ENOKI'S INJECTIVITY THEOREM (PRIVATE NOTE)

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Contents

1. Preliminaries

Let us recall the basic notion of the complex geometry. For details, see, for example, [D].

Definition 1.1 (Chern connection and its curvature form)**.** Let X be a complex manifold and let (E, h) be a holomorphic hermitian vector bundle on X. Then there exists the *Chern connection* $D = D_{(E,h)}$, which can be split in a unique way as a sum of a $(1, 0)$ and of a $(0, 1)$-connection, $D = D'_{(E,h)} + D''_{(E,h)}$. By the definition of the Chern connection, $D'' = D''_{(E,h)} = \bar{\partial}$. We obtain the *curvature form* $\Theta_h(E) := D^2_{(E,h)}$. The subscripts might be suppressed if there is no danger of confusion.

Definition 1.2 (Inner product)**.** Let X be an n-dimensional complex manifold with the hermitian metric g. We denote by ω the *fundamental form* of g. Let (E, h) be a holomorphic hermitian vector bundle on X, and u, v are E-valued (p, q)-forms with measurable coefficients, we set

$$\|u\|^2 = \int_X |u|^2 dV_\omega, \ \langle\langle u, v\rangle\rangle = \int_X \langle u, v\rangle dV_\omega,$$

where $|u|$ (resp. $\langle u, v\rangle$) is the pointwise norm (resp. inner product) induced by g and h on $\Lambda^{p,q}T^*_X \otimes E$, and $dV_\omega = \frac{1}{n!}\omega^n$.

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2. Enoki's injectivity theorem

In this section, we discuss Enoki's injectivity theorem (cf. [E, Theorem 0.2]), which contains Kollár's original injectivity theorem. We recommend the reader to compare the proof of Theorem 2.1 with the arguments in [K1, Section 2] and [K2, Chapter 9].

Theorem 2.1 (Enoki's injectivity theorem). *Let X be a compact Kähler manifold and let L be a semi-positive line bundle on X. Then, for any non-zero holomorphic section s of $L^{\otimes k}$ with some positive integer k, the multiplication homomorphism*

$$\times s : H^q(X, \omega_X \otimes L^{\otimes l}) \longrightarrow H^q(X, \omega_X \otimes L^{\otimes (l+k)}),$$

which is induced by $\otimes s$, is injective for every $q \geq 0$ and $l > 0$.

Proof. Throughout this proof, we fix a Kähler metric g on X. Let h be a smooth hermitian metric of L such that the curvature $\sqrt{-1}\Theta_h(L) = \sqrt{-1}\bar{\partial}\partial \log h$ is a smooth semi-positive $(1,1)$-form on X. We put $n = \dim X$. We introduce the space of $L^{\otimes l}$-valued harmonic (n,q)-forms as follows,

$$\mathcal{H}^{n,q}(X, L^{\otimes l}) := \{u \in C^{n,q}(X, L^{\otimes l}) | \Delta'' u = 0\}$$

for every $q \geq 0$, where

$$\Delta'' := \Delta''_{(L^{\otimes l}, h^l)} := D''^{*}_{(L^{\otimes l}, h^l)} \bar{\partial} + \bar{\partial} D''^{*}_{(L^{\otimes l}, h^l)}$$

and $C^{n,q}(X, L^{\otimes l})$ is the space of $L^{\otimes l}$-valued smooth (n,q)-forms on X. We note that $D''_{(L^{\otimes l}, h^l)} = \bar{\partial}$ and that $D''^{*}_{(L^{\otimes l}, h^l)}$ is the formal adjoint of $D''_{(L^{\otimes l}, h^l)}$. It is easy to see that $\Delta'' u = 0$ if and only if $D''^{*}_{(L^{\otimes l}, h^l)} u = \bar{\partial} u = 0$ for $u \in C^{n,q}(X, L^{\otimes l})$ since X is compact. It is well known that

$$C^{n,q}(X, L^{\otimes l}) = \mathrm{Im}\bar{\partial} \oplus \mathcal{H}^{n,q}(X, L^{\otimes l}) \oplus \mathrm{Im} D''^{*}_{(L^{\otimes l}, h^l)}$$

and

$$\mathrm{Ker}\bar{\partial} = \mathrm{Im}\bar{\partial} \oplus \mathcal{H}^{n,q}(X, L^{\otimes l}).$$

