

Fundamental Properties of Basic slc-trivial Fibrations II

by

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Abstract

We prove that if the moduli \mathbb{Q} -b-divisor of a basic slc-trivial fibration is b-numerically trivial then it is \mathbb{Q} -b-linearly trivial. As a consequence, we prove that the moduli part of a basic slc-trivial fibration is semi-ample when the base space is a curve.

2010 Mathematics Subject Classification: Primary 14N30; Secondary 14D07, 32G20, 14E30.
Keywords: basic slc-trivial fibrations, variation of mixed Hodge structure, b-semi-ampleness conjecture.

§1. Introduction

This paper is a continuation of the first author's paper: [Fu5]. We strongly recommend the reader to look at [Fu5, 1. Introduction] before starting to read this paper. In [Fu5], we introduced the notion of basic slc-trivial fibrations, which is a kind of canonical bundle formula for reducible varieties, and investigated some fundamental properties. For the precise definition of basic slc-trivial fibrations, see [Fu5, Definition 4.1] or Definition 3.1 below. The following statement is one of the main results of [Fu5].

Theorem 1.1 ([Fu5, Theorem 1.2]). *Let $f: (X, B) \rightarrow Y$ be a basic slc-trivial fibration and let \mathbf{B} and \mathbf{M} be the induced discriminant and moduli \mathbb{Q} -b-divisors of Y respectively. Then we have the following properties:*

Communicated by Y. Namikawa. Received June 9, 2019. Revised March 16, 2020; March 23, 2020; April 21, 2020.

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- (i) $\mathbf{K} + \mathbf{B}$ is \mathbb{Q} -b-Cartier, and
- (ii) \mathbf{M} is b-potentially nef, that is, there exists a proper birational morphism $\sigma: Y' \rightarrow Y$ from a normal variety Y' such that $\mathbf{M}_{Y'}$ is a potentially nef \mathbb{Q} -divisor on Y' and that $\mathbf{M} = \overline{\mathbf{M}_{Y'}}$.

For the definition and some basic properties of b-divisors, see [C, 2.3.2 b-divisors] and [Fu5, Section 2].

On moduli \mathbb{Q} -b-divisors, we have the following conjecture, which is still widely open.

Conjecture 1.2 (b-semi-ampleness conjecture, see [Fu5, Conjecture 1.4]). *Let $f: (X, B) \rightarrow Y$ be a basic slc-trivial fibration. Then the moduli \mathbb{Q} -b-divisor \mathbf{M} is b-semi-ample.*

The main purpose of this paper is to prove the following theorem.

Theorem 1.3 (Main Theorem). *Let $f: (X, B) \rightarrow Y$ be a basic slc-trivial fibration such that Y is complete. Let \mathbf{M} be the moduli \mathbb{Q} -b-divisor associated to $f: (X, B) \rightarrow Y$. Assume that there exists a proper birational morphism $\sigma: Y' \rightarrow Y$ from a normal variety Y' such that $\mathbf{M} = \overline{\mathbf{M}_{Y'}}$ with $\mathbf{M}_{Y'} \equiv 0$. Then $\mathbf{M}_{Y'} \sim_{\mathbb{Q}} 0$ holds.*

Theorem 1.3 solves Conjecture 1.2 when the moduli \mathbb{Q} -b-divisor \mathbf{M} is b-numerically trivial. It is obviously a generalization of [A2, Theorem 3.5] and [F1, Theorem 1.3]. More precisely, Florin Ambro and Enrica Floris proved Theorem 1.3 for klt-trivial fibrations and lc-trivial fibrations, respectively.

As a direct consequence of Theorem 1.3, we have the following result: Corollary 1.4. It says that the b-semi-ampleness conjecture (see Conjecture 1.2) holds true when the base space is a curve. Note that Corollary 1.4 was already proved for klt-trivial fibrations by Florin Ambro (see [A1, Theorem 0.1]).

Corollary 1.4. *Let $f: (X, B) \rightarrow Y$ be a basic slc-trivial fibration with $\dim Y = 1$. Then the moduli \mathbb{Q} -divisor M_Y of $f: (X, B) \rightarrow Y$ is semi-ample.*

For the proof of Theorem 1.3, we closely follow Floris's arguments in [F1]. We adapt her proof of Theorem 1.3 for lc-trivial fibrations to our setting. As is well known, the main ingredient of [A1, Theorem 0.1], [A2, Theorem 3.5], and [F1, Theorem 1.3] is Deligne's result on local subsystems of polarizable variations of \mathbb{Q} -Hodge structure (see [D1, Corollaire (4.2.8)]).

In [FF1], the first and the second authors discussed variations of mixed Hodge structure toward applications for higher-dimensional algebraic varieties (see also [FFS]). One of the most important applications of [FF1] is the proof of the projectivity of the coarse moduli spaces of stable varieties in [Fu4]. Then the first author

introduced the notion of basic slc-trivial fibrations in [Fu5] in order to make results in [FF1] useful for various geometric applications. The first and the third authors established that every quasi-log canonical pair has only Du Bois singularities in [FL] by using [Fu5]. We strongly recommend the reader to look at [Fu5, 1. Introduction] for more details. In this paper, we prove [Fu5, Conjecture 1.4] under some special assumption. We freely use the formulation introduced in [Fu5] and the arguments in this paper heavily depend on [FF1].

We briefly explain the organization of this paper. In Section 2, we fix the notation and recall some definitions for the reader's convenience. In Section 3, we quickly recall the notion of basic slc-trivial fibrations and some definitions following [Fu5]. In Section 4, we see that the cyclic group action constructed in [Fu5, Section 6] preserves some parts of weight filtrations of the variation of mixed Hodge structure. Section 5 is devoted to the proof of Theorem 1.3. By using the result obtained in Section 4, we reduce Theorem 1.3 to Deligne's result on local subsystems of polarizable variations of \mathbb{Q} -Hodge structure.

Conventions. We work over \mathbb{C} , the complex number field, throughout this paper. We freely use the basic notation of the minimal model program as in [Fu1] and [Fu3]. A *scheme* means a separated scheme of finite type over \mathbb{C} . A *variety* means a reduced scheme, that is, a reduced separated scheme of finite type over \mathbb{C} . In this paper, a variety may be reducible. However, we sometimes assume that a variety is irreducible without mentioning it explicitly if there is no danger of confusion. The set of integers (resp. rational numbers) is denoted by \mathbb{Z} (resp. \mathbb{Q}). The set of positive rational numbers (resp. integers) is denoted by $\mathbb{Q}_{>0}$ (resp. $\mathbb{Z}_{>0}$).

In this paper, we do not use \mathbb{R} -divisors. We only use \mathbb{Q} -divisors.

§2. Preliminaries

In this section, we quickly recall some basic definitions and notation for the reader's convenience. For the details, see [Fu5, Section 2].

Let us start with the definition of *simple normal crossing pairs*.

Definition 2.1 (Simple normal crossing pairs). We say that the pair (X, B) is *simple normal crossing* at a point $a \in X$ if X has a Zariski open neighborhood U of a that can be embedded in a smooth variety M , where M has a regular system of parameters $(x_1, \dots, x_p, y_1, \dots, y_r)$ at $a = 0$ in which U is defined by a monomial equation

$$x_1 \cdots x_p = 0$$

and

$$B = \sum_{i=1}^r b_i (y_i = 0)|_U, \quad b_i \in \mathbb{Q}.$$

We say that (X, B) is a *simple normal crossing pair* if it is simple normal crossing at every point of X . If $(X, 0)$ is a simple normal crossing pair, then X is called a *simple normal crossing variety*. If (X, B) is a simple normal crossing pair and B is reduced, then B is called a *simple normal crossing divisor* on X .

Let (X, B) be a simple normal crossing pair such that all the coefficients of B are less than or equal to one. Let $\nu: X^\nu \rightarrow X$ be the normalization of X . We put $K_{X^\nu} + \Theta = \nu^*(K_X + B)$, that is, Θ is the sum of the inverse images of B and the singular locus of X . By assumption, all the coefficients of Θ are less than or equal to one. Therefore, it is easy to see that (X^ν, Θ) is sub log canonical. In this situation, we simply say that W is a *stratum* of (X, B) if W is an irreducible component of X or W is the ν -image of some log canonical center of (X^ν, Θ) . We note that a stratum of a simple normal crossing variety X means a stratum of a simple normal crossing pair $(X, 0)$.

We write the precise definition of *semi-log canonical pairs*, *slc centers*, and *slc strata* for the reader's convenience. For the details of semi-log canonical pairs, we recommend the reader to see [Fu2].

