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Canonical bundle formula and vanishing theorem

By

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Abstract

In this paper, we treat two different topics. We give sample computations of our canonical bundle formula. They help us understand our canonical bundle formula, Fujita–Kawamata's semi-positivity theorem, and Viehweg's weak positivity theorem. We also treat Viehweg's vanishing theorem.

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§ 1. Introduction

In this paper, we treat two different topics. In Section 2, we give sample computations of our canonical bundle formula (cf. [FM]). The examples constructed in Section 2 help us understand our canonical bundle formula, Fujita–Kawamata's semi-positivity theorem (cf. [K1, Theorem 5]), and Viehweg's weak positivity theorem (cf. [V2, Theorem III and Theorem 4.1]). There are no new results in Section 2. In Section 3,

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we treat Viehweg's vanishing theorem (cf. [V1, Theorem IV]) which was obtained by Viehweg as a byproduct of his original proof of the Kawamata–Viehweg vanishing theorem. It is a consequence of Bogomolov's vanishing theorem. There exists a generalization of Viehweg's vanishing theorem. See, for example, [EV1, (2.13) Theorem] and [EV2, Corollary 5.12 d)]. Here, we quickly give a proof of the generalized Viehweg vanishing theorem (cf. Theorem 3.1) as an application of the usual Kawamata–Viehweg vanishing theorem (cf. Theorem 3.3). By our proof, we see that the generalized Viehweg vanishing theorem is essentially the same as the usual Kawamata–Viehweg vanishing theorem. There are no new results in Section 3. We note that Sections 2 and 3 are mutually independent. Although this paper contains no new results, we hope that the examples and the arguments will be useful. It seems to be the first time that the generalized Viehweg vanishing theorem is treated in the relative setting (cf. Theorem 3.1). In Section 4, which is an appendix, we discuss Miyaoka's vanishing theorem. Section 4 is a memorandum for the author's talk in Professor Miyaoka's sixtieth birthday celebration symposium: Invariant in Algebraic Geometry, in November 2009.

Notation. For a $\mathbb{Q}$-divisor $D = \sum_{j=1}^{r} d_j D_j$ such that D_j is a prime divisor for every j and $D_i \neq D_j$ for $i \neq j$, we define the *round-down* $\llcorner D \lrcorner = \sum_{j=1}^{r} \llcorner d_j \lrcorner D_j$, where for every rational number x, $\llcorner x \lrcorner$ is the integer defined by $x - 1 < \llcorner x \lrcorner \leq x$. The *fractional part* $\{D\}$ of D denotes $D - \llcorner D \lrcorner$.

In Sections 3 and 4, κ (resp. ν) denotes the *Kodaira dimension* (resp. *numerical Kodaira dimension*).

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We will work over $\mathbb{C}$, the complex number field, throughout this paper.

§ 2. Sample computations of canonical bundle formula

We give sample computations of our canonical bundle formula obtained in [FM]. We will freely use the notation in [FM]. For details of our canonical bundle formula, see [FM], [F1, §3], and [F2, §3, §4, §5, and §6].

2.1 (Kummer manifolds). Let E be an elliptic curve and E^n the n-times direct product of E. Let G be the cyclic group of order two of analytic automorphisms of E^n generated by an automorphism $g : E^n \to E^n : (z_1, \cdots, z_n) \mapsto (-z_1, \cdots, -z_n)$. The automorphism g has 2^{2n} fixed points. Each singular point is terminal for $n \geq 3$ and is canonical for $n \geq 2$.

2.2 (Kummer surfaces). First, we consider $q : E^2/G \to E/G \simeq \mathbb{P}^1$, which is induced by the first projection, and $g = q \circ \mu : Y \to \mathbb{P}^1$, where $\mu : Y \to E^2/G$ is the minimal resolution of sixteen A_1-singularities. It is easy to see that Y is a $K3$ surface. In this case, it is obvious that

$$g_*\mathcal{O}_Y(mK_{Y/\mathbb{P}^1}) \simeq \mathcal{O}_{\mathbb{P}^1}(2m)$$

for every $m \geq 1$. Thus, we can put $L_{Y/\mathbb{P}^1} = D$ for any degree two Weil divisor D on $\mathbb{P}^1$. We obtain $K_Y = g^*(K_{\mathbb{P}^1} + L_{Y/\mathbb{P}^1})$. Let Q_i be the branch point of $E \to E/G \simeq \mathbb{P}^1$ for $1 \leq i \leq 4$. Then we have

$$L^{ss}_{Y/\mathbb{P}^1} = D - \sum_{i=1}^{4}(1-\frac{1}{2})Q_i = D - \sum_{i=1}^{4}\frac{1}{2}Q_i$$

by the definition of the semi-stable part $L^{ss}_{Y/\mathbb{P}^1}$. Therefore, we obtain

$$K_Y = g^*(K_{\mathbb{P}^1} + L^{ss}_{Y/\mathbb{P}^1} + \sum_{i=1}^{4}\frac{1}{2}Q_i).$$

