Fundamental theorems for semi log canonical pairs

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Dedicated to Professor Shigeru Mukai on the occasion of his sixtieth birthday

Abstract

We prove that every quasi-projective semi log canonical pair has a natural quasi-log structure with several good properties. It implies that various vanishing theorems, torsion-free theorem, and the cone and contraction theorem hold for semi log canonical pairs.

1. Introduction

In this paper, we give a *natural* quasi-log structure (cf. [Am03]) to an arbitrary quasi-projective semi log canonical pair. Note that a stable pointed curve is a typical example of semi log canonical pairs. As applications, we obtain various Kodaira type vanishing theorems, the cone and contraction theorem, and so on, for semi log canonical pairs. The notion of semi log canonical singularities was introduced in [KSB88] in order to investigate deformations of surface singularities and compactifications of moduli spaces for surfaces of general type. By the recent developments of the minimal model program, we know that the appropriate singularities to permit on the varieties at the boundaries of moduli spaces are semi log canonical (see, for example, [Al96a], [Al96b], [Ko13a], [HK10, Part III], [Kv05], [Kv13], and so on). We note that the approach to the moduli problems in [KSB88] is not directly related to Mumford's geometric invariant theory. However, the notion of semi log canonical singularities appears to be natural from the geometric invariant theoretic viewpoint by [O13]. Moreover, semi log canonical pairs play crucial roles in our inductive treatment of the log abundance conjecture (see, for example, [F00b] and [FG11]). Therefore, it is very important to establish some foundational techniques to investigate semi log canonical pairs. To the best knowledge of the author, there were no attempts to prove the fundamental theorems of the log minimal model program, for example, the cone and contraction theorem, various Kodaira type vanishing theorems, and so on, for *semi log canonical pairs*. For a different approach to semi log canonical pairs by János Kollár, see [Ko13b], where he discusses his gluing theory for stable pairs, that is, semi log canonical pairs with ample log canonical divisor. We prove the following theorem.

THEOREM 1.1. *Let (X, Δ) be a quasi-projective semi log canonical pair. Then $[X, K_X + \Delta]$ has a quasi-log structure with only qlc singularities.*

Our proof of Theorem 1.1 heavily depends on the recent developments of the theory of partial

resolution of singularities for *reducible* varieties (see, for example, [Ko13b, Section 10.4], [BM12], [BP13], and so on). Precisely speaking, we prove the following theorem.

Theorem 1.2 (Main theorem). *Let (X, Δ) be a quasi-projective semi log canonical pair. Then we can construct a smooth quasi-projective variety M with $\dim M = \dim X + 1$, a simple normal crossing divisor Z on M, a subboundary $\mathbb{R}$-Cartier $\mathbb{R}$-divisor B on M, and a projective surjective morphism $h : Z \to X$ with the following properties.*

(1) *B and Z have no common irreducible components.*

(2) *$\mathrm{Supp}(Z + B)$ is a simple normal crossing divisor on M.*

(3) *$K_Z + \Delta_Z \sim_{\mathbb{R}} h^*(K_X + \Delta)$ such that $\Delta_Z = B|_Z$.*

(4) *$h_* \mathcal{O}_Z(\lceil -\Delta_Z^{<1} \rceil) \simeq \mathcal{O}_X$.*

By the properties (1), (2), (3), and (4), $[X, K_X + \Delta]$ has a quasi-log structure with only qlc singularities.

(5) *The set of slc strata of (X, Δ) gives the set of qlc centers of $[X, K_X + \Delta]$. This means that W is an slc stratum of (X, Δ) if and only if W is the h-image of some stratum of the simple normal crossing pair (Z, Δ_Z).*

By the property (5), the above quasi-log structure of $[X, K_X + \Delta]$ is compatible with the original semi log canonical structure of (X, Δ).

We note that $h_* \mathcal{O}_Z \simeq \mathcal{O}_X$ by the condition (4).

Remark 1.3. In Theorem 1.2, if $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then we can make B a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on M satisfying

$$K_Z + \Delta_Z \sim_{\mathbb{Q}} h^*(K_X + \Delta).$$

It is obvious by the construction of B in the proof of Theorem 1.2.

By Theorem 1.2, we can prove the fundamental theorems, that is, various Kodaira type vanishing theorems, the base point free theorem, the rationality theorem, the cone theorem, and so on, for semi log canonical pairs. Note that all the fundamental theorems for log canonical pairs can be proved without using the theory of quasi-log varieties (see [F11b] and [F11c]). We also note that all the results in this section except Theorem 1.8 are new even for semi log canonical *surfaces*.

Example 1.4. Let X be an equidimensional projective variety having only normal crossing points and pinch points. Then X is a semi log canonical variety. By Theorem 1.2, X has a natural quasi-log structure. Therefore, all the theorems in this section hold for X.

Note that h is not necessarily birational in Theorem 1.2. It is a key point of the theory of quasi-log varieties.

Remark 1.5 (Double covering trick due to Kollár). If the irreducible components of X have no self-intersection in codimension one, then we can make $h : Z \to X$ birational in Theorem 1.2. For some applications, by using Kollár's double covering trick (see Lemma 5.1), we can reduce the problem to the case when the irreducible components of X have no self-intersection in codimension one. This reduction sometimes makes the problem much easier not only technically but also psychologically.

Let us quickly recall a very important example. We recommend the reader to see [F07, Section 3.6] for related topics.

1.6 (Whitney umbrella). Let us consider the Whitney umbrella $X = (x^2 - y^2 z = 0) \subset \mathbb{A}^3$. In this case, we take a blow-up $Bl_C \mathbb{A}^3 \to \mathbb{A}^3$ of $\mathbb{A}^3$ along $C = (x = y = 0) \subset \mathbb{A}^3$ and set $M = Bl_C \mathbb{A}^3$ and $Z = X' + E$, where X' is the strict transform of X on M and E is the exceptional divisor of the blow-up. Then the projective surjective morphism $h : Z \to X$ gives a quasi-log structure on the pair $(X, 0)$. Since Z is a quasi-projective simple normal crossing variety, we can easily use the theory of mixed Hodge structures and obtain various vanishing theorems for X. It is a key point of the theory of quasi-log varieties. Note that $K_Z = h^* K_X$ and $h_* \mathcal{O}_Z \simeq \mathcal{O}_X$. Although $g = h|_{X'} : X' \to X$ is a resolution of singularities, it does not have good properties. This is because X is not normal and $\mathcal{O}_X \subsetneq g_* \mathcal{O}_{X'}$.

By Theorem 1.2, we can prove the following vanishing theorem (see [KMM87, Theorem 1-2-5]). It is a generalization of the Kawamata–Viehweg vanishing theorem.

THEOREM 1.7 (Vanishing theorem I). *Let (X, Δ) be a semi log canonical pair and let $\pi : X \to S$ be a projective morphism onto an algebraic variety S. Let D be a Cartier divisor on X, or a Weil divisor on X whose support does not contain any irreducible components of the conductor of X and which is $\mathbb{Q}$-Cartier. Assume that $D - (K_X + \Delta)$ is π-ample. Then $R^i \pi_* \mathcal{O}_X(D) = 0$ for every $i > 0$.*

As a special case of Theorem 1.7, we have the Kodaira vanishing theorem for semi log canonical varieties (cf. [KSS10, Corollary 6.6]).

THEOREM 1.8 (Kodaira vanishing theorem). *Let X be a projective semi log canonical variety and let $\mathcal{L}$ be an ample line bundle on X. Then $H^i(X, \omega_X \otimes \mathcal{L}) = 0$ for every $i > 0$.*

Note that the dual form of the Kodaira vanishing theorem, that is, $H^i(X, \mathcal{L}^{-1}) = 0$ for $i < \dim X$, is treated by Kovács–Schwede–Smith. For the details, see [KSS10, Corollary 6.6]. In general, X is not Cohen–Macaulay. Therefore, the dual form of the Kodaira vanishing theorem does not always hold. The arguments in [KSS10] are based on the theory of Du Bois singularities (see, for example, [KSS10], [KK10], and [Ko13b, Chapter 6]). In this paper, we do not use the notion of Du Bois singularities.

To the best knowledge of the author, even the following basic vanishing result for stable n-folds with $n \geqslant 2$ is new. It is a direct consequence of Theorem 1.7.

COROLLARY 1.9 (Vanishing theorem for stable varieties). *Let X be a stable variety, that is, a projective semi log canonical variety such that K_X is ample. Then $H^i(X, \mathcal{O}_X(mK_X)) = 0$ for every $i > 0$ and $m \geqslant 2$. In particular,*

$$\chi(X, \mathcal{O}_X(mK_X)) = \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(mK_X)) \geqslant 0$$

for every $m \geqslant 2$.

Theorem 1.7 is a special case of the following theorem: Theorem 1.10. It is a generalization of the vanishing theorem of Reid–Fukuda type. The proof of Theorem 1.10 is much harder than that of Theorem 1.7.

THEOREM 1.10 (Vanishing theorem II). *Let (X, Δ) be a semi log canonical pair and let $\pi : X \to S$ be a projective morphism onto an algebraic variety S. Let D be a Cartier divisor on X, or a Weil divisor on X whose support does not contain any irreducible components of the conductor of X and which is $\mathbb{Q}$-Cartier. Assume that $D - (K_X + \Delta)$ is nef and log big over S with respect to (X, Δ). Then $R^i \pi_* \mathcal{O}_X(D) = 0$ for every $i > 0$.*

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For applications to the study of linear systems on semi log canonical pairs, Theorem 1.11, which is a generalization of the Kawamata–Viehweg–Nadel vanishing theorem, is more convenient (see, for example, [F11c, Theorem 8.1]). See also Remark 5.2 below.

THEOREM 1.11 (Vanishing theorem III). *Let (X, Δ) be a semi log canonical pair and let $\pi :$ $X \to S$ be a projective morphism onto an algebraic variety S. Let D be a Cartier divisor on X such that $D - (K_X + \Delta)$ is nef and log big over S with respect to (X, Δ). Assume that X' is a union of some slc strata of (X, Δ) with the reduced structure. Let $\mathcal{I}_{X'}$ be the defining ideal sheaf of X' on X. Then $R^i \pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(D)) = 0$ for every $i > 0$.*

Note that our proof of the vanishing theorems uses the theory of the mixed Hodge structures on cohomology groups with compact support (cf. [F09c, Chapter 2]). Therefore, Theorems 1.7, 1.8, 1.10, and 1.11 are Hodge theoretic (see also [F09a], [F11c], [F12c], and [F13]).

We can also prove a generalization of Kollár's torsion-free theorem for semi log canonical pairs (see [KMM87, Theorem 1-2-7], [F04, Theorem 2.2], [F11c, Theorem 6.3 (iii)], and so on).

THEOREM 1.12 (Torsion-free theorem). *Let (X, Δ) be a semi log canonical pair and let $\pi : X \to S$ be a projective morphism onto an algebraic variety S. Let D be a Cartier divisor on X, or a Weil divisor on X whose support does not contain any irreducible components of the conductor of X and which is $\mathbb{Q}$-Cartier. Assume that $D - (K_X + \Delta)$ is π-semi-ample. Then every associated prime of $R^i \pi_* \mathcal{O}_X(D)$ is the generic point of the π-image of some slc stratum of (X, Δ) for every i.*

By the following adjunction formula, which is a direct consequence of Theorem 1.2, we can apply the theory of quasi-log varieties to any union of some slc strata of a quasi-projective semi log canonical pair (X, Δ).

THEOREM 1.13 (Adjunction). *Let (X, Δ) be a quasi-projective semi log canonical pair and let X' be a union of some slc strata of (X, Δ) with the reduced structure. Then $[X', (K_X + \Delta)|_{X'}]$ has a natural quasi-log structure with only qlc singularities induced by the quasi-log structure on $[X, K_X + \Delta]$ constructed in Theorem 1.2. Therefore, W is a qlc center of $[X', (K_X + \Delta)|_{X'}]$ if and only if W is an slc stratum of (X, Δ) contained in X'. In particular, X' is semi-normal.*

Theorem 1.14, which is a vanishing theorem for a union of some slc strata, is very powerful for various applications (cf. [F11c, Theorem 11.1]). See Remark 1.17 below.

THEOREM 1.14 (Vanishing theorem IV). *Let (X, Δ) be a semi log canonical pair and let $\pi :$ $X \to S$ be a projective morphism onto an algebraic variety S. Assume that X' is a union of some slc strata of (X, Δ) with the reduced structure. Let L be a Cartier divisor on X' such that $L - (K_X + \Delta)|_{X'}$ is nef over S. Assume that $(L - (K_X + \Delta)|_{X'})|_W$ is big over S where W is any slc stratum of (X, Δ) contained in X'. Then $R^i (\pi|_{X'})_* \mathcal{O}_{X'}(L) = 0$ for every $i > 0$.*

Theorem 1.14 directly follows from Theorem 1.13 by the theory of quasi-log varieties.

By Theorem 1.2, we can use the theory of quasi-log varieties to investigate semi log canonical pairs. The base point free theorem holds for semi log canonical pairs (cf. [KMM87, Theorem 3-1-1]).

THEOREM 1.15 (Base point free theorem). *Let (X, Δ) be a semi log canonical pair and let $\pi : X \to S$ be a projective morphism onto an algebraic variety S. Let D be a π-nef Cartier divisor on X. Assume that $aD - (K_X + \Delta)$ is π-ample for some real number $a > 0$. Then*

$\mathcal{O}_X(mD)$ is π-generated for every $m \gg 0$, that is, there exists a positive integer m_0 such that $\mathcal{O}_X(mD)$ is π-generated for every $m \geqslant m_0$.

We can prove the base point free theorem of Reid–Fukuda type for semi log canonical pairs (see also [F00a, Theorem 0.1], [F12b, Section 5], and so on). It is a slight generalization of Theorem 1.15. Note that Theorem 1.15 is sufficient for the contraction theorem in Theorem 1.19.

THEOREM 1.16 (Base point free theorem II). *Let (X, Δ) be a semi log canonical pair and let $\pi : X \to S$ be a projective morphism onto an algebraic variety S. Let D be a π-nef Cartier divisor on X. Assume that $aD - (K_X + \Delta)$ is nef and log big over S with respect to (X, Δ) for some real number $a > 0$. Then $\mathcal{O}_X(mD)$ is π-generated for every $m \gg 0$, that is, there exists a positive integer m_0 such that $\mathcal{O}_X(mD)$ is π-generated for every $m \geqslant m_0$.*

From some technical viewpoints, we give an important remark.