Definition 2.2 (Semi-log canonical pairs). Let X be an equidimensional scheme which satisfies Serre's S_2 condition and is normal crossing in codimension one. Let Δ be an effective \mathbb{Q} -divisor on X such that no irreducible component of $\text{Supp } \Delta$ is contained in the singular locus of X and that $K_X + \Delta$ is \mathbb{Q} -Cartier. We say that (X, Δ) is a *semi-log canonical pair* if (X^ν, Δ_{X^ν}) is log canonical in the usual sense, where $\nu: X^\nu \rightarrow X$ is the normalization of X and $K_{X^\nu} + \Delta_{X^\nu} = \nu^*(K_X + \Delta)$, that is, Δ_{X^ν} is the sum of the inverse images of Δ and the conductor of X . An *slc center* of (X, Δ) is the ν -image of an lc center of (X^ν, Δ_{X^ν}) . An *slc stratum* of (X, Δ) means either an slc center of (X, Δ) or an irreducible component of X .

We recall various definitions and operations of (\mathbb{Q} -)divisors. We note that we are mainly interested in *reducible* varieties in this paper.

2.3 (Divisors). Let X be a scheme with structure sheaf \mathcal{O}_X and let \mathcal{K}_X be the sheaf of total quotient rings of \mathcal{O}_X . Let \mathcal{K}_X^* denote the (multiplicative) sheaf of invertible elements in \mathcal{K}_X , and \mathcal{O}_X^* the sheaf of invertible elements in \mathcal{O}_X . We note that $\mathcal{O}_X \subset \mathcal{K}_X$ and $\mathcal{O}_X^* \subset \mathcal{K}_X^*$ hold. A *Cartier divisor* D on X is a global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$, that is, D is an element of $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. A *\mathbb{Q} -Cartier divisor* is an element of $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let D_1 and D_2 be two \mathbb{Q} -Cartier divisors on

X . Then D_1 is *linearly* (resp. \mathbb{Q} -*linearly*) *equivalent* to D_2 , denoted by $D_1 \sim D_2$ (resp. $D_1 \sim_{\mathbb{Q}} D_2$), if

$$D_1 = D_2 + \sum_{i=1}^k r_i(f_i)$$

such that $f_i \in \Gamma(X, \mathcal{K}_X^*)$ and $r_i \in \mathbb{Z}$ (resp. $r_i \in \mathbb{Q}$) for every i . We note that (f_i) is a *principal Cartier divisor* associated to f_i , that is, the image of f_i by

$$\Gamma(X, \mathcal{K}_X^*) \rightarrow \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*).$$

Let $f: X \rightarrow Y$ be a morphism between schemes. If there exists a \mathbb{Q} -Cartier divisor B on Y such that $D_1 \sim_{\mathbb{Q}} D_2 + f^*B$, then D_1 is said to be *relatively \mathbb{Q} -linearly equivalent* to D_2 . It is denoted by $D_1 \sim_{\mathbb{Q},f} D_2$ or $D_1 \sim_{\mathbb{Q},Y} D_2$.

From now on, let X be an equidimensional scheme. We note that X is not necessarily regular in codimension one. A (*Weil*) *divisor* D on X is a finite formal sum

$$D = \sum_i d_i D_i,$$

where D_i is an irreducible reduced closed subscheme of X of pure codimension one and d_i is an integer for every i such that $D_i \neq D_j$ for every $i \neq j$. If $d_i \in \mathbb{Q}$ for every i , then D is called a \mathbb{Q} -*divisor*. Let $D = \sum_i d_i D_i$ be a \mathbb{Q} -divisor as above. We put

$$D^{\leq 1} = \sum_{d_i \leq 1} d_i D_i, \quad D^{< 1} = \sum_{d_i < 1} d_i D_i, \quad D^{=1} = \sum_{d_i=1} D_i, \quad \text{and} \quad [D] = \sum_i [d_i] D_i,$$

where $[d_i]$ is the integer defined by $d_i \leq [d_i] < d_i + 1$. Let D be a \mathbb{Q} -divisor. We also put

$$[D] = -[-D].$$

We call D a *subboundary* \mathbb{Q} -divisor if $D = D^{\leq 1}$ holds. When D is effective and $D = D^{\leq 1}$ holds, we call D a *boundary* \mathbb{Q} -divisor.

We further assume that $f: X \rightarrow Y$ is a surjective morphism onto an irreducible variety Y . Then we put

$$D^v = \sum_{f(D_i) \subsetneq Y} d_i D_i \quad \text{and} \quad D^h = D - D^v,$$

and call D^v the *vertical part* and D^h the *horizontal part* of D with respect to $f: X \rightarrow Y$, respectively.

Finally, let D be a \mathbb{Q} -Cartier divisor on a complete normal irreducible variety X . If $D \cdot C = 0$ for any complete curve C on X , then D is said to be *numerically trivial*. When D is numerically trivial, we simply write $D \equiv 0$.

Let us recall the definition of *potentially nef divisors* introduced by the first author in [Fu5].

Definition 2.4 (Potentially nef divisors, see [Fu5, Definition 2.5]). Let X be a normal irreducible variety and let D be a divisor on X . If there exist a completion X^\dagger of X , that is, X^\dagger is a complete normal variety and contains X as a dense Zariski open set, and a nef divisor D^\dagger on X^\dagger such that $D = D^\dagger|_X$, then D is called a *potentially nef* divisor on X . A finite $\mathbb{Q}_{>0}$ -linear combination of potentially nef divisors is called a *potentially nef* \mathbb{Q} -divisor.

Although it is dispensable, the following definition is very useful when we state our results (see Theorems 1.1 and 1.3). We note that the \mathbb{Q} -Cartier closure of a \mathbb{Q} -Cartier \mathbb{Q} -divisor D on a normal variety X is the \mathbb{Q} -b-divisor \overline{D} with trace

$$\overline{D}_Y = f^*D,$$

where $f: Y \rightarrow X$ is a proper birational morphism from a normal variety Y .

Definition 2.5 (see [Fu5, Definition 2.12]). Let X be a normal irreducible variety. A \mathbb{Q} -b-divisor \mathbf{D} of X is *b-potentially nef* (resp. *b-semi-ample*) if there exists a proper birational morphism $X' \rightarrow X$ from a normal variety X' such that $\mathbf{D} = \overline{\mathbf{D}_{X'}}$, that is, \mathbf{D} is the \mathbb{Q} -Cartier closure of $\mathbf{D}_{X'}$, and that $\mathbf{D}_{X'}$ is potentially nef (resp. semi-ample). A \mathbb{Q} -b-divisor \mathbf{D} of X is *\mathbb{Q} -b-Cartier* if there is a proper birational morphism $X' \rightarrow X$ from a normal variety X' such that $\mathbf{D} = \overline{\mathbf{D}_{X'}}$.

Let X be a complete normal irreducible variety. A \mathbb{Q} -b-divisor \mathbf{D} of X is *b-numerically trivial* (resp. *\mathbb{Q} -b-linearly trivial*) if there exists a proper birational morphism $X' \rightarrow X$ from a complete normal variety X' such that $\mathbf{D} = \overline{\mathbf{D}_{X'}}$ with $\mathbf{D}_{X'} \equiv 0$ (resp. $\mathbf{D}_{X'} \sim_{\mathbb{Q}} 0$).

For the details of (b-)potentially nef divisors, we recommend the reader to see [Fu5, Section 2].

§3. Quick review of basic slc-trivial fibrations

In this section, we quickly recall some definitions of *basic slc-trivial fibrations* in [Fu5, Section 4]. We recommend the reader to see [Fu5, 1.15] for some historical comments.

We introduce the notion of basic slc-trivial fibrations.

Definition 3.1 (Basic slc-trivial fibrations, see [Fu5, Definition 4.1]). A *pre-basic slc-trivial fibration* $f: (X, B) \rightarrow Y$ consists of a projective surjective morphism

$f: X \rightarrow Y$ and a simple normal crossing pair (X, B) satisfying the following properties:

- (1) Y is a normal irreducible variety,
- (2) every stratum of X is dominant onto Y and $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$,
- (3) B is a \mathbb{Q} -divisor such that $B = B^{\leq 1}$ holds over the generic point of Y , and
- (4) there exists a \mathbb{Q} -Cartier \mathbb{Q} -divisor D on Y such that

$$K_X + B \sim_{\mathbb{Q}} f^*D.$$

If a pre-basic slc-trivial fibration $f: (X, B) \rightarrow Y$ also satisfies

- (5) $\text{rank } f_*\mathcal{O}_X([-B^{<1}]) = 1$,

then it is called a *basic slc-trivial fibration*.