Thus,

$$L^{ss}_{Y/\mathbb{P}^1} = D - \sum_{i=1}^{4}\frac{1}{2}Q_i \not\sim 0$$

but

$$2L^{ss}_{Y/\mathbb{P}^1} = 2D - \sum_{i=1}^{4}Q_i \sim 0.$$

Note that $L^{ss}_{Y/\mathbb{P}^1}$ is not a Weil divisor but a $\mathbb{Q}$-Weil divisor on $\mathbb{P}^1$.

2.3 (Elliptic fibrations). Next, we consider E^3/G and E^2/G. We consider the morphism $p : E^3/G \to E^2/G$ induced by the projection $E^3 \to E^2 : (z_1, z_2, z_3) \to (z_1, z_2)$. Let $\nu : X' \to E^3/G$ be the weighted blow-up of E^3/G at sixty-four $\frac{1}{2}(1,1,1)$-singularities. Thus

$$K_{X'} = \nu^* K_{E^3/G} + \sum_{j=1}^{64}\frac{1}{2}E_j,$$

where $E_j \simeq \mathbb{P}^2$ is the exceptional divisor for every j. Let P_i be an A_1-singularity of E^2/G for $1 \leq i \leq 16$. Let $\psi : X \to X'$ be the blow-up of X' along the strict transform of $p^{-1}(P_i)$, which is isomorphic to $\mathbb{P}^1$, for every i. Then we obtain the following commutative diagram.

$$\begin{array}{ccc} E^3/G & \xleftarrow{\phi:=\nu\circ\psi} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ E^2/G & \xleftarrow[\mu]{} & Y \end{array}$$

Note that

$$K_X=\phi^*K_{E^3/G}+\sum_{j=1}^{64}\frac{1}{2}E_j+\sum_{k=1}^{16}F_k,$$

where E_j is the strict transform of E_j on X and F_k is the ψ-exceptional prime divisor for every k. We can check that X is a smooth projective threefold. We put $C_i=\mu^{-1}(P_i)$ for every i. It can be checked that C_i is a (-2)-curve for every i. It is easily checked that f is smooth outside $\sum_{i=1}^{16}C_i$ and that the degeneration of f is of type I_0^* along C_i for every i. We renumber $\{E_j\}_{j=1}^{64}$ as $\{E_i^j\}$, where $f(E_i^j)=C_i$ for every $1\le i\le 16$ and $1\le j\le 4$. We note that f is flat since f is equi-dimensional.

Let us recall the following theorem (cf. [K2, Theorem 20] and [N, Corollary 3.2.1 and Theorem 3.2.3]).

Theorem 2.4 (..., Kawamata, Nakayama, ...)**.** *We have the following isomorphism.*

$$(f_*\omega_{X/Y})^{\otimes 12}\simeq \mathcal{O}_Y(\sum_{i=1}^{16}6C_i),$$

where $\omega_{X/Y}\simeq\mathcal{O}_X(K_{X/Y})=\mathcal{O}_X(K_X-f^*K_Y)$.

The proof of Theorem 2.4 depends on the investigation of the upper canonical extension of the Hodge filtration and the period map. It is obvious that

$$2K_X=f^*(2K_Y+\sum_{i=1}^{16}C_i)$$

and

$$2mK_X=f^*(2mK_Y+m\sum_{i=1}^{16}C_i)$$

for all $m\ge 1$ since $f^*C_i=2F_i+\sum_{j=1}^{4}E_i^j$. Therefore, we have $2L_{X/Y}\sim\sum_{i=1}^{16}C_i$. On the other hand, $f_*\omega_{X/Y}\simeq\mathcal{O}_Y(\llcorner L_{X/Y}\lrcorner)$. Note that Y is a smooth surface and f is flat. Since

$$\mathcal{O}_Y(12\llcorner L_{X/Y}\lrcorner)\simeq(f_*\omega_{X/Y})^{\otimes 12}\simeq\mathcal{O}_Y(\sum_{i=1}^{16}6C_i),$$

we have

$$12L_{X/Y}\sim 6\sum_{i=1}^{16}C_i\sim 12\llcorner L_{X/Y}\lrcorner.$$

Thus, $L_{X/Y}$ is a Weil divisor on Y. It is because the fractional part $\{L_{X/Y}\}$ is effective and linearly equivalent to zero. So, $L_{X/Y}$ is numerically equivalent to $\frac{1}{2}\sum_{i=1}^{16}C_i$. We have $g^*Q_i=2G_i+\sum_{j=1}^{4}C_i^j$. Here, we renumbered $\{C_j\}_{j=1}^{16}$ as $\{C_i^j\}_{i,j=1}^{4}$ such that