REMARK 1.17. We can prove Theorem 1.15 without using the theory of quasi-log varieties. The proofs of the non-vanishing theorem and the base point free theorem in [F11c] can be adapted to our situation in Theorem 1.15 once we adopt Theorem 1.14. For the details, see [F11c, Sections 12 and 13]. On the other hand, the theory of quasi-log varieties seems to be indispensable for the proof of Theorem 1.16. Therefore, the proof of Theorem 1.16 is much harder than that of Theorem 1.15.

It is known that the rationality theorem holds for quasi-log varieties. Therefore, as a consequence of Theorem 1.2, we obtain the rationality theorem for semi log canonical pairs (cf. [KMM87, Theorem 4-1-1]). Note that we can obtain Theorem 1.18 as an application of Theorem 1.11 and that the proof of Theorem 1.18 does not need the theory of quasi-log varieties (see [F11c, Theorem 8.1 and the proof of Theorem 15.1]).

THEOREM 1.18 (Rationality theorem). *Let (X, Δ) be a semi log canonical pair and let $\pi : X \to S$ be a projective morphism onto an algebraic variety S. Let H be a π-ample Cartier divisor on X. Assume that $K_X + \Delta$ is not π-nef and that there is a positive integer a such that $a(K_X + \Delta)$ is $\mathbb{R}$-linearly equivalent to a Cartier divisor. Let r be a positive real number such that $H + r(K_X + \Delta)$ is π-nef but not π-ample. Then r is a rational number, and in reduced form, it has denominator at most $a(\dim X + 1)$.*

By using Theorem 1.15 and Theorem 1.18, we obtain the cone and contraction theorem for semi log canonical pairs.

THEOREM 1.19 (Cone and contraction theorem). *Let (X, Δ) be a semi log canonical pair and let $\pi : X \to S$ be a projective morphism onto an algebraic variety S. Then we have the following properties.*

(1) *There are (countably many) rational curves $C_j \subset X$ such that $0 < -(K_X + \Delta) \cdot C_j \leqslant 2 \dim X$, $\pi(C_j)$ is a point, and*

$$\overline{NE}(X/S) = \overline{NE}(X/S)_{(K_X + \Delta) \geqslant 0} + \sum \mathbb{R}_{\geqslant 0}[C_j].$$

(2) *For any $\varepsilon > 0$ and any π-ample $\mathbb{R}$-divisor H,*

$$\overline{NE}(X/S) = \overline{NE}(X/S)_{(K_X + \Delta + \varepsilon H) \geqslant 0} + \sum_{finite} \mathbb{R}_{\geqslant 0}[C_j].$$

(3) *Let $F \subset \overline{NE}(X/S)$ be a $(K_X + \Delta)$-negative extremal face. Then there is a unique morphism $\varphi_F : X \to Z$ over S such that $(\varphi_F)_* \mathcal{O}_X \simeq \mathcal{O}_Z$, Z is projective over S, and that an irreducible curve $C \subset X$ where $\pi(C)$ is a point is mapped to a point by φ_F if and only if $[C] \in F$. The map φ_F is called the contraction associated to F.*

(4) *Let F and φ_F be as in (3). Let L be a line bundle on X such that $L \cdot C = 0$ for every curve $[C] \in F$. Then there is a line bundle M on Z such that $L \simeq \varphi_F^* M$.*

Although we have established the cone and contraction theorem for semi log canonical pairs, a simple example (see Example 5.4) shows that we can not always run the minimal model program even for semi log canonical *surfaces*. However, we have some nontrivial applications of Theorem 1.19 (see Section 6). Moreover, Kento Fujita has recently constructed *semi-terminal modifications* for quasi-projective *demi-normal* pairs by running a variant of the minimal model program for *semi-terminal* pairs. His arguments use Theorem 1.19 and Kollár's gluing theory. For the details, see [Ft13].

We can prove many other powerful results by translating the results for quasi-log varieties (see, for example, Corollary 3.5). For the details of the theory of quasi-log varieties, see [F09c] and [F11a]. We recommend the reader to see [F11c] for various vanishing theorems, the non-vanishing theorem, the base point free theorem, the cone theorem, and so on, for pairs (X, Δ), where X is a *normal* variety and Δ is an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. The arguments in [F11c] are independent of the theory of quasi-log varieties and only use *normal* varieties for the above fundamental theorems. In this paper, we do not need the recent advances in the minimal model program mainly due to Birkar–Cascini–Hacon–McKernan (cf. [HK10, Part II]).

For the abundance conjecture for semi log canonical pairs, see [F00b], [G13], [FG11], and [HX11]. These papers are independent of the techniques discussed in this paper. We give some results supplementary to [FG11] in Section 6. In this introduction, we explain only one result on the finiteness of automorphisms.

THEOREM 1.20 (see Theorem 6.16). *Let (X, Δ) be a complete semi log canonical pair such that $K_X + \Delta$ is a big $\mathbb{Q}$-Carteir $\mathbb{Q}$-divisor. Then*

$$\mathrm{Bir}(X, \Delta) = \{f \,|\, f : (X, \Delta) \dashrightarrow (X, \Delta) \text{ is B-birational}\}$$

is a finite group. In particular,

$$\mathrm{Aut}(X, \Delta) = \{f \,|\, f : X \to X \text{ is an isomorphism such that } \Delta = f_*^{-1}\Delta\}$$

is a finite group.

For the details, see Theorem 6.16 below. Theorem 1.20 seems to be an important property when we consider moduli spaces of *stable pairs*.

By combining the arguments in the proof of Theorem 1.2 with our new semi-positivity theorem in [FF11] (see also [FFS13]), we obtain the following semi-positivity theorem in [F12d].

THEOREM 1.21 (see [F12d, Theorem 1.8]). *Let X be an equidimensional variety which satisfies Serre's S_2 condition and is Gorenstein in codimension one. Let $f : X \to C$ be a projective surjective morphism onto a smooth projective curve C such that every irreducible component of X is dominant onto C. Assume that there exists a non-empty Zariski open set U of C such that $f^{-1}(U)$ has only semi log canonical singularities. Then $f_* \omega_{X/C}$ is semi-positive.*

Assume further that $\omega_{X/C}^{[k]} := (\omega_{X/C}^{\otimes k})^{**}$ is locally free and f-generated for some positive integer k. Then $f_*\omega_{X/C}^{[m]}$ is semi-positive for every $m \geqslant 1$.

Theorem 1.21 implies that the moduli functor of stable varieties is semi-positive in the sense of Kollár (see [Ko90, 2.4. Definition]). Therefore, Theorem 1.21 plays crucial roles for the projectivity of the moduli spaces of stable varieties. For the details, see [Ko90], [FF11], and [F12d]. The reader can find some generalizations of Theorem 1.21 in [F12d].

Finally, in this paper, we are mainly interested in *non-normal* algebraic varieties. So we have to be careful about some basic definitions.

1.22 (Big $\mathbb{R}$-Cartier $\mathbb{R}$-divisors). Let X be a *non-normal* complete irreducible algebraic variety and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $m_0 D$ is Cartier for some positive integer m_0. We can consider the asymptotic behavior of $\dim H^0(X, \mathcal{O}_X(mm_0 D))$ for $m \to \infty$ since $\mathcal{O}_X(mm_0 D)$ is a well-defined line bundle on X associated to $mm_0 D$. Therefore, there are no difficulties to define big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. Let B be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, that is, a finite $\mathbb{R}$-linear combination of Cartier divisors, on X. In this case, there are some difficulties to consider the asymptotic behavior of $\dim H^0(X, \mathcal{O}_X(mB))$ for $m \to \infty$. It is because the meaning of $\mathcal{O}_X(mB)$ is not clear. It may happen that the support of mB is contained in the singular locus of X. Therefore, we have to discuss the definition and the basic properties of big $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on *non-normal* complete irreducible varieties.

We summarize the contents of this paper. In Section 2, we collect some basic definitions and results. Section 3 contains supplementary results for the theory of quasi-log varieties. Section 4 is devoted to the proof of the main theorem: Theorem 1.2. The proof heavily depends on the recent developments of the theory of partial resolution of singularities for *reducible* varieties (cf. [Ko13b, Section 10.4], [BM12], [BP13]). In Section 5, we treat the fundamental theorems in Section 1 as applications of Theorem 1.2. In Section 6, we discuss miscellaneous applications, for example, the base point free theorem for $\mathbb{R}$-divisors, a generalization of Kollár's effective base point free theorem for semi log canonical pairs, Shokurov's polytope for semi log canonical pairs, depth of sheaves on slc pairs, semi log canonical morphisms, the finiteness of B-birational automorphisms for stable pairs, and so on. In Section 7, which is an appendix, we discuss the notion of big $\mathbb{R}$-divisors on *non-normal* algebraic varieties because there are no good references on this topic.

We fix the basic notation. For the standard notation of the log minimal model program, see, for example, [F11c].

NOTATION. Let B_1 and B_2 be two $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on a variety X. Then B_1 is linearly (resp. $\mathbb{Q}$-linearly, or $\mathbb{R}$-linearly) equivalent to B_2, denoted by $B_1 \sim B_2$ (resp. $B_1 \sim_{\mathbb{Q}} B_2$, or $B_1 \sim_{\mathbb{R}} B_2$) if

$$B_1 = B_2 + \sum_{i=1}^{k} r_i(f_i)$$

such that $f_i \in \Gamma(X, \mathcal{K}_X^*)$ and $r_i \in \mathbb{Z}$ (resp. $r_i \in \mathbb{Q}$, or $r_i \in \mathbb{R}$) for every i. Here, $\mathcal{K}_X$ is the sheaf of total quotient rings of $\mathcal{O}_X$ and $\mathcal{K}_X^*$ is the sheaf of invertible elements in the sheaf of rings $\mathcal{K}_X$. We note that (f_i) is a *principal Cartier divisor* associated to f_i, that is, the image of f_i by $\Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ is the sheaf of invertible elements in $\mathcal{O}_X$.

Let $f : X \to Y$ be a morphism. If there is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor B on Y such that

$$B_1 \sim_{\mathbb{R}} B_2 + f^*B,$$

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then B_1 is said to be *relatively $\mathbb{R}$-linearly equivalent* to B_2. It is denoted by $B_1 \sim_{\mathbb{R},f} B_2$.

When X is complete, B_1 is *numerically equivalent* to B_2, denoted by $B_1 \equiv B_2$, if $B_1 \cdot C = B_2 \cdot C$ for every curve C on X.

Let D be a $\mathbb{Q}$-divisor (resp. an $\mathbb{R}$-divisor) on an equidimensional variety X, that is, D is a finite formal $\mathbb{Q}$-linear (resp. $\mathbb{R}$-linear) combination

$$D = \sum_i d_i D_i$$

of irreducible reduced subschemes D_i of codimension one. We define the *round-up* $\lceil D \rceil = \sum_i \lceil d_i \rceil D_i$ (resp. *round-down* $\lfloor D \rfloor = \sum_i \lfloor d_i \rfloor D_i$), where every real number x, $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \leqslant \lceil x \rceil < x + 1$ (resp. $x - 1 < \lfloor x \rfloor \leqslant x$). The *fractional part* $\{D\}$ of D denotes $D - \lfloor D \rfloor$. We set

$$D^{<1} = \sum_{d_i < 1} d_i D_i, \quad \text{and} \quad D^{=1} = \sum_{d_i = 1} D_i.$$

We call D a *boundary* (resp. *subboundary*) $\mathbb{R}$-divisor if $0 \leqslant d_i \leqslant 1$ (resp. $d_i \leqslant 1$) for every i.

Let X be a normal variety and let Δ be an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a resolution such that $\mathrm{Exc}(f) \cup f_*^{-1}\Delta$, where $\mathrm{Exc}(f)$ is the exceptional locus of f and $f_*^{-1}\Delta$ is the strict transform of Δ on Y, has a simple normal crossing support. We can write

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i.$$

We say that (X, Δ) is *sub log canonical* (*sub lc*, for short) if $a_i \geqslant -1$ for every i. We usually write $a_i = a(E_i, X, \Delta)$ and call it the *discrepancy coefficient* of E_i with respect to (X, Δ). If (X, Δ) is sub log canonical and Δ is effective, then (X, Δ) is called *log canonical* (*lc*, for short). We note that we can define $a(E_i, X, \Delta)$ in more general settings (see [Ko13b, Definition 2.4]).

If (X, Δ) is sub log canonical and there exist a resolution $f : Y \to X$ and a divisor E on Y such that $a(E, X, \Delta) = -1$, then $f(E)$ is called a *log canonical center* (an *lc center*, for short) with respect to (X, Δ).

Let X be a smooth projective variety and let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then $\kappa(X, D)$ denotes *Iitaka's D-dimension* of D (see, for example, [N04, Chapter II. 3.2. Definition]).

A pair $[X, \omega]$ consists of a scheme X and an $\mathbb{R}$-Carteir $\mathbb{R}$-divisor ω on X. In this paper, X is always a *variety*, that is, X is a reduced separated scheme of finite type over $\mathrm{Spec}\,\mathbb{C}$.

We will work over $\mathbb{C}$, the field of complex numbers, throughout this paper. Note that, by the Lefschetz principle, all the results hold over any algebraically closed field k of characteristic zero. In this paper, we will use the notion of quasi-log varieties introduced by Florin Ambro in [Am03], which has not yet been so familiar even to the experts of the log minimal model program. Therefore we recommend the reader to take a glance at [F11a] for a gentle introduction to the theory of quasi-log varieties before reading this paper.

2. Preliminaries

In this section, we collect some basic definitions and results. First, let us recall the definition of *conductors*.

DEFINITION 2.1 (Conductor). Let X be an equidimensional variety which satisfies Serre's S_2 condition and is normal crossing in codimension one and let $\nu : X^\nu \to X$ be the normalization. Then the *conductor ideal* of X is defined by

$$\mathfrak{cond}_X := \mathcal{H}om_{\mathcal{O}_X}(\nu_* \mathcal{O}_{X^\nu}, \mathcal{O}_X) \subset \mathcal{O}_X.$$

The *conductor* $\mathcal{C}_X$ of X is the subscheme defined by $\mathfrak{cond}_X$. Since X satisfies Serre's S_2 condition and X is normal crossing in codimension one, $\mathcal{C}_X$ is a reduced closed subscheme of pure codimension one in X.