Roughly speaking, if X is irreducible and (X, B) is sub kawamata log terminal (resp. sub log canonical) over the generic point of Y , then it is a klt-trivial fibration (resp. an lc-trivial fibration).

In order to define discriminant \mathbb{Q} -b-divisors and moduli \mathbb{Q} -b-divisors for basic slc-trivial fibrations, we need the notion of induced (pre-)basic slc-trivial fibrations.

3.2 (Induced (pre-)basic slc-trivial fibrations, see [Fu5, 4.3]). Let $f: (X, B) \rightarrow Y$ be a (pre-)basic slc-trivial fibration and let $\sigma: Y' \rightarrow Y$ be a generically finite surjective morphism from a normal irreducible variety Y' . Then we have an *induced (pre-)basic slc-trivial fibration* $f': (X', B_{X'}) \rightarrow Y'$, where $B_{X'}$ is defined by $\mu^*(K_X + B) = K_{X'} + B_{X'}$, with the following commutative diagram:

$$\begin{array}{ccc} (X', B_{X'}) & \xrightarrow{\mu} & (X, B) \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{\sigma} & Y, \end{array}$$

where X' coincides with $X \times_Y Y'$ over a nonempty Zariski open set of Y' . More precisely, X' is a simple normal crossing variety with a morphism $X' \rightarrow X \times_Y Y'$ that is an isomorphism over a nonempty Zariski open set of Y' such that X' is projective over Y' and that every stratum of X' is dominant onto Y' .

Now we are ready to define *discriminant \mathbb{Q} -b-divisors* and *moduli \mathbb{Q} -b-divisors* for basic slc-trivial fibrations.

3.3 (Discriminant and moduli \mathbb{Q} -b-divisors, see [Fu5, 4.5]). Let $f: (X, B) \rightarrow Y$ be a (pre-)basic slc-trivial fibration as in Definition 3.1. Let P be a prime divisor on Y . By shrinking Y around the generic point of P , we assume that P is Cartier. We set

$$b_P = \max \left\{ t \in \mathbb{Q} \mid \begin{array}{l} (X^\nu, \Theta + t\nu^* f^* P) \text{ is sub log canonical} \\ \text{over the generic point of } P \end{array} \right\},$$

where $\nu: X^\nu \rightarrow X$ is the normalization and $K_{X^\nu} + \Theta = \nu^*(K_X + B)$, that is, Θ is the sum of the inverse images of B and the singular locus of X , and set

$$B_Y = \sum_P (1 - b_P)P,$$

where P runs over prime divisors on Y . Then it is easy to see that B_Y is a well-defined \mathbb{Q} -divisor on Y and is called the *discriminant \mathbb{Q} -divisor* of $f: (X, B) \rightarrow Y$. We set

$$M_Y = D - K_Y - B_Y$$

and call M_Y the *moduli \mathbb{Q} -divisor* of $f: (X, B) \rightarrow Y$. By definition, we have

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Y + B_Y + M_Y).$$

Let $\sigma: Y' \rightarrow Y$ be a proper birational morphism from a normal variety Y' and let $f': (X', B_{X'}) \rightarrow Y'$ be an induced (pre-)basic slc-trivial fibration by $\sigma: Y' \rightarrow Y$. We can define $B_{Y'}$, $K_{Y'}$ and $M_{Y'}$ such that $\sigma^*D = K_{Y'} + B_{Y'} + M_{Y'}$, $\sigma_*B_{Y'} = B_Y$, $\sigma_*K_{Y'} = K_Y$ and $\sigma_*M_{Y'} = M_Y$. We note that $B_{Y'}$ is independent of the choice of $(X', B_{X'})$, that is, $B_{Y'}$ is well defined. Hence there exist a unique \mathbb{Q} -b-divisor \mathbf{B} such that $\mathbf{B}_{Y'} = B_{Y'}$ for every $\sigma: Y' \rightarrow Y$ and a unique \mathbb{Q} -b-divisor \mathbf{M} such that $\mathbf{M}_{Y'} = M_{Y'}$ for every $\sigma: Y' \rightarrow Y$. Note that \mathbf{B} is called the *discriminant \mathbb{Q} -b-divisor* and that \mathbf{M} is called the *moduli \mathbb{Q} -b-divisor* associated to $f: (X, B) \rightarrow Y$. We sometimes simply say that \mathbf{M} is the *moduli part* of $f: (X, B) \rightarrow Y$.

For the full details of this section, we recommend the reader to see [Fu5, Section 4].

§4. On variation of mixed Hodge structure

This section heavily depends on [FF1, Sections 4 and 7]. We strongly recommend the reader to take a quick look at [FF1, Section 4] before reading this section.

Let us quickly recall [FF1, Theorem 7.1], which is one of the main ingredients of [Fu5] (see [Fu5, Section 3]).

Theorem 4.1 ([FF1, Theorem 7.1]). *Let (V, T) be a simple normal crossing pair such that T is reduced and let $h: V \rightarrow Y$ be a projective surjective morphism onto a smooth variety Y . Assume that every stratum of (V, T) is dominant onto Y . Let Σ be a simple normal crossing divisor on Y such that every stratum of (V, T) is smooth over $Y^* = Y \setminus \Sigma$. We put $V^* = h^{-1}(Y^*)$, $T^* = T|_{V^*}$, and $d = \dim V - \dim Y$. Let $\iota: V^* \setminus T^* \hookrightarrow V^*$ be the natural open immersion. Then the local system $R^k(h|_{V^*})_* \iota! \mathbb{Q}_{V^* \setminus T^*}$ underlies a graded polarizable admissible variation of \mathbb{Q} -mixed Hodge structure on Y^* for every k . We put $\mathcal{V}_{Y^*}^k = R^k(h|_{V^*})_* \iota! \mathbb{Q}_{V^* \setminus T^*} \otimes \mathcal{O}_{Y^*}$ for every k . Let*

$$\dots \subset F^{p+1}(\mathcal{V}_{Y^*}^k) \subset F^p(\mathcal{V}_{Y^*}^k) \subset F^{p-1}(\mathcal{V}_{Y^*}^k) \subset \dots$$

be the Hodge filtration. We assume that all the local monodromies on the local system $R^k(h|_{V^})_* \iota! \mathbb{Q}_{V^* \setminus T^*}$ around Σ are unipotent for every k . Then $R^k h_* \mathcal{O}_V(-T)$ is isomorphic to the canonical extension of*

$$\mathrm{Gr}_F^0(\mathcal{V}_{Y^*}^k) = F^0(\mathcal{V}_{Y^*}^k)/F^1(\mathcal{V}_{Y^*}^k),$$

which is denoted by $\mathrm{Gr}_F^0(\mathcal{V}_Y^k)$, for every k . By taking the dual, we have

$$R^{d-k} h_* \omega_{V/Y}(T) \simeq (\mathrm{Gr}_F^0(\mathcal{V}_Y^k))^*$$

for every k .

For the details of Theorem 4.1, we recommend the reader to see [FF1, Sections 4 and 7] (see also [FFS]). We note that the reader can find basic definitions of variations of mixed Hodge structure in [FF1, Section 3].

Let us introduce the notion of *birational maps* of simple normal crossing pairs.

Definition 4.2 (Birational maps of simple normal crossing pairs). Let (V_1, T_1) and (V_2, T_2) be simple normal crossing pairs such that T_1 and T_2 are reduced. Let $\alpha: V_1 \dashrightarrow V_2$ be a proper birational map. Assume that there exist Zariski open sets U_1 and U_2 of V_1 and V_2 respectively such that U_1 contains the generic point of any stratum of (V_1, T_1) , U_2 contains the generic point of any stratum of (V_2, T_2) , and α induces an isomorphism between $(U_1, T_1|_{U_1})$ and $(U_2, T_2|_{U_2})$. Then we call α a *birational map between (V_1, T_1) and (V_2, T_2)* .

As an easy application of [FF1, Lemma 6.2] and [BVP, Theorem 1.4], we can prove the following useful lemma.

