2.4 E_1-DEGENERATIONS OF HODGE TO DE RHAM TYPE SPECTRAL SEQUENCES

OSAMU FUJINO

2. E_1-degenerations of Hodge to de Rham type spectral sequences

sec3

From 2.27 to 2.29, we recall some well-known results on mixed Hodge structures. We use the notations in [D2] freely. The basic references on this topic are [D2, Section 8], [E1, Part II], and [E2, Chapitres 2 and 3]. The recent book [PS] may be useful. First, we start with the pure Hodge structures on proper smooth algebraic varieties.

s1 **2.27.** (Hodge structures for proper smooth varieties). Let X be a proper smooth algebraic variety over $\mathbb{C}$. Then the triple $(\mathbb{Z}_X, (\Omega^\bullet_X, F), \alpha)$, where $\Omega^\bullet_X$ is the holomorphic de Rham complex with the filtration bête F and $\alpha : \mathbb{C}_X \to \Omega^\bullet_X$ is the inclusion, is a cohomological Hodge complex (CHC, for short) of weight zero.

If we define weight filtrations as follows:

$$W_m\mathbb{Q}_X = \begin{cases} 0 & \text{if} \quad m < 0 \\ \mathbb{Q}_X & \text{if} \quad m \geq 0 \end{cases}$$

and

$$W_m\Omega^\bullet_X = \begin{cases} 0 & \text{if} \quad m < 0 \\ \Omega^\bullet_X & \text{if} \quad m \geq 0, \end{cases}$$

then we can see that $(\mathbb{Z}_X, (\mathbb{Q}_X, W), (\Omega^\bullet_X, F, W))$ is a cohomological mixed Hodge complex (CMHC, for short). We need these weight filtrations in the following arguments.

The next one is also a fundamental example. For the details, see [E1, I.1.] or [E2, 3.5].

s2 **2.28.** (Mixed Hodge structures for proper simple normal crossing varieties). Let D be a proper simple normal crossing algebraic variety

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Now I am revising Section 2.4 in my book.

over $\mathbb{C}$. Let $\varepsilon : D^\bullet \to D$ be the Mayer–Vietoris simplicial resolution (cf. Definition 011 ??). The following complex of sheaves, denoted by $\mathbb{Q}_{D^\bullet}$,

$$\varepsilon_{0*}\mathbb{Q}_{D^0} \to \varepsilon_{1*}\mathbb{Q}_{D^1} \to \cdots \to \varepsilon_{k*}\mathbb{Q}_{D^k} \to \cdots ,$$

is a resolution of $\mathbb{Q}_D$. More explicitly, the differential $d_k : \varepsilon_{k*}\mathbb{Q}_{D^k} \to \varepsilon_{k+1*}\mathbb{Q}_{D^{k+1}}$ is $\sum_{j=0}^{k+1}(-1)^j\lambda^*_{j,k+1}$ for every $k \geq 0$. The weight filtration W on $\mathbb{Q}_{D^\bullet}$ is defined by

$$\begin{aligned} W_{-q}(\mathbb{Q}_{D^\bullet}) &= \bigoplus_{m\geq q} \varepsilon_{m*}\mathbb{Q}_{D^m} \\ &= (0 \to \cdots \to \varepsilon_{q*}\mathbb{Q}_{D^q} \to \varepsilon_{q+1*}\mathbb{Q}_{D^{q+1}} \to \cdots). \end{aligned}$$

We obtain the resolution $\Omega^\bullet_{D^\bullet}$ of $\mathbb{C}_D$ as follows:

$$\varepsilon_{0*}\Omega^\bullet_{D^0} \to \varepsilon_{1*}\Omega^\bullet_{D^1} \to \cdots \to \varepsilon_{k*}\Omega^\bullet_{D^k} \to \cdots .$$