DEFINITION 2.2 (Double normal crossing points and pinch points). An n-dimensional singularity $(x \in X)$ is called a *double normal crossing point* if it is analytically (or formally) isomorphic to

$$(0 \in (x_0 x_1 = 0)) \subset (0 \in \mathbb{C}^{n+1}).$$

It is called a *pinch point* if it is analytically (or formally) isomorphic to

$$(0 \in (x_0^2 = x_1 x_2^2)) \subset (0 \in \mathbb{C}^{n+1}).$$

We recall the definition of *semi log canonical pairs*.

DEFINITION 2.3 (Semi log canonical pairs). Let X be an equidimensional algebraic variety that satisfies Serre's S_2 condition and is normal crossing in codimension one. Let Δ be an effective $\mathbb{R}$-divisor whose support does not contain any irreducible components of the conductor of X. The pair (X, Δ) is called a *semi log canonical pair* (an *slc pair*, for short) if

(1) $K_X + \Delta$ is $\mathbb{R}$-Cartier, and

(2) (X^ν, Θ) is log canonical, where $\nu : X^\nu \to X$ is the normalization and $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$.

We introduce the notion of *semi log canonical centers*. It is a direct generalization of the notion of log canonical centers for log canonical pairs.

DEFINITION 2.4 (Slc center). Let (X, Δ) be a semi log canonical pair and let $\nu : X^\nu \to X$ be the normalization. We set

$$K_{X^\nu} + \Theta = \nu^*(K_X + \Delta).$$

A closed subvariety W of X is called a *semi log canonical center* (an *slc center*, for short) *with respect to* (X, Δ) if there exist a resolution of singularities $f : Y \to X^\nu$ and a prime divisor E on Y such that the discrepancy coefficient $a(E, X^\nu, \Theta) = -1$ and $\nu \circ f(E) = W$.

For our purposes, it is very convenient to introduce the notion of *slc strata* for semi log canonical pairs.

DEFINITION 2.5 (Slc stratum). Let (X, Δ) be a semi log canonical pair. A subvariety W of X is called an *slc stratum* of the pair (X, Δ) if W is a semi log canonical center with respect to (X, Δ) or W is an irreducible component of X.

In this paper, we mainly discuss *non-normal* algebraic varieties and divisors on them. We have to be careful when we use Weil divisors on non-normal varieties.

2.6 (Divisorial sheaves). Let D be a Weil divisor on a semi log canonical pair (X, Δ) whose support does not contain any irreducible components of the conductor of X. Then the reflexive sheaf $\mathcal{O}_X(D)$ is well-defined. In this paper, we do not discuss Weil divisors whose supports contain some irreducible components of the conductor of X. Note that if D is a Cartier divisor on X then $\mathcal{O}_X(D)$ is a well-defined invertible sheaf on X without any assumptions on the support of D.

For the details, we recommend the reader to see [Ko13b, 5.6] and [K+92, Chapter 16] by Alesio Corti. See also [Ha94, Sections 1 and 2]. The remarks in 2.6 are sufficient for our purposes in this paper. So we do not pursue the definition of $\mathcal{O}_X(D)$ any more.

Next, let us recall the definition of *nef and log big* $\mathbb{R}$-*Cartier* $\mathbb{R}$-*divisors* on semi log canonical pairs. For the details of big $\mathbb{R}$-Cartier $\mathbb{R}$-divisors, see Section 7.

DEFINITION 2.7 (Nef and log big divisors on slc pairs). Let (X, Δ) be a semi log canonical pair and let $\pi : X \to S$ be a proper surjective morphism onto an algebraic variety S. Let D be a π-nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then D is *nef and log big over* S *with respect to* (X, Δ) if $D|_W$ is big over S for every slc stratum of (X, Δ).

Finally, let us recall the definition of *simple normal crossing pairs*. In [Ko13b] and [BP13], a simple normal crossing pair is called a *semi-snc pair*.

DEFINITION 2.8 (Simple normal crossing pairs). We say that the pair (X, D) is *simple normal crossing* at a point $a \in X$ if X has a Zariski open neighborhood U of a that can be embedded in a smooth variety Y, where Y has regular system of parameters $(x_1, \cdots, x_p, y_1, \cdots, y_r)$ at $a = 0$ in which U is defined by a monomial equation

$$x_1 \cdots x_p = 0$$

and

$$D = \sum_{i=1}^{r} \alpha_i (y_i = 0)|_U, \quad \alpha_i \in \mathbb{R}.$$

We say that (X, D) is a *simple normal crossing pair* if it is simple normal crossing at every point of X. We say that a simple normal crossing pair (X, D) is *embedded* if there exists a closed embedding $\iota : X \to M$, where M is a smooth variety of dim $X + 1$. If $(X, 0)$ is a simple normal crossing pair, then X is called a *simple normal crossing variety*. If X is a simple normal crossing variety, then X has only Gorenstein singularities. Thus, it has an invertible dualizing sheaf ω_X. Therefore, we can define the *canonical divisor* K_X such that $\omega_X \simeq \mathcal{O}_X(K_X)$. It is a Cartier divisor on X and is well-defined up to linear equivalence.

Let X be a simple normal crossing variety and let $X = \bigcup_{i \in I} X_i$ be the irreducible decomposition of X. A *stratum* of X is an irreducible component of $X_{i_1} \cap \cdots \cap X_{i_k}$ for some $\{i_1, \cdots, i_k\} \subset I$.

Let X be a simple normal crossing variety and let D be a Cartier divisor on X. If (X, D) is a simple normal crossing pair and D is reduced, then D is called a *simple normal crossing divisor* on X.

Let (X, D) be a simple normal crossing pair such that D is a subboundary $\mathbb{R}$-divisor on X. Let $\nu : X^\nu \to X$ be the normalization. We define Ξ by the formula

$$K_{X^\nu} + \Xi = \nu^*(K_X + D).$$

Then a *stratum* of (X, D) is an irreducible component of X or the ν-image of a log canonical center of (X^ν, Ξ). We note that (X^ν, Ξ) is sub log canonical. When $D = 0$, this definition is

compatible with the above definition of the strata of X. When D is a boundary $\mathbb{R}$-divisor, W is a stratum of (X, D) if and only if W is an slc stratum of (X, D). Note that (X, D) is semi log canonical if D is a boundary $\mathbb{R}$-divisor.

The author learned the following interesting example from Kento Fujita (cf. [Ko13b, Remark 1.9]).

EXAMPLE 2.9. Let $X_1 = \mathbb{P}^2$ and let C_1 be a line on X_1. Let $X_2 = \mathbb{P}^2$ and let C_2 be a smooth conic on X_2. We fix an isomorphism $\tau : C_1 \to C_2$. By gluing X_1 and X_2 along $\tau : C_1 \to C_2$, we obtain a simple normal crossing surface X such that $-K_X$ is ample (cf. [Ft12]). We can check that X can not be embedded into any smooth varieties as a simple normal crossing divisor.

The reader can find various vanishing theorems and a generalization of the Fujita–Kawamata semi-positivity theorem for simple normal crossing pairs in [F09c], [F12c], [F13], and [FF11] (see also [FFS13]). All of them depend on the theory of the mixed Hodge structures on cohomology groups with compact support.

3. Supplements to the theory of quasi-log varieties

In this section, let us give supplementary arguments to the theory of quasi-log varieties (cf. [Am03]). For the details of the theory of quasi-log varieties, see [F09c, Chapter 3] and [F11a].

Let us introduce the notion of *globally embedded simple normal crossing pairs*, which is much easier than the notion of embedded simple normal crossing pairs from some technical viewpoints. It is obvious that a globally embedded simple normal crossing pair is an embedded simple normal crossing pair.

DEFINITION 3.1 (Globally embedded simple normal crossing pairs). Let Y be a simple normal crossing divisor on a smooth variety M and let B be an $\mathbb{R}$-divisor on M such that $\mathrm{Supp}(B + Y)$ is a simple normal crossing divisor and that B and Y have no common irreducible components. We set $\Delta_Y = B|_Y$ and consider the pair (Y, Δ_Y). We call (Y, Δ_Y) a *globally embedded simple normal crossing pair*.

Let us recall the definition of *quasi-log varieties with only qlc singularities*.

DEFINITION 3.2 (Quasi-log varieties with only qlc singularities). A *quasi-log variety with only qlc singularities* is a (not necessarily equidimensional) variety X with an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor ω, and a finite collection $\{C\}$ of reduced and irreducible subvarieties of X such that there is a proper morphism $f : (Y, \Delta_Y) \to X$ from a globally embedded simple normal crossing pair satisfying the following properties.

(1) $f^*\omega \sim_{\mathbb{R}} K_Y + \Delta_Y$ such that Δ_Y is a subboundary $\mathbb{R}$-divisor.
(2) There is an isomorphism
$$\mathcal{O}_X \simeq f_*\mathcal{O}_Y(\lceil -\Delta_Y^{<1} \rceil).$$
(3) The collection of subvarieties $\{C\}$ coincides with the image of the (Y, Δ_Y)-strata.

We simply write $[X, \omega]$ to denote the above data
$$\left(X, \omega, f : (Y, \Delta_Y) \to X \right)$$
if there is no risk of confusion. The subvarieties C are called the *qlc centers* of $[X, \omega]$, and $f : (Y, \Delta_Y) \to X$ is called a *quasi-log resolution* of $[X, \omega]$. We sometimes simply say that $[X, \omega]$

is a *qlc pair*, or the pair $[X, \omega]$ is *qlc*. We call ω the *quasi-log canonical class* of $[X, \omega]$. Note that ω is defined up to $\mathbb{R}$-linear equivalence.

The notion of *crepant log structures* introduced by Kollár–Kovács, which is a very special but important case of quasi-log structures, is also useful for various applications (see, for example, [Ko13b, 4.4 Crepant log structures]). For a prototype of quasi-log structures and crepant log structures, see [F99, Theorem 4.1].

Let us recall the following very useful lemma. By this lemma, it is sufficient to treat globally embedded simple normal crossing pairs for the theory of qlc pairs.

LEMMA 3.3 (cf. [F09c, Proposition 3.57]). *Let (Y, Δ_Y) be an embedded simple normal crossing pair such that Δ_Y is a subboundary $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y. Let M be the ambient space of Y. Then we can construct a sequence of blow-ups*

$$M_k \xrightarrow{\sigma_k} M_{k-1} \xrightarrow{\sigma_{k-1}} \cdots \xrightarrow{\sigma_0} M_0 = M$$

with the following properties.

(1) *$\sigma_{i+1} : M_{i+1} \to M_i$ is the blow-up along a smooth irreducible component of $\operatorname{Supp} \Delta_{Y_i}$ for every i.*

(2) *We set $Y_0 = Y$ and $\Delta_{Y_0} = \Delta_Y$. We define Y_{i+1} as the strict transform of Y_i on M_{i+1} for every i. Note that Y_i is a simple normal crossing divisor on M_i for every i.*

(3) *We define $\Delta_{Y_{i+1}}$ by*

$$K_{Y_{i+1}} + \Delta_{Y_{i+1}} = \sigma_{i+1}^*(K_{Y_i} + \Delta_{Y_i})$$

for every i.

(4) *There exists an $\mathbb{R}$-divisor B on M_k such that $\operatorname{Supp}(B + Y_k)$ is a simple normal crossing divisor on M_k, B and Y_k have no common irreducible components, and $B|_{Y_k} = \Delta_{Y_k}$.*

(5) *$\sigma_* \mathcal{O}_{Y_k}(\lceil -\Delta_{Y_k}^{<1} \rceil) \simeq \mathcal{O}_Y(\lceil -\Delta_Y^{<1} \rceil)$ where $\sigma = \sigma_1 \circ \cdots \circ \sigma_k : M_k \to M$.*

Proof. All we have to do is to check the property (5). The other properties are obvious by the construction of blow-ups. By

$$K_{Y_{i+1}} + \Delta_{Y_{i+1}} = \sigma_{i+1}^*(K_{Y_i} + \Delta_{Y_i}),$$

we have

$$\begin{aligned} K_{Y_{i+1}} =& \sigma_{i+1}^*(K_{Y_i} + \{\Delta_{Y_i}\} + \Delta_{Y_i}^{=1}) \\ &+ \sigma_{i+1}^* \lfloor \Delta_{Y_i}^{<1} \rfloor - \lfloor \Delta_{Y_{i+1}}^{<1} \rfloor - \Delta_{Y_{i+1}}^{=1} - \{\Delta_{Y_{i+1}}\}. \end{aligned}$$

We can easily check that $\sigma_{i+1}^* \lfloor \Delta_{Y_i}^{<1} \rfloor - \lfloor \Delta_{Y_{i+1}}^{<1} \rfloor$ is an effective σ_{i+1}-exceptional Cartier divisor on Y_{i+1}. This is because $a(\nu, Y_i, \{\Delta_{Y_i}\} + \Delta_{Y_i}^{=1}) = -1$ for a prime divisor ν over Y_i implies $a(\nu, Y_i, \Delta_{Y_i}) = -1$ (cf. [Ko13b, Definition 2.4]). Thus, we can write

$$\sigma_{i+1}^* \lceil -\Delta_{Y_i}^{<1} \rceil + E = \lceil -\Delta_{Y_{i+1}}^{<1} \rceil$$

where E is an effective σ_{i+1}-exceptional Cartier divisor on Y_{i+1}. This implies that

$$\sigma_{i+1*} \mathcal{O}_{Y_{i+1}}(\lceil -\Delta_{Y_{i+1}}^{<1} \rceil) \simeq \mathcal{O}_{Y_i}(\lceil -\Delta_{Y_i}^{<1} \rceil)$$

for every i. Thus, $\sigma_* \mathcal{O}_{Y_k}(\lceil -\Delta_{Y_k}^{<1} \rceil) \simeq \mathcal{O}_Y(\lceil -\Delta_Y^{<1} \rceil)$. $\square$

Although we do not need the following theorem explicitly in this paper, it is very important and useful. It helps the reader to understand quasi-log structures.

