A short remark on [2]

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Abstract

In this note, we give a short remark on Section 4 of [2].

1 Remark

In Section 4 of [2], the lemma on two filtrations [1, Propositions (7.2.5) and (7.2.8)] (see also [3, Theorem 3.12]) was used several times (explicitly stated at p. 608, the proof of Lemma 4.5, p. 610, Remark 4.6, p. 618, Step 1 of the proof of Lemma 4.10 and p. 623, the proof of Lemma 4.12, and implicitly used at p. 611, the proof of Lemma 4.8). However, there are missing points in the arguments.

Let K be a complex, W a finite increasing filtration on K and F a finite decreasing filtration on K. In order to apply the lemma on two filtrations for the spectral sequence

$$(E_r^{p,q}(K, W), F_{\text{rec}}),$$

it is necessary to discuss about the E_0-terms. More precisely, it has to be checked that the strictness of the filtration F on the complex $\operatorname{Gr}_m^W K$ holds true for all m. Here we will explain how to check this strictness for the case of Lemma 4.10 of [2] mentioned above. For the other cases, the similar arguments are valid.

In Step 1 of the proof of Lemma 4.10, the bifiltered complex

$$(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L, F)$$

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is studied. Thus the strictness of the filtration F on the complex

$$\operatorname{Gr}_m^L Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet)$$

has to be checked for all m. Under the canonical isomorphism

$$\begin{aligned}\operatorname{Gr}_m^L Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet) &\simeq Rf_* \operatorname{Gr}_m^L \Omega_{X_\bullet/\Delta}(\log E_\bullet) \\ &\simeq Rf_{-m*}\Omega_{X_{-m}/\Delta}(\log E_{-m})[m],\end{aligned}$$

the filtration F coincides with the filtration induced from the stupid filtration on $\Omega_{X_{-m}/\Delta}(\log E_{-m})$ denoted by F again. Therefore it suffices to prove the strictness of the filtration F on $Rf_{-m*}\Omega_{X_{-m}/\Delta}(\log E_{-m})$. The strictness of F on $Rf_{-m*}\Omega_{X_{-m}/\Delta}(\log E_{-m})$ is equivalent to the E_1-degeneracy of the spectral sequence $E_r^{p,q}(Rf_{-m*}\Omega_{X_{-m}/\Delta}(\log E_{-m}), F)$. We note that the morphism of E_r-terms

$$\begin{aligned}d_r : E_r^{p,q}&(Rf_{-m*}\Omega_{X_{-m}/\Delta}(\log E_{-m}), F) \\ &\longrightarrow E_r^{p+r,q-r+1}(Rf_{-m*}\Omega_{X_{-m}/\Delta}(\log E_{-m}), F)\end{aligned}$$

is zero on Δ^* for all p, q and for all $r \geq 1$ because $X_{-m} \longrightarrow \Delta$ is smooth and projective over Δ^*. On the other hand,

$$\begin{aligned}E_1^{p,q}(Rf_{-m*}\Omega_{X_{-m}/\Delta}(\log E_{-m}), F) &= R^{p+q}f_{-m*} \operatorname{Gr}_F^p \Omega_{X_{-m}/\Delta}(\log E_{-m}) \\ &= R^q f_{-m*}\Omega^p_{X_{-m}/\Delta}(\log E_{-m}),\end{aligned}$$

is a locally free $\mathcal{O}_\Delta$-module of finite rank by [4, (2.11) Theorem]. Therefore the morphism of E_1-terms d_1 is zero on the whole Δ for all p, q. Inductively on r, we obtain that $E_r^{p,q}(Rf_{-m*}\Omega_{X_{-m}/\Delta}(\log E_{-m}), F)$ is a locally free $\mathcal{O}_\Delta$-module of finite rank and that d_r is zero on the whole Δ for all p, q and for all $r \geq 1$. Thus the E_1-degeneracy is proved.

References

[1] P. Deligne, *Théorie de Hodge* III, Inst. Hautes Études Sci. Publ. Math. **44** (1974), 5–77.

[2] O. Fujino and T. Fujisawa, *Variations of mixed Hodge structure and semi-positivity theorems*, Publ. RIMS Kyoto Univ. **50** (2014), no. 4, 589–661.

[3] C. A. M. Peters and J. H. M. Steenbrink, *Mixed Hodge Structures*, A Series of Modern Surveys in Mathematics, vol. 52, Springer, 2008.

[4] J. H. M. Steenbrink, *Mixed Hodge structure on the vanishing cohomology*, Real and complex singularities. Oslo 1976 (Alphen a/d Rijn), Sijthoff-Noordhoff, 1977, pp. 525–563.