Corrigendum: On subadditivity of the logarithmic Kodaira dimension

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Abstract. John Lesieutre constructed an example satisfying $\kappa_\sigma \neq \kappa_\nu$. This says that the proof of the inequalities in Theorems 1.3, 1.9, and Remark 3.8 in [O. Fujino, On subadditivity of the logarithmic Kodaira dimension, J. Math. Soc. Japan **69** (2017), no. 4, 1565–1581] is insufficient. We claim that some weaker inequalities still hold true and they are sufficient for various applications.

1. Introduction

In [Les], John Lesieutre constructs a smooth projective threefold X and a pseudo-effective $\mathbb{R}$-divisor D on X such that $\kappa_\sigma(D) = 1$ and $\kappa_\nu(D) = 2$. This means that the equality $\kappa_\sigma = \kappa_\nu$ does not always hold true. In the proof of [F2, Theorem 1.3], we used the following lemma (see [F2, Lemma 2.8]), which is a special case of [Leh, Theorem 6.7 (7)].

Lemma 1.1. *Let D be a pseudo-effective Cartier divisor on a smooth projective variety X. We fix some sufficiently ample Cartier divisor A on X. Then there exist positive constants C_1 and C_2 such that*

$$C_1 m^{\kappa_\sigma(X,D)} \leq \dim H^0(X, \mathcal{O}_X(mD + A)) \leq C_2 m^{\kappa_\sigma(X,D)}$$

for every sufficiently large m.

Unfortunately, the proof of [Leh, Theorem 6.7 (7)] in [Leh] (see also [E]) depends on the wrong fact that $\kappa_\sigma = \kappa_\nu$ always holds. Moreover, Lesieutre's example says that [Leh, Theorem 6.7 (7)] is not true when D is an $\mathbb{R}$-divisor. Therefore, this trouble damages [F2, Theorems 1.3, 1.9, and Remark 3.8]. This means that the proof of the inequalities in [F2, Theorems 1.3 and 1.9] and [N, Chapter V, 4.1. Theorem (1)] is incomplete.

In this paper, we explain that slightly weaker inequalities than the original ones in [F2, Theorems 1.3 and 1.9] and [N, Chapter V, 4.1. Theorem (1)] still hold true. Fortunately, these weaker inequalities are sufficient for [F2, Corollaries 1.5 and 1.6] and some other applications. Note that one of the main purposes of [F2] is to reduce Iitaka's subadditivity conjecture on the logarithmic Kodaira dimension $\overline{\kappa}$ (see [F2, Conjecture 1.1]) to a special case of the generalized abundance conjecture (see [F2, Conjecture 1.4]). For that purpose, one of the weaker inequalities in Theorem 2.1 below is sufficient.

Remark 1.2.   Kenta Hashizume reduces Iitaka's subadditivity conjecture on the logarithmic Kodaira dimension $\overline{\kappa}$ to the generalized abundance conjecture for sufficiently general fibers (see [H, Theorem 1.2]). In some sense, his result is better than the one in [F2]. We note that his proof uses [GL, Theorem 4.3] (see Theorem 3.2 below) and that the proof of [GL, Theorem 4.3] uses [N, Chapter V, 4.2. Corollary] which follows from [N, Chapter V, 4.1. Theorem (1)]. Fortunately, the inequality (3.3) below, which is weaker than the one in [N, Chapter V, 4.1. Theorem (1)], is sufficient for our purpose. So there are no troubles in [H].

Remark 1.3.   We note that [F1, Lemma 2.4.9] is nothing but [Leh, Theorem 6.7 (7)]. Fortunately, however, we do not use it directly in [F1].

It is highly desirable to solve the following conjecture.

Conjecture 1.4.   *Let X be a smooth projective variety and let D be a pseudo-effective $\mathbb{R}$-divisor on X. Then there exist a positive integer m_0, a positive rational number C, and an ample Cartier divisor A on X such that*

$$(1.1) \qquad Cm^{\kappa_\sigma(X,D)} \le \dim H^0(X, \mathcal{O}_X(\lfloor mm_0 D \rfloor + A))$$

holds for every large positive integer m.

If Conjecture 1.4 is true, then there are no troubles in [F2, Theorems 1.3 and 1.9] and [N, Chapter V, 4.1. Theorem (1)].