$g(C_i^j) = Q_i$ for every i and j. More precisely, we put $2G_i = g^*Q_i - \sum_{j=1}^4 C_i^j$ for every i. We note that we used notations in 2.2. We consider $A := g^*D - \sum_{i=1}^4 G_i$. Then A is a Weil divisor and $2A \sim \sum_{i=1}^{16} C_i$. Thus, A is numerically equivalent to $\frac{1}{2}\sum_{i=1}^{16} C_i$. Since $H^1(Y, \mathcal{O}_Y) = 0$, we can put $L_{X/Y} = A$. So, we have

$$L^{ss}_{X/Y} = g^*D - \sum_{i=1}^{4} G_i - \sum_{j=1}^{16} \frac{1}{2} C_j.$$

We obtain the following canonical bundle formula.

Theorem 2.5. *The next formula holds.*

$$K_X = f^*(K_Y + L^{ss}_{X/Y} + \sum_{j=1}^{16} \frac{1}{2} C_j),$$

where $L^{ss}_{X/Y} = g^*D - \sum_{i=1}^4 G_i - \sum_{j=1}^{16} \frac{1}{2} C_j$.

We note that $2L^{ss}_{X/Y} \sim 0$ but $L^{ss}_{X/Y} \not\sim 0$. The semi-stable part $L^{ss}_{X/Y}$ is not a Weil divisor but a $\mathbb{Q}$-divisor on Y.

The next lemma is obvious since the index of $K_{E^3/G}$ is two. We give a direct proof here.

Lemma 2.6. $H^0(Y, L_{X/Y}) = 0$.

Proof. If there exists an effective Weil divisor B on Y such that $L_{X/Y} \sim B$. Since $B \cdot C_i = -1$, we have $B \geq \frac{1}{2} C_i$ for all i. Thus $B \geq \sum_{i=1}^{16} \frac{1}{2} C_i$. This implies that $B - \sum_{i=1}^{16} \frac{1}{2} C_i$ is an effective $\mathbb{Q}$-divisor and is numerically equivalent to zero. Thus $B = \sum_{i=1}^{16} \frac{1}{2} C_i$. It is a contradiction. □

We can easily check the following corollary.

Corollary 2.7. *We have*

$$f_*\omega_{X/Y}^{\otimes m} \simeq \begin{cases} \mathcal{O}_Y(\sum_{i=1}^{16} nC_i) & \text{if } m = 2n, \\ \mathcal{O}_Y(L_{X/Y} + \sum_{i=1}^{16} nC_i) & \text{if } m = 2n+1. \end{cases}$$

In particular, $f_*\omega_{X/Y}^{\otimes m}$ *is not nef for any* $m \geq 1$*. We can also check that*

$$H^0(Y, f_*\omega_{X/Y}^{\otimes m}) \simeq \begin{cases} \mathbb{C} & \text{if } m \text{ is even}, \\ 0 & \text{if } m \text{ is odd}. \end{cases}$$

2.8 (Weak positivity). Let us recall the definition of Viehweg's weak positivity (cf. [V2, Definition 1.2] and [V4, Definition 2.11]).

Definition 2.9 (Weak positivity). Let W be a smooth quasi-projective variety and $\mathcal{F}$ a locally free sheaf on W. Let U be an open subvariety of W. Then, $\mathcal{F}$ is *weakly positive over* U if for every ample invertible sheaf $\mathcal{H}$ and every positive integer α there exists some positive integer β such that $S^{\alpha\cdot\beta}(\mathcal{F})\otimes\mathcal{H}^{\beta}$ is generated by global sections over U. This means that the natural map

$$H^0(W, S^{\alpha\cdot\beta}(\mathcal{F})\otimes\mathcal{H}^{\beta})\otimes\mathcal{O}_W\to S^{\alpha\cdot\beta}(\mathcal{F})\otimes\mathcal{H}^{\beta}$$

is surjective over U.

Remark 2.10 (cf. [V2, (1.3) Remark. iii)]). In Definition 2.9, it is enough to check the condition for one invertible sheaf $\mathcal{H}$, not necessarily ample, and all $\alpha>0$. For details, see [V4, Lemma 2.14 a)].

Remark 2.11. In [V3, Definition 3.1], $S^{\alpha\cdot\beta}(\mathcal{F})\otimes\mathcal{H}^{\otimes\beta}$ is only required to be generically generated. See also [Mo, (5.1) Definition].