THEOREM 3.4 (cf. [Am03, Proposition 4.8], [F09c, Theorem 3.45]). *Let $[X, \omega]$ be a qlc pair. Then we have the following properties.*

(i) *The intersection of two qlc centers is a union of qlc centers.*

(ii) *For any point $P \in X$, the set of all qlc centers passing through P has a unique minimal element W. Moreover, W is normal at P.*

By Theorem 1.2 (5) and Theorem 3.4, we have an obvious corollary.

COROLLARY 3.5. *Let (X, Δ) be a quasi-projective semi log canonical pair and let W be a minimal slc stratum of the pair (X, Δ). Then W is normal.*

The following result is a key lemma for the proof of Theorem 3.4 (ii). We contain it for the reader's convenience.

LEMMA 3.6. *Let $f : X \to Y$ be a proper surjective morphism from a simple normal crossing variety X to an irreducible variety Y. Assume that every stratum of X is dominant onto Y and that $f_* \mathcal{O}_X \simeq \mathcal{O}_Y$. Then Y is normal.*

Proof. Let $\nu : Y^\nu \to Y$ be the normalization. By applying [BM12, Theorem 1.5] to the graph of the rational map $\nu^{-1} \circ f : X \dashrightarrow Y^\nu$, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
Z & \xrightarrow{\ \alpha\ } & X \\
{\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle f} \\
Y^\nu & \xrightarrow[\ \nu\ ]{} & Y
\end{array}
$$

such that

(i) Z *is a simple normal crossing variety, and*

(ii) *there is a Zariski open set U (resp. V) of Z (resp. X) such that U (resp. V) contains the generic point of any stratum of Z (resp. X) and that $\alpha|_U : U \to V$ is an isomorphism.*

Then it is easy to see that $\alpha_* \mathcal{O}_Z \simeq \mathcal{O}_X$. Therefore,

$$\mathcal{O}_Y \simeq f_* \mathcal{O}_X \simeq f_* \alpha_* \mathcal{O}_Z \simeq \nu_* \beta_* \mathcal{O}_Z \supset \nu_* \mathcal{O}_{Y^\nu}.$$

This implies that $\mathcal{O}_Y \simeq \nu_* \mathcal{O}_{Y^\nu}$. So, we obtain that Y is normal. $\qquad\square$

We recommend the reader to see [F11a] for the basic properties of qlc pairs. Note that adjunction and vanishing theorem (see, for example, [F11a, Theorem 3.6]) for qlc pairs is one of the most important properties of qlc pairs.

4. Proof of the main theorem

Let us start the proof of the main theorem: Theorem 1.2.

Proof of Theorem 1.2. We divide the proof into several steps. We repeatedly use [BM12], [BP13], and [Ko13b, 10.4. Semi-log-resolution]. We prove Theorem 1.2 simultaneously with Remark 1.5.

STEP 1. Let X^{ncp} denote the open subset of X consisting of smooth points, double normal crossing points and pinch points. Then, by [BM12, Theorem 1.18], there exists a morphism $f_1 : X_1 \to X$ which is a finite composite of admissible blow-ups, such that

(i) $X_1 = X_1^{\mathrm{ncp}}$,

(ii) f_1 is an isomorphism over X^{ncp}, and

(iii) $\mathrm{Sing}\, X_1$ maps birationally onto the closure of $\mathrm{Sing}\, X^{\mathrm{ncp}}$.

Since X satisfies Serre's S_2 condition and $\mathrm{codim}_X(X \setminus X^{\mathrm{ncp}}) \geqslant 2$, we can easily check that $f_{1*}\mathcal{O}_{X_1} \simeq \mathcal{O}_X$.

REMARK 4.1 (see [Ko13b, Corollary 10.55]). In Step 1, we assume that the irreducible components of X have no self-intersection in codimension one. Let $X^{\mathrm{snc}2}$ be the open subset of X which has only smooth points and simple normal crossing points of multiplicity $\leqslant 2$. Then there is a projective birational morphism $f_1 : X_1 \to X$ such that

(i) $X_1 = X_1^{\mathrm{snc}2}$,

(ii) f_1 is an isomorphism over $X^{\mathrm{snc}2}$, and

(iii) $\mathrm{Sing}\, X_1$ maps birationally onto the closure of $\mathrm{Sing}\, X^{\mathrm{snc}2}$.

STEP 2 (cf. [Ko13b, Proposition 10.59]). By the construction in Step 1, X_1 is quasi-projective. Therefore, we can embed X_1 into $\mathbb{P}^N$. We pick a finite set $W \subset X_1$ such that each irreducible component of $\mathrm{Sing}\, X_1$ contains a point of W. We take a sufficiently large positive integer d such that the scheme theoretic base locus of $|\mathcal{O}_{\mathbb{P}^N}(d) \otimes \mathcal{I}_{\overline{X}_1}|$ is X_1 near every point of W, where $\overline{X}_1$ is the closure of X_1 in $\mathbb{P}^N$ and $\mathcal{I}_{\overline{X}_1}$ is the defining ideal sheaf of $\overline{X}_1$ in $\mathbb{P}^N$. By taking a complete intersection of $(N - \dim X_1 - 1)$ general members of $|\mathcal{O}_{\mathbb{P}^N}(d) \otimes \mathcal{I}_{\overline{X}_1}|$, we obtain $Y \supset X_1$ such that Y is smooth at every point of W. Note that we used the fact that X_1 has only hypersurface singularities near W. By replacing Y with $Y \setminus (\overline{X}_1 \setminus X_1)$, we may assume that X_1 is closed in Y.

STEP 3. Let $g : Y_2 \to Y$ be a resolution, which is a finite composite of admissible blow-ups. Let X_2 be the strict transform of X_1 on Y_2. Note that $f_2 = g|_{X_2} : X_2 \to X_1$ is an isomorphism over the generic point of any irreducible component of $\mathrm{Sing}\, X_1$ because Y is smooth at every point of W.

STEP 4. Apply [BM12, Theorem 1.18] to $X_2 \subset Y_2$ (see also Proof of Theorem 1.18 in [BM12]). We obtain a projective birational morphism $g_3 : Y_3 \to Y_2$, which is a finite composite of admissible blow-ups, from a smooth variety Y_3 with the following properties (i), (ii), and (iii). Note that X_3 is the strict transform of X_2 on Y_3 and $f_3 = g_3|_{Y_3} : X_3 \to X_2$.

(i) $X_3 = X_3^{\mathrm{ncp}}$,

(ii) f_3 is an isomorphism over X_2^{ncp}, and

(iii) $\mathrm{Sing}\, X_3$ maps birationally onto the closure of $\mathrm{Sing}\, X_2^{\mathrm{ncp}}$.

Let E be an irreducible component of $\mathrm{Sing}\, X_3$. If $E \to (f_2 \circ f_3)(E)$ is not birational, then we take a blow-up of Y_3 along E and replace X_3 with its strict transform. After finitely many blow-ups, we may assume that X_3 satisfies (i) and

(iv) $\mathrm{Sing}\, X_3$ maps birationally onto $\mathrm{Sing}\, X_1$ by $f_2 \circ f_3$.

From now on, we do not require the properties (ii) and (iii) above. By the above constructions, we can easily check that $(f_2 \circ f_3)_* \mathcal{O}_{X_3} \simeq \mathcal{O}_{X_1}$ since X_1 satisfies Serre's S_2 condition.

REMARK 4.2. When X_1 is a simple normal crossing variety, we apply Szabó's resolution lemma to the pair (Y_2, X_2) in Step 4. Then we have the following properties.

(i) $X_3 = X_3^{\mathrm{snc}}$, and

(ii) f_3 is an isomorphism over X_2^{snc}.

By taking more blow-ups if necessary, we may assume (i) and

(iv) $\mathrm{Sing}\, X_3$ maps birationally onto $\mathrm{Sing}\, X_1$ by $f_2 \circ f_3$.

STEP 5. We set

$$K_{X_1} + \Delta_1 = f_1^*(K_X + \Delta)$$

and

$$K_{X_3} + \Delta_3 = (f_1 \circ f_2 \circ f_3)^*(K_X + \Delta).$$

Note that X_1 and X_3 have only Gorenstein singularities. Therefore, Δ_1 and Δ_3 are $\mathbb{R}$-Cartier $\mathbb{R}$-divisors. We also note that the support of Δ_1 (resp. Δ_3) does not contain any irreducible components of the conductor of X_1 (resp. X_3). Let $\nu_3 : X_3^\nu \to X_3$ be the normalization. We set

$$K_{X_3^\nu} + \Theta_3 = \nu_3^*(K_{X_3} + \Delta_3).$$

Then the pair (X_3^ν, Θ_3) is sub log canonical because (X, Δ) is semi log canonical.

STEP 6. Let X_3^{snc} denote the simple normal crossing locus of X_3. Let C be an irreducible component of $X_3 \setminus X_3^{\mathrm{snc}}$. Then C is smooth and $\dim C = \dim X_3 - 1$. Let $\alpha : W \to Y_3$ be the blow-up along C and let V be $\alpha^{-1}(X_3)$ with the reduced structure. Then we can directly check that $\beta_* \mathcal{O}_V \simeq \mathcal{O}_{X_3}$ where $\beta = \alpha|_V$. We set

$$K_V + \Delta_V = \beta^*(K_{X_3} + \Delta_3).$$

Note that $K_V = \beta^* K_{X_3}$ and $\Delta_V = \beta^* \Delta_3$. Let $\nu : V^\nu \to V$ be the normalization of V. Then (V^ν, Θ_{V^ν}) is sub log canonical, where $K_{V^\nu} + \Theta_{V^\nu} = \nu^*(K_V + \Delta_V)$. When C is a double normal crossing points locus, it is almost obvious. If C is a pinch points locus, then it follows from Lemma 4.4 below. By repeating this process finitely many times, we obtain a projective birational morphism $g_4 : Y_4 \to Y_3$ from a smooth variety Y_4 and a simple normal crossing divisor X_4 on Y_4 with the following properties.

(i) $f_{4*} \mathcal{O}_{X_4} \simeq \mathcal{O}_{X_3}$ where $f_4 = g_4|_{X_4}$.
(ii) We set

$$K_{X_4} + \Delta_4 = f_4^*(K_{X_3} + \Delta_3).$$

Then (X_4^ν, Θ_4) is sub log canonical where $\nu_4 : X_4^\nu \to X_4$ is the normalization and $K_{X_4^\nu} + \Theta_4 = \nu_4^*(K_{X_4} + \Delta_4)$.

REMARK 4.3. We can skip Step 6 if $X_3 = X_3^{\mathrm{snc}}$. Therefore, we can make $h : Z \to X$ birational when the irreducible components of X have no self-intersection in codimension one (see Remarks 4.1 and 4.2). This is because f_5 in Step 7 below is always birational.

STEP 7 (cf. [BP13, Section 4]). Let U be the largest Zariski open subset of X_4 such that $(U, \Delta_4|_U)$ is a simple normal crossing pair. Then there is a projective birational morphism $g_5 : Y_5 \to Y_4$ given by a composite of blow-ups with smooth centers with the following properties.

(i) Let X_5 be the strict transform of X_4 on Y_5. Then $f_5 = g_5|_{X_5} : X_5 \to X_4$ is an isomorphism over U.
(ii) $(X_5, f_{5*}^{-1}\Delta_4 + \mathrm{Exc}(f_5))$ is a simple normal crossing pair, where $\mathrm{Exc}(f_5)$ is the exceptional locus of f_5. By the construction, we can check that $f_{5*} \mathcal{O}_{X_5} \simeq \mathcal{O}_{X_4}$.

STEP 8. We set $M = Y_5$, $Z = X_5$, and $h = f_1 \circ f_2 \circ f_3 \circ f_4 \circ f_5 : Z = X_5 \to X$. Note that M is a smooth quasi-projective variety and Z is a simple normal crossing divisor on M. We set

$$K_Z + \Delta_Z = h^*(K_X + \Delta).$$

Then (Z, Δ_Z) is a simple normal crossing pair by the above construction. Note that Δ_Z is a subboundary $\mathbb{R}$-divisor on Z.

For the proof of Theorem 1.2, we have to see that $h_* \mathcal{O}_Z(\lceil -\Delta_Z^{<1} \rceil) \simeq \mathcal{O}_X$. We will prove it in the subsequent steps.

STEP 9. It is obvious that

$$f_{1*} \mathcal{O}_{X_1}(\lceil -\Delta_1^{<1} \rceil) \simeq \mathcal{O}_X.$$

This is because $\lceil -\Delta_1^{<1} \rceil$ is effective and f_1-exceptional. Note that $f_{1*} \mathcal{O}_{X_1} \simeq \mathcal{O}_X$.

STEP 10. We can easily check that

$$\mathcal{O}_{X_1} \subset (f_2 \circ f_3)_* \mathcal{O}_{X_3}(\lceil -\Delta_3^{<1} \rceil) \subset \mathcal{O}_{X_1}(\lceil -\Delta_1^{<1} \rceil).$$

We note that $\lceil -\Delta_3^{<1} \rceil$ is effective. Therefore,

$$(f_1 \circ f_2 \circ f_3)_* \mathcal{O}_{X_3}(\lceil -\Delta_3^{<1} \rceil) \simeq \mathcal{O}_X.$$

STEP 11. We use the notation in Step 6. Let $\alpha : W \to Y_3$ be the blow-up in Step 6. Note that $\Delta_V = \beta^* \Delta_3$ and $K_V = \beta^* K_{X_3}$. Therefore, we have

$$0 \leqslant \lceil -\Delta_V^{<1} \rceil \leqslant \beta^*(\lceil -\Delta_3^{<1} \rceil).$$

See the description of the blow-up in Lemma 4.4 when $\alpha : W \to Y_3$ is a blow-up along a pinch points locus. Thus

$$\mathcal{O}_{X_3} \subset \beta_* \mathcal{O}_V(\lceil -\Delta_V^{<1} \rceil) \subset \mathcal{O}_{X_3}(\lceil -\Delta_3^{<1} \rceil)$$

since $\beta_* \mathcal{O}_V \simeq \mathcal{O}_{X_3}$. Therefore, we obtain that

$$\mathcal{O}_{X_3} \subset f_{4*} \mathcal{O}_{X_4}(\lceil -\Delta_4^{<1} \rceil) \subset \mathcal{O}_{X_3}(\lceil -\Delta_3^{<1} \rceil).$$

This implies that $(f_1 \circ f_2 \circ f_3 \circ f_4)_* \mathcal{O}_{X_4}(\lceil -\Delta_4^{<1} \rceil) \simeq \mathcal{O}_X$.