The following observation may help the reader understand this paper, the trouble in [N, Chapter V, 4.1. Theorem (1)], and Conjecture 1.4.

1.5 (Observation).   Let us consider

$$f, g : \mathbb{Z}_{\ge 0} \longrightarrow \mathbb{Z}_{\ge 0}$$

such that

$$(1.2) \qquad \limsup_{m \to \infty} f(m) > 0 \qquad \text{and} \qquad \limsup_{m \to \infty} g(m) > 0.$$

We want to prove

$$(1.3) \qquad \limsup_{m \to \infty} (f(m)g(m)) > 0.$$

In general, (1.3) does not follow from (1.2). It may happen that $f(m)g(m) = 0$ holds for every m. If there exists a positive constant C such that $f(m) \ge C$ for every large positive integer m, then we have

$$\limsup_{m \to \infty} (f(m)g(m)) > 0.$$

We will freely use the notation in [F2] and work over $\mathbb{C}$, the complex number field, throughout this paper.

2. On [F2, Theorems 1.3 and 1.9]

The proof of the main theorem of [F2], that is, [F2, Theorem 1.3], is incomplete. Here we will prove slightly weaker inequalities.

THEOREM 2.1 (SEE [F2, Theorem 1.3]). *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Let D_X (resp. D_Y) be a simple normal crossing divisor on X (resp. Y). Assume that $\mathrm{Supp} f^* D_Y \subset \mathrm{Supp} D_X$. Then we have*

$$\kappa_\sigma(X, K_X + D_X) \geq \kappa_\sigma(F, K_F + D_X|_F) + \kappa(Y, K_Y + D_Y)$$

and

$$\kappa_\sigma(X, K_X + D_X) \geq \kappa(F, K_F + D_X|_F) + \kappa_\sigma(Y, K_Y + D_Y),$$

where F is a sufficiently general fiber of $f : X \to Y$.

The inequalities in Theorem 2.1 follow from the proof of [F2, Theorem 1.3] without any difficulties.

Before we prove Theorem 2.1, we give a small remark on κ_σ.

REMARK 2.2. Let D be a pseudo-effective $\mathbb{R}$-divisor on a smooth projective variety X. Then $\kappa_\sigma(X, D) = \kappa_\sigma(X, lD)$ holds for every positive integer l. We note that $\kappa_\sigma(X, lD) \geq \kappa_\sigma(X, D)$ holds by [N, Chapter V, 2.7. Proposition (1)] since $lD - D = (l-1)D$ is pseudo-effective. By definition, $\kappa_\sigma(X, lD) \leq \kappa_\sigma(X, D)$ always holds.

Let us prove Theorem 2.1.

Proof of Theorem 2.1. In this proof, we will freely use the notation in the proof of [F2, Theorem 1.3]. In the proof of [F2, Theorem 1.3], we have

$$(2.1) \quad \begin{aligned} &\dim H^0(X, \mathcal{O}_X(mk(K_X + D_X) + A + 2f^* H)) \\ &\geq r(mD; A) \cdot \dim H^0(Y, \mathcal{O}_Y(mk(K_Y + D_Y) + H)) \end{aligned}$$

for every positive integer m, where

$$(2.2) \qquad\qquad D = k(K_{X/Y} + D_X - f^* D_Y)$$

and

$$(2.3) \qquad\qquad r(mD; A) = \mathrm{rank} f_* \mathcal{O}_X(mD + A).$$

We can take a positive integer m_0 and a positive real number C_0 such that

$$(2.4) \qquad\qquad C_0 m^{\kappa(F, D|_F)} \leq r(mm_0 D; A)$$

for every large positive integer m. Since $\kappa(F, D|_F) = \kappa(F, K_F + D_X|_F)$, we have

$$(2.5) \quad \begin{aligned} &\dim H^0(X, \mathcal{O}_X(mm_0 k(K_X + D_X) + A + 2f^*H)) \\ &\geq C_0 m^{\kappa(F, K_F + D_X|_F)} \cdot \dim H^0(Y, \mathcal{O}_Y(mm_0 k(K_Y + D_Y) + H)) \end{aligned}$$

for every positive integer m by (2.1) and (2.4). We may assume that H is sufficiently ample. Then we get