Therefore, we have the following isomorphisms,

$$H^q(X, \omega_X \otimes L^{\otimes l}) \simeq H^{n,q}(X, L^{\otimes l}) = \frac{\mathrm{Ker}\bar{\partial}}{\mathrm{Im}\bar{\partial}} \simeq \mathcal{H}^{n,q}(X, L^{\otimes l}).$$

We obtain $H^q(X, \omega_X \otimes L^{\otimes (l+k)}) \simeq \mathcal{H}^{n,q}(X, L^{\otimes (l+k)})$ similarly.

Claim. *The multiplication map*

$$\times s : \mathcal{H}^{n,q}(X, L^{\otimes l}) \longrightarrow \mathcal{H}^{n,q}(X, L^{\otimes (l+k)})$$

is well-defined.

If the claim is true, then the therorem is obvious. It is because $su = 0$ in $\mathcal{H}^{n,q}(X, L^{\otimes(l+k)})$ implies $u = 0$ for $u \in \mathcal{H}^{n,q}(X, L^{\otimes l})$. This implies the desired injectivity. Thus, it is sufficient to prove the above claim. By the Nakano identity (cf. [D, (4.6)]), we have

$$\|D''^{*}_{(L^{\otimes l},h^l)}u\|^2 + \|D''u\|^2 = \|D'^{*}u\|^2 + \langle\langle \sqrt{-1}\Theta_{h^l}(L^{\otimes l})\Lambda u, u\rangle\rangle$$

holds for $L^{\otimes l}$-valued smooth (n,q)-form u, where Λ is the adjoint of $\omega \wedge \cdot$ and ω is the fundamental form of g. If $u \in \mathcal{H}^{n,q}(X, L^{\otimes l})$, then the left hand side is zero by the definition of $\mathcal{H}^{n,q}(X, L^{\otimes l})$. Thus we obtain $\|D'^{*}u\|^2 = \langle\langle \sqrt{-1}\Theta_{h^l}(L^{\otimes l})\Lambda u, u\rangle\rangle = 0$ since $\sqrt{-1}\Theta_{h^l}(L^{\otimes l}) = \sqrt{-1}l\Theta_h(L)$ is a smooth semi-positive $(1,1)$-form on X. Therefore, $D'^{*}u = 0$ and $\langle \sqrt{-1}\Theta_{h^l}(L^{\otimes l})\Lambda u, u\rangle_{h^l} = 0$, where $\langle\ ,\ \rangle_{h^l}$ is the pointwise inner product with respect to h^l and g. By Nakano's identity again,

$$\begin{aligned}&\|D''^{*}_{(L^{\otimes(l+k)},h^{l+k})}(su)\|^2 + \|D''(su)\|^2 \\ &= \|D'^{*}(su)\|^2 + \langle\langle \sqrt{-1}\Theta_{h^{l+k}}(L^{\otimes(l+k)})\Lambda su, su\rangle\rangle\end{aligned}$$

Note that we assumed $u \in \mathcal{H}^{n,q}(X, L^{\otimes l})$. Since s is holomorphic, $D''(su) = \bar\partial(su) = 0$ by the Leibnitz rule. We know that $D'^{*}(su) = -*\bar\partial*(su) = sD'^{*}u = 0$ since s is a holomorphic $L^{\otimes k}$-valued $(0,0)$-form and $D'^{*}u = 0$, where $*$ is the Hodge star operator with respect to g. Note that D'^{*} is independent of the fiber metrics. So, we have

$$\|D''^{*}_{(L^{\otimes(l+k)},h^{l+k})}(su)\|^2 = \langle\langle \sqrt{-1}\Theta_{h^{l+k}}(L^{\otimes(l+k)})\Lambda su, su\rangle\rangle.$$

