ON FINITE GENERATION OF ADJOINT RINGS

OSAMU FUJINO

Abstract. In this short paper, we prove that adjoint rings are finitely generated even in the complex analytic setting.

1. Finite generation of adjoint rings

The following theorem was first obtained in [DHP], whose argument is a complex analytic generalization of [CL]. When $\pi\colon X \to Y$ is a projective morphism between algebraic varieties, Theorems 1.1 and 1.2 below are well known (see [BCHM, Corollary 1.1.9]). In this paper, we see that they easily follow from [F]. In [F, Definition 2.23], we defined *locally finitely generated graded $\mathcal{O}_X$-algebras* on a complex analytic space X. Similarly, we can define *locally finitely generated $\mathbb{N}^k$-graded $\mathcal{O}_X$-algebras*.

Theorem 1.1. *Let X be a smooth complex variety and let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces. Let $B_1, \cdots, B_k$ be $\mathbb{Q}$-divisors on X with $\lfloor B_i \rfloor = 0$ for all i such that the support of $\sum_{i=1}^k B_i$ is a simple normal crossing divisor on X. Let A be a π-nef and π-big $\mathbb{Q}$-divisor on X. We put $D_i = K_X + A + B_i$ for every i. Then the relative adjoint ring*

$$R(X/Y, D_1, \cdots, D_k) := \bigoplus_{(m_1,\cdots,m_k)\in\mathbb{N}^k} \pi_*\mathcal{O}_X\left(\left\lfloor \sum m_i D_i \right\rfloor\right)$$

is a locally finitely generated $\mathbb{N}^k$-graded $\mathcal{O}_Y$-algebra.

Although Theorem 1.2 is essentially equivalent to Theorem 1.1, the following formulation may be useful for some applications.

Theorem 1.2. *Let X be a normal complex variety and let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces. Let $B_1, \cdots, B_k$ be $\mathbb{Q}$-divisors on X such that (X, B_i) is divisorial log terminal for every i. Let A be a π-ample $\mathbb{Q}$-divisor on X. We put $D_i = K_X + A + B_i$ for every i. Then the relative adjoint ring*

$$R(X/Y, D_1, \cdots, D_k) := \bigoplus_{(m_1,\cdots,m_k)\in\mathbb{N}^k} \pi_*\mathcal{O}_X\left(\left\lfloor \sum m_i D_i \right\rfloor\right)$$

is a locally finitely generated $\mathbb{N}^k$-graded $\mathcal{O}_Y$-algebra.

We make an easy remark.

Remark 1.3. If (X, B_i) is kawamata log terminal for every i in Theorem 1.2, then it is sufficient to assume that A is π-nef and π-big. This is obvious by the proof of Theorem 1.2.

Date: 2023/5/6, version 0.01.

2010 *Mathematics Subject Classification.* Primary 14E30; Secondary 32C15.

Key words and phrases. minimal model program, complex analytic spaces, adjoint rings, finite generation.

This note will be contained in [F].

Acknowledgments. The author was partially supported by JSPS KAKENHI Grant Numbers JP19H01787, JP20H00111, JP21H00974, JP21H04994.

In this paper, we will freely use [F]. We note that $\mathbb{N}$ denotes the set of non-negative integers.

2. Proof of theorems

In this section, we will prove Theorems 1.1 and 1.2. Let us start with an easy lemma.

Lemma 2.1. *Let X be a smooth complex variety and let $\mathcal{L}_1, \cdots, \mathcal{L}_k$ be line bundles on X. We put $\mathcal{E} := \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_k$ and consider the projective bundle $f\colon Z := \mathbb{P}_X(\mathcal{E}) \to X$ associated to $\mathcal{E}$. Let T_i be the divisor on $\mathbb{P}_X(\mathcal{E})$ associated to the quotient $\mathcal{E} \to \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_{i-1} \oplus \mathcal{L}_{i+1} \oplus \cdots \oplus \mathcal{L}_k$ for every i. Then $\sum_{i=1}^k T_i$ is a simple normal crossing divisor on Z such that*

$$\mathcal{O}_Z\left(K_Z + \sum_{i=1}^k T_i\right) \simeq f^*\mathcal{O}_X(K_X)$$

holds.