$$(2.6) \quad \limsup_{m \to \infty} \frac{\dim H^0(X, \mathcal{O}_X(mm_0 k(K_X + D_X) + A + 2f^*H))}{m^{\kappa(F, K_F + D_X|_F) + \kappa_\sigma(Y, K_Y + D_Y)}} > 0$$

by (2.5) and the definition of $\kappa_\sigma(Y, K_Y + D_Y)$. This means that the following inequality

$$(2.7) \quad \kappa_\sigma(X, K_X + D_X) \geq \kappa(F, K_F + D_X|_F) + \kappa_\sigma(Y, K_Y + D_Y)$$

holds.

Similarly, we can take a positive integer m_1 and a positive real number C_1 such that

$$(2.8) \quad \begin{aligned} C_1 m^{\kappa(Y, K_Y + D_Y)} &\leq \dim H^0(Y, \mathcal{O}_Y(mm_1 k(K_Y + D_Y))) \\ &\leq \dim H^0(Y, \mathcal{O}_Y(mm_1 k(K_Y + D_Y) + H)) \end{aligned}$$

for every large positive integer m by the definition of $\kappa(Y, K_Y + D_Y)$ if H is a sufficiently ample Cartier divisor. Then, by (2.1) and (2.8), we have

$$(2.9) \quad \begin{aligned} &\dim H^0(X, \mathcal{O}_X(mm_1 k(K_X + D_X) + A + 2f^*H)) \\ &\geq C_1 m^{\kappa(Y, K_Y + D_Y)} \cdot r(mm_1 D; A) \end{aligned}$$

for every large positive integer m. Therefore, we get

$$(2.10) \quad \limsup_{m \to \infty} \frac{\dim H^0(X, \mathcal{O}_X(mm_1 k(K_X + D_X) + A + 2f^*H))}{m^{\kappa_\sigma(F, K_F + D_X|_F) + \kappa(Y, K_Y + D_Y)}} > 0$$

when A is sufficiently ample. Note that

$$(2.11) \quad \sigma(m_1 D|_F; A|_F) = \max\left\{ k \in \mathbb{Z}_{\geq 0} \cup \{-\infty\} \;\middle|\; \limsup_{m \to \infty} \frac{r(mm_1 D; A)}{m^k} > 0 \right\}$$

for a sufficiently general fiber F of $f : X \to Y$ and that

$$(2.12) \quad \begin{aligned} \kappa_\sigma(F, K_F + D_X|_F) &= \kappa_\sigma(F, D|_F) \\ &= \kappa_\sigma(F, m_1 D|_F) \\ &= \max\{\sigma(m_1 D|_F; A|_F) \,|\, A \text{ is very ample}\}. \end{aligned}$$

Hence we have the inequality

$$(2.13) \quad \kappa_\sigma(X, K_X + D_X) \geq \kappa_\sigma(F, K_F + D_X|_F) + \kappa(Y, K_Y + D_Y)$$

by (2.10). $\square$

Of course, we have to weaken inequalities in [F1, Theorem 4.12.1 and Corollary 4.12.2]

following Theorem 2.1.

Theorem 1.9 in [F2] has the same trouble as [F2, Theorem 1.3]. Of course, we can prove slightly weaker inequalities.

THEOREM 2.3 (SEE [F2, Theorem 1.9]). *Let $f : X \to Y$ be a proper surjective morphism from a normal variety X onto a smooth complete variety Y with connected fibers. Let D_X be an effective $\mathbb{Q}$-divisor on X such that (X, D_X) is lc and let D_Y be a simple normal crossing divisor on Y. Assume that $\mathrm{Supp} f^* D_Y \subset \lfloor D_X \rfloor$, where $\lfloor D_X \rfloor$ is the round-down of D_X. Then we have*

$$\kappa_\sigma(X, K_X + D_X) \geq \kappa_\sigma(F, K_F + D_X|_F) + \kappa(Y, K_Y + D_Y)$$

and

$$\kappa_\sigma(X, K_X + D_X) \geq \kappa(F, K_F + D_X|_F) + \kappa_\sigma(Y, K_Y + D_Y),$$

where F is a sufficiently general fiber of $f : X \to Y$.