Of course, $d_k : \varepsilon_{k*}\Omega^\bullet_{D^k} \to \varepsilon_{k+1*}\Omega^\bullet_{D^{k+1}}$ is $\sum_{j=0}^{k+1}(-1)^j\lambda^*_{j,k+1}$. Let $s(\Omega^\bullet_{D^\bullet})$ be the single complex associated to the double complex $\Omega^\bullet_{D^\bullet}$. The Hodge filtration F on $s(\Omega^\bullet_{D^\bullet})$ is defined by

$$F^p = s(0 \to \cdots \to 0 \to \varepsilon_*\Omega^p_{D^\bullet} \to \varepsilon_*\Omega^{p+1}_{D^\bullet} \to \cdots).$$

We note that

$$\varepsilon_*\Omega^p_{D^\bullet} = (\varepsilon_{0*}\Omega^p_{D^0} \to \varepsilon_{1*}\Omega^p_{D^1} \to \cdots \to \varepsilon_{k*}\Omega^p_{D^k} \to \cdots)$$

for every p. The weight filtration W on $s(\Omega^\bullet_{D^\bullet})$ is defined by

$$\begin{aligned} W_{-q}(s(\Omega^\bullet_{D^\bullet})) &= s(\bigoplus_{m\geq q} \varepsilon_{m*}\Omega^\bullet_{D^m}) \\ &= s(0 \to \cdots \to 0 \to \varepsilon_{q*}\Omega^\bullet_{D^q} \to \varepsilon_{q+1*}\Omega^\bullet_{D^{q+1}} \to \cdots). \end{aligned}$$

We note that

$$\mathrm{Gr}^W_{-q}\mathbb{Q}_{D^\bullet} \simeq \varepsilon_{q*}\mathbb{Q}_{D^q}[-q],$$

and

$$\mathrm{Gr}^W_{-q}(s(\Omega^\bullet_{D^\bullet})) \simeq \varepsilon_{q*}\Omega^\bullet_{D^q}[-q].$$

Then $(\mathbb{Z}_D, (\mathbb{Q}_{D^\bullet}, W), (s(\Omega^\bullet_{D^\bullet}), W, F))$ is a CMHC. Here, we omitted the quasi-isomorphisms $\alpha : \mathbb{Z}_D \otimes \mathbb{Q} \to \mathbb{Q}_{D^\bullet}$ and $\beta : (\mathbb{Q}_{D^\bullet}, W) \to (s(\Omega^\bullet_{D^\bullet}), W)$ since there is no danger of confusion. This CMHC induces a natural mixed Hodge structure on $H^\bullet(D, \mathbb{Z})$. We note that the spectral sequence with respect to W on $\mathbb{Q}_{D^\bullet}$ is

$$\begin{aligned} {}_WE_1^{p,q} = H^{p+q}(D, \mathrm{Gr}^W_{-p}\mathbb{Q}_{D^\bullet}) &= H^{p+q}(D, \varepsilon_{p*}\mathbb{Q}_{D^p}[-p]) \\ &= H^q(D^p, \mathbb{Q}) \\ &\Longrightarrow H^{p+q}(D, \mathbb{Q}) \end{aligned}$$

such that the differential $d_1^{p,q} : {}_WE_1^{p,q} \to {}_WE_1^{p+1,q}$ is given by

$$d_1^{p,q} = \sum_{j=0}^{p+1} (-1)^j \lambda_{j,p+1}^* : H^q(D^p, \mathbb{Q}) \to H^q(D^{p+1}, \mathbb{Q})$$

and it degenerates in E_2. The spectral sequence with respect to F is

$$\begin{aligned} {}_FE_1^{p,q} = \mathbb{H}^{p+q}(D, \mathrm{Gr}_F^p(s(\Omega^\bullet_{D^\bullet}))) &= H^q(D^\bullet, \Omega^p_{D^\bullet}) \\ &\Longrightarrow H^{p+q}(D, \mathbb{C}) \end{aligned}$$

and it degenerates in E_1.