STEP 12. It is easy to see that

$$\mathcal{O}_{X_4} \subset f_{5*} \mathcal{O}_{X_5}(\lceil -\Delta_5^{<1} \rceil) \subset \mathcal{O}_{X_4}(\lceil -\Delta_4^{<1} \rceil)$$

because f_5 is a birational map. Thus

$$(f_1 \circ f_2 \circ f_3 \circ f_4 \circ f_5)_* \mathcal{O}_{X_5}(\lceil -\Delta_5^{<1} \rceil) \simeq \mathcal{O}_X.$$

So we obtain $f_* \mathcal{O}_Z(\lceil -\Delta_Z^{<1} \rceil) \simeq \mathcal{O}_X$.

STEP 13. By the construction, it is easy to see that $K_Z + \Delta_Z \sim_{\mathbb{R}} h^*(K_X + \Delta)$ and that W is an slc stratum of (X, Δ) if and only if W is the h-image of some stratum of the simple normal crossing pair (Z, Δ_Z) (cf. Lemma 4.4).

STEP 14. By applying Lemma 3.3, we may assume that there is a subboundary $\mathbb{R}$-Cartier $\mathbb{R}$-divisor B on M such that B and Z have no common irreducible components, $\mathrm{Supp}(B + Z)$ is a simple normal crossing divisor on M, and $B|_Z = \Delta_Z$ after taking some blow-ups.

Therefore, $h : (Z, \Delta_Z) \to X$ gives the pair $[X, K_X + \Delta]$ a quasi-log structure with the desired properties (1), (2), (3), (4), and (5). $\qquad\square$

The following easy local calculation played a crucial role in the proof of Theorem 1.2.

LEMMA 4.4. *We consider*

$$V = (x_1^2 - x_2^2 x_3 = 0) \subset \mathbb{A}^{n+1} = \mathrm{Spec}\,\mathbb{C}[x_1, \cdots, x_{n+1}]$$

and

$$C = (x_1 = x_2 = 0) \subset V \subset \mathbb{A}^{n+1}.$$

Let $\varphi : Bl_C \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ be the blow-up whose center is C. Let $W \simeq C \times \mathbb{P}^1$ be the exceptional divisor of the above blow-up and let $\pi = \varphi|_W : W \to C$ be the natural projection. We set $D = V'|_W$ where V' is the strict transform of V on $Bl_C \mathbb{A}^{n+1}$. Assume that B is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on C such that $(D, \pi^* B|_D)$ is sub log canonical. Then the pair $(W, D + \pi^* B)$ is sub log canonical.

Furthermore, we obtain the following description. A closed subset Q of C is the π-image of some lc center of $(W, D + \pi^* B)$ if and only if $Q = C$ or Q is the $\pi|_D$-image of some lc center of $(D, \pi^* B|_D)$.

Proof. We can check that $K_W + D = \pi^*(K_V|_C)$ because

$$K_{Bl_C \mathbb{A}^{n+1}} + V' + W = \varphi^*(K_{\mathbb{A}^{n+1}} + V).$$

Therefore, $K_W + D + \pi^* B = \pi^*(K_V|_C + B)$. Note that it is easy to see that D is a smooth divisor on W and that $\pi|_D : D \to C$ is a finite morphism with $\deg \pi|_D = 2$ which ramifies only over A, where

$$A = (x_1 = x_2 = x_3 = 0) \subset C \subset V \subset \mathbb{A}^{n+1}.$$

By adjunction, $K_D = (\pi|_D)^*(K_V|_C)$. We consider the following base change diagram

$$
\begin{array}{ccc}
W & \xleftarrow{\ q\ } & \widetilde{W} \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \\
C & \xleftarrow[\ \pi|_D\ ]{} & D
\end{array}
$$

where $\widetilde{W} = W \times_C D$. Then we obtain

$$K_{\widetilde{W}} - q^* \left(\frac{1}{2} \pi^* A \right) + q^* D = p^* K_D$$

by $K_W + D = \pi^*(K_V|_C)$ and $K_D = (\pi|_D)^*(K_V|_C)$, and have

$$K_{\widetilde{W}} - q^* \left(\frac{1}{2} \pi^* A \right) + q^* D + q^* \pi^* B = p^*(K_D + \pi^* B|_D). \qquad (\heartsuit)$$

Note that $q^* D = D_1 + D_2$ such that D_1 and D_2 are sections of $p : \widetilde{W} \to D$. By the construction, we can check that $D_1|_{D_2} = q^* \left(\frac{1}{2} \pi^* A \right)|_{D_2}$ and $D_2|_{D_1} = q^* \left(\frac{1}{2} \pi^* A \right)|_{D_1}$. We also note that p is smooth and $p : D_1 \cap D_2 \simeq \frac{1}{2}(\pi|_D)^* A$. We take a resolution of singularities $\alpha : D^\dagger \to D$ of the pair $(D, \pi^*(A + B)|_D)$, which is a finite composite of blow-ups whose centers are smooth. We consider the base change of $p : \widetilde{W} \to D$ by α.

$$
\begin{array}{ccc}
\widetilde{W} & \longleftarrow & \widetilde{W} \times_D D^\dagger \\
{\scriptstyle p}\downarrow & & \downarrow \\
D & \xleftarrow[\ \alpha\ ]{} & D^\dagger
\end{array}
$$

Then $W^\dagger = \widetilde{W} \times_D D^\dagger$ is smooth since p is smooth. By the above construction, we can easily see that all the discrepancy coefficients of $(\widetilde{W}, -q^*\left(\frac{1}{2}\pi^*A\right) + q^*D + q^*\pi^*B)$ are $\geqslant -1$ since $(D, \pi^*B|_D)$ is sub log canonical and the equation ($\heartsuit$) holds. Therefore, $(\widetilde{W}, -q^*\left(\frac{1}{2}\pi^*A\right) + q^*D + q^*\pi^*B)$ is sub log canonical. Since

$$K_{\widetilde{W}} - q^*\left(\frac{1}{2}\pi^*A\right) + q^*D + q^*\pi^*B = q^*(K_W + D + \pi^*B),$$

we have that $(W, D + \pi^*B)$ has only sub log canonical singularities.

The description of the π-images of lc centers of $(W, D + \pi^*B)$ is almost obvious by the above discussions. $\qquad\square$

5. Proofs of the fundamental theorems

In this section, we prove the theorems in Section 1. First, let us recall Kollár's double covering trick.

Lemma 5.1 (A natural double cover due to Kollár). *Let (X, Δ) be a semi log canonical pair. Then we can construct a finite morphism $p : \widetilde{X} \to X$ with the following properties.*

(1) *Let X^0 be the largest Zariski open subset whose singularities are double normal crossing points only. Then*

$$p^0 = p|_{p^{-1}(X^0)} : \widetilde{X}^0 := p^{-1}(X^0) \to X^0$$

is an étale double cover.

(2) *$\widetilde{X}$ satisfies Serre's S_2 condition, p is étale in codimension one, the normalization of $\widetilde{X}$ is a disjoint union of two copies of the normalization of X.*

(3) *The irreducible components of $\widetilde{X}$ are smooth in codimension one.*

In particular, $(\widetilde{X}, \widetilde{\Delta})$ is semi log canonical where

$$K_{\widetilde{X}} + \widetilde{\Delta} = p^*(K_X + \Delta).$$

For the construction and related topics, see [Ko13b, 5.23]. Let us start the proofs of the fundamental theorems in Section 1.

Proof of Theorem 1.7 *and* Theorem 1.10. It is sufficient to prove Theorem 1.10. This is because Theorem 1.7 is a special case of Theorem 1.10. By Lemma 5.1, we can take a double cover $p : \widetilde{X} \to X$. Since $\mathcal{O}_X(D)$ is a direct summand of $p_*\mathcal{O}_{\widetilde{X}}(p^*D)$, we may assume that the irreducible components of X are smooth in codimension one by replacing X with $\widetilde{X}$. Without loss of generality, we may assume that S is affine by shrinking S. Therefore, X is quasi-projective. By Theorem 1.2, we can construct a quasi-log resolution $h : Z \to X$. Note that we may assume that h is birational by Remark 1.5. We may further assume that $\text{Supp } h^*D \cup \text{Supp } \Delta_Z$ is a simple normal crossing divisor on Z by [BP13, Theorem 1.4] when D is not a Cartier divisor. By the construction,

$$h^*D + \lceil -\Delta_Z^{\leq 1} \rceil - (K_Z + \Delta_Z^{=1} + \{\Delta_Z\}) \sim_{\mathbb{R}} h^*(D - (K_X + \Delta)).$$

If D is Cartier, then

$$R^i\pi_*h_*\mathcal{O}_Z(h^*D + \lceil -\Delta_Z^{\leq 1} \rceil) \simeq R^i\pi_*\mathcal{O}_X(D) = 0.$$

for every $i > 0$ by [F09c, Theorem 2.47 or Theorem 3.38]. We note that $h_*\mathcal{O}_Z(h^*D + \lceil -\Delta_Z^{\leq 1} \rceil) \simeq \mathcal{O}_X(D)$. From now on, we assume that D is not Cartier. Let $\{h^*D\} = \sum_i b_i B_i$ and $\{\Delta_Z\} = \sum_i c_i B_i$ be the irreducible decompositions. If $c_i \geqslant b_i$, then we set $d_i = 0$ and $e_i = c_i - b_i \geqslant 0$. If $c_i < b_i$, then we set $d_i = 1$ and $e_i = c_i + 1 - b_i < 1$. We define $E = \lceil -\Delta_Z^{\leq 1} \rceil + \sum_i d_i B_i$ and $F = \sum_i e_i B_i$. Then we have

$$\lfloor h^*D \rfloor + E - (K_Z + \Delta_Z^{=1} + F) \sim_{\mathbb{R}} h^*(D - (K_X + \Delta)).$$

By the construction, E is an effective h-exceptional divisor on X and $\{F\} = 0$. Note that E and $\lfloor h^*D \rfloor$ are both Cartier divisors on Z. This is because $\operatorname{Supp} h^*D \cup \operatorname{Supp} \Delta_Z$ is a simple normal crossing divisor on Z and h^*D and Δ_Z are $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on Z. By [F09c, Theorem 2.47 or Theorem 3.38], we obtain that

$$R^i \pi_* h_* \mathcal{O}_Z(\lfloor h^*D \rfloor + E) = 0$$

for every $i > 0$. Therefore, $R^i \pi_* \mathcal{O}_X(D) = 0$ for every $i > 0$ since $h_*\mathcal{O}_Z(\lfloor h^*D \rfloor + E) \simeq \mathcal{O}_X(D)$. $\square$

Proof of Theorem 1.8. We take a Cartier divisor L on X such that $\mathcal{O}_X(L) \simeq \mathcal{L}$. Without loss of generality, we may assume that the supports of the Weil divisor K_X and L do not contain any irreducible components of the conductor of X. Since $(K_X + L) - K_X$ is ample by the assumption, we obtain $H^i(X, \omega_X \otimes \mathcal{L}) = 0$ for every $i > 0$ by Theorem 1.7. Note that $\omega_X \otimes \mathcal{L} \simeq \mathcal{O}_X(K_X + L)$. $\square$

Proof of Corollary 1.9. Without loss of generality, we may assume that the support of the Weil divisor K_X does not contain any irreducible components of the conductor of X. By the assumption, $mK_X - K_X$ is ample if $m \geqslant 2$. Therefore, we obtain $H^i(X, \mathcal{O}_X(mK_X)) = 0$ for every $i > 0$ and $m \geqslant 2$ by Theorem 1.7. $\square$

Proof of Theorem 1.11. Since the claim is local, we may assume that S is quasi-projective by shrinking S. By Theorem 1.2, $[X, K_X + \Delta]$ has a quasi-log structure induced by the semi log canonical structure of (X, Δ) since X is quasi-projective. Therefore, $R^i \pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(D)) = 0$ for every $i > 0$ by [F09c, Theorem 3.39]. $\square$

REMARK 5.2. Let $\{C_i\}_{i \in I}$ be the set of slc strata of (X, Δ). We set

$$I_1 = \{i \in I \mid C_i \subset X'\}$$

and

$$I_2 = \{i \in I \mid C_i \not\subset X'\}.$$

Then, for the vanishing theorem: Theorem 1.11, the following weaker assumption is sufficient.

(♣) $D - (K_X + \Delta)$ is nef over S and $(D - (K_X + \Delta))|_{C_i}$ is big over S for every $i \in I_2$.

It is obvious by the proof given in [F09c, Theorem 3.39].