It is obvious how to modify the proof of [F2, Theorem 1.9] in order to get Theorem 2.3. For the details, see the proof of Theorem 2.1 above.

As we said above, the inequalities in Theorem 2.1 are sufficient for [F2, Corollaries 1.5 and 1.6]. The reader can check it without any difficulties.

3. On [F2, Remark 3.8], [N, Chapter V, 4.1. Theorem (1)], and so on

In [F2, Remark 3.8], the author pointed out a gap in the proof of [N, Chapter V, 4.1. Theorem (1)] and filled it by using [Leh, Theorem 6.7 (7)] (see also [F1, Remark 4.11.6]). Therefore, the proof of the inequalities in [N, Chapter V, 4.1. Theorem (1)] is still incomplete.

3.1 (NAKAYAMA'S INEQUALITY FOR κ_σ). Here, we will freely use the notation in [N, Chapter V, 4.1. Theorem]. In [N, Chapter V, 4.1. Theorem (1)], Nakayama claims that the inequality

$$(3.1) \qquad \kappa_\sigma(D + f^* Q) \geq \kappa_\sigma(D; X/Y) + \kappa_\sigma(Q)$$

holds. Unfortunately, this inequality (3.1) does not follow directly from the inequality

$$(3.2) \qquad h^0(X, \lceil m(D + f^* Q) \rceil + A + 2f^* H) \geq r(mD; A) \cdot h^0(Y, \lfloor mQ \rfloor + H)$$

established in the proof of [N, Chapter V, 4.1. Theorem (1)]. By the same argument as in the proof of Theorem 2.1 above, by using [N, Chapter II, 3.7. Theorem], we can prove

$$(3.3) \qquad \kappa_\sigma(D + f^* Q) \geq \kappa_\sigma(D; X/Y) + \kappa(Q)$$

and

$$(3.4) \qquad \kappa_\sigma(D + f^* Q) \geq \kappa(D; X/Y) + \kappa_\sigma(Q).$$

If we put $D = K_{X/Y} + \Delta$ and $Q = K_Y$, then we have

$$(3.5) \qquad \kappa_\sigma(K_X + \Delta) \geq \kappa_\sigma(K_{X_y} + \Delta|_{X_y}) + \kappa(K_Y)$$

and

$$(3.6) \qquad \kappa_\sigma(K_X + \Delta) \geq \kappa(K_{X_y} + \Delta|_{X_y}) + \kappa_\sigma(K_Y).$$

Lesieutre's example does not affect [N, Chapter V, 4.1. Theorem (2)] because we do not use κ_σ for the proof of [N, Chapter V, 4.1. Theorem (2)].

We note that the inequalities (3.3) and (3.4) are sufficient for the proof of [F1, Theorem 4.12.8].

The inequality (3.1) has already played an important role in the theory of minimal models. The following result is very well known and has already been used in various papers.

THEOREM 3.2 ([DHP, Remark 2.6] AND [GL, Theorem 4.3]). *Let (X, Δ) be a projective klt pair such that Δ is a $\mathbb{Q}$-divisor. Then (X, Δ) has a good minimal model if and only if $\kappa_\sigma(X, K_X + \Delta) = \kappa(X, K_X + \Delta)$.*

In the proof of Theorem 3.2, the inequality (3.1) plays an important role in [DHP, Remark 2.6]. Fortunately, in [DHP, Remark 2.6], the inequality (3.3) is sufficient because we need the inequality (3.1) in the case where Q is a big divisor. In [GL, Theorem 4.3], Gongyo and Lehmann need [N, Chapter V, 4.2. Corollary] in the proof of [GL, Theorem 4.3]. We note that Nakayama uses the inequality (3.1) in the proof of [N, Chapter V, 4.2. Corollary]. Fortunately, we can easily see that [N, Chapter V, 4.2. Corollary] holds true because the inequality (3.3) is sufficient for that proof. We strongly recommend the reader to see [HH, Subsection 2.2] for some related topics. We can find a generalization of Theorem 3.2 (see [HH, Lemma 2.13]).

REMARK 3.3. In a recent preprint [F3], we introduce the notion of mixed-ω-sheaves and discuss some topics related to Nakayama's inequality for κ_σ. For the details, see [F3, Section 11].

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