Lemma 4.3. *Let (V_1, T_1) and (V_2, T_2) be simple normal crossing pairs such that T_1 and T_2 are reduced. Let $\alpha: V_1 \dashrightarrow V_2$ be a birational map between (V_1, T_1) and*

(V_2, T_2) . Then there exists a commutative diagram

$$(4.1) \quad \begin{array}{ccc} & (V', T') & \\ p_1 \swarrow & & \searrow p_2 \\ (V_1, T_1) & \overset{\alpha}{\dashrightarrow} & (V_2, T_2), \end{array}$$

where (V', T') is a simple normal crossing pair such that T' is reduced, and p_i is a proper birational morphism between (V', T') and (V_i, T_i) for $i = 1, 2$. In this situation, p_i induces a natural one-to-one correspondence between the set of strata of (V', T') and that of (V_i, T_i) for $i = 1, 2$. Let S be any stratum of (V', T') . Then we have

$$Rp_{i*}\mathcal{O}_S \simeq \mathcal{O}_{p_i(S)}$$

for $i = 1, 2$. Moreover, we have

$$Rp_{i*}\mathcal{O}_{V'}(-T') \simeq \mathcal{O}_{V_i}(-T_i)$$

for $i = 1, 2$.

Proof. By [BVP, Theorem 1.4], we can take a desired commutative diagram (4.1), where p_i is a proper birational morphism between (V', T') and (V_i, T_i) such that p_i is an isomorphism over U_i for $i = 1, 2$. By [FF1, Lemma 6.2], we have

$$Rp_{i*}\mathcal{O}_{V'}(-T') \simeq \mathcal{O}_{V_i}(-T_i)$$

for $i = 1, 2$. Let S be a stratum of (V', T') . Then $p_i(S)$ is a stratum of (V_i, T_i) since p_i is a birational morphism between (V', T') and (V_i, T_i) for $i = 1, 2$. Therefore, $p_i(S)$ is a smooth irreducible variety and $p_i: S \rightarrow p_i(S)$ is obviously birational for $i = 1, 2$. This implies that

$$Rp_{i*}\mathcal{O}_S \simeq \mathcal{O}_{p_i(S)}$$

for $i = 1, 2$. Since $p_i: V' \rightarrow V_i$ is a proper birational morphism between (V', T') and (V_i, T_i) , it is easy to see that there exists a natural one-to-one correspondence between the set of strata of (V', T') and that of (V_i, T_i) for $i = 1, 2$. \square

Remark 4.4. In Lemma 4.3, we assume that $\alpha: (V_1, T_1) \dashrightarrow (V_2, T_2)$ is projective over a fixed scheme Y , that is, there exists the following commutative diagram

$$\begin{array}{ccc} (V_1, T_1) & \overset{\alpha}{\dashrightarrow} & (V_2, T_2) \\ h_1 \searrow & & \swarrow h_2 \\ & Y & \end{array}$$

such that h_1 and h_2 are projective. Then we see that we can make V' projective over Y by the proof of Lemma 4.3.

We define a somewhat artificial condition for birational maps of simple normal crossing pairs. We will use it in Lemma 4.6 below. For the basic definitions of semi-simplicial varieties, see, for example, [PS, Section 5.1].

Definition 4.5. Let (V, T) be a simple normal crossing pair such that T is reduced. Let $\alpha: V \dashrightarrow V$ be a birational map between (V, T) and (V, T) in the sense of Definition 4.2. We say that α satisfies condition (\star) if there exists a commutative diagram

$$(4.2) \quad \begin{array}{ccc} & (V', T') & \\ p_1 \swarrow & & \searrow p_2 \\ (V, T) & \dashrightarrow \alpha \dashrightarrow & (V, T) \end{array}$$

with the following properties:

- (1) (V', T') is a simple normal crossing pair such that T' is reduced.
- (2) p_i is a proper birational morphism between (V', T') and (V, T) in the sense of Definition 4.2 for $i = 1, 2$.
- (3) There are semi-simplicial resolutions $\varepsilon_T: T_\bullet \rightarrow T$ and $\varepsilon_V: V_\bullet \rightarrow V$, that is, T_\bullet and V_\bullet are semi-simplicial varieties, ε_T and ε_V are argumentations and of cohomological descent, such that V_p and T_q are disjoint unions of some strata of (V, T) for all p and q and that they fit in the following commutative diagram

$$(4.3) \quad \begin{array}{ccc} T_\bullet & \xrightarrow{\phi} & V_\bullet \\ \varepsilon_T \downarrow & & \downarrow \varepsilon_V \\ T & \xrightarrow{j} & V, \end{array}$$

where ϕ is a morphism of semi-simplicial varieties and j is the natural closed embedding. Moreover, $\varepsilon_T: S \rightarrow \varepsilon_T(S)$ (resp. $\varepsilon_V: S \rightarrow \varepsilon_V(S)$) is a natural isomorphism for any irreducible component S of T_\bullet (resp. V_\bullet). We note that S is a stratum of (V, T) .

- (4) There are semi-simplicial varieties $\varepsilon_{T'}: T'_\bullet \rightarrow T'$ and $\varepsilon_{V'}: V'_\bullet \rightarrow V'$ such that $\varepsilon_{T'}$ and $\varepsilon_{V'}$ are argumentations, V'_p and T'_q are disjoint unions of some strata of (V', T') for all p and q and that they fit in the following commutative diagram

$$(4.4) \quad \begin{array}{ccc} T'_\bullet & \xrightarrow{\phi'} & V'_\bullet \\ \varepsilon_{T'} \downarrow & & \downarrow \varepsilon_{V'} \\ T' & \xrightarrow{j'} & V', \end{array}$$

where ϕ' is a morphism of semi-simplicial varieties and j' is the natural closed embedding. As in (3), $\varepsilon_{T'}: S' \rightarrow \varepsilon_{T'}(S')$ (resp. $\varepsilon_{V'}: S' \rightarrow \varepsilon_{V'}(S')$) is a natural isomorphism for any irreducible component S' of T'_\bullet (resp. V'_\bullet). We note that S' is a stratum of (V', T') .

(5) The following commutative diagram

$$(4.5) \quad \begin{array}{ccccc} & & T' & & \\ & p_1|_{T'} \swarrow & \downarrow & \searrow p_2|_{T'} & \\ T & \xrightarrow{\alpha|_T} & T & \xrightarrow{\alpha|_T} & T \\ \downarrow j & & \downarrow j' & & \downarrow j \\ & p_1 \swarrow & V' & \searrow p_2 & \\ V & \xrightarrow{\alpha} & V & \xrightarrow{\alpha} & V \end{array}$$

can be lifted to a commutative diagram

$$(4.6) \quad \begin{array}{ccccc} & & T'_\bullet & & \\ & p_1|_{T'_\bullet} \swarrow & \downarrow & \searrow p_2|_{T'_\bullet} & \\ T_\bullet & \xrightarrow{\alpha|_{T_\bullet}} & T_\bullet & \xrightarrow{\alpha|_{T_\bullet}} & T_\bullet \\ \downarrow \phi & & \downarrow \phi' & & \downarrow \phi \\ & p_1|_{V'_\bullet} \swarrow & V'_\bullet & \searrow p_2|_{V'_\bullet} & \\ V_\bullet & \xrightarrow{\alpha_\bullet} & V_\bullet & \xrightarrow{\alpha_\bullet} & V_\bullet \end{array}$$

over (4.5) by (4.3) and (4.4) such that $p_1|_{V'_p}, p_2|_{V'_p}, \alpha_p, p_1|_{T'_q}, p_2|_{T'_q}$, and $\alpha|_{T'_q}$ are birational maps of smooth varieties for all p and q .

(6) If $\alpha: (V, T) \dashrightarrow (V, T)$ is projective over a fixed scheme Y , that is, there exists the following commutative diagram

$$\begin{array}{ccc} (V, T) & \xrightarrow{\alpha} & (V, T) \\ & \searrow h & \swarrow h \\ & & Y \end{array}$$

such that h is projective, then V' is also projective over Y .

The main purpose of this section is to establish the following result, which will play a crucial role in the proof of Theorem 1.3 in Section 5.

Lemma 4.6. *We use the same notation and assumption as in Theorem 4.1. We assume that Y is a curve. We further assume that $(V, T + \text{Supp } h^*\Sigma)$ is a simple normal crossing pair and that all the local monodromies on the local system $R^j h_* \mathbb{Q}_{S^*}$ around Σ are unipotent for any stratum S of (V, T) and all j , where $S^* = S|_{V^*}$. Let $\alpha: V \dashrightarrow V$ be a birational map between (V, T) and (V, T) over Y . We assume that α satisfies condition (\star) in Definition 4.5. Then α induces isomorphisms*

$$\alpha^*: W_m \text{Gr}_F^0(\mathcal{V}_Y^k) \xrightarrow{\sim} W_m \text{Gr}_F^0(\mathcal{V}_Y^k)$$

for all m and k , where W denotes the canonical extension of the weight filtration.

Let G be a finite group which acts on (V, T) birationally over Y such that every element $\alpha \in G$ satisfies condition (\star) in Definition 4.5. Then G acts on $W_m \text{Gr}_F^0(\mathcal{V}_Y^k)$ for all m and k .

In the proof of Lemma 4.6, we will use some arguments and constructions in [FF1, Section 4].

Proof of Lemma 4.6. By assumption, α satisfies condition (\star) in Definition 4.5. Therefore, we can take a commutative diagram

$$\begin{array}{ccc} & (V', T') & \\ p_1 \swarrow & & \searrow p_2 \\ (V, T) & \dashrightarrow \alpha \dashrightarrow & (V, T) \end{array}$$

as in (4.2). We note that V' is projective over Y . From now on, we will use the same notation as in Definition 4.5. We put $u = h \circ j \circ \varepsilon_T: T_\bullet \rightarrow Y$ and $v = h \circ \varepsilon_V: V_\bullet \rightarrow Y$. We set $E_\bullet = v^{-1}(\Sigma)_{\text{red}}$ and $F_\bullet = u^{-1}(\Sigma)_{\text{red}}$. Since $(V, T + \text{Supp } h^*\Sigma)$ is a simple normal crossing pair by assumption, E_\bullet and F_\bullet are simple normal crossing divisors on V_\bullet and T_\bullet , respectively. As in the proof of [FF1, Lemma 4.12], we can construct a complex $C(\phi^*)$ on Y equipped with filtrations W and F such that $H^k(C(\phi^*)) \simeq \mathcal{V}_Y^k$, where \mathcal{V}_Y^k is the canonical extension of $\mathcal{V}_{Y^*}^k = R^k(h|_{V^*})_* \iota^! \mathbb{Q}_{V^* \setminus T^*} \otimes \mathcal{O}_{Y^*}$, for every k . We note that the filtration W is denoted by L in [FF1, Lemma 4.12].

Step 1. The spectral sequence

$$E_1^{p,q}(C(\phi^*), F) = H^{p+q}(\text{Gr}_F^p C(\phi^*)) \Rightarrow H^{p+q}(C(\phi^*))$$

degenerates at E_1 (see the proof of [FF1, Lemma 4.12] and [Fu5, 13.3]). Therefore, we have the following short exact sequences

$$0 \longrightarrow H^{p+q}(F^1 C(\phi^*)) \xrightarrow{s^{p+q}} H^{p+q}(C(\phi^*)) \xrightarrow{t^{p+q}} H^{p+q}(\mathrm{Gr}_F^0 C(\phi^*)) \longrightarrow 0$$

for all p and q . We note that $F^0 C(\phi^*) = C(\phi^*)$ by construction. Let us consider the following commutative diagram.

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{p+q}(F^1 C(\phi^*)) & \xrightarrow{s^{p+q}} & H^{p+q}(C(\phi^*)) & \xrightarrow{t^{p+q}} & H^{p+q}(\mathrm{Gr}_F^0 C(\phi^*)) & \longrightarrow & 0 \\ & & & & \uparrow a_{-p}^{p+q} & & \uparrow b_{-p}^{p+q} & & \\ & & & & H^{p+q}(W_{-p} C(\phi^*)) & \longrightarrow & H^{p+q}(W_{-p} \mathrm{Gr}_F^0 C(\phi^*)) & & \end{array}$$

By definition, we have

$$F^1 H^{p+q}(C(\phi^*)) = \mathrm{Im} s^{p+q}$$

and

$$W_q H^{p+q}(C(\phi^*)) = \mathrm{Im} a_{-p}^{p+q}$$

for all p and q . We put

$$(4.7) \quad W_q H^{p+q}(\mathrm{Gr}_F^0 C(\phi^*)) := \mathrm{Im} b_{-p}^{p+q}$$

for all p and q . Then the map t^{p+q} induces

$$(4.8) \quad \begin{array}{ccc} \mathrm{Gr}_F^0 H^{p+q}(C(\phi^*)) & \xrightarrow{\sim} & H^{p+q}(\mathrm{Gr}_F^0 C(\phi^*)) \\ \uparrow & & \uparrow \\ W_q \mathrm{Gr}_F^0 H^{p+q}(C(\phi^*)) & \xrightarrow{i_q^{p+q}} & W_q H^{p+q}(\mathrm{Gr}_F^0 C(\phi^*)) \end{array}$$

for all p and q . We will prove that i_q^{p+q} are isomorphisms for all p and q in Step 2.

Step 2. Let us analyse the spectral sequence

$$(4.9) \quad E_1^{p,q}(C(\phi^*), W) \Rightarrow H^{p+q}(C(\phi^*))$$

in detail. Let $\Omega_{V_{p+1}/Y}(\log E_{p+1})$ and $\Omega_{T_p/Y}(\log F_p)$ be relative logarithmic de Rham complexes of $v_{p+1}: V_{p+1} \rightarrow Y$ and $u_p: T_p \rightarrow Y$, respectively. Then we have

$$\begin{aligned} & (E_1^{p,q}(C(\phi^*), W), F) \\ &= (R^q(v_{p+1})_* \Omega_{V_{p+1}/Y}(\log E_{p+1}), F) \oplus (R^q(u_p)_* \Omega_{T_p/Y}(\log F_p), F) \end{aligned}$$

by construction. We note that the differentials of the spectral sequence (4.9) are strictly compatible with the filtration induced by F (see [D1, (1.1.5)], [FF1, Remark 3.2], and [PS, A. 3.1]) and that the spectral sequence (4.9) degenerates at

E_2 . We do not repeat the proof of the above facts here. For the proof, see the first part of the proof of [FF1, Lemma 4.12] and [Fu5, 13.3].

The following argument corresponds to the strictness of the filtration F on the E_0 -term of the spectral sequence $E_r^{p,q}(C(\phi^*), W)$ (see [Fu5, 13.3]). By [S, (2.11) Theorem], $R^b(u_p)_* \Omega_{T_p/Y}^a(\log F_p)$ is locally free for any a, b , and p . Therefore, the spectral sequence

$$R^b(u_p)_* \Omega_{T_p/Y}^a(\log F_p) \Rightarrow R^{a+b}(u_p)_* \Omega_{T_p/Y}(\log F_p)$$

degenerates at E_1 . In particular,

$$\mathrm{Gr}_F^0 R^q(u_p)_* \Omega_{T_p/Y}(\log F_p) \simeq R^q(u_p)_* \mathcal{O}_{T_p}$$

holds for any p, q . By the same way, we see that

$$\mathrm{Gr}_F^0 R^q(v_{p+1})_* \Omega_{V_{p+1}/Y}(\log E_{p+1}) \simeq R^q(v_{p+1})_* \mathcal{O}_{V_{p+1}}$$

holds for any p, q . Thus we have

$$\begin{aligned} & \mathrm{Gr}_F^0 E_1^{p,q}(C(\phi^*), W) \\ (4.10) \quad &= \mathrm{Gr}_F^0 R^q(v_{p+1})_* \Omega_{V_{p+1}/Y}(\log E_{p+1}) \oplus \mathrm{Gr}_F^0 R^q(u_p)_* \Omega_{T_p/Y}(\log F_p) \\ & \simeq R^q(v_{p+1})_* \mathcal{O}_{V_{p+1}} \oplus R^q(u_p)_* \mathcal{O}_{T_p}. \end{aligned}$$

By taking Gr_F^0 of the spectral sequence (4.9), we obtain the following spectral sequence

$$E_1^{p,q}(\mathrm{Gr}_F^0 C(\phi^*), W) \Rightarrow H^{p+q}(\mathrm{Gr}_F^0 C(\phi^*)).$$

Note that

$$\mathrm{Gr}_F^0 E_1^{p,q}(C(\phi^*), W) \simeq E_1^{p,q}(\mathrm{Gr}_F^0 C(\phi^*), W)$$

holds as we saw in (4.10). Moreover,

$$\mathrm{Gr}_F^0 E_r^{p,q}(C(\phi^*), W) \simeq E_r^{p,q}(\mathrm{Gr}_F^0 C(\phi^*), W)$$

holds for every $r \geq 0$ by the lemma on two filtrations (see [D2, Propositions (7.2.5) and (7.2.8)] and [PS, Theorem 3.12]). Hence, we obtain

$$\begin{aligned} (4.11) \quad & \mathrm{Gr}_F^0 \mathrm{Gr}_q^W H^{p+q}(C(\phi^*)) \simeq \mathrm{Gr}_F^0 E_2^{p,q}(C(\phi^*), W) \\ & \simeq E_2^{p,q}(\mathrm{Gr}_F^0 C(\phi^*), W) \simeq \mathrm{Gr}_q^W H^{p+q}(\mathrm{Gr}_F^0 C(\phi^*)) \end{aligned}$$

for all p and q . We note that the filtration W on $H^{p+q}(\mathrm{Gr}_F^0 C(\phi^*))$ is the one defined in (4.7). We also note that $\mathrm{Gr}_F^0 \mathrm{Gr}_q^W H^{p+q}(C(\phi^*))$ is canonically isomorphic to $\mathrm{Gr}_q^W \mathrm{Gr}_F^0 H^{p+q}(C(\phi^*))$. Thus, we can check that

$$(4.12) \quad i_q^{p+q}: W_q \mathrm{Gr}_F^0 H^{p+q}(C(\phi^*)) \longrightarrow W_q H^{p+q}(\mathrm{Gr}_F^0 C(\phi^*))$$

in (4.8) are isomorphisms for all p and q inductively by using (4.8) and (4.11).

Step 3. In this proof, we did not define the filtration W on $C(\phi^*)$ explicitly. For the details of the filtration W on $C(\phi^*)$, which is denoted by L in [FF1, Section 4], see (4.2.1) and (4.8.2) in [FF1, Section 4]. By construction, we have

$$\begin{aligned} W_{-p} \mathrm{Gr}_F^0 C(\phi^*)^n &= W_{-p-1} (Rv_* \mathcal{O}_{V_\bullet})^{n+1} \oplus W_{-p} (Ru_* \mathcal{O}_{T_\bullet})^n \\ &= \bigoplus_{s \geq p+1} (R(v_s)_* \mathcal{O}_{V_s})^{n+1-s} \oplus \bigoplus_{t \geq p} (R(u_t)_* \mathcal{O}_{T_t})^{n-t}. \end{aligned}$$

Therefore, by Lemma 4.3 and the commutative diagram (4.6) in Definition 4.5, α induces isomorphisms

$$\alpha^*: W_{-p} \mathrm{Gr}_F^0 C(\phi^*) \xrightarrow{\sim} W_{-p} \mathrm{Gr}_F^0 C(\phi^*)$$

for all p . Thus α induces isomorphisms

$$\alpha^*: W_q H^{p+q}(\mathrm{Gr}_F^0 C(\phi^*)) \xrightarrow{\sim} W_q H^{p+q}(\mathrm{Gr}_F^0 C(\phi^*))$$

for all p and q by the following commutative diagram

$$\begin{array}{ccc} H^{p+q}(W_{-p} \mathrm{Gr}_F^0 C(\phi^*)) & \longrightarrow & H^{p+q}(\mathrm{Gr}_F^0 C(\phi^*)) \\ \alpha^* \downarrow \wr & & \alpha^* \downarrow \wr \\ H^{p+q}(W_{-p} \mathrm{Gr}_F^0 C(\phi^*)) & \longrightarrow & H^{p+q}(\mathrm{Gr}_F^0 C(\phi^*)) \end{array}$$

and the definition of the filtration W in (4.7). Hence, we obtain isomorphisms

$$\alpha^*: W_m H^k(\mathrm{Gr}_F^0 C(\phi^*)) \xrightarrow{\sim} W_m H^k(\mathrm{Gr}_F^0 C(\phi^*))$$

for all m and k by putting $p = k - m$ and $q = m$. By (4.12) and the fact that $\mathcal{V}_Y^k \simeq H^k(C(\phi^*))$, we obtain the desired isomorphisms

$$\alpha^*: W_m \mathrm{Gr}_F^0(\mathcal{V}_Y^k) \xrightarrow{\sim} W_m \mathrm{Gr}_F^0(\mathcal{V}_Y^k)$$

for all m and k .

When the group G acts on (V, T) birationally over Y such that every element $\alpha \in G$ satisfies condition (\star) in Definition 4.5, it is easy to see that G also acts on $W_m \mathrm{Gr}_F^0(\mathcal{V}_Y^k)$ for all m and k by the above result. \square

We make an important remark on dual variations of mixed Hodge structure. We will use it in Step 4 in the proof of Theorem 1.3.

Remark 4.7 (see [FF1, Remarks 3.15 and 7.4]). In this remark, we use the same notation and assumption as in Lemma 4.6. Let us consider the dual local system of $R^k(h|_{V^*})_* \iota^! \mathbb{Q}_{V^* \setminus T^*}$ and the dual variation of mixed Hodge structure on it. Then the locally free sheaf $(\mathcal{V}_{Y^*}^k)^*$ carries the Hodge filtration F and the weight filtration W defined as in [FF1, Remark 3.15]. By the construction of the Hodge filtration F ,

$$\mathrm{Gr}_F^0((\mathcal{V}_Y^k)^*) \simeq (\mathrm{Gr}_F^0(\mathcal{V}_Y^k))^*$$

holds, where $(\mathcal{V}_Y^k)^*$ is the canonical extension of $(\mathcal{V}_{Y^*}^k)^*$. More generally,

$$\mathrm{Gr}_F^{-p}((\mathcal{V}_Y^k)^*) \simeq (\mathrm{Gr}_F^p(\mathcal{V}_Y^k))^*$$

holds for every p . We note that $\mathrm{Gr}_F^0((\mathcal{V}_Y^k)^*) = F^0((\mathcal{V}_Y^k)^*)$, the canonical extension of the lowest piece of the Hodge filtration. By taking the dual of Lemma 4.6, G acts on $W_m \mathrm{Gr}_F^0((\mathcal{V}_Y^k)^*)$ for every m , where W denotes the canonical extension of the weight filtration of $(\mathcal{V}_{Y^*}^k)^*$. We note that we have

$$\mathrm{Gr}_m^W \mathrm{Gr}_F^p((\mathcal{V}_Y^k)^*) \simeq (\mathrm{Gr}_{-m}^W \mathrm{Gr}_F^{-p}(\mathcal{V}_Y^k))^*$$

for all p and m by construction.

We close this section with the following lemma, which is more or less well known to the experts (see [Z], [P], [K], and [FF2]). We will use it in the proof of Theorem 1.3 in Section 5.

Lemma 4.8. *Let C be a smooth projective curve and let C_0 be a non-empty Zariski open set of C . Let V_0 be a polarizable variation of \mathbb{Q} -Hodge structure over C_0 with unipotent monodromies around $\Sigma = C \setminus C_0$. Let F^b be the canonical extension of the lowest piece of the Hodge filtration. Let \mathcal{L} be a line bundle on C which is a direct summand of F^b . Assume that $\deg_C \mathcal{L} = 0$. Then $\mathcal{L}|_{C_0}$ is a flat subbundle of $F^b|_{C_0}$.*

Proof. Let h_0 be the smooth hermitian metric on $\mathcal{L}|_{C_0}$ induced by the Hodge metric of $F^b|_{C_0}$. Then $\frac{\sqrt{-1}}{2\pi} \Theta_{h_0}(\mathcal{L}|_{C_0})$ is a semipositive smooth $(1, 1)$ -form on C_0 . We note that $\Theta_{h_0}(\mathcal{L}|_{C_0})$ is the curvature tensor of the Chern connection of $(\mathcal{L}|_{C_0}, h_0)$. Then

$$\deg_C \mathcal{L} = \frac{\sqrt{-1}}{2\pi} \int_{C_0} \Theta_{h_0}(\mathcal{L}|_{C_0})$$

holds (see, for example, [K, Theorem 5.1]). Note that the right hand side is an improper integral. By assumption, $\deg_C \mathcal{L} = 0$. This implies that $\Theta_{h_0}(\mathcal{L}|_{C_0}) = 0$. Therefore, $\mathcal{L}|_{C_0}$ is a flat subbundle of $F^b|_{C_0}$. \square

Remark 4.9. In Lemma 4.8, the smooth hermitian metric h_0 on $\mathcal{L}|_{C_0}$ can be extended naturally to a singular hermitian metric h on \mathcal{L} in the sense of Demailly such that $\sqrt{-1}\Theta_h(\mathcal{L})$ is positive in the sense of currents and that the Lelong numbers of h are zero everywhere. For the details, see [FF2, Theorem 1.1].

§5. Proof of Theorem 1.3

In this section, we prove Theorem 1.3 and Corollary 1.4.

Let us prepare an easy lemma. By this lemma, we can reduce the problem to the case where the base space is a curve.

Lemma 5.1. *Let Y be a smooth projective irreducible variety with $\dim Y \geq 2$ and let N be a numerically trivial Cartier divisor on Y . Let H be a smooth ample Cartier divisor on Y such that H contains no irreducible components of $\text{Supp } N$. Then $N \sim 0$ if and only if $N|_H \sim 0$.*

Proof. We consider the following long exact sequence

$$\begin{aligned} 0 \rightarrow H^0(Y, \mathcal{O}_X(N - H)) \rightarrow H^0(Y, \mathcal{O}_Y(N)) \rightarrow H^0(H, \mathcal{O}_H(N|_H)) \\ \rightarrow H^1(Y, \mathcal{O}_Y(N - H)) \rightarrow \dots \end{aligned}$$

It is obvious that $H^0(Y, \mathcal{O}_Y(N - H)) = 0$. By the Kodaira vanishing theorem, we have $H^1(Y, \mathcal{O}_Y(N - H)) = 0$. Therefore, $H^0(Y, \mathcal{O}_Y(N)) \simeq H^0(H, \mathcal{O}_H(N|_H))$ holds. In particular, $N \sim 0$ if and only if $N|_H \sim 0$. \square

Let us start the proof of Theorem 1.3. We adapt Floris's proof of Theorem 1.3 for lc-trivial fibrations (see [Fl]) to our setting, that is, basic slc-trivial fibrations.

Proof of Theorem 1.3. This proof heavily depends on [Fu5, Section 6]. Let $\sigma: Y' \rightarrow Y$ be a projective birational morphism from a smooth projective variety Y' . By considering the induced basic slc-trivial fibration by $\sigma: Y' \rightarrow Y$, we may assume that Y is a smooth projective variety.

Step 1. In this step, we construct a cyclic cover of the generic fiber of $f: X \rightarrow Y$ following [Fu5, 6.1 and 6.2]. Let $f: (X, B) \rightarrow Y$ be a basic slc-trivial fibration. Let F be a general fiber of $f: X \rightarrow Y$. We put

$$b = \min\{m \in \mathbb{Z}_{>0} \mid m(K_F + B_F) = m(K_X + B)|_F \sim 0\}.$$

Then we can write

$$(5.1) \quad K_X + B + \frac{1}{b}(\varphi) = f^*(K_Y + B_Y + M_Y)$$

with $\varphi \in \Gamma(X, \mathcal{K}_X^*)$, where B_Y is the discriminant \mathbb{Q} -divisor and M_Y is the moduli \mathbb{Q} -divisor of $f: (X, B) \rightarrow Y$. By taking some suitable blow-ups (see [BVP, Theorem 1.4 and Section 8] and [Fu4, Lemma 2.11]), we may assume that $\text{Supp}(B - f^*(B_Y + M_Y))$ is a simple normal crossing divisor on X , $(B^h)^{=1}$ is Cartier, and every stratum of $(X, (B^h)^{=1})$ is dominant onto Y . We take the b -fold cyclic cover $\pi: \tilde{X} \rightarrow X$ associated to (5.1), that is,

$$\tilde{X} = \text{Spec}_X \bigoplus_{i=0}^{b-1} \mathcal{O}_X(\lfloor i\Delta \rfloor),$$

where $\Delta = K_{X/Y} + B - f^*(B_Y + M_Y)$. We note that $\pi: \tilde{X} \rightarrow X$ is a finite Galois cover by construction (see [Fu5, Proposition 6.3 (i)]). We put $K_{\tilde{X}} + B_{\tilde{X}} = \pi^*(K_X + B)$. By construction, it is easy to see that $(B_{\tilde{X}}^h)^{=1} = \pi^*((B^h)^{=1})$ and that $(\tilde{X}, (B_{\tilde{X}}^h)^{=1})$ is semi-log canonical. Moreover, every slc stratum of $(\tilde{X}, (B_{\tilde{X}}^h)^{=1})$ is dominant onto Y . We take a projective birational morphism $d: V \rightarrow \tilde{X}$ from a simple normal crossing variety V such that d is an isomorphism over the generic point of every slc stratum of $(\tilde{X}, (B_{\tilde{X}}^h)^{=1})$ by [BVP, Theorem 1.4]. We put $K_V + B_V = d^*(K_{\tilde{X}} + B_{\tilde{X}})$. Then we get the following commutative diagram

$$(5.2) \quad \begin{array}{ccccc} (X, B) & \xleftarrow{\pi} & \tilde{X} & \xleftarrow{d} & (V, B_V) \\ & & \searrow \tilde{f} & & \searrow h \\ & & & & \searrow h \\ & & & & Y \end{array}$$

with $g = \pi \circ d$. By taking a suitable birational modification of Y and considering induced (pre-)basic slc-trivial fibrations as in [Fu5, 6.2], we may further assume that the following properties hold for

$$K_X + B + \frac{1}{b}(\varphi) = f^*(K_Y + B_Y + M_Y)$$

and

$$h: (V, B_V) \xrightarrow{g} (X, B) \xrightarrow{f} Y.$$

- (a) Y is a smooth projective irreducible variety, and X and V are projective simple normal crossing varieties.
- (b) There exist simple normal crossing divisors Σ_X , Σ_V , and Σ_Y on X , V , and Y , respectively.
- (c) f and h are projective surjective morphisms.
- (d) The supports of B , B_V , and B_Y , M_Y are contained in Σ_X , Σ_V , and Σ_Y , respectively.

- (e) Every stratum of (X, Σ_X^h) and (V, Σ_V^h) is smooth over $Y \setminus \Sigma_Y$.
- (f) $f^{-1}(\Sigma_Y) \subset \Sigma_X$, $f(\Sigma_X^v) \subset \Sigma_Y$, and $h^{-1}(\Sigma_Y) \subset \Sigma_V$, $h(\Sigma_V^v) \subset \Sigma_Y$.
- (g) $(B^h)^{=1}$ and $(B_V^h)^{=1}$ are Cartier.

We note that conditions (a)–(g) above are nothing but the conditions stated just before [Fu5, Proposition 6.3]. As we saw in the proof of [Fu5, Theorem 1.2] (see [Fu5, Section 9]), $\mathbf{M} = \overline{\mathbf{M}}_Y$ holds and \mathbf{M}_Y is a nef \mathbb{Q} -divisor on Y . By assumption, $\mathbf{M}_Y \equiv 0$. If $\nu: Y'' \rightarrow Y$ is a finite surjective morphism from a smooth projective irreducible variety Y'' , then it is easy to see that $\mathbf{M}_Y \sim_{\mathbb{Q}} 0$ if and only if $\nu^* \mathbf{M}_Y \sim_{\mathbb{Q}} 0$. Therefore, by taking a unipotent reduction (see [Fu5, Lemma 7.3]), we may further assume that

- (A) for any irreducible component P of $\text{Supp } \Sigma_Y$, there exists a prime divisor Q on V such that $\text{mult}_Q(-B_V + h^* B_Y) = 0$, $h(Q) = P$, and $\text{mult}_Q h^* P = 1$,
- (B) all the local monodromies on the local system

$$R^{\dim V - \dim Y} (h|_{V^*})_* \iota^! \mathbb{Q}_{V^* \setminus (B_{V^*}^h)^{=1}} = 1$$

around Σ_Y are unipotent, where $Y^* = Y \setminus \Sigma_Y$, $V^* = h^{-1}(Y^*)$, $B_{V^*} = (B_V)|_{V^*}$, and $\iota: V^* \setminus (B_{V^*}^h)^{=1} \hookrightarrow V^*$ is the natural open immersion, and

- (C) all the local monodromies on the local system $R^k h_* \mathbb{Q}_{S^*}$ around Σ_Y are unipotent for any stratum S of $(V, (B_V^h)^{=1})$ and every k , where $S^* = S|_{V^*}$.

Note that the above assumptions (A) and (B) are nothing but the assumptions in (iv) and (v) in [Fu5, Proposition 6.3]. We also note that we do not treat the assumption (C) in the original statement of [Fu5, Lemma 7.3]. Therefore, we have to make N_j in the proof of [Fu5, Lemma 7.3] sufficiently divisible in order to make the monodromy on the local system $R^k h_* \mathbb{Q}_{S^*}$ around P_j , an irreducible component of Σ_Y , unipotent for any stratum S of $(V, (B_V^h)^{=1})$ and every k when we take a finite cover $\nu: Y'' \rightarrow Y$ for a unipotent reduction (see [Fu5, Lemma 7.3]).

Step 2. We assume that $\dim Y \geq 2$. Then we take a general ample Cartier divisor H on Y and put $Z = f^* H$ and $W = h^* H$. In this situation,

$$K_X + Z + B + \frac{1}{b}(\varphi) = f^*(K_Y + H + B_Y + M_Y).$$

By adjunction,

$$K_Z + B|_Z + \frac{1}{b}(\varphi|_Z) = f^*(K_H + B_Y|_H + M_Y|_H)$$

holds. It is not difficult to see that $f|_Z: (Z, B|_Z) \rightarrow H$ is a basic slc-trivial fibration and

$$h|_W: (W, B_V|_W) \xrightarrow{g|_W} (Z, B|_Z) \xrightarrow{f|_Z} H$$

satisfies conditions (a)–(g), (A), (B), and (C) in Step 1. We note that $B_Y|_H = B_H$ and $M_Y|_H = M_H$ hold, where B_H (resp. M_H) is the discriminant (resp. moduli) \mathbb{Q} -divisor of $f|_Z: (Z, B|_Z) \rightarrow H$. By Lemma 5.1, $M_Y \sim_{\mathbb{Q}} 0$ if and only if $M_Y|_H \sim_{\mathbb{Q}} 0$. Therefore, we can replace $f: (X, B) \rightarrow Y$ with $f|_Z: (Z, B|_Z) \rightarrow H$. By repeating this reduction finitely many times, we may assume that Y is a smooth projective curve.

Step 3. In Step 1, we have already seen that $\pi: \tilde{X} \rightarrow X$ is Galois. Let $G = \mathbb{Z}/b\mathbb{Z}$ be the Galois group of the b -fold cyclic cover $\pi: \tilde{X} \rightarrow X$. The action of G on \tilde{X} preserves the slc strata of $(\tilde{X}, (B_{\tilde{X}}^h)^{-1})$ by construction. Therefore, any element α of G induces a birational map between (V, T) and (V, T) over X , where $T = (B_V^h)^{-1}$. From now on, we will check that α satisfies condition (\star) in Definition 4.5. As usual, we can take a commutative diagram

$$\begin{array}{ccccc}
 & & (V', T') & & \\
 & \swarrow p_1 & \downarrow g' & \searrow p_2 & \\
 (V, T) & \overset{\alpha}{\dashrightarrow} & & \dashrightarrow & (V, T) \\
 & \searrow g & \downarrow & \swarrow g & \\
 & & X & & \\
 & \swarrow h & \downarrow f & \searrow h & \\
 & & Y & &
 \end{array}$$

by using [BVP, Theorem 1.4], where (V', T') is a simple normal crossing pair such that T' is reduced, and p_i is a projective birational morphism between (V', T') and (V, T) for $i = 1, 2$. We put $C = (B^h)^{-1}$. The irreducible decomposition of X and C are given by

$$X = \bigcup_{i \in I} X_i, \quad \text{and} \quad C = \bigcup_{\lambda \in \Lambda} C_\lambda$$

respectively as in [FF1, 4.14]. We put $V = \bigcup_{i \in I} V_i$ and $V_i = \bigcup_j V_{i,j}$, where $V_{i,j}$ runs over irreducible components of V such that $g(V_{i,j}) = X_i$. We put $T = \bigcup_{\lambda \in \Lambda} T_\lambda$ and $T_\lambda = \bigcup_l T_{\lambda,l}$, where $T_{\lambda,l}$ runs over irreducible components of T such that $g(T_{\lambda,l}) = C_\lambda$. Note that T_λ and V_i are disjoint unions of some strata of (V, T) . By applying the same construction as above to (V', T') and $g' := g \circ p_1 = g \circ p_2: V' \rightarrow X$, we get $V' = \bigcup_{i \in I} V'_i$ and $T' = \bigcup_{\lambda \in \Lambda} T'_\lambda$. We apply the same construction as in [FF1, 4.14] to $V = \bigcup_{i \in I} V_i$ and $T = \bigcup_{\lambda \in \Lambda} T_\lambda$ (resp. $V' = \bigcup_{i \in I} V'_i$ and $T' = \bigcup_{\lambda \in \Lambda} T'_\lambda$) instead of $X = \bigcup_{i \in I} X_i$ and $D = \bigcup_{\lambda \in \Lambda} D_\lambda$ in [FF1, 4.14]. Then we can construct semi-simplicial resolutions $\varepsilon_T: T_\bullet \rightarrow T$ and $\varepsilon_V: V_\bullet \rightarrow V$ (resp. $\varepsilon_{T'}: T'_\bullet \rightarrow T'$ and $\varepsilon_{V'}: V'_\bullet \rightarrow V'$). By construction, these semi-simplicial resolutions satisfy the

conditions stated in Definition 4.5. Therefore, α satisfies condition (\star) . This is what we wanted.

Step 4. We note that M_Y is a Cartier divisor on Y and that $\mathcal{O}_Y(M_Y)$ is a direct summand of

$$(\mathrm{Gr}_F^0(\mathcal{V}_Y^d))^* \simeq \mathrm{Gr}_F^0((\mathcal{V}_Y^d)^*),$$

where $d = \dim X - \dim Y$ (see [Fu5, Proposition 6.3]). More precisely, by construction, $\mathcal{O}_Y(M_Y)$ is an eigensheaf of rank one corresponding to the eigenvalue ζ^{-1} of

$$h_*\omega_{V/Y}((B_V^h)^{-1}) \simeq \mathrm{Gr}_F^0((\mathcal{V}_Y^d)^*)$$

by the group action of $G = \mathbb{Z}/b\mathbb{Z}$, where ζ is a fixed primitive b -th root of unity (see the proof of [Fu5, Proposition 6.3]). We take an integer l such that

$$\mathcal{O}_Y(M_Y) \subset W_l \mathrm{Gr}_F^0((\mathcal{V}_Y^d)^*) \quad \text{and} \quad \mathcal{O}_Y(M_Y) \not\subset W_{l-1} \mathrm{Gr}_F^0((\mathcal{V}_Y^d)^*)$$

hold. Thus we can easily see that $\mathcal{O}_Y(M_Y)$ is an eigensheaf of rank one corresponding to the eigenvalue ζ^{-1} of $W_l \mathrm{Gr}_F^0((\mathcal{V}_Y^d)^*)$ and that

$$\mathcal{O}_Y(M_Y) \cap W_{l-1} \mathrm{Gr}_F^0((\mathcal{V}_Y^d)^*) = \{0\}$$

in $W_l \mathrm{Gr}_F^0((\mathcal{V}_Y^d)^*)$. We note that G acts on $W_m \mathrm{Gr}_F^0((\mathcal{V}_Y^d)^*)$ for every m by Lemma 4.6 and Remark 4.7. Since $\deg M_Y = 0$ by assumption, $\mathcal{O}_Y(M_Y)|_{Y^*}$ defines a local subsystem of $\mathrm{Gr}_l^W((\mathcal{V}_{Y^*}^d)^*)$ by Lemma 4.8. We note that

$$\mathrm{Gr}_l^W \mathrm{Gr}_F^0((\mathcal{V}_{Y^*}^d)^*) \simeq \mathrm{Gr}_F^0 \mathrm{Gr}_l^W((\mathcal{V}_{Y^*}^d)^*) = F^0 \mathrm{Gr}_l^W((\mathcal{V}_{Y^*}^d)^*) \subset \mathrm{Gr}_l^W((\mathcal{V}_{Y^*}^d)^*)$$

holds since we have $F^1 \mathrm{Gr}_l^W((\mathcal{V}_{Y^*}^d)^*) = 0$ by the construction of the dual Hodge filtration (see [FF1, Remark 3.15] and Remark 4.7). Therefore, there exists a positive integer a such that $\mathcal{O}_Y(aM_Y)|_{Y^*} \simeq \mathcal{O}_{Y^*}$ by [D1, Corollaire (4.2.8) (iii) b)]. This is because $\mathrm{Gr}_l^W((\mathcal{V}_{Y^*}^d)^*)$ is a polarizable variation of \mathbb{Q} -Hodge structure. Thus we get $\mathcal{O}_Y(aM_Y) \simeq \mathcal{O}_Y$ by taking the canonical extension. This is what we wanted.

Hence, we obtain $\mathbf{M}_{Y'} \sim_{\mathbb{Q}} 0$. \square

We close this section with the proof of Corollary 1.4.

Proof of Corollary 1.4. By [Fu5, Lemma 4.12], we may assume that Y is a smooth projective curve. We always have $\deg M_Y \geq 0$ since M_Y is nef by [Fu5, Theorem 1.2]. If $\deg M_Y > 0$, then it is obvious that M_Y is ample. If $\deg M_Y = 0$, then M_Y is numerically trivial. In this case, by Theorem 1.3, $M_Y \sim_{\mathbb{Q}} 0$ holds. Therefore, we see that M_Y is always semi-ample. \square

Acknowledgements

The first author was partially supported by JSPS KAKENHI Grant Numbers JP16H03925, JP16H06337. The second author was partially supported by JSPS KAKENHI Grant Number JP16K05107. The authors would like to thank Takeshi Abe for useful discussions and comments.

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