We explicitly check the weak positivity for the elliptic fibration constructed in 2.3 (cf. [V2, Theorem 4.1 and Theorem III] and [V4, Theorem 2.41 and Corollary 2.45]).

Proposition 2.12. *Let $f:X\to Y$ be the elliptic fibration constructed in* 2.3*. Then $f_*\omega_{X/Y}^{\otimes m}$ is weakly positive over $Y_0=Y\setminus\sum_{i=1}^{16}C_i$. Let U be a Zariski open set such that $U\not\subset Y_0$. Then $f_*\omega_{X/Y}^{\otimes m}$ is not weakly positive over U.*

Proof. Let H be a very ample Cartier divisor on E^2/G and H' a very ample Cartier divisor on Y such that $L_{X/Y}+H'$ is very ample. We put $\mathcal{H}=\mathcal{O}_Y(\mu^*H+H')$. Let α be an arbitrary positive integer. Then

$$S^{\alpha}(f_*\omega_{X/Y}^{\otimes m})\otimes\mathcal{H}\simeq\mathcal{O}_Y(\alpha\sum_{i=1}^{16}nC_i+\mu^*H+H')$$

if $m=2n$. When $m=2n+1$, we have

$$\begin{aligned}&S^{\alpha}(f_*\omega_{X/Y}^{\otimes m})\otimes\mathcal{H}\\ \simeq&\begin{cases}\mathcal{O}_Y(\alpha\sum_{i=1}^{16}nC_i+\mu^*H+H'+L_{X/Y}+\llcorner\frac{\alpha}{2}\lrcorner\sum_{i=1}^{16}C_i) & \text{if } \alpha \text{ is odd},\\ \mathcal{O}_Y(\alpha\sum_{i=1}^{16}nC_i+\mu^*H+H'+\frac{\alpha}{2}\sum_{i=1}^{16}C_i) & \text{if } \alpha \text{ is even}.\end{cases}\end{aligned}$$

Thus, $S^{\alpha}(f_*\omega_{X/Y}^{\otimes m})\otimes\mathcal{H}$ is generated by global sections over Y_0 for every $\alpha>0$. Therefore, $f_*\omega_{X/Y}^{\otimes m}$ is weakly positive over Y_0.

Let $\mathcal{H}$ be an ample invertible sheaf on Y. We put $k=\max_j(C_j\cdot\mathcal{H})$. Let α be a positive integer with $\alpha>k/2$. We note that

$$S^{2\alpha\cdot\beta}(f_*\omega_{X/Y}^{\otimes m})\otimes\mathcal{H}^{\otimes\beta}\simeq(\mathcal{O}_Y(\alpha\sum_{i=1}^{16}mC_i)\otimes\mathcal{H})^{\otimes\beta}.$$

If $H^0(Y, S^{2\alpha\cdot\beta}(f_*\omega_{X/Y}^{\otimes m}) \otimes \mathcal{H}^{\otimes\beta}) \neq 0$, then we can take

$$G \in |(\mathcal{O}_Y(\alpha \sum_{i=1}^{16} mC_i) \otimes \mathcal{H})^{\otimes\beta}|.$$

In this case, $G \cdot C_i < 0$ for every i because $\alpha > k/2$. Therefore, we obtain $G \geq \sum_{i=1}^{16} C_i$. Thus, $S^{2\alpha\cdot\beta}(f_*\omega_{X/Y}^{\otimes m}) \otimes \mathcal{H}^{\otimes\beta}$ is not generated by global sections over U for any $\beta \geq 1$. This means that $f_*\omega_{X/Y}^{\otimes m}$ is not weakly positive over U. □

Proposition 2.12 implies that [V4, Corollary 2.45] is the best result.

2.13 (Semi-positivity). We give a supplementary example for Fujita–Kawamata's semi-positivity theorem (cf. [K1, Theorem 5]). For details of Fujita–Kawamata's semi-positivity theorem, see, for example, [Mo, §5] and [F3, Section 5].

Definition 2.14. A locally free sheaf $\mathcal{E}$ on a projective variety V is (*numerically*) *semi-positive* (or *nef*) if the tautological line bundle $\mathcal{O}_{\mathbb{P}_V(\mathcal{E})}(1)$ is nef on $\mathbb{P}_V(\mathcal{E})$.

For details of semi-positive locally free sheaves, see [V4, Proposition 2.9].