For the precise definitions of CHC and CMHC (CHMC, in French), see [D2, Section 8] or [E2, Chapitre 3]. See also [PS, 2.3.3 and 3.3]. The third example is not so standard but is indispensable for our injectivity theorems.

s3 **2.29.** (Mixed Hodge structures on compact support cohomology groups). Let X be a proper smooth algebraic variety over $\mathbb{C}$ and D a simple normal crossing divisor on X. We consider the mixed cones of $\phi : \mathbb{Q}_X \to \mathbb{Q}_{D^\bullet}$ and $\psi : \Omega^\bullet_X \to \Omega^\bullet_{D^\bullet}$ with suitable shifts of complexes and weight filtrations (for the details, see [E1, 1.3.], [E2, 3.7.14] or [PS, Theorem 3.22]), where ϕ and ψ are induced by the natural restriction map. More precisely, we define a complex

$$\mathbb{Q}_{X-D^\bullet} = \mathrm{Cone}^\bullet(\phi)[-1].$$

Then we have

$$(\mathbb{Q}_{X-D^\bullet})^p = (\mathbb{Q}_X)^p \oplus (\mathbb{Q}_{D^\bullet})^{p-1}.$$

The weight filtration on $\mathbb{Q}_{X-D^\bullet}$ is defined as follows:

$$(W_m\mathbb{Q}_{X-D^\bullet})^p = (W_m\mathbb{Q}_X)^p \oplus (W_{m+1}(\mathbb{Q}_{D^\bullet}))^{p-1}.$$

We note that $\mathbb{Q}_{X-D^\bullet}$ is quasi-isomorphic to $j_!\mathbb{Q}_{X-D}$, where $j : X-D \to X$ is the natural open immersion. We put

$$\Omega^\bullet_{X-D^\bullet} = \mathrm{Cone}^\bullet(\psi)[-1].$$

We note that

$$\Omega^p_{X-D^\bullet} = \Omega^p_X \oplus (s\Omega^\bullet_{D^\bullet})^{p-1}.$$

We define filtrations on $\Omega^\bullet_{X-D^\bullet}$ as follows:

$$(W_m\Omega^\bullet_{X-D^\bullet})^p = (W_m\Omega^\bullet_X)^p \oplus (W_{m+1}(s\Omega^\bullet_{D^\bullet}))^{p-1}$$

and

$$(F^r\Omega^\bullet_{X-D^\bullet})^p = (F^r\Omega^\bullet_X)^p \oplus (F^r(s\Omega^\bullet_{D^\bullet}))^{p-1}.$$

Then we obtain that the triple $(j_!\mathbb{Z}_{X-D}, (\mathbb{Q}_{X-D^\bullet}, W), (\Omega^\bullet_{X-D^\bullet}, W, F))$ is a CMHC. It defines a natural mixed Hodge structure on $H^\bullet_c(X-D, \mathbb{Z})$. We note that

$$\mathrm{Gr}^W_0 \mathbb{Q}_{X-D^\bullet} = \mathbb{Q}_X$$

and

$$\mathrm{Gr}^W_{-p} \mathbb{Q}_{X-D^\bullet} = \mathrm{Gr}^W_{1-p}\mathbb{Q}_{D^\bullet} = \varepsilon_{p-1*}\mathbb{Q}_{D^{p-1}}[-(p-1)]$$

for $p \geq 1$. Therefore, the spectral sequence with respect to W

$${}_WE_1^{p,q} = H^{p+q}(X, \mathrm{Gr}^W_{-p}\mathbb{Q}_{X-D^\bullet}) \Longrightarrow H^{p+q}_c(X-D, \mathbb{Q})$$

degenerates in E_2, where

$${}_WE_1^{0,q} = H^q(X, \mathbb{Q})$$

and

$${}_WE_1^{p,q} = H^q(D^{p-1}, \mathbb{Q})$$

for every $p \geq 1$. Since we can check that the complex

$$\begin{aligned} 0 \to \Omega^\bullet_X(\log D)(-D) \to \Omega^\bullet_X \to \varepsilon_{0*}\Omega^\bullet_{D^0} \\ \to \varepsilon_{1*}\Omega^\bullet_{D^1} \to \cdots \to \varepsilon_{k*}\Omega^\bullet_{D^k} \to \cdots \end{aligned}$$

is exact by direct local calculations, we see that $(\Omega^\bullet_{X-D^\bullet}, F)$ is quasi-isomorphic to $(\Omega^\bullet_X(\log D)(-D), F)$ in $D^+F(X, \mathbb{C})$, where

$$\begin{aligned} &F^p\Omega^\bullet_X(\log D)(-D) \\ &\quad = (0 \to \cdots \to 0 \to \Omega^p_X(\log D)(-D) \to \Omega^{p+1}_X(\log D)(-D) \to \cdots). \end{aligned}$$