Proof of Theorem 1.12. It is obvious that the claim holds for $\pi_* \mathcal{O}_X(D)$. By Lemma 5.1, we can take a natural double cover $p : \widetilde{X} \to X$. Since $\mathcal{O}_X(D)$ is a direct summand of $p_* \mathcal{O}_{\widetilde{X}}(p^*D)$, we may assume that the irreducible components of X have no self-intersection in codimension one by replacing X with $\widetilde{X}$. Without loss of generality, we may assume that S is affine by shrinking S. Therefore, X is quasi-projective and we can apply Theorem 1.2. Let $h : Z \to X$ be a morphism constructed in Theorem 1.2. We may assume that h is birational (see Remark 1.5). Note that

$$h^*D + \lceil -\Delta_Z^{\leq 1} \rceil - (K_Z + \{\Delta_Z\} + \Delta_Z^{=1}) \sim_{\mathbb{R}} h^*(D - (K_X + \Delta))$$

is $(\pi \circ h)$-semi-ample. As in the proof of Theorem 1.7 and Theorem 1.10, we can write

$$\lfloor h^*D \rfloor + E - (K_Z + \Delta_Z^{=1} + F) \sim_{\mathbb{R}} h^*(D - (K_X + \Delta))$$

when D is not Cartier. Therefore, every associated prime of $R^i(\pi \circ h)_* \mathcal{O}_Z(\lfloor h^*D \rfloor + E)$ is the generic point of the π-image of some slc stratum of (X, Δ) for every i by Theorem 1.2 (5) (see, for example, [F09c, Theorem 2.39 (i)] and [F12c, Theorem 1.1 (i)]). Since

$$R^1\pi_* \mathcal{O}_X(D) \simeq R^1\pi_*(h_* \mathcal{O}_Z(\lfloor h^*D \rfloor + E))$$
$$\subset R^1(\pi \circ h)_* \mathcal{O}_Z(\lfloor h^*D \rfloor + E),$$

the claim holds for $R^1\pi_* \mathcal{O}_X(D)$. When D is Cartier, it is sufficient to replace $\lfloor h^*D \rfloor + E$ with $h^*D + \lceil -\Delta_Z^{\leq 1} \rceil$ in the above arguments. Let A be a sufficiently ample general effective Cartier divisor on X. By considering the short exact sequence

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D + A) \to \mathcal{O}_A(D + A) \to 0,$$

we obtain

$$R^i\pi_* \mathcal{O}_X(D) \simeq R^{i-1}(\pi|_A)_* \mathcal{O}_A(D + A)$$

for every $i \geq 2$ since $R^i\pi_* \mathcal{O}_X(D + A) = 0$ for $i \geq 1$. Note that

$$R^i\pi_* \mathcal{O}_X(D) \simeq R^{i-1}(\pi|_{A'})_* \mathcal{O}_{A'}(D + A')$$

holds for every $i \geq 2$ and every general member A' of $|A|$. By induction on dimension, every associated prime of $R^{i-1}(\pi|_A)_* \mathcal{O}_A(D+A)$ is the generic point of the $\pi|_A$-image of some slc stratum of $(A, \Delta|_A)$ for every i. Note that $(A, \Delta|_A)$ is semi log canonical with $(K_X + A + \Delta)|_A = K_A + \Delta|_A$ and that $h^*A = h_*^{-1}A$ and $\mathrm{Supp}(h_*^{-1}A + \Delta_Z)$ are simple normal crossing divisors on Z since A is general. The above statements also hold for any general member A' of $|A|$. Therefore, the claim holds for $R^i\pi_* \mathcal{O}_X(D)$, which is isomorphic to $R^{i-1}(\pi|_{A'})_* \mathcal{O}_{A'}(D + A')$ for every $i \geq 2$ and every general member A' of $|A|$. It completes the proof. $\square$

Proof of Theorem 1.13. By Theorem 1.2, $[X, K_X + \Delta]$ has a quasi-log structure. Note that W is an slc stratum of (X, Δ) if and only if W is a qlc center of $[X, K_X + \Delta]$ by Theorem 1.2 (5). Therefore, by adjunction for quasi-log varieties (see, for example, [F09c, Theorem 3.39] and [F11a, Theorem 3.6]), $[X', (K_X + \Delta)|_{X'}]$ has a natural quasi-log structure induced by the quasi-log structure of $[X, K_X + \Delta]$. Since $[X', (K_X + \Delta)|_{X'}]$ is a qlc pair, X' is semi-normal (see, for example, [F09c, Remark 3.33] and [F11a, Remark 3.2]). $\square$

Proof of Theorem 1.14. Without loss of generality, we may assume that S is affine by shrinking S. Therefore, we may assume that X is quasi-projective and $[X, K_X + \Delta]$ has a quasi-log structure by Theorem 1.2. By Theorem 1.13, $[X', (K_X + \Delta)|_{X'}]$ has a natural quasi-log structure induced by that of $[X, K_X + \Delta]$. Therefore, this theorem is a special case of the vanishing theorem for quasi-log varieties (see, for example, [F09c, Theorem 3.39 (ii)]). $\square$

REMARK 5.3. In Theorems 1.10, 1.11, 1.12, 1.13, and 1.14, if (X, Δ) is log canonical, then it is sufficient to assume that π is *proper*. This is because (X, Δ) has a natural quasi-log structure when (X, Δ) is log canonical (see, for example, [F09c, Example 3.42] and [F11a, Proposition 3.3]).

Proof of Theorem 1.15 *and* Theorem 1.16. By shrinking S, we may assume that S is affine and X is quasi-projective. Therefore, by applying Theorem 1.2, (X, Δ) has a natural quasi-log structure. Thus, by [F09c, Theorem 3.36] and [F09c, Theorem 4.1], we obtain that $\mathcal{O}_X(mD)$ is π-generated for every $m \gg 0$. $\square$

Proof of Theorem 1.18. The proof of [F11c, Theorem 15.1] works with only minor modifications if we adopt Theorem 1.11. We do not need the theory of quasi-log varieties for the proof of the rationality theorem. □

Proof of Theorem 1.19. The proof of [F11c, Theorem 16.1] works with only minor modifications by Theorem 1.18 and Theorem 1.15. Here we only give a supplementary argument on (1). Let R be a $(K_X + \Delta)$-negative extremal ray. Then there is a contraction morphism $\varphi_R : X \to Z$ over S associated to R (cf. (3)). Note that $-(K_X + \Delta)$ is φ_R-ample. Let $\nu : X^\nu \to X$ be the normalization. We set $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$. Then $-(K_{X^\nu} + \Theta)$ is $(\varphi_R \circ \nu)$-ample and $\varphi_R \circ \nu$ is nontrivial. Note that (X^ν, Θ) is log canonical. By [F11c, Theorem 18.2], we can find a rational curve C' on X^ν such that $-(K_{X^\nu} + \Theta) \cdot C' \leqslant 2 \dim X^\nu$ and $(\varphi_R \circ \nu)(C')$ is a point. We set $C = \nu(C')$. Then C is a rational curve on X and $-(K_X + \Delta) \cdot C \leqslant 2 \dim X$ such that $\varphi_R(C)$ is a point. Therefore, C is a desired curve in (1). □

We close this section with an important example. This example shows that we can not always run the minimal model program even for semi log canonical *surfaces*. For some related examples, see [Ko11a]. However, Kento Fujita ([Ft13]) establishes a variant of the minimal model program for *semi-terminal* pairs in order to construct *semi-terminal modifications* for quasi-projective *demi-normal* pairs. His arguments use not only Theorem 1.19, but also Kollár's gluing theory (see [Ko13b, Section 5]). For the details, see [Ft13].

EXAMPLE 5.4 (see [F09c, Example 3.76]). We consider the first projection $p : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. We take a blow-up $\mu : Z \to \mathbb{P}^1 \times \mathbb{P}^1$ at $(0, \infty)$. Let A_∞ (resp. A_0) be the strict transform of $\mathbb{P}^1 \times \{\infty\}$ (resp. $\mathbb{P}^1 \times \{0\}$) on Z. We define $M = \mathbb{P}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z(A_0))$ and X is the restriction of M on $(p \circ \mu)^{-1}(0)$. Then X is a simple normal crossing divisor on M. More explicitly, X is a $\mathbb{P}^1$-bundle over $(p \circ \mu)^{-1}(0)$ and is obtained by gluing $X_1 = \mathbb{P}^1 \times \mathbb{P}^1$ and $X_2 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ along a fiber. In particular, $(X, 0)$ is a semi log canonical surface. By the construction, $M \to Z$ has two sections. Let D^+ (resp. D^-) be the restriction of the section of $M \to Z$ corresponding to $\mathcal{O}_Z \oplus \mathcal{O}_Z(A_0) \to \mathcal{O}_Z(A_0) \to 0$ (resp. $\mathcal{O}_Z \oplus \mathcal{O}_Z(A_0) \to \mathcal{O}_Z \to 0$). Then it is easy to see that D^+ is a nef Cartier divisor on X and that the linear system $|mD^+|$ is free for every $m > 0$. Note that M is a projective toric variety. Let E be the section of $M \to Z$ corresponding to $\mathcal{O}_Z \oplus \mathcal{O}_Z(A_0) \to \mathcal{O}_Z(A_0) \to 0$. Then, it is easy to see that E is a nef Cartier divisor on M. Therefore, the linear system $|E|$ is free. In particular, $|D^+|$ is free on X since $D^+ = E|_X$. So, $|mD^+|$ is free for every $m > 0$. We take a general member $B_0 \in |mD^+|$ with $m \geqslant 2$. We consider $K_X + B$ with $B = D^- + B_0 + B_1 + B_2$, where B_1 and B_2 are general fibers of $X_1 = \mathbb{P}^1 \times \mathbb{P}^1 \subset X$. We note that B_0 does not intersect D^-. Then (X, B) is an embedded simple normal crossing pair. In particular, (X, B) is a semi log canonical surface. It is easy to see that there exists only one integral curve C on $X_2 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \subset X$ such that $C \cdot (K_X + B) < 0$. Note that C is nothing but the negative section of $X_2 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$. We also note that $(K_X + B)|_{X_1}$ is ample on X_1. By the cone theorem (see Theorem 1.19), we obtain

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+B)\geqslant 0} + \mathbb{R}_{\geqslant 0}[C].$$

By the contraction theorem (see Theorem 1.19), we have $\varphi : X \to W$ which contracts C. We can easily see that $K_W + B_W$, where $B_W = \varphi_* B$, is not $\mathbb{Q}$-Cartier because C is not $\mathbb{Q}$-Cartier on X. Therefore, we can not always run the minimal model program for semi log canonical surfaces.

For a new framework of the minimal model program for *log surfaces*, see [F12a], [FT12], [T12a], and [T12b].

6. Miscellaneous applications

In this paper, we adopt the following definition of *stable pairs*. It is a generalization of the notion of *stable pointed curves*.

DEFINITION 6.1 (Stable pairs). Let (X, Δ) be a projective semi log canonical pair such that $K_X + \Delta$ is ample. Then we call (X, Δ) a *stable pair*.

Stable pairs will play important roles in the theory of moduli of canonically polarized varieties.

6.1 Base point free theorems revisited

First, we prove the base point free theorem for $\mathbb{R}$-divisors (see [F11c, Theorem 17.1]). It is an easy consequence of the base point free theorem (see Theorem 1.15) and the cone theorem (see Theorem 1.19). For the definition and basic properties of π-semi-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisors, see, for example, [F11c, Definition 4.11, Lemmas 4.13 and 4.14].

THEOREM 6.2 (Base point free theorem for $\mathbb{R}$-divisors). *Let (X, Δ) be a semi log canonical pair and let $\pi : X \to S$ be a projective morphism onto an algebraic variety S. Let D be a π-nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Assume that $D - (K_X + \Delta)$ is π-ample. Then D is π-semi-ample.*

Proof. This theorem is an easy consequence of Theorem 1.15 and Theorem 1.19 (2). For the details, see the proof of [F11c, Theorem 17.1]. □

Next, we discuss a generalization of Kollár's effective base point free theorem for semi log canonical pairs (cf. [F09b]).

THEOREM 6.3 (Effective base point free theorem). *Let (X, Δ) be a projective semi log canonical pair such that Δ is a $\mathbb{Q}$-divisor and let L be a nef Cartier divisor on X. Assume that $aL - (K_X + \Delta)$ is nef and log big with respect to (X, Δ) for some real number $a > 0$. Then there exists a positive integer $m = m(n, a)$, which only depends on $n = \dim X$ and a, such that $|mL|$ is free.*

REMARK 6.4. We can take $m(n, a) = 2^{n+1}(n+1)!(\lceil a \rceil + n)$ in Theorem 6.3. For the details, see [F09b].

We give a remark on [F09b].

REMARK 6.5. In this remark, we use the same notation as in [F09b, 2.1.1]. By the vanishing theorem [F09b, Theorem 3.2 (b)], we have

$$h^i(S, h_* \mathcal{O}_Y(N' - F)) = h^i(S, h_* \mathcal{O}_Y(N')) = 0$$

for all $i > 0$. This implies that

$$h^i(S, h_* \mathcal{O}_F(N')) = 0$$

for all $i > 0$. Therefore, we do not need the vanishing theorem [F09b, Theorem 3.2 (b)] for a simple normal crossing variety F. The vanishing theorem for Y is sufficient. Note that Y is a smooth variety. The vanishing theorem [F09b, Theorem 3.2 (b)] is much simpler for smooth varieties than for simple normal crossing varieties.

Sketch of the proof of Theorem 6.3. The arguments in [F09b] work for our situation by suitable modifications. We use a quasi-log resolution constructed in Theorem 1.2 instead of taking a resolution of singularities (cf. [F09b, 2.1.1]). We also use the vanishing theorem for simple normal crossing pairs (see, for example, [F09c, Theorem 2.39] or [F12c, Theorem 1.1]) and Theorem 1.11. All the other modifications we need are more or less routine works. We leave the details for the reader's exercise. □

6.2 Shokurov's polytope

Let us introduce the notion of Shokurov's polytope for semi log canonical pairs. It is useful for reducing the problems for $\mathbb{R}$-divisors to ones for $\mathbb{Q}$-divisors.

6.6 (Shokurov's polytope). Let X be an equidimensional algebraic variety which satisfies Serre's S_2 condition and is normal crossing in codimension one. Let B be a reduced Weil divisor on X whose support does not contain any irreducible components of the conductor of X. Let $B = \sum_i B_i$ be the irreducible decomposition. We define a finite-dimensional $\mathbb{R}$-vector space $V = \bigoplus_i \mathbb{R} B_i$. Then it is easy to see that

$$\mathcal{L} = \{D \in V \,|\, (X, D) \text{ is semi log canonical}\}$$

is a rational polytope in V. Let $\pi : X \to S$ be a projective morphism onto an algebraic variety S. We can also check that

$$\mathcal{N} = \{D \in \mathcal{L} \,|\, K_X + D \text{ is } \pi\text{-nef}\}$$

is a rational polytope (see, for example, the proof of [B11, Proposition 3.2]). A key point is the boundedness of lengths of extremal rays in Theorem 1.19 (1). We note that $\mathcal{N}$ is known as Shokurov's polytope when X is normal. Assume that Δ is an $\mathbb{R}$-divisor on X such that $\operatorname{Supp} \Delta \subset \operatorname{Supp} B$, (X, Δ) is semi log canonical, and $K_X + \Delta$ is π-nef. Then $\Delta \in \mathcal{N}$. In this case, we can write

$$K_X + \Delta = \sum_{i=1}^{k} r_i (K_X + D_i)$$

such that

(i) $D_i \in \mathcal{N}$ is an effective $\mathbb{Q}$-divisor on X for every i,

(ii) (X, D_i) is semi log canonical for every i, and

(iii) $0 < r_i < 1$, $r_i \in \mathbb{R}$ for every i, and $\sum_{i=1}^{k} r_i = 1$.

If Δ is contained in a face $\mathcal{F}$ of $\mathcal{N}$, then we can choose D_i such that $D_i \in \mathcal{F}$ for every i. Moreover, we can make D_i arbitrarily close to Δ in a given norm on V for every i.