Example 2.15. Let $f : X \to Y$ be the elliptic fibration constructed in 2.3. Let $Z := C \times X$, where C is a smooth projective curve with the genus $g(C) = r \geq 2$. Let $\pi_1 : Z \to C$ (resp. $\pi_2 : Z \to X$) be the first (resp. second) projection. We put $h := f \circ \pi_2 : Z \to Y$. In this case, $K_Z = \pi_1^* K_C \otimes \pi_2^* K_X$. Therefore, we obtain

$$h_*\omega_{Z/Y}^{\otimes m} = f_*\pi_{2*}(\pi_1^*\omega_C^{\otimes m} \otimes \pi_2^*\omega_X^{\otimes m}) \otimes \omega_Y^{\otimes -m} = (f_*\omega_{X/Y}^{\otimes m})^{\oplus l},$$

where $l = \dim H^0(C, \mathcal{O}_C(mK_C))$. Thus, $l = (2m-1)r - 2m + 1$ if $m \geq 2$ and $l = r$ if $m = 1$. So, $h_*\omega_{Z/Y}$ is a rank $r \geq 2$ vector bundle on Y such that $h_*\omega_{Z/Y}$ is not semi-positive. We note that h is smooth over $Y_0 = Y \setminus \sum_{i=1}^{16} C_i$. We also note that $h_*\omega_{Z/Y}^{\otimes m}$ is weakly positive over Y_0 for every $m \geq 1$ by [V4, Theorem 2.41 and Corollary 2.45].

Example 2.15 shows that the assumption on the local monodromies around $\sum_{i=1}^{16} C_i$ is indispensable for Fujita–Kawamata's semi-positivity theorem (cf. [K1, Theorem 5 (iii)]).

We close this section with a comment on [FM].

2.16 (Comment). We give a remark on [FM, Section 4]. In [FM, 4.4], $g : Y \to X$ is a log resolution of (X, Δ). However, it is better to assume that g is a log resolution of $(X, \Delta - (1/b)B^{\Delta})$ for the proof of [FM, Theorem 4.8].

§ 3. Viehweg's vanishing theorem

In this section, we quickly give a proof of the generalized Viehweg vanishing theorem (cf. [EV1, (2.13) Theorem]) as an application of the usual Kawamata–Viehweg vanishing theorem. See also [EV2, Corollary 5.12 d)]. Our proof is different from the proofs given in [EV1] and [EV2]. We treat it in the relative setting.

Theorem 3.1 (cf. [EV1, (2.13) Theorem]). *Let $\pi : X \to S$ be a proper surjective morphism from a smooth variety X, $\mathcal{L}$ an invertible sheaf on X, and D an effective Cartier divisor on X such that* Supp$D$ *is normal crossing. Assume that $\mathcal{L}^N(-D)$ is π-nef for some positive integer N and that $\kappa(X_\eta, (\mathcal{L}^{(1)})_\eta) = m$, where X_η is the generic fiber of π, $(\mathcal{L}^{(1)})_\eta = \mathcal{L}^{(1)}|_{X_\eta}$, and*

$$\mathcal{L}^{(1)} = \mathcal{L}(-\llcorner \frac{D}{N} \lrcorner).$$

Then we have

$$R^i\pi_*(\mathcal{L}^{(1)} \otimes \omega_X) = 0$$

for $i > \dim X - \dim S - m$.

We note that Supp$D$ is not necessarily *simple* normal crossing. We only assume that Supp$D$ is normal crossing.

Remark 3.2. In Theorem 3.1, we assume that S is a point for simplicity. We note that $\kappa(X, \mathcal{L}^{(1)}) = m$ does not necessarily imply $\kappa(X, \mathcal{L}^{(i)}) = m$ for $2 \leq i \leq N-1$, where

$$\mathcal{L}^{(i)} = \mathcal{L}^{\otimes i}(-\llcorner \frac{iD}{N} \lrcorner).$$

Therefore, the original arguments in [V1] depending on Bogomolov's vanishing theorem do not seem to work in our setting.

Let us recall the Kawamata–Viehweg vanishing theorem. Although there are many formulations of the Kawamata–Viehweg vanishing theorem, the following one is the most convenient one for various applications of the log minimal model program.

Theorem 3.3 (Kawamata–Viehweg's vanishing theorem). *Let $f : Y \to X$ be a projective morphism from a smooth variety Y and M a Cartier divisor on Y. Let Δ be an effective $\mathbb{Q}$-divisor on Y such that* SuppΔ *is simple normal crossing and $\llcorner \Delta \lrcorner = 0$. Assume that $M - (K_X + \Delta)$ is f-nef and f-big. Then*

$$R^i f_* \mathcal{O}_Y(M) = 0$$

for all $i > 0$.