Therefore, the spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_X(\log D)(-D)) \Longrightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_X(\log D)(-D))$$

degenerates in E_1 and the right hand side is isomorphic to $H^{p+q}_c(X - D, \mathbb{C})$.

From here, we treat mixed Hodge structures on much more complicated algebraic varieties (cf. elzein2 [E2, 3.9]).

s4 **2.30.** (Mixed Hodge structures for proper simple normal crossing pairs). Let (X, D) be a proper simple normal crossing pair over $\mathbb{C}$ such that D is reduced. Let $\varepsilon : X^\bullet \to X$ be the Mayer–Vietoris simplicial resolution of X. As we saw in the previous step, we have a CMHC

$$(j_{n!}\mathbb{Z}_{X^n-D^n}, (\mathbb{Q}_{X^n-(D^n)^\bullet}, W), (\Omega^\bullet_{X^n-(D^n)^\bullet}, W, F))$$

on X^n, where $j_n : X^n - D^n \to X^n$ is the natural open immersion with $D^n = \varepsilon_n^* D$, and we know that $(\Omega^\bullet_{X^n-(D^n)^\bullet}, F)$ is quasi-isomorphic to $(\Omega^\bullet_{X^n}(\log D^n)(-D^n), F)$ in $D^+F(X^n, \mathbb{C})$ for every $n \geq 0$. Therefore, by using the Mayer–Vietoris simplicial resolution $\varepsilon : X^\bullet \to X$, we can construct a CMHC $(j_!\mathbb{Z}_{X-D}, (K_\mathbb{Q}, W), (K_\mathbb{C}, W, F))$ on X that induces a natural mixed Hodge structure on $H^\bullet_c(X - D, \mathbb{Z})$. More explicitly, we put

$$K_{\mathbb{Q}} = s(\varepsilon_{0*}\mathbb{Q}_{X^0-(D^0)^\bullet} \to \varepsilon_{1*}\mathbb{Q}_{X^1-(D^1)^\bullet} \to \cdots \to \varepsilon_{k*}\mathbb{Q}_{X^k-(D^k)^\bullet} \to \cdots)$$

and

$$K_{\mathbb{C}} = s(\varepsilon_{0*}\Omega^\bullet_{X^0-(D^0)^\bullet} \to \varepsilon_{1*}\Omega^\bullet_{X^1-(D^1)^\bullet} \to \cdots \to \varepsilon_{k*}\Omega^\bullet_{X^k-(D^k)^\bullet} \to \cdots).$$

We define filtrations as follows:

$$\begin{aligned} W_m K_{\mathbb{Q}} = s(&\varepsilon_{0*}W_m\mathbb{Q}_{X^0-(D^0)^\bullet} \to \varepsilon_{1*}W_{m+1}\mathbb{Q}_{X^1-(D^1)^\bullet} \to \cdots \\ &\to \varepsilon_{k*}W_{m+k}\mathbb{Q}_{X^k-(D^k)^\bullet} \to \cdots), \end{aligned}$$

$$\begin{aligned} W_m K_{\mathbb{C}} = s(&\varepsilon_{0*}W_m\Omega^\bullet_{X^0-(D^0)^\bullet} \to \varepsilon_{1*}W_{m+1}\Omega^\bullet_{X^1-(D^1)^\bullet} \to \cdots \\ &\to \varepsilon_{k*}W_{m+k}\Omega^\bullet_{X^k-(D^k)^\bullet} \to \cdots), \end{aligned}$$

and

$$\begin{aligned} F^p K_{\mathbb{C}} = s(&\varepsilon_{0*}F^p\Omega^\bullet_{X^0-(D^0)^\bullet} \to \varepsilon_{1*}F^p\Omega^\bullet_{X^1-(D^1)^\bullet} \to \cdots \\ &\to \varepsilon_{k*}F^p\Omega^\bullet_{X^k-(D^k)^\bullet} \to \cdots). \end{aligned}$$