The abundance conjecture is one of the most important conjectures in the minimal model theory.

Conjecture 6.7 ((Log) abundance conjecture). *Let (X, Δ) be a semi log canonical pair and let $\pi : X \to S$ be a projective morphism. Suppose that $K_X + \Delta$ is π-nef. Then $K_X + \Delta$ is π-semi-ample.*

By the arguments in 6.6, we may assume that Δ is a $\mathbb{Q}$-divisor on X. This reduction seems to be very important because we do not know how to use the gluing arguments for $\mathbb{R}$-divisors (cf. [F00b], [FG11], [HX11]). We note that if Δ is a $\mathbb{Q}$-divisor then Conjecture 6.7 can be reduced to the case when (X, Δ) is log canonical, that is, X is normal (cf. [FG11], [HX11]).

From now on, we treat the two extreme cases of Conjecture 6.7.

Theorem 6.8 (Numerically trivial case). *Let (X, Δ) be a semi log canonical pair and let $\pi : X \to S$ be a projective morphism onto an algebraic variety S. Assume that $K_X + \Delta$ is numerically trivial over S. Then $K_X + \Delta$ is π-semi-ample.*

Proof. We set $B = \lceil \Delta \rceil$ and apply the arguments in 6.6. Then we can write

$$K_X + \Delta = \sum_{i=1}^{k} r_i(K_X + D_i)$$

as in 6.6. Since $K_X + \Delta$ is numerically π-trivial and $K_X + D_i$ is π-nef for every i, $K_X + D_i$ is numerically π-trivial for every i. Therefore, we can reduce the problem to the case when Δ is a $\mathbb{Q}$-divisor. If Δ is a $\mathbb{Q}$-divisor, then the statement is nothing but [FG11, Corollary 4.11] (see also [FG11, Subsection 4.1]). Therefore, $K_X + \Delta$ is π-semi-ample. □

THEOREM 6.9 (Nef and log big case). *Let (X, Δ) be a semi log canonical pair and let $\pi : X \to S$ be a projective morphism onto an algebraic variety S. Assume that $K_X + \Delta$ is nef and log big over S with respect to (X, Δ). Then $K_X + \Delta$ is π-semi-ample.*

Proof. We set $B = \lceil \Delta \rceil$ and apply the arguments in 6.6. Then we can write

$$K_X + \Delta = \sum_{i=1}^{k} r_i(K_X + D_i)$$

as in 6.6. If D_i is sufficiently close to Δ, then $K_X + D_i$ is nef and log big over S with respect to (X, Δ). This is because the bigness is an open condition. It is easy to see that $K_X + D_i$ is nef and log big over S with respect to (X, D_i) if D_i is sufficiently close to Δ. Therefore, we may assume that Δ is a $\mathbb{Q}$-divisor. In this case, we can check that $K_X + \Delta$ is semi-ample over S by Theorem 1.16. □

6.3 Depth of sheaves on slc pairs

The following theorem is an $\mathbb{R}$-divisor version of Kollár's result (see [Ko11b, Theorem 3]), which is a generalization of [Al08, Lemma 3.2] and [F09c, Theorem 4.21]. It can be proved by the method of two spectral sequences of local cohomology groups (cf. [F09c, 4.2.1 Appendix and Section 4.3]). For the details and some interesting examples, see [Ko11b]. For some related topics, see [Kv11] and [AH12].

THEOREM 6.10. *Let (X, Δ) be a semi log canonical pair and let $x \in X$ be a scheme theoretic point. Assume that x is not the generic point of any slc center of (X, Δ). Then we have the following properties.*

(1) *Let D be a Weil divisor on X whose support does not contain any irreducible components of the conductor of X. Let Δ' be an effective $\mathbb{R}$-divisor on X such that $\Delta' \leqslant \Delta$ and that $D \sim_{\mathbb{R},loc} \Delta'$, that is, D is locally $\mathbb{R}$-linearly equivalent to Δ'. Then*

$$\operatorname{depth}_x \mathcal{O}_X(-D) \geqslant \min\{3, \operatorname{codim}_X x\}.$$

(2) *Let X' be any reduced closed subscheme of X that is a union of some slc centers of (X, Δ). Then*

$$\operatorname{depth}_x \mathcal{I}_{X'} \geqslant \min\{3, 1 + \operatorname{codim}_{X'} x\},$$

where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X' on X.

Sketch of the proof of Theorem 6.10. First, we consider (2). The proof of Theorem 3 (2) in [Ko11b] works without any changes. Next we consider (1). Since the problem is local, we may assume that X is affine and $D \sim_{\mathbb{R}} \Delta'$. By considering the real vector space spanned by the irreducible components of $\operatorname{Supp} \Delta$ and perturbing Δ and Δ', we can find effective $\mathbb{Q}$-divisors Δ'_0

and Δ_0 on X such that $\Delta_0' \leqslant \Delta_0$, $D \sim_{\mathbb{Q}} \Delta_0'$, (X, Δ_0) is semi log canonical, and x is not the generic point of any slc center of (X, Δ_0). Therefore, by Theorem 3 (1) in [Ko11b], we obtain the desired inequality. $\square$

6.4 Slc morphisms

In this subsection, we introduce the notion of *slc morphisms* and prove some basic properties.

DEFINITION 6.11 (Slc morphisms). Let (X, Δ) be a semi log canonical pair and let $f : X \to C$ be a flat morphism onto a smooth curve C. We say that $f : (X, \Delta) \to C$ is *semi log canonical* (*slc*, for short) if $(X, \Delta + f^*c)$ is semi log canonical for every closed point $c \in C$.

The following lemma is almost obvious by the definition of slc morphisms. See [KM98, Lemma 7.2].

LEMMA 6.12. *Assume that $f : (X, \Delta) \to C$ is slc. Then we have the following properties.*

(1) *Every fiber of f is reduced.*

(2) *Δ is horizontal, that is, none of the irreducible components of Δ is contained in a fiber of f.*

(3) *If E is an exceptional divisor over X such that the center $c_X(E)$ is contained in a fiber, then $a(E, X, \Delta) \geqslant 0$.*

By the same arguments as in the proof of [KM98, Lemma 7.6], we know that the notion of slc morphisms is stable under base changes.

LEMMA 6.13. *Assume that $f : (X, \Delta) \to C$ is slc. Let $g : C' \to C$ be a non-constant morphism from a smooth curve C', $X' = X \times_C C'$ with projections $h : X' \to X$ and $f' : X' \to C'$. We set $K_{X'} + \Delta' = h^*(K_X + \Delta)$. Then $f' : (X', \Delta') \to C'$ is also slc.*

The following theorem is the main result of this subsection. It is a consequence of Theorem 1.12.

THEOREM 6.14. *Let $f : X \to C$ be a projective semi log canonical morphism. Then $R^i f_* \mathcal{O}_X(K_X)$ is locally free for every i. Therefore, for every i, we obtain that $R^i f_* \mathcal{O}_X(K_{X/C})$ is locally free and that*

$$R^i f_* \mathcal{O}_X(K_{X/C}) \otimes \mathbb{C}(c) \simeq H^i(X_c, \mathcal{O}_{X_c}(K_{X_c}))$$

for all $c \in C$, where $X_c = f^{-1}(c)$. In particular, $\dim_{\mathbb{C}} H^i(X_c, \mathcal{O}_{X_c}(K_{X_c}))$ is independent of $c \in C$.
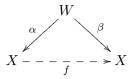
Proof. By Theorem 1.12, $R^i f_* \mathcal{O}_X(K_X)$ is torsion-free because every slc stratum of X is dominant onto C (see Lemma 6.12 (3)). The other claims are obvious by the base change theorem (cf. [Ko11b, (4.3)]). $\square$

6.5 Finiteness of birational automorphisms

This subsection is a supplement to [FG11]. Let us introduce the notion of *B-birational maps* for semi log canonical pairs (cf. [F00b], [FG11]).

DEFINITION 6.15 (cf. [F00b, Definition 3.1], [FG11, Definition 2.11]). Let (X, Δ) be a semi log canonical pair. We say that a proper birational map $f : (X, \Delta) \dashrightarrow (X, \Delta)$ is *B-birational* if there

exists a common resolution

such that
$$\alpha^*(K_X + \Delta) = \beta^*(K_X + \Delta).$$
This means that it holds that $E_\alpha = E_\beta$ when we write
$$K_W = \alpha^*(K_X + \Delta) + E_\alpha$$
and
$$K_W = \beta^*(K_X + \Delta) + E_\beta.$$
We define
$$\mathrm{Bir}(X, \Delta) = \{f \mid f : (X, \Delta) \dashrightarrow (X, \Delta) \text{ is } B\text{-birational}\}.$$
It is obvious that $\mathrm{Bir}(X, \Delta)$ has a natural group structure. We also define
$$\mathrm{Aut}(X, \Delta) = \{f \mid f : X \to X \text{ is an isomorphism such that } \Delta = f_*^{-1}\Delta\}.$$
We can easily see that $\mathrm{Aut}(X, \Delta)$ is a subgroup of $\mathrm{Bir}(X, \Delta)$.

The following theorem is the main theorem of this subsection. It is essentially contained in [FG11, Corollary 3.13].

THEOREM 6.16 (Finiteness of B-birational maps). *Let (X, Δ) be a complete semi log canonical pair such that Δ is a $\mathbb{Q}$-divisor. Assume that $K_X + \Delta$ is big, that is, $K_{X_i^\nu} + \Theta_i$ is big for every i, where $\nu : X^\nu \to X$ is the normalization, $X^\nu = \cup_i X_i^\nu$ is the irreducible decomposition, and $K_{X_i^\nu} + \Theta_i = \nu^*(K_X + \Delta)|_{X_i^\nu}$. Then $\mathrm{Bir}(X, \Delta)$ is a finite group. In particular, $\mathrm{Aut}(X, \Delta)$ is a finite group.*

Proof. Let $f : Y \to X$ be a resolution such that Y is projective, $K_Y + \Delta_Y = f^*(K_X + \Delta)$, and $\mathrm{Supp}\,\Delta_Y$ is a simple normal crossing divisor on Y. It is easy to see that $\mathrm{Bir}(X, \Delta)$ is isomorphic to $\mathrm{Bir}(Y, \Delta_Y)$ because f is birational. By [FG11, Corollary 3.13 and Remark 3.16], we know that $\mathrm{Bir}(Y, \Delta_Y)$ is a finite group. Therefore, so is $\mathrm{Bir}(X, \Delta)$. Since $\mathrm{Aut}(X, \Delta)$ is a subgroup of $\mathrm{Bir}(X, \Delta)$, $\mathrm{Aut}(X, \Delta)$ is also a finite group. $\qquad\square$

As a direct consequence of Theorem 6.16, we obtain the following corollary.

COROLLARY 6.17. *Let (X, Δ) be a stable pair such that Δ is a $\mathbb{Q}$-divisor. Then $\mathrm{Bir}(X, \Delta)$ and $\mathrm{Aut}(X, \Delta)$ are finite groups.*

Proof. Since $K_{X^\nu} + \Theta$ is ample, where $\nu : X^\nu \to X$ is the normalization and $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$, $\mathrm{Bir}(X, \Delta)$ is a finite group by Theorem 6.16. Therefore, so is $\mathrm{Aut}(X, \Delta)$. $\qquad\square$

Corollary 6.17 seems to be indispensable when we consider moduli problems for stable pairs.

7. Appendix: Big $\mathbb{R}$-divisors

In this appendix, we discuss the notion of big $\mathbb{R}$-divisors on singular varieties. The basic references of big $\mathbb{R}$-divisors are [L04, 2.2] and [N04, II. §3 and §5]. Since we have to consider big $\mathbb{R}$-divisors on *non-normal* varieties, we give supplementary definitions and arguments to [L04] and [N04].

First, let us quickly recall the definition of big Cartier divisors on normal complete irreducible varieties. For details, see, for example, [KMM87, § 0-3].

DEFINITION 7.1 (Big Cartier divisors). Let X be a normal complete irreducible variety and let D be a Cartier divisor on X. Then D is *big* if one of the following equivalent conditions holds.

(1) $\max\limits_{m \in \mathbb{N}} \{\dim \Phi_{|mD|}(X)\} = \dim X$, where $\Phi_{|mD|} : X \dashrightarrow \mathbb{P}^N$ is the rational map associated to the linear system $|mD|$ and $\Phi_{|mD|}(X)$ is the image of $\Phi_{|mD|}$.

(2) There exist a rational number α and a positive integer m_0 such that
$$\alpha m^{\dim X} \leqslant \dim H^0(X, \mathcal{O}_X(mm_0 D))$$
for every $m \gg 0$.

It is well known that we can take $m_0 = 1$ in the condition (2).

One of the most important properties of big Cartier divisors is known as Kodaira's lemma.

LEMMA 7.2 (Kodaira's lemma). *Let X be a normal complete irreducible variety and let D be a big Cartier divisor on X. Then, for an arbitrary Cartier divisor M, we have $H^0(X, \mathcal{O}_X(lD - M)) \neq 0$ for every $l \gg 0$.*

Proof. By replacing X with its resolution, we may assume that X is smooth and projective. Then it is sufficient to show that for a sufficiently ample Cartier divisor A, $H^0(X, \mathcal{O}_X(lD - A)) \neq 0$ for every $l \gg 0$. Since we have the exact sequence
$$0 \to \mathcal{O}_X(lD - A) \to \mathcal{O}_X(lD) \to \mathcal{O}_Y(lD) \to 0,$$
where Y is a general member of $|A|$, and since there exist positive rational numbers α, β such that $\alpha l^{\dim X} \leqslant \dim H^0(X, \mathcal{O}_X(lD))$ and $\dim H^0(Y, \mathcal{O}_Y(lD)) \leqslant \beta l^{\dim Y}$ for every $l \gg 0$, we have $H^0(X, \mathcal{O}_X(lD - A)) \neq 0$ for every $l \gg 0$. $\square$

For non-normal varieties, we need the following definition.

DEFINITION 7.3 (Big Cartier divisors on non-normal varieties). Let X be a complete irreducible variety and let D be a Cartier divisor on X. Then D is *big* if $\nu^* D$ is big on X^ν, where $\nu : X^\nu \to X$ is the normalization.

Before we define big $\mathbb{R}$-divisors, let us recall the definition of big $\mathbb{Q}$-divisors.