We note that we can prove Theorem 3.3 without using Viehweg's vanishing theorem. See, for example, [KMM, Theorem 1-2-3]. The reader can find a log canonical generalization of the Kawamata–Viehweg vanishing theorem in [F4, Theorem 2.48].

Remark 3.4. In the proof of [KMM, Theorem 1-2-3], when we construct completions $\pi' : X' \to S'$ with $\pi'|_X = \pi$ and a π-ample $\mathbb{Q}$-divisor D' on X' with $D'|_X = D$, it seems to be better to use Szabó's resolution lemma. It is because we have to make the support of the fractional part of D' have only simple normal crossings.

Remark 3.5. It is obvious that Theorem 3.3 is a special case of Theorem 3.1. By applying Theorem 3.1, the assumption in Theorem 3.3 can be weaken as follows: $M - (K_X + \Delta)$ is f-nef and $M - K_X$ is f-big. We note that $M - K_X$ is f-big if so is $M - (K_X + \Delta)$. In this section, we give a quick proof of Theorem 3.1 only by using Theorem 3.3 and Hironaka's resolution. Therefore, Theorem 3.1 is essentially the same as Theorem 3.3.

Let us start the proof of Theorem 3.1.

Proof. Without loss of generality, we can assume that S is affine. Let $f : Y \to X$ be a proper birational morphism from a smooth quasi-projective variety Y such that $\mathrm{Supp} f^*D \cup \mathrm{Exc}(f)$ is simple normal crossing. We write

$$K_Y = f^*(K_X + (1-\varepsilon)\{\frac{D}{N}\}) + E_\varepsilon.$$

Then $F = \ulcorner E_\varepsilon \urcorner$ is an effective exceptional Cartier divisor on Y and independent of ε for $0 < \varepsilon \ll 1$. Therefore, the coefficients of $F - E_\varepsilon$ are continuous for $0 < \varepsilon \ll 1$. Let L be a Cartier divisor on X such that $\mathcal{L} \simeq \mathcal{O}_X(L)$. We can assume that $\kappa(X_\eta, (L - \llcorner \frac{D}{N} \lrcorner)_\eta) = m \geq 0$. Let $\Phi : X \dashrightarrow Z$ be the relative Iitaka fibration over S with respect to $l(L - \llcorner \frac{D}{N} \lrcorner)$, where l is a sufficiently large and divisible integer. We can further assume that

$$f^*(L - \llcorner \frac{D}{N} \lrcorner) \sim_{\mathbb{Q}} \varphi^* A + E,$$

where E is an effective $\mathbb{Q}$-divisor such that $\mathrm{Supp} E \cup \mathrm{Supp} f^*D \cup \mathrm{Exc}(f)$ is simple normal crossing, $\varphi = \Phi \circ f : Y \to Z$ is a morphism, and A is a ψ-ample $\mathbb{Q}$-divisor on Z with $\psi : Z \to S$. Let $\sum_i E_i = \mathrm{Supp} E \cup \mathrm{Supp} f^*D \cup \mathrm{Exc}(f)$ be the irreducible decomposition. We can write $E_\varepsilon = \sum_i a_i^\varepsilon E_i$ and $E = \sum_i b_i E_i$ and note that a_i^ε is continuous for $0 < \varepsilon \ll 1$. We put $\Delta_\varepsilon = F - E_\varepsilon + \varepsilon E$. By definition, we can see that every coefficient of Δ_ε is in $[0, 2)$ for $0 < \varepsilon \ll 1$. Thus, $\llcorner \Delta_\varepsilon \lrcorner$ is reduced. If $a_i^\varepsilon < 0$, then $a_i^\varepsilon \geq -1 + \frac{1}{N}$ for $0 < \varepsilon \ll 1$. Therefore, if $\ulcorner a_i^\varepsilon \urcorner - a_i^\varepsilon + \varepsilon b_i \geq 1$ for $0 < \varepsilon \ll 1$, then $a_i^\varepsilon > 0$. Thus, $F' = F - \llcorner \Delta_\varepsilon \lrcorner$ is effective and f-exceptional for $0 < \varepsilon \ll 1$. On the other hand,