We note that

$$\begin{aligned}&\langle \sqrt{-1}\Theta_{h^{l+k}}(L^{\otimes(l+k)})\Lambda su, su\rangle_{h^{l+k}} \\ &= \frac{l+k}{k}|s|^2_{h^k}\langle \sqrt{-1}\Theta_{h^l}(L^{\otimes l})\Lambda u, u\rangle_{h^l} = 0\end{aligned}$$

where $\langle\ ,\ \rangle_{h^{l+k}}$ (resp. $|s|_{h^k}$) is the pointwise inner product (resp. the pointwise norm of s) with respect to h^{l+k} and g (resp. with respect to h^k). Thus, we obtain $D''^{*}_{(L^{\otimes(l+k)},h^{l+k})}(su) = 0$. Therefore, we know that $\Delta''_{(L^{\otimes(l+k)},h^{l+k})}(su) = 0$, equivalently, $su \in \mathcal{H}^{n,q}(X, L^{\otimes(l+k)})$. We finish the proof of the claim. This implies the desired injectivity. □

We contain Kodaira's vanishing theorem and its proof based on Bochner's technique for the reader's convenience.

Theorem 2.2 (Kodaira vanishing theorem). *Let X be a compact complex manifold and let L be a positive line bundle on X. Then $H^q(X, \omega_X \otimes L) = 0$ for every $q > 0$.*

Proof. We take a smooth hermitian metric h of L such that $\sqrt{-1}\Theta_h(L) = \sqrt{-1}\bar{\partial}\partial \log h$ is a smooth positive $(1,1)$-form on X. We define a Kähler metric g on X associated to $\omega := \sqrt{-1}\Theta_h(L)$. As we saw in the proof of Theorem 2.1, we have

$$H^q(X, \omega_X \otimes L) \simeq \mathcal{H}^{n,q}(X, L)$$

where $n = \dim X$ and $\mathcal{H}^{n,q}(X, L)$ is the space of L-valued harmonic (n,q)-forms on X. We take $u \in \mathcal{H}^{n,q}(X, L)$. By Nakano's identity, we have

$$\begin{aligned} 0 &= \|D''^{*}_{(L,h)}u\|^2 + \|D''u\|^2 \\ &= \|D'^{*}u\|^2 + \langle\langle \sqrt{-1}\Theta_h(L)\Lambda u, u\rangle\rangle. \end{aligned}$$

On the other hand, we have

$$\langle \sqrt{-1}\Theta_h(L)\Lambda u, u\rangle_h = q|u|_h^2.$$

Therefore, we obtain $0 = \|u\|^2$. Thus, we have $u = 0$. This means that $\mathcal{H}^{n,q}(X, L) = 0$ for every $q \geq 1$. Therefore, we have $H^q(X, \omega_X \otimes L) = 0$ for every $q \geq 1$. □

It is a routine work to prove Theorem 2.3 by using Theorem 2.1.

Theorem 2.3 (Torsion-freeness and vanishing theorem)**.** *Let X be a compact Kähler manifold and let Y be a projective variety. Let $\pi : X \to Y$ be a surjective morphism. Then we obtain the following properties.*

(i) *$R^i\pi_*\omega_X$ is torsion-free for every $i \geq 0$.*
(ii) *If H is an ample line bundle on Y, then*

$$H^j(Y, H \otimes R^i\pi_*\omega_X) = 0$$

for every $i \geq 0$ and $j > 0$.

For related topics, see [T], [O], [F1], and [F2]. We close this section with a conjecture.

Conjecture 2.4. *Let X be a compact Kähler manifold* (*or a smooth projective variety*) *and let D be a reduced simple normal crossing divisor on X. Let L be a semi-positive line bundle on X and let s be a non-zero holomorphic section of $L^{\otimes k}$ on X for some positive integer k. Assume that* $(s = 0)$ *contains no strata of D. Then the multiplication homomorphism*

$$\times s : H^q(X, \omega_X \otimes \mathcal{O}_X(D) \otimes L^{\otimes l}) \to H^q(X, \omega_X \otimes \mathcal{O}_X(D) \otimes L^{\otimes(l+k)}),$$

which is induced by $\otimes s$, is injective for every $q \geq 0$ and $l > 0$.

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