Then we obtain

$$\operatorname{Gr}^W_m K_{\mathbb{Q}} = \bigoplus_q \varepsilon_{q*}\operatorname{Gr}^W_{m+q}(\mathbb{Q}_{X^q-(D^q)^\bullet})[-q],$$

and

$$(\operatorname{Gr}^W_m K_{\mathbb{C}}, F) = (\bigoplus_q \varepsilon_{q*}\operatorname{Gr}^W_{m+q}(\Omega^\bullet_{X^q-(D^q)^\bullet})[-q], F).$$

The descriptions of W in 2.29 help us understand $\operatorname{Gr}^W_m K_{\mathbb{Q}}$ and $(\operatorname{Gr}^W_m K_{\mathbb{C}}, F)$. We can see that $(K_{\mathbb{C}}, F)$ is quasi-isomorphic to $(s(\Omega^\bullet_{X^\bullet}(\log D^\bullet)(-D^\bullet)), F)$ in $D^+F(X, \mathbb{C})$, where

$$\begin{aligned} F^p = s(0 \to \cdots \to 0 \to &\varepsilon_*\Omega^p_{X^\bullet}(\log D^\bullet)(-D^\bullet) \\ &\to \varepsilon_*\Omega^{p+1}_{X^\bullet}(\log D^\bullet)(-D^\bullet) \to \cdots). \end{aligned}$$

We note that $\Omega^\bullet_{X^\bullet}(\log D^\bullet)(-D^\bullet)$ is the double complex

$$\begin{aligned} 0 \to \varepsilon_{0*}\Omega^\bullet_{X^0}(\log D^0)(-D^0) \to \varepsilon_{1*}\Omega^\bullet_{X^1}(\log D^1)(-D^1) \to \cdots \\ \to \varepsilon_{k*}\Omega^\bullet_{X^k}(\log D^k)(-D^k) \to \cdots. \end{aligned}$$

Therefore, the spectral sequence

$$E^{p,q}_1 = H^q(X^\bullet, \Omega^p_{X^\bullet}(\log D^\bullet)(-D^\bullet)) \Longrightarrow \mathbb{H}^{p+q}(X, s(\Omega^\bullet_{X^\bullet}(\log D^\bullet)(-D^\bullet)))$$

degenerates in E_1 and the right hand side is isomorphic to $H^{p+q}_c(X - D, \mathbb{C})$.

Let us start the proof of the E_1-degeneration that we already used in the proof of Proposition ??.

s5 **2.31** (E_1-degeneration for Proposition ??)**.** Here, we use the notation in the proof of Proposition ??. In this case, Y has only quotient singularities. Then $(\mathbb{Z}_Y, (\widetilde{\Omega}^\bullet_Y, F), \alpha)$ is a CHC, where F is the filtration bête and $\alpha : \mathbb{C}_Y \to \widetilde{\Omega}^\bullet_Y$ is the inclusion. For the details, see [St, (1.6)]. It is easy to see that T is a divisor with V-normal crossings on Y (see ?? or [St, (1.16) Definition]). We can easily check that Y is singular only over the singular locus of Supp$B$. Let $\varepsilon : T^\bullet \to T$ be the Mayer–Vietoris simplicial resolution. Though T has singularities, Definition ?? makes sense without any modifications. We note that T^n has only quotient singularities for every $n \geq 0$ by the construction of $\pi : Y \to X$. We can also check that the same construction in 2.28 works with minor modifications and we have a CMHC $(\mathbb{Z}_T, (\mathbb{Q}_{T^\bullet}, W), (s(\widetilde{\Omega}^\bullet_{T^\bullet}), W, F))$ that induces a natural mixed Hodge structure on $H^\bullet(T, \mathbb{Z})$. By the same arguments as in 2.29, we can construct a triple $(j_!\mathbb{Z}_{Y-T}, (\mathbb{Q}_{Y-T^\bullet}, W), (K_\mathbb{C}, W, F))$, where $j : Y - T \to Y$ is the natural open immersion. It is a CMHC that induces a natural mixed Hodge structure on $H^\bullet_c(Y - T, \mathbb{Z})$ and $(K_\mathbb{C}, F)$ is quasi-isomorphic to $(\widetilde{\Omega}^\bullet_Y(\log T)(-T), F)$ in $D^+F(Y, \mathbb{C})$, where