DEFINITION 7.4 (Big $\mathbb{Q}$-divisors). Let X be a complete irreducible variety and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then D is *big* if mD is a big Cartier divisor for some positive integer m.

We note the following obvious lemma.

LEMMA 7.5. *Let $f : W \to V$ be a birational morphism between normal complete irreducible varieties and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on V. Then D is big if and only if so is $f^* D$.*

Next, let us start to consider big $\mathbb{R}$-divisors.

DEFINITION 7.6 (Big $\mathbb{R}$-divisors on complete varieties). An $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on a complete irreducible variety X is *big* if it can be written in the form
$$D = \sum_i a_i D_i$$
where each D_i is a big Cartier divisor and a_i is a positive real number for every i.

Let us recall an easy but very important lemma.

LEMMA 7.7 (see [N04, 2.11. Lemma]). *Let $f : Y \to X$ be a proper surjective morphism between normal irreducible varieties with connected fibers. Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then we have a canonical isomorphism*

$$\mathcal{O}_X(\lfloor D \rfloor) \simeq f_* \mathcal{O}_Y(\lfloor f^* D \rfloor).$$

LEMMA 7.8. *Let D be a big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on a smooth projective irreducible variety X. Then there exist a positive rational number α and a positive integer m_0 such that*

$$\alpha m^{\dim X} \leqslant \dim H^0(X, \mathcal{O}_X(\lfloor mm_0 D \rfloor))$$

for every $m \gg 0$.

Proof. By using Lemma 7.2, we can find an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor E on X such that $D - E$ is ample. Therefore, there exists a positive integer m_0 such that $A = \lfloor m_0 D - m_0 E \rfloor$ is ample. We note that $m_0 D = A + \{m_0 D - m_0 E\} + m_0 E$. This implies that $mA \leqslant mm_0 D$ for any positive integer m. Therefore,

$$\dim H^0(X, \mathcal{O}_X(mA)) \leqslant \dim H^0(X, \mathcal{O}_X(\lfloor mm_0 D \rfloor)).$$

So, we can find a positive rational number α such that

$$\alpha m^{\dim X} \leqslant \dim H^0(X, \mathcal{O}_X(\lfloor mm_0 D \rfloor)).$$

It is the desired inequality. $\qquad\square$

REMARK 7.9. By Lemma 7.5 and Lemma 7.8, Definition 7.6 is compatible with Definition 7.4.

LEMMA 7.10 (Weak Kodaira's lemma). *Let X be a projective irreducible variety and let D be a big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then we can write*

$$D \sim_{\mathbb{R}} A + E,$$

where A is an ample $\mathbb{Q}$-divisor on X and E is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X.

Proof. Let B be a big Cartier divisor on X and let H be a general very ample Cartier divisor on X. We consider the short exact sequence

$$0 \to \mathcal{O}_X(lB - H) \to \mathcal{O}_X(lB) \to \mathcal{O}_H(lB) \to 0$$

for every l. It is easy to see that $\dim H^0(X, \mathcal{O}_X(lB)) \geqslant \alpha l^{\dim X}$ and $\dim H^0(H, \mathcal{O}_H(lB)) \leqslant \beta l^{\dim H}$ for some positive rational numbers α, β, and for every $l \gg 0$. Therefore, $H^0(X, \mathcal{O}_X(lB - H)) \neq 0$ for some large l. This means that $lB \sim H + G$ for some effective Cartier divisor G. By Definition 7.6, we can write $D = \sum_i a_i D_i$ where a_i is a positive real number and D_i is a big Cartier divisor for every i. By applying the above argument to each D_i, we can easily obtain the desired decomposition $D \sim_{\mathbb{R}} A + E$. $\qquad\square$

We prepare an important lemma.

LEMMA 7.11. *Let X be a complete irreducible variety and let N be a numerically trivial $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then N can be written in the form*

$$N = \sum_i r_i N_i$$

where each N_i is a numerically trivial Cartier divisor and r_i is a real number for every i.

Proof. Let Z_j be an integral 1-cycle on X for $1 \leqslant j \leqslant \rho = \rho(X)$ such that $\{[Z_1], \cdots, [Z_\rho]\}$ is a basis of the vector space $N_1(X)$. The condition that an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $B = \sum_i b_i B_i$, where b_i is a real number and B_i is Cartier for every i, is numerically trivial is given by the integer linear equations

$$\sum_i b_i(B_i \cdot Z_j) = 0$$

on b_i for $1 \leqslant j \leqslant \rho$. Any real solution to these equations is an $\mathbb{R}$-linear combination of integral ones. Thus, we obtain the desired expression $N = \sum_i r_i N_i$. $\qquad \square$

The following proposition seems to be very important.

PROPOSITION 7.12. *Let X be a complete irreducible variety. Let D and D' be $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on X. If $D \equiv D'$, then D is big if and only if so is D'.*

Proof. We set $N = D' - D$. Then N is a numerically trivial $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. By Lemma 7.11, we can write $N = \sum_i r_i N_i$, where r_i is a real number and N_i is a numerically trivial Cartier divisor for every i. By Definition 7.6, we are reduced to showing that if B is a big Cartier divisor and G is a numerically trivial Cartier divisor, then $B + rG$ is big for any real number r. If r is not a rational number, we can write

$$B + rG = t(B + r_1 G) + (1 - t)(B + r_2 G)$$

where r_1 and r_2 are rational, $r_1 < r < r_2$, and t is a real number with $0 < t < 1$. Therefore, we may assume that r is rational. Let $f : Y \to X$ be a resolution. Then it is sufficient to check that $f^*B + rf^*G$ is big by Lemma 7.5 and Definitions 7.3. So, we may assume that X is smooth and projective. By Kodaira's lemma (see Lemma 7.2), we can write $lB \sim A + E$, where A is an ample Cartier divisor, E is an effective Cartier divisor, and l is a positive integer. Thus, $l(B + rG) \sim (A + lrG) + E$. We note that $A + lrG$ is an ample $\mathbb{Q}$-divisor. This implies that $B + rG$ is a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. We finish the proof. $\qquad \square$

We give a small remark on Iitaka's D-dimension for $\mathbb{R}$-divisors. Please compare it with Proposition 7.12.

REMARK 7.13. We consider $X = \mathbb{P}^1$ and take $P, Q \in X$ with $P \neq Q$. We set $D = \sqrt{2}P - \sqrt{2}Q$. Then it is obvious that $D \sim_{\mathbb{R}} 0$. However, $\kappa(X, D) = -\infty$ because $\deg \lfloor mD \rfloor < 0$ for every positive integer m. Note that $\mathbb{R}$-linear equivalence does not always preserve Iitaka's D-dimension.

Proposition 7.14 seems to be missing in the literature. We note that X is not assumed to be *projective* in Proposition 7.14.

PROPOSITION 7.14. *Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on a normal complete irreducible variety X. Then the following conditions are equivalent.*

(1) *D is big.*

(2) *There exist a positive rational number α and a positive integer m_0 such that*

$$\alpha m^{\dim X} \leqslant \dim H^0(X, \mathcal{O}_X(\lfloor mm_0 D \rfloor))$$

for every $m \gg 0$.

Proof. First, we assume (2). Let $f : Y \to X$ be a resolution such that Y is projective. By Lemma 7.7, we have

$$\alpha m^{\dim X} \leqslant \dim H^0(X, \mathcal{O}_X(\lfloor mm_0 f^*D \rfloor)).$$

By the usual argument as in the proof of Kodaira's lemma (cf. Lemma 7.2), we can write $f^*D \equiv A + E$, where A is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and E is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y. By using Lemma 7.16 below, we can write $A + E \equiv \sum a_i G_i$ where a_i is a positive real number and G_i is a big Cartier divisor for every i. By Proposition 7.12, f^*D is a big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y. Let D' be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X whose coefficients are very close to those of D. Then $A + f^*D' - f^*D$ is an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y. Therefore, $f^*D' \equiv (A + f^*D' - f^*D) + E$ is also a big $\mathbb{Q}$-divisor on Y as above. By Lemma 7.5, D' is a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. This means that there exists a big Cartier divisor M on X (see Example 7.17 below). By the assumption, we can write $lD \sim M + E'$, where E' is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor (see, for example, the usual proof of Kodaira's lemma: Lemma 7.2). By using Lemma 7.15 and Lemma 7.16 below, we can write $M + E' \equiv \sum a_i' G_i'$, where a_i' is a positive real number and G_i' is a big Cartier divisor for every i. By Proposition 7.12, D is a big $\mathbb{R}$-divisor on X.

Next, we assume (1). Let $f : Y \to X$ be a resolution. Then f^*D is big by Definition 7.6 and Lemma 7.5. By Lemma 7.7 and Lemma 7.8, we obtain the desired estimate in (2). $\square$

We have already used the following lemmas in the proof of Proposition 7.14.

LEMMA 7.15. *Let X be a normal irreducible variety and let B be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then B can be written in the form*

$$B = \sum_i b_i B_i$$

where each B_i is an effective Cartier divisor and b_i is a positive real number for every i.

Proof. We can write $B = \sum_{j=1}^l d_j D_j$, where d_j is a real number and D_j is Cartier for every j. We set $E = \cup_j \operatorname{Supp} D_j$. Let $E = \sum_{k=1}^m E_k$ be the irreducible decomposition. We can write $D_j = \sum_{k=1}^m a_k^j E_k$ for every j. Note that a_k^j is integer for every j and k. We can also write $B = \sum_{k=1}^m c_k E_k$ with $c_k \geqslant 0$ for every k. We consider

$$\mathcal{E} = \left\{ (r_1, \cdots, r_l) \in \mathbb{R}^l \ \middle| \ \sum_{j=1}^l r_j a_k^j \geqslant 0 \ \text{ for every } k \right\} \subset \mathbb{R}^l.$$

Then $\mathcal{E}$ is a rational convex polyhedral cone and $(d_1, \cdots, d_l) \in \mathcal{E}$. Therefore, we can find effective Cartier divisors B_i and positive real numbers b_i such that $B = \sum_i b_i B_i$. $\square$

LEMMA 7.16. *Let B be a big Cartier divisor on a normal irreducible variety X and let G be an effective Cartier divisor on X. Then $B + rG$ is big for any positive real number r.*

Proof. If r is rational, then this lemma is obvious by the definition of big $\mathbb{Q}$-divisors. If r is not rational, then we can write

$$B + rG = t(B + r_1 G) + (1 - t)(B + r_2 G)$$

where r_1 and r_2 are rational, $0 < r_1 < r < r_2$, and t is a real number with $0 < t < 1$. By Definition 7.6, $B + rG$ is a big $\mathbb{R}$-divisor. $\square$

Example 7.17 implies that a normal complete algebraic variety does not always have big Cartier divisors even when the Picard number is one. For the details of Example 7.17, see [F05, Section 4].

EXAMPLE 7.17. Let Δ be the fan in $\mathbb{R}^3$ whose rays are generated by $v_1 = (1, 0, 1)$, $v_2 = (0, 1, 1)$, $v_3 = (-1, -2, 1)$, $v_4 = (1, 0, -1)$, $v_5 = (0, 1, -1)$, $v_6 = (-1, -1, -1)$ and whose maximal cones are

$$\langle v_1, v_2, v_4, v_5 \rangle, \langle v_2, v_3, v_5, v_6 \rangle, \langle v_1, v_3, v_4, v_6 \rangle, \langle v_1, v_2, v_3 \rangle, \langle v_4, v_5, v_6 \rangle.$$

Then the associated toric threefold X is complete with $\rho(X) = 0$. More precisely, every Cartier divisor on X is linearly equivalent to zero.

Let $f : Y \to X$ be the blow-up along $v_7 = (0, 0, -1)$ and let E be the f-exceptional divisor on Y. Then we can check that $\rho(Y) = 1$ and that $\mathcal{O}_Y(E)$ is a generator of $\mathrm{Pic}(Y)$. Therefore, there are no big Cartier divisors on Y.

The next lemma is almost obvious.

LEMMA 7.18. *Let V be a complete irreducible variety and let D be a big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on V. Let $g : W \to V$ be an arbitrary proper birational morphism from an irreducible variety W. Then g^*D is big.*

Proof. By Definition 7.6, we may assume that D is Cartier. We obtain the following commutative diagram.

$$
\begin{array}{ccc}
W & \xleftarrow{\ \mu\ } & W^\nu \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\
V & \xleftarrow{\ \nu\ } & V^\nu
\end{array}
$$

Here, $\mu : W^\nu \to W$ and $\nu : V^\nu \to V$ are the normalizations. Since ν^*D is big, $h^*\nu^*D = \mu^*g^*D$ is also big. We note that h is a birational morphism between normal irreducible varieties (see Lemma 7.5). Thus, g^*D is big by Definition 7.3. $\qquad \square$

Kodaira's lemma for big $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on normal varieties is also obvious (cf. the proof of Lemma 7.2).

LEMMA 7.19 (Kodaira's lemma for big $\mathbb{R}$-divisors on normal varieties). *Let X be a complete irreducible normal variety and let D be a big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Let M be an arbitrary Cartier divisor on X. Then there exist a positive integer l and an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor E on X such that $lD - M \sim E$.*

Finally, we discuss relatively big $\mathbb{R}$-divisors.

DEFINITION 7.20 (Relatively big $\mathbb{R}$-divisors). Let $\pi : X \to S$ be a proper morphism from an irreducible variety X onto a variety S and let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then D is called π-*big* (or, *big over S*) if $D|_{X_\eta}$ is big on X_η, where X_η is the generic fiber of π.

We need the following lemma for the proof of the Kawamata–Viehweg vanishing theorem for $\mathbb{R}$-divisors.

LEMMA 7.21 (cf. [KMM87, Corollary 0-3-6]). *Let $\pi : X \to S$ be a proper surjective morphism from an irreducible variety X onto a quasi-projective variety S and let D be a π-nef and π-big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then there exist a proper birational morphism $\mu : Y \to X$ from a smooth variety Y projective over S and divisors F_α's on Y such that $\mathrm{Supp}\, \mu^*D \cup (\cup F_\alpha)$ is a simple normal crossing divisor and that $\mu^*D - \sum_\alpha \delta_\alpha F_\alpha$ is $\pi \circ \mu$-ample for some δ_α with $0 < \delta_\alpha \ll 1$.*

We can check Lemma 7.21 by Lemma 7.19 and Hironaka's resolution theorem.

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