Proof. We put $\mathcal{E}_i := \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_{i-1} \oplus \mathcal{L}_{i+1} \oplus \cdots \oplus \mathcal{L}_k$. Then we have $T_i = \mathbb{P}_X(\mathcal{E}_i)$ by definition. It is almost obvious that $\sum_{i=1}^k T_i$ is a simple normal crossing divisor on Z. We set $\mathcal{O}_Z(1) := \mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ and $\mathcal{O}_{T_i}(1) := \mathcal{O}_{\mathbb{P}_X(\mathcal{E}_i)}(1)$. Then we consider the following short exact sequence:

$$0 \to \mathcal{O}_Z(1) \otimes \mathcal{O}_Z(-T_i) \to \mathcal{O}_Z(1) \to \mathcal{O}_{T_i}(1) \to 0.$$

By taking the pushforward by f, we obtain that

$$0 \to f_*\left(\mathcal{O}_Z(1) \otimes \mathcal{O}_Z(-T_i)\right) \to \mathcal{E} \to \mathcal{E}_i \to 0$$

is exact. This implies that $f_*\left(\mathcal{O}_Z(1) \otimes \mathcal{O}_Z(-T_i)\right) \simeq \mathcal{L}_i$ and $\mathcal{O}_Z(1) \otimes \mathcal{O}_Z(-T_i) \simeq f^*\mathcal{L}_i$ for every i. We note that

$$\mathcal{O}_Z(K_Z) \simeq f^*\mathcal{O}_X(K_X) \otimes f^*\det\mathcal{E} \otimes \mathcal{O}_Z(-k)$$

since $f\colon Z = \mathbb{P}_X(\mathcal{E}) \to X$. Hence, we obtain

$$\mathcal{O}_Z(K_Z) \simeq f^*\mathcal{O}_X(K_X) \otimes \mathcal{O}_Z\left(-\sum_{i=1}^k T_i\right).$$

This is what we wanted. $\square$

For the proof of Theorem 1.1, we need the following lemma.

Lemma 2.2. *Let X be a normal complex variety and let $D_1, \cdots, D_k$ be $\mathbb{Q}$-divisors on X. Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces. Let d be any positive integer. Then the relative adjoint ring*

$$\mathcal{A} := \bigoplus_{(m_1, \cdots, m_k) \in \mathbb{N}^k} \pi_*\mathcal{O}_X\left(\left\lfloor \sum_{i=1}^k m_i D_i \right\rfloor\right)$$

is locally finitely generated $\mathbb{N}^k$-graded $\mathcal{O}_Y$-algebra if and only if so is the truncation

$$\mathcal{A}^{(d)} := \bigoplus_{(m_1, \cdots, m_k) \in \mathbb{N}^k} \pi_*\mathcal{O}_X\left(\left\lfloor \sum_{i=1}^k m_i d D_i \right\rfloor\right).$$

Proof. Although [F, Lemma 2.26] only treats $\mathbb{N}$-graded $\mathcal{O}_Y$-algebras, the proof of [F, Lemma 2.26] works for this lemma. □

Let us prove Theorem 1.1. The proof given below is essentially the same as **Aliter** in the proof of [BCHM, Corollary 1.1.9].

Proof of Theorem 1.1. The problem is local. Hence we take an arbitrary point $P \in Y$ and will replace Y with a small open neighborhood of P in Y freely. Let U be any relatively compact open neighborhood of P in Y. By [KM, Proposition 2.36 (1)], we take a suitable finite composite of blow-ups $f\colon X' \to X$. Then, over some open neighborhood of $\overline{U}$, we can write

$$K_{X'} + B'_i = f^*(K_X + B_i) + E_i$$

such that B'_i and E_i have no common irreducible components, $f_*B'_i = B_i$, $f_*E_i = 0$, and $\operatorname{Supp} B'_i$ is smooth for every i. We may further assume that the support of $\sum_{i=1}^k B'_i$ is a smooth divisor. Then we have