$(Y, \{\Delta_\varepsilon\})$ is obviously klt for $0 < \varepsilon \ll 1$. We note that

$$\begin{aligned}
&f^*(K_X + L - \llcorner \frac{D}{N} \lrcorner) + F' - (K_Y + \{\Delta_\varepsilon\}) \\
&= f^*(K_X + L - \llcorner \frac{D}{N} \lrcorner) + F - f^*(K_X + (1-\varepsilon)\{\frac{D}{N}\}) - E_\varepsilon \\
&\quad - (F - E_\varepsilon + \varepsilon E) \\
&\sim_{\mathbb{Q}} (1-\varepsilon) f^*(L - \frac{D}{N}) + \varepsilon \varphi^* A
\end{aligned}$$

for a rational number ε with $0 < \varepsilon \ll 1$. We put

$$M = f^*(K_X + L - \llcorner \frac{D}{N} \lrcorner) + F'.$$

By combining the long exact sequence

$$\cdots \to R^i p_* \mathcal{O}_Y(M) \to R^i p_* \mathcal{O}_Y(M+H) \to R^i p_* \mathcal{O}_H(M+H) \to \cdots$$

obtained from

$$0 \to \mathcal{O}_Y(M) \to \mathcal{O}_Y(M+H) \to \mathcal{O}_H(M+H) \to 0$$

for a p-ample general smooth Cartier divisor H on Y, where $p = \psi \circ \varphi = \pi \circ f : Y \to S$, and the induction on the dimension, we obtain

$$R^i p_* \mathcal{O}_Y(M) = R^i p_* \mathcal{O}_Y(f^*(K_X + L - \llcorner \frac{D}{N} \lrcorner) + F') = 0$$

for every $i > \dim Y - \dim S - m = \dim X - \dim S - m$ by Theorem 3.3 (cf. [V1, Remark 0.2]). We note that

$$M - (K_Y + \{\Delta_\varepsilon\}) \sim_{\mathbb{Q}} (1-\varepsilon) f^*(L - \frac{D}{N}) + \varepsilon \varphi^* A,$$

$$(M+H) - (K_Y + \{\Delta_\varepsilon\}) \sim_{\mathbb{Q}} (1-\varepsilon) f^*(L - \frac{D}{N}) + \varepsilon \varphi^* A + H,$$

and

$$(M+H)|_H - (K_H + \{\Delta_\varepsilon\}|_H) \sim_{\mathbb{Q}} (1-\varepsilon) f^*(L - \frac{D}{N})|_H + \varepsilon \varphi^* A|_H.$$

We also note that $(H, \{\Delta_\varepsilon\}|_H)$ is klt and

$$\kappa(H_\eta, (\varphi^* A)|_{H_\eta}) \geq \min\{m, \dim H_\eta\}.$$

On the other hand,

$$R^i f_* \mathcal{O}_Y(M) = R^i f_* \mathcal{O}_Y(f^*(K_X + L - \llcorner \frac{D}{N} \lrcorner) + F') = 0$$

for every $i > 0$ by Theorem 3.3. We note that

$$f_*\mathcal{O}_Y(f^*(K_X + L - \llcorner \frac{D}{N} \lrcorner) + F') \simeq \mathcal{O}_X(K_X + L - \llcorner \frac{D}{N} \lrcorner)$$

by the projection formula because F' is effective and f-exceptional. Therefore, we obtain

$$R^i\pi_*\mathcal{O}_X(K_X + L - \llcorner \frac{D}{N} \lrcorner) = R^i p_*\mathcal{O}_Y(M) = 0$$

for every $i > \dim X - \dim S - m$. $\square$

We close this section with an obvious corollary.

Corollary 3.6. *Let X be an n-dimensional smooth complete variety, $\mathcal{L}$ an invertible sheaf on X. Assume that $D \in |\mathcal{L}^N|$ for some positive integer N and that* $\mathrm{Supp}D$ *is simple normal crossing. Then we have*

$$H^i(X, \mathcal{L}^{(1)} \otimes \omega_X) = 0$$

for $i > n - \kappa(X, \{\frac{D}{N}\})$.

§ 4. Appendix: Miyaoka's vanishing theorem

The following statement is a correct formulation of Miyaoka's vanishing theorem (cf. [Mi, Proposition 2.3]) from our modern viewpoint. Miyaoka's vanishing theorem seems to be the first vanishing theorem for the integral part of $\mathbb{Q}$-divisors.

Theorem 4.1. *Let X be a smooth complete variety with $\dim X \geq 2$ and D a Cartier divisor on X. Assume that D is numerically equivalent to $M + B$, where M is a nef $\mathbb{Q}$-divisor on X with $\nu(X, M) \geq 2$ and B is an effective $\mathbb{Q}$-divisor with $\llcorner B \lrcorner = 0$. Then $H^1(X, \mathcal{O}_X(-D)) = 0$.*

Proof. By the Serre duality, it is sufficient to see that $H^{n-1}(X, \mathcal{O}_X(K_X + D)) = 0$, where $n = \dim X$. Let $\mathcal{J}(X, B)$ be the multiplier ideal sheaf of (X, B). We consider

$$\begin{aligned} \cdots \to H^{n-1}(X, \mathcal{O}_X(K_X + D) \otimes \mathcal{J}(X, B)) \to H^{n-1}(X, \mathcal{O}_X(K_X + D)) \\ \to H^{n-1}(X, \mathcal{O}_X(K_X + D) \otimes \mathcal{O}_X/\mathcal{J}(X, B)) \to \cdots . \end{aligned}$$