$$\begin{aligned} &F^p\widetilde{\Omega}^\bullet_Y(\log T)(-T) \\ &\quad = (0 \to \cdots \to 0 \to \widetilde{\Omega}^p_Y(\log T)(-T) \to \widetilde{\Omega}^{p+1}_Y(\log T)(-T) \to \cdots). \end{aligned}$$

Therefore, the spectral sequence

$$E_1^{p,q} = H^q(Y, \widetilde{\Omega}^p_Y(\log T)(-T)) \Longrightarrow \mathbb{H}^{p+q}(Y, \Omega^\bullet_Y(\log T)(-T))$$

degenerates in E_1 and the right hand side is isomorphic to $H^{p+q}_c(Y - T, \mathbb{C})$.

The final one is the E_1-degeneration that we used in the proof of Proposition ??. It may be one of the main contributions of this chapter.

s6 **2.32** (E_1-degeneration for Proposition ??)**.** We use the notation in the proof of Proposition ??. Let $\varepsilon : Y^\bullet \to Y$ be the Mayer–Vietoris simplicial resolution. By the previous step, we can obtain a CMHC

$$(j_{n!}\mathbb{Z}_{Y^n - T^n}, (\mathbb{Q}_{Y^n-(T^n)^\bullet}, W), (K_\mathbb{C}, W, F))$$

for each $n \geq 0$. Of course, $j_n : Y^n - T^n \to Y^n$ is the natural open immersion for every $n \geq 0$. Therefore, we can construct a CMHC

$$(j_!\mathbb{Z}_{Y-T}, (K_\mathbb{Q}, W), (K_\mathbb{C}, W, F))$$

on Y as in 2.30. It induces a natural mixed Hodge structure on $H^{\bullet}_c(Y - T, \mathbb{Z})$. We note that $(K_{\mathbb{C}}, F)$ is quasi-isomorphic to $(s(\widetilde{\Omega}^{\bullet}_{Y^{\bullet}}(\log T^{\bullet})(-T^{\bullet})), F)$ in $D^{+}F(Y, \mathbb{C})$, where

$$F^p = s(0 \to \cdots \to 0 \to \varepsilon_* \widetilde{\Omega}^{p}_{Y^{\bullet}}(\log T^{\bullet})(-T^{\bullet}) \to \varepsilon_* \widetilde{\Omega}^{p+1}_{Y^{\bullet}}(\log T^{\bullet})(-T^{\bullet}) \to \cdots).$$

For the details, see 2.30 above. Thus, the desired spectral sequence

$$E_1^{p,q} = H^q(Y^{\bullet}, \widetilde{\Omega}^{p}_{Y^{\bullet}}(\log T^{\bullet})(-T^{\bullet})) \Longrightarrow \mathbb{H}^{p+q}(Y, s(\widetilde{\Omega}^{\bullet}_{Y^{\bullet}}(\log T^{\bullet})(-T^{\bullet})))$$

degenerates in E_1. It is what we need in the proof of Proposition **??**. Note that $\mathbb{H}^{p+q}(Y, s(\widetilde{\Omega}^{\bullet}_{Y^{\bullet}}(\log T^{\bullet})(-T^{\bullet}))) \simeq H^{p+q}_c(Y - T, \mathbb{C})$.

References

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Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan

E-mail address: fujino@math.kyoto-u.ac.jp