$$R(X/Y, D_1, \cdots, D_k) \simeq R(X'/Y, D'_1, \cdots, D'_k)$$

over some open neighborhood of $\overline{U}$, where we put $D'_i := K_{X'} + f^*A + B'_i$ for every i. Therefore, by shrinking Y suitably and replacing X, B_i, A, and $\pi\colon X \to Y$ with X', B'_i, f^*A, and $\pi \circ f\colon X' \to Y$, respectively, we may assume that the support of $\sum_{i=1}^k B_i$ is smooth. We take a positive integer $d \geq 2$ such that dB_i is integral for every i and that dA is also integral. We put

$$\mathcal{E} := \mathcal{O}_X(dB_1) \oplus \cdots \oplus \mathcal{O}_X(dB_k).$$

We consider the projective bundle $f\colon Z := \mathbb{P}_X(\mathcal{E}) \to X$ associated to $\mathcal{E}$. We take a global section σ_i of $\mathcal{O}_X(dB_i)$ with $(\sigma_i = 0) = dB_i$ for every i. Then $\sigma = (\sigma_1, \cdots, \sigma_k)$ is a global section of $\mathcal{E}$. By the natural surjection

$$f^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1) := \mathcal{O}_Z(1),$$

we obtain a global section σ_Z of $\mathcal{O}_Z(1)$, that is, the image of $f^*\sigma$. Let S be the divisor corresponding to σ_Z on Z. Let T_i be the divisor on Z associated to the quotient

$$\mathcal{E} \to \mathcal{O}_X(dB_1) \oplus \cdots \oplus \mathcal{O}_X(dB_{i-1}) \oplus \mathcal{O}_X(dB_{i+1}) \oplus \cdots \oplus \mathcal{O}_X(dB_k).$$

It is easy to see that $T := \sum_{i=1}^k T_i$ is a simple normal crossing divisor on Z as in Lemma 2.1.

Claim. *The pair $(Z, T + S/d)$ is divisorial log terminal. Moreover, the pair $(Z, T + S/d)$ is kawamata log terminal outside T.*

Proof of Claim. We can directly check that the support of S is a simple normal crossing divisor on Z and the coefficients of S/d is less than one. Hence $(Z, T + S/d)$ is obviously kawamata log terminal outside T. From now on, we will use induction on k to prove that $(Z, T + S/d)$ is divisorial log terminal. If $k = 1$, then the statement is obvious. By adjunction and induction, for every j, we have

$$\left(K_Z + T + \frac{1}{d}S\right)\bigg|_{T_j} = K_{T_j} + (T - T_j)|_{T_j} + \frac{1}{d}S|_{T_j}$$

and Claim holds true for $(T_j, (T - T_j)|_{T_j} + S|_{T_j}/d)$. By inversion of adjunction, we know that $(Z, T + S/d)$ is divisorial log terminal. This is what we wanted. We finish the proof. □

Let us go back to the proof of Theorem 1.1. We put

$$\Gamma := \sum_{i=1}^{k} T_i + f^*A + \frac{1}{d}S.$$

Then, by Lemma 2.1,

$$\begin{aligned}\mathcal{O}_Z(md(K_Z+\Gamma)) &\simeq \mathcal{O}_Z\left(mdf^*(K_X+A)+mS\right) \\ &\simeq \mathcal{O}_Z(m)\otimes f^*\mathcal{O}_X(md(K_X+A)).\end{aligned}$$

Therefore, we obtain

$$f_*\mathcal{O}_Z(md(K_Z+\Gamma)) \simeq S^m(\mathcal{E})\otimes \mathcal{O}_X(md(K_X+A)),$$

where $S^m(\mathcal{E})$ denotes the mth symmetric product of $\mathcal{E}$. Hence we have

$$\bigoplus_{m\in\mathbb{N}}(\pi\circ f)_*\mathcal{O}_Z(md(K_Z+\Gamma)) \simeq \bigoplus_{(m_1,\cdots,m_k)\in\mathbb{N}^k} \pi_*\mathcal{O}_X\left(\sum m_i d(K_X+A+B_i)\right).$$

