (K_X + \Delta) = \sum_i b_i(f_i)$$

where b_i is a real number and $f_i \in \Gamma(X, \mathcal{K}_X^*)$ for every i. We put

$$E = \sum_i b_i(f_i) - (\overline{L} - (K_{\overline{X}} + \overline{\Delta})).$$

Then we have

$$\overline{L} + \lceil E \rceil \sim_{\mathbb{R}} K_{\overline{X}} + \overline{\Delta} + \{-E\}.$$

By the above construction, it is obvious that $\mathrm{Supp}\, E \subset \overline{X} \setminus X$. Let $\overline{D}$ be the closure of D in $\overline{X}$. It is sufficient to prove that the map

$$\varphi^q : R^q\overline{\pi}_*\mathcal{O}_{\overline{X}}(\overline{L} + \lceil E \rceil) \to R^q\overline{\pi}_*\mathcal{O}_{\overline{X}}(\overline{L} + \lceil E \rceil + \overline{D})$$

induced by the natural inclusion $\mathcal{O}_{\overline{X}} \to \mathcal{O}_{\overline{X}}(\overline{D})$ is injective for every q. Suppose that φ^q is not injective for some q. Let A be a sufficiently ample general Cartier divisor on $\overline{V}$ such that $H^0(\overline{V}, \mathrm{Ker}\varphi^q \otimes \mathcal{O}_{\overline{V}}(A)) \neq 0$. In this case, the map

$$H^0(\overline{V}, R^q\overline{\pi}_*\mathcal{O}_{\overline{X}}(\overline{L} + \lceil E \rceil) \otimes \mathcal{O}_{\overline{V}}(A))$$
$$\to H^0(\overline{V}, R^q\overline{\pi}_*\mathcal{O}_{\overline{X}}(\overline{L} + \lceil E \rceil + \overline{D}) \otimes \mathcal{O}_{\overline{V}}(A))$$

induced by φ^q is not injective. Since A is sufficiently ample, this implies that

$$H^q(\overline{X}, \mathcal{O}_{\overline{X}}(\overline{L} + \lceil E \rceil + \overline{\pi}^*A))$$
$$\to H^q(\overline{X}, \mathcal{O}_{\overline{X}}(\overline{L} + \lceil E \rceil + \overline{\pi}^*A + \overline{D}))$$

is not injective. Since

$$\overline{L} + \lceil E \rceil + \overline{\pi}^*A \sim_{\mathbb{R}} K_{\overline{X}} + \overline{\Delta} + \{-E\} + \overline{\pi}^*A,$$

it contradicts Theorem 1.1. Hence φ^q is injective for every q.   Q.E.D.

The following theorem is the relative version of Theorem 5.3.3. It is obvious by the proof of Theorem 5.3.3 and Theorem 6.1.

Theorem 6.2. *Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $\pi : X \to V$ be a proper morphism between algebraic varieties and let L be a Cartier divisor on X such that $L-(K_X+\Delta)$ is semi-ample over V. Let $\mathcal{J}'(X,\Delta)$ be the maximal non-lc ideal sheaf associated to (X,Δ) and let Y be the closed subscheme defined by $\mathcal{J}'(X,\Delta)$. Then we have a short exact sequence*

$$0 \to \pi_*(\mathcal{J}'(X,\Delta) \otimes \mathcal{O}_X(L)) \to \pi_*\mathcal{O}_X(L) \to \pi_*\mathcal{O}_Y(L) \to 0.$$

Proof. It is sufficient to prove that $\pi_*\mathcal{O}_X(L) \to \pi_*\mathcal{O}_Y(L)$ is surjective. Since the problem is local, we may assume that V is affine by shrinking V. Then the proof of Theorem 5.3.3 works without any modifications if we use Theorem 6.1.   Q.E.D.

The relative version of Theorem 5.3.4 is:

Theorem 6.3. *Let X be a normal variety and let $\pi : X \to V$ be a proper morphism between algebraic varieties with $\pi_*\mathcal{O}_X \simeq \mathcal{O}_V$. Let Δ be an effective $\mathbb{R}$-divisor on X such that $-(K_X + \Delta)$ is semi-ample over V. Let x be a closed point of V. Suppose that*

$$\mathrm{Nlc}(X,\Delta) \cap \pi^{-1}(x) \neq \emptyset.$$

Then $\mathrm{Nlc}(X,\Delta) \cap \pi^{-1}(x)$ is connected and intersects every log canonical center C of (X,Δ) with $C \cap \pi^{-1}(x) \neq \emptyset$.

Proof. By shrinking V, we may assume that V is affine. As in the proof of Theorem 5.3.4, we may assume that $K_X + \Delta \sim_{\mathbb{R}} 0$. From now on, we use the same notation as in the proof of Theorem 5.3.4. Since $\mathcal{O}_V \simeq \pi_*\mathcal{O}_X \to \pi_*\mathcal{O}_Y$ is surjective by Theorem 6.2, $Y \cap \pi^{-1}(x)$ is connected. By Theorem 6.1,

$$R^i(\pi \circ f)_*\mathcal{O}_Z(P - N - W) \to R^i(\pi \circ f)_*\mathcal{O}_Z(P - W)$$

is injective for every i. Thus the natural restriction map

$$(\pi \circ f)_*\mathcal{O}_Z(P - W) \to (\pi \circ f)_*\mathcal{O}_N(P - W)$$

is surjective. Since $(\pi \circ f)_*\mathcal{O}_Z(P - W) \subset \mathcal{I}_x \subsetneq \mathcal{O}_V$, where $\mathcal{I}_x$ is the defining ideal sheaf of x on V, we obtain

$$(\pi \circ f)_*\mathcal{O}_N(P - W) \subsetneq (\pi \circ f)_*\mathcal{O}_N \subset (\pi \circ f)_*\mathcal{O}_N(P)$$

at x. This implies $N \cap W \cap (\pi \circ f)^{-1}(x) \neq \emptyset$. Therefore, $C \cap Y \cap \pi^{-1}(x) \neq \emptyset$.   Q.E.D.

Theorem 6.4, which is the relative version of Theorem 5.2.1, directly follows from Theorem 6.2. See the proof of Theorem 5.2.1 by Theorem 5.3.3 in Subsection 5.3.

Theorem 6.4 (Relative extension theorem). *Let (X, Δ) be a log canonical pair and let $\pi : X \to V$ be a proper morphism. Let L be a Cartier divisor on X such that $H = L - (K_X + \Delta)$ is a π-semi-ample $\mathbb{R}$-divisor on X. Let D be an effective $\mathbb{R}$-divisor on X such that $D \sim_{\mathbb{R},\pi} tH$, that is, there is an $\mathbb{R}$-Cartier divisor B on V with $D \sim_{\mathbb{R}} tH + \pi^* B$, for some positive real number t and let Z be the union of the log canonical centers of (X, Δ) contained in $\mathrm{Supp} D$. Then the natural restriction map*

$$\pi_* \mathcal{O}_X(L) \to \pi_* \mathcal{O}_Z(L)$$

is surjective.

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