Since $\llcorner B \lrcorner = 0$, we see that $\dim \mathrm{Supp}\mathcal{O}_X/\mathcal{J}(X, B) \leq n-2$. Therefore, $H^{n-1}(X, \mathcal{O}_X(K_X + D) \otimes \mathcal{O}_X/\mathcal{J}(X, B)) = 0$. Thus, it is enough to see that $H^{n-1}(X, \mathcal{O}_X(K_X + D) \otimes \mathcal{J}(X, B)) = 0$. Let $f : Y \to X$ be a resolution such that $\mathrm{Supp}f^*B$ is a simple normal crossing divisor. Then we have $\mathcal{J}(X, B) = f_*\mathcal{O}_Y(K_{Y/X} - \llcorner f^*B \lrcorner)$ and $R^i f_*\mathcal{O}_Y(K_{Y/X} - \llcorner f^*B \lrcorner) = 0$ for every $i > 0$. So, we obtain $H^{n-1}(X, \mathcal{O}_X(K_X + D) \otimes \mathcal{J}(X, B)) \simeq$

$H^{n-1}(Y, \mathcal{O}_Y(K_Y + f^*D - \llcorner f^*B \lrcorner)) = 0$ by the usual general hyperplane cutting technique (cf. the proof of Theorem 3.1) and Kawamata–Viehweg's vanishing theorem (cf. Theorem 3.3). $\square$

Remark 4.2. In Theorem 4.1, we can replace the assumption $\nu(X, M) \geq 2$ with $\kappa(Y, f^*D - \llcorner f^*B \lrcorner) \geq 2$ by Theorem 3.1.

References

[EV1] H. Esnault, E. Viehweg, Logarithmic de Rham complexes and vanishing theorems, Invent. Math. **86** (1986), no. 1, 161–194.

[EV2] H. Esnault, E. Viehweg, *Lectures on vanishing theorems*, DMV Seminar, **20**. Birkhäuser Verlag, Basel, 1992.

[F1] O. Fujino, Algebraic fiber spaces whose general fibers are of maximal Albanese dimension, Nagoya Math. J. **172** (2003), 111–127.

[F2] O. Fujino, A canonical bundle formula for certain algebraic fiber spaces and its applications, Nagoya Math. J. **172** (2003), 129–171.

[F3] O. Fujino, Remarks on algebraic fiber spaces, J. Math. Kyoto Univ. **45** (2005), no. 4, 683–699.

[F4] O. Fujino, Introduction to the log minimal model program for log canonical pairs, preprint (2009).

[FM] O. Fujino, S. Mori, A canonical bundle formula, J. Differential Geom. **56** (2000), no. 1, 167–188.

[K1] Y. Kawamata, Characterization of abelian varieties, Compositio Math. **43** (1981), no. 2, 253–276.

[K2] Y. Kawamata, Kodaira dimension of certain algebraic fiber spaces, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **30** (1983), no. 1, 1–24.

[KMM] Y. Kawamata, K. Matsuda, and K. Matsuki, Introduction to the minimal model problem, Algebraic geometry, Sendai, 1985, 283–360, Adv. Stud. Pure Math., **10**, North-Holland, Amsterdam, 1987.

[Mi] Y. Miyaoka, On the Mumford-Ramanujam vanishing theorem on a surface, Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, pp. 239–247, Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980.

[Mo] S. Mori, Classification of higher-dimensional varieties, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 269–331, Proc. Sympos. Pure Math., **46**, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[N] N. Nakayama, Local structure of an elliptic fibration, Higher dimensional birational geometry (Kyoto, 1997), 185–295, Adv. Stud. Pure Math., **35**, Math. Soc. Japan, Tokyo, 2002.

[V1] E. Viehweg, Vanishing theorems, J. Reine Angew. Math. **335** (1982), 1–8.

[V2] E. Viehweg, Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces, Algebraic varieties and analytic varieties (Tokyo, 1981), 329–353, Adv. Stud. Pure Math., **1**, North-Holland, Amsterdam, 1983.

[V3] E. Viehweg, Weak positivity and the additivity of the Kodaira dimension. II. The local Torelli map, Classification of algebraic and analytic manifolds (Katata, 1982), 567–589, Progr. Math., **39**, Birkhäuser Boston, Boston, MA, 1983.

[V4] E. Viehweg, *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], **30**. Springer-Verlag, Berlin, 1995.