Thus, by Lemma 2.2, it is sufficient to prove that

$$\bigoplus_{m\in\mathbb{N}}(\pi\circ f)_*\mathcal{O}_Z(md(K_Z+\Gamma))$$

is a locally finitely generated graded $\mathcal{O}_Y$-algebra. Since T is f-ample by construction and every log canonical center of $(Z, T+S/d)$ is dominant onto X by Claim, we can find an effective $\mathbb{Q}$-divisor Δ on Z such that $K_Z+\Gamma\sim_{\mathbb{Q}} K_Z+\Delta$ and that (Z,Δ) is kawamata log terminal after replacing Y with a suitable open neighborhood of P in Y. Therefore, we obtain that

$$\bigoplus_{m\in\mathbb{N}}(\pi\circ f)_*\mathcal{O}_Z(md(K_Z+\Gamma))$$

is a locally finitely generated graded $\mathcal{O}_Y$-algebra by [F, Theorem 1.18] and Lemma 2.2 (see also [F, Lemma 2.26]). We finish the proof. □

We see that Theorem 1.2 easily follows from Theorem 1.1.

Proof of Theorem 1.2. We take an arbitrary point $P\in Y$. Throughout this proof, we will freely shrink Y around P without mentioning it explicitly. By [KM, Proposition 2.43], we can take an effective $\mathbb{Q}$-divisor B'_i such that (X, B'_i) is kawamata log terminal with $B_i + A\sim_{\mathbb{Q}} B'_i+(1-\varepsilon)A$ for every i, where ε is a small positive rational number. We put $D'_i := K_X+(1-\varepsilon)A+B'_i$ and consider the relative adjoint ring $R(X/Y, D'_1,\cdots,D'_k)$. Then $R(X/Y, D_1,\cdots,D_k)$ and $R(X/Y,D'_1,\cdots,D'_k)$ have isomorphic truncation. Hence, by Lemma 2.2, it is sufficient to prove the finite generation of $R(X/Y,D'_1,\cdots,D'_k)$. Therefore, by replacing B_i and A with B'_i and $(1-\varepsilon)A$, respectively, we may assume that (X,B_i) is kawamata log terminal for every i. We take a resolution $f\colon X'\to X$ such that

$$K_{X'}+B'_i = f^*(K_X+B_i)+E_i,$$

where B'_i and E_i have no common irreducible components, $f_*B'_i = B_i$, and $f_*E_i=0$. We may further assume that the support of $\sum_{i=1}^k B'_i$ is a simple normal crossing divisor on X'. Since (X,B_i) is kawamata log terminal, we have $\lfloor B'_i\rfloor = 0$. As in the proof of Theorem 1.1, by replacing (X,B_i), A, and $\pi\colon X\to Y$ with (X',B'_i), f^*A, and $\pi\circ f\colon X'\to Y$, respectively, we may assume that $\lfloor B_i\rfloor=0$, the support of $\sum_{i=1}^k B_i$ is a simple normal crossing divisor, and A is π-nef and π-big. Thus, by Theorem 1.1, we obtain the desired finite generation of $R(X/Y,D_1,\cdots,D_k)$. We finish the proof. □

References

[BCHM] C. Birkar, P. Cascini, C. D. Hacon, J. McKernan, Existence of minimal models for varieties of log general type, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468.

[CL] P. Cascini, V. Lazić, New outlook on the minimal model program, I, Duke Math. J. **161** (2012), no. 12, 2415–2467.

[DHP] O. Das, C. Hacon, M. Păun, On the 4-dimensional minimal model program for Kähler varieties, preprint (2022). arXiv:2205.12205 [math.AG]

[F] O. Fujino, Minimal model program for projective morphisms between complex analytic spaces, preprint (2022). arXiv:2201.11315 [math.AG]

[KM] J. Kollár, S. Mori, *Birational geometry of algebraic varieties*, With the collaboration of C. H. Clemens and A. Corti. Translated from the 1998 Japanese original. Cambridge Tracts in Mathematics, **134**. Cambridge University Press, Cambridge, 1998.

Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan

E-mail address: `fujino@math.kyoto-u.ac.jp`