VANISHING THEOREMS FOR PROJECTIVE MORPHISMS BETWEEN COMPLEX ANALYTIC SPACES

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ABSTRACT. We discuss vanishing theorems for projective morphisms between complex analytics spaces and some related results. They will play a crucial role in the minimal model theory for projective morphisms of complex analytic spaces. Roughly speaking, we establish an ultimate generalization of Kollár's package from the minimal model theoretic viewpoint.

CONTENTS

1. INTRODUCTION

In [F16], we have already discussed the minimal model program for kawamata log terminal pairs in a complex analytic setting. Roughly speaking, we showed that [BCHM] and [HM] can work for projective morphisms between complex analytic spaces. We note that one of the main ingredients of [BCHM] and [HM] is the Kawamata–Viehweg vanishing theorem and that the Kawamata–Viehweg vanishing theorem can be formulated and proved for projective morphisms of complex analytic spaces. Hence the results in [F16] are not surprising although they are nontrivial. In [F3] and [F9, Chapter 6], we established the fundamental theorems of the minimal model program for log canonical pairs and quasi-log schemes, respectively. It is highly desirable to generalize [F3] and [F9, Chapter 6] into the complex analytic setting. For that purpose, we have to establish suitable vanishing theorems and some related results for projective morphisms of complex analytic spaces. Therefore, in this paper, we prove the following theorems (see Theorems 1.1 and 1.2), which give a complete answer to [F9, Remark 5.8.3]. They will play a crucial role for the study of complex analytic log canonical pairs and quasi-log structures on complex analytic spaces (see [F3] and [F9, Chapter 6]). In [F17], we will establish the cone and contraction theorem of normal pairs for projective morphisms between complex analytic

spaces as an application of this paper. In [F18], we will discuss quasi-log structures for complex analytic spaces by using Theorems 1.1 and 1.2.

Theorem 1.1 (Main theorem). *Let (X, Δ) be an analytic simple normal crossing pair such that Δ is a boundary $\mathbb{R}$-divisor on X. Let $f \colon X \to Y$ be a projective morphism to a complex analytic space Y and let $\mathcal{L}$ be a line bundle on X. Let q be an arbitrary nonnegative integer. Then we have the following properties.*

   (i) (*Strict support condition*). *If $\mathcal{L} - (\omega_X + \Delta)$ is f-semiample, then every associated subvariety of $R^q f_* \mathcal{L}$ is the f-image of some stratum of (X, Δ).*
   (ii) (*Vanishing theorem*). *If $\mathcal{L} - (\omega_X + \Delta) \sim_{\mathbb{R}} f^* \mathcal{H}$ holds for some π-ample $\mathbb{R}$-line bundle $\mathcal{H}$ on Y, where $\pi \colon Y \to Z$ is a projective morphism to a complex analytic space Z, then we have $R^p \pi_* R^q f_* \mathcal{L} = 0$ for every $p > 0$.*

Theorem 1.2 (Vanishing theorem of Reid–Fukuda type). *Let (X, Δ) be an analytic simple normal crossing pair such that Δ is a boundary $\mathbb{R}$-divisor on X. Let $f \colon X \to Y$ and $\pi \colon Y \to Z$ be projective morphisms between complex analytic spaces and let $\mathcal{L}$ be a line bundle on X. If $\mathcal{L} - (\omega_X + \Delta) \sim_{\mathbb{R}} f^* \mathcal{H}$ holds such that $\mathcal{H}$ is an $\mathbb{R}$-line bundle, which is nef and log big over Z with respect to $f \colon (X, \Delta) \to Y$, on Y, then $R^p \pi_* R^q f_* \mathcal{L} = 0$ holds for every $p > 0$ and every q.*

We make an important remark on Theorems 1.1 and 1.2.

Remark 1.3. (i) Let X be a complex analytic space and let $\mathrm{Pic}(X)$ be the group of line bundles on X, that is, the *Picard group* of X. An element of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$) is called an $\mathbb{R}$-*line bundle* (resp. a $\mathbb{Q}$-*line bundle*) on X. In this paper, we write the group law of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ additively for simplicity of notation.

(ii) In Theorems 1.1 and 1.2, we always assume that Δ is *globally* $\mathbb{R}$-*Cartier*, that is, Δ is a finite $\mathbb{R}$-linear combination of Cartier divisors. We note that if the number of the irreducible components of $\mathrm{Supp}\, \Delta$ is finite then Δ is globally $\mathbb{R}$-Cartier. This condition is harmless to applications because the restriction of Δ to a relatively compact open subset of X has only finitely many irreducible components in its support.

(iii) Under the assumption that Δ is globally $\mathbb{R}$-Cartier, we can obtain an $\mathbb{R}$-line bundle $\mathcal{N}$ naturally associated to $\mathcal{L} - (\omega_X + \Delta)$, which is a hybrid of line bundles $\mathcal{L}$ and ω_X and a globally $\mathbb{R}$-Cartier divisor Δ. The assumption in Theorem 1.1 (i) means that $\mathcal{N}$ is a finite positive $\mathbb{R}$-linear combination of f-semiample line bundles on X. The assumption in Theorem 1.1 (ii) and Theorem 1.2 means that $\mathcal{N} = f^* \mathcal{H}$ holds in $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

We will use Theorems 1.1 and 1.2 in order to translate various results on log canonical pairs in [F3] and quasi-log schemes in [F9, Chapter 6] into the complex analytic setting (see [F17] and [F18]). The proof of Theorems 1.1 and 1.2 is different from the corresponding one in [F3] and [F9, Chapter 5] in the algebraic setting. In this paper, we use a spectral sequence coming from Saito's theory of mixed Hodge modules (see [Sa1], [Sa2], [Sa3], [FFS], and [Sa5]). Roughly speaking, we reduce the problem to a well-known simpler case by semisimplicity of polarizable Hodge modules. Then we use Takegoshi's complex analytic generalization of Kollár's package (see [T]). The approach in this paper clarifies the meaning of the strict support condition in Theorem 1.1 (i). We strongly recommend the interested reader to compare this paper with [F9, Chapter 5]. We note that the reader can find an alternative approach to Theorems 1.1 and 1.2, which is free from Saito's theory of mixed Hodge modules, in [FF].

Let us quickly explain various related vanishing theorems. We recommend the interested reader to see [F9, Chapter 3]. There are many results on vanishing theorems. Our

choice of topics here is biased and reflects author's personal taste. The author apologizes for many important omitted references.

1.4 (Kodaira vanishing theorem)**.** The Kodaira vanishing theorem (see [Kod]) is a monumental result and is very important in the study of complex algebraic varieties. We have many useful and powerful generalizations.

1.5 (Kawamata–Viehweg vanishing theorem)**.** The Kawamata–Viehweg vanishing theorem (see [Ka] and [V]) is one of the most famous generalizations of the Kodaira vanishing theorem. It plays a crucial role in the minimal model theory for higher-dimensional complex algebraic varieties with only mild singularities.

1.6 (Kollár's injectivity, torsion-free, and vanishing theorem)**.** In [Kol1], János Kollár obtained a very powerful generalization of the Kodaira vanishing theorem. The reader can find simpler approaches and some generalizations in [EV] and [Kol3]. The original approach in [Kol1] depends on the theory of pure Hodge structures. By using the theory of mixed Hodge structures, we can prove some generalizations (see [EV], [F1], [A], [F9, Chapter 5], and so on). They have already had many applications in the study of minimal models of complex algebraic varieties with bad singularities. For the details, see [F1], [A], [F2], [F3], [F6], [F7], [F8], [F9, Chapters 5 and 6], [F11], [F12], [F14], and so on.

1.7 (Kollár's conjecture)**.** In [Kol2, Section 5], Kollár discussed some conjecture about abstract variations of Hodge structure, which is now usually called *Kollár's conjecture.* In [Sa4], Morihiko Saito showed that it naturally follows from the general theory of Hodge modules (see [Sa1] and [Sa2]). Note that [F1, Section 4], which was written by Saito, and [FFS] are closely related to Kollár's conjecture. We also note that the main result in this paper heavily depends on [FFS] (see also [Sa5]).

1.8 (Kodaira–Saito vanishing theorem)**.** Saito established a powerful generalization of the Kodaira vanishing theorem in the framework of mixed Hodge modules (see [Sa2, (2.g) Kodaira vanishing]). For the details and some further generalizations, see [P], [Sc1], [Su], [W], and so on. Although we use the result in [FFS] for the proof of the main theorem of this paper, we do not directly use the Kodaira–Saito vanishing theorem.

1.9 (Takegoshi's analytic generalization of Kollár's theorem)**.** In [T], Kensho Takegoshi established Kollár's injectivity, torsion-free, and vanishing theorem in a suitable complex analytic setting. We use Takegoshi's result in this paper. For various generalizations of Kollár's package in the complex analytic setting, see [F4], [F5], [FM], [F13], [M], [F15], and so on.

We look at the organization of this paper. In Section 2, we collect some basic definitions and necessary results. In Section 3, we treat a very standard setting of proper Kähler morphisms. We establish the strict support condition, the vanishing theorem, and the injectivity theorem in this standard setting. In Section 4, we prove Theorem 1.1 under the extra assumption that X is irreducible. We note that the proof of Theorem 1.1 for the case where X is irreducible is essential. In Section 5, we prove Theorem 1.1 in full generality. More precisely, we prepare a technical but important lemma and show that essentially the same argument as in Section 4 works. In Section 6, we prove Theorem 1.2. Although it looks very similar to Theorem 1.1 (ii), the proof of Theorem 1.2 is much harder than that of Theorem 1.1 (ii).

In this paper, every complex analytic space is assumed to be *Hausdorff* and *second-countable*. We will freely use the standard notation in [F3], [F9], [F16], and so on. We will also freely use the basic results on complex analytic geometry in [BS] and [Fi]. For the minimal model program for projective morphisms between complex analytic spaces, see [N1], [N2], and [F16].

2. Preliminaries

In this section, we will collect some basic definitions and necessary results. Let us start with the definition of *analytic simple normal crossing pairs*.

Definition 2.1 (Analytic simple normal crossing pairs)**.** Let X be a simple normal crossing divisor on a smooth complex analytic space M and let B be an $\mathbb{R}$-divisor on M such that $\mathrm{Supp}(B + X)$ is a simple normal crossing divisor on M and that B and X have no common irreducible components. Then we put $D := B|_X$ and consider the pair (X, D). We call (X, D) an *analytic globally embedded simple normal crossing pair* and M the *ambient space* of (X, D).

If the pair (X, D) is locally isomorphic to an analytic globally embedded simple normal crossing pair and the irreducible components of X and D are all smooth, then (X, D) is called an *analytic simple normal crossing pair*. If $(X, 0)$ is an analytic simple normal crossing pair, then we simply say that X is *simple normal crossing*.

When X is simple normal crossing, it has an invertible dualizing sheaf ω_X. We sometimes use the symbol K_X as a formal divisor class with an isomorphism $\mathcal{O}_X(K_X) \simeq \omega_X$ if there is no danger of confusion. We note that we can not always define K_X globally with $\mathcal{O}_X(K_X) \simeq \omega_X$. In general, it only exists locally on X.

We need the following definition in order to state [FFS, Corollary 1 and 4.7. Remark].

Definition 2.2. Let (X, D) be an analytic simple normal crossing pair such that D is reduced. For any positive integer k, we put

$$X^{[k]} := \left\{ x \in X \,|\, \mathrm{mult}_x X \geq k \right\}^{\sim},$$

where $Z^{\sim}$ denotes the normalization of Z. Then $X^{[k]}$ is the disjoint union of the intersections of k irreducible components of X, and is smooth. We have a reduced simple normal crossing divisor $D^{[k]} \subset X^{[k]}$ defined by the pull-back of D by the natural morphism $X^{[k]} \to X$. For any nonnegative integer l, we put

$$D^{[k,l]} := \left\{ x \in X^{[k]} \,\big|\, \mathrm{mult}_x D^{[k]} \geq l \right\}^{\sim}.$$

We note that $D^{[k,0]} = X^{[k]}$ holds by definition. We also note that $\dim D^{[k,l]} = n+1-k-l$, where $n = \dim X$.

We recall the notion of *strata* of analytic simple normal crossing pairs.

Definition 2.3 (Strata)**.** Let (X, D) be an analytic simple normal crossing pair. Let $\nu \colon X^\nu \to X$ be the normalization. We put $K_{X^\nu} + \Theta = \nu^*(K_X + D)$. This means that Θ is the union of $\nu_*^{-1}D$ and the inverse image of the singular locus of X. If W is an irreducible component of X or the ν-image of some log canonical center of (X^ν, Θ), then W is called a *stratum* of (X, D).

When D is reduced, W is a stratum of (X, D) if and only if W is the image of an irreducible component of $D^{[k,l]}$ for some $k > 0$ and $l \geq 0$.

The following easy remark may help the reader understand the notion of strata.

Remark 2.4. Let (X, D) be an analytic simple normal crossing pair. Let $D = \sum_i a_i D_i$ be the irreducible decomposition. We put $G := \sum_{a_i=1} D_i$. Then (X, G) is an analytic simple normal crossing pair such that G is reduced. We can easily check that W is a stratum of (X, D) if and only if W is a stratum of (X, G). Therefore, a stratum W of (X, D) is the image of an irreducible component of $G^{[k,l]}$ for some $k > 0$ and $l \geq 0$.

For Theorem 1.2, the notion of *nef and log big* $\mathbb{R}$-*line bundles* is necessary.

Definition 2.5 (Nef and log bigness). Let $f: (X, \Delta) \to Y$ be a projective morphism from an analytic simple normal crossing pair (X, Δ) to a complex analytic space Y and let $\pi: Y \to Z$ be a projective morphism between complex analytic spaces. Let $\mathcal{H}$ be an $\mathbb{R}$-line bundle on Y. We say that $\mathcal{H}$ is *nef and log big over* Z *with respect to* $f: (X, \Delta) \to Y$ if $\mathcal{H}|_{f(W)}$ is nef and big over $\pi \circ f(W)$ for every stratum W of (X, Δ), equivalently, $\mathcal{H} \cdot C \geq 0$ holds for every projective integral curve C on Y such that $\pi(C)$ is a point and $\mathcal{H}|_{f(W)}$ can be written as a finite positive $\mathbb{R}$-linear combination of π-big line bundles on $f(W)$ for every stratum W of (X, Δ). When f is the identity morphism, we simply say that $\mathcal{H}$ is *nef and log big over* Z *with respect to* (X, Δ).

One of the main ingredients of the proof of Theorem 1.1 is a deep result coming from Saito's theory of mixed Hodge modules (see [Sa1], [Sa2], [Sc2], and so on). Roughly speaking, we reduce the problem to the case where X is smooth by using Theorem 2.6. Then we use Takegoshi's results (see [T] and Theorem 2.9 below) in order to obtain Theorem 1.1.

Theorem 2.6 ([FFS, Corollary 1 and 4.7. Remark] and [Sa5]). *Let (X, D) be an analytic simple normal crossing pair with $\dim X = n$ such that D is reduced and let $f: X \to Y$ be a proper morphism to a smooth complex analytic space Y. Assume that f is Kähler on each irreducible component of X. Then there is the weight spectral sequence*

$$_F E_1^{-q,i+q} = \bigoplus_{k+l=n+q+1} R^i f_* \omega_{D^{[k,l]}/Y} \Rightarrow R^i f_* \omega_{X/Y}(D),$$

degenerating at E_2, and its E_1-differential d_1 splits so that the $_F E_2^{-q,i+q}$ are direct factors of $_F E_1^{-q,i+q}$.

We note that in [Sa5] a proper morphism $f: X \to Y$ of smooth complex analytic spaces is said to be *Kähler* if there exists a relative Kähler form ξ_f, which is a closed real 2-form on X, satisfying the following condition that locally on Y there is a Kähler form ξ_Y such that $\xi_f + f^* \xi_Y$ is a Kähler form on X.

Remark 2.7. Although we assume that $f: X \to Y$ is a projective morphism of complex algebraic varieties such that Y is smooth in [FFS, Corollary 1], the results in [FFS] are also valid in the analytic case where f is a proper Kähler morphism on each irreducible component of X, Y is a complex manifold, and (X, D) is an analytic simple normal crossing pair. For the details, see [FFS, 4.7. Remark]. If $f: X \to Y$ is projective, then everything works well in Saito's usual framework of mixed Hodge modules (see [Sa1] and [Sa2]). When f is only a proper Kähler morphism on each irreducible component of X, we have to use the decomposition theorem for proper Kähler morphisms announced in [Sa3].

The reader can find some remarks and supplementary results in [Sa5]. Theorem 2.6 for projective morphisms is sufficient for the proof of Theorems 1.1 and 1.2. Therefore, the reader who is only interested in Theorems 1.1 and 1.2 can assume that f is projective.

The following remark is due to Morihiko Saito.

Remark 2.8. In [FFS], there are some changes of English from the original version (see, for example, arXiv:1302.6180v2 [math.AG]) by the printer unexpectedly. Some of them are misleading.

- On page 86 from lines 3 to 4, the reader can find, "There is no shift of one in [De2], [SZ] although different notation is used."; however, this is misleading. We claim, "There is no shift *compared* with the one in [De2], [SZ] although different notation is used."
- In [FFS, 4.6. Remarks.(iv)], the reader can find, "This can be obtained by applying the standard cohomological functors ..."; however, this is not correct. In the original version of [FFS], we claim, "This can be obtained by *repeating* the standard cohomological functors ..."

We need a special case of Takegoshi's theorem (see [T]), which is a complex analytic generalization of Kollár's torsion-freeness and vanishing theorem. For the details of Theorem 2.9, see also [N2, Chapter V. 3.7.Theorem].

Theorem 2.9 (see [T]). *Let $f\colon X \to Y$ be a proper surjective morphism from a smooth irreducible complex analytic space X such that X is Kähler. Then $R^q f_* \omega_X$ is torsion-free for every q.*

Let $\pi\colon Y \to Z$ be a projective morphism between complex analytic spaces and let $\mathcal{A}$ be a π-ample line bundle on Y. Then $R^p \pi_* (\mathcal{A} \otimes R^q f_* \omega_X) = 0$ holds for every $p > 0$ and every q.

We recall Siu's theorem on complex analytic sheaves, which is a special case of [Si, Theorem 4]. We need it in order to state Theorem 1.1 (i).

Theorem 2.10. *Let $\mathcal{F}$ be a coherent sheaf on a complex analytic space X. Then there exists a locally finite family $\{Y_i\}_{i \in I}$ of complex analytic subvarieties of X such that*

$$\mathrm{Ass}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) = \{\mathfrak{p}_{x,1}, \ldots, \mathfrak{p}_{x,r(x)}\}$$

holds for every point $x \in X$, where $\mathfrak{p}_{x,1}, \ldots, \mathfrak{p}_{x,r(x)}$ are the prime ideals of $\mathcal{O}_{X,x}$ associated to the irreducible components of the germs $Y_{i,x}$ of Y_i at x with $x \in Y_i$. We note that each Y_i is called an associated subvariety *of $\mathcal{F}$.*

In this paper, we do not treat the precise definition of *proper Kähler morphisms of complex analytic spaces*, which is somewhat complicated. Here, we collect all the necessary properties in the following proposition. For the details, see, for example, [Fk, Definition 4.1 and Lemma 4.4].

Proposition 2.11. *Let $f\colon X \to Y$ be a proper morphism of complex analytic spaces such that X is smooth. Then we have:*

(i) *If X is Kähler, then f is Kähler.*

On the other hand, if we assume that f is Kähler, then we have the following properties.

(ii) *Let $U^\dagger$ be any open subset of Y such that $U^\dagger$ is a closed analytic subspace of a polydisc Δ^m for some m. Then $X_U := f^{-1}(U)$ is Kähler, where U is any relatively compact open subset of $U^\dagger$.*

(iii) *Let $\pi\colon Y \to Z$ be a projective morphism of complex analytic spaces and let $V^\dagger$ be any open subset of Z such that $V^\dagger$ is a closed analytic subspace of a polydisc Δ^m for some m. Then, for any relatively compact open subset V of $V^\dagger$, there exists the following commutative diagram*

$$
\begin{array}{ccc}
Y_V & \overset{\iota}{\hookrightarrow} & V \times \mathbb{P}^n \\
{\scriptstyle\pi}\downarrow & \swarrow {\scriptstyle p_1} & \\
V & &
\end{array}
$$

where $Y_V := \pi^{-1}(V)$, ι is a closed embedding, and p_1 is the first projection, and $X_V := (\pi \circ f)^{-1}(V)$ is Kähler.

We will freely use Proposition 2.11 in the subsequent sections. Let us recall some standard definitions.

Definition 2.12. Let D be an $\mathbb{R}$-*divisor* (resp. $\mathbb{Q}$-*divisor*) on an equidimensional complex analytic space X. This means that D is a locally finite formal sum $\sum_i a_i D_i$, where D_i is an irreducible and reduced closed analytic subspace of X of codimension one with $D_i \neq D_j$ for $i \neq j$ and $a_i \in \mathbb{R}$ (resp. $a_i \in \mathbb{Q}$) for every i. We note that the *support* $\operatorname{Supp} D := \bigcup_{a_i \neq 0} D_i$ of D is a closed analytic subspace of X. If $0 \le a_i \le 1$ holds for every i, then D is called a *boundary* $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor). We put

$$
\lfloor D \rfloor = \sum_i \lfloor a_i \rfloor D_i, \quad \lceil D \rceil := -\lfloor -D \rfloor, \quad \text{and} \quad \{D\} := D - \lfloor D \rfloor
$$

as usual, where $\lfloor a_i \rfloor$ is the integer defined by $a_i - 1 < \lfloor a_i \rfloor \le a_i$. We also put

$$
D^{<1} := \sum_{a_i < 1} a_i D_i.
$$

We need the following special case of the vanishing theorem of Reid–Fukuda type, which is an easy consequence of the Kawamata–Viehweg vanishing theorem for projective morphisms between complex analytic spaces.

Lemma 2.13 (Vanishing lemma of Reid–Fukuda type). *Let (X, Δ) be a simple normal crossing pair such that Δ is a boundary $\mathbb{R}$-divisor and let $\mathcal{L}$ be a line bundle on X such that $\mathcal{L} - (\omega_X + \Delta)$ is nef and log big over Y with respect to (X, Δ), where $f\colon X \to Y$ is a projective morphism between complex analytic spaces. Then $R^i f_* \mathcal{L} = 0$ holds for every $i > 0$.*

Before we prove Lemma 2.13, we make an important remark.

Remark 2.14. In Lemma 2.13, we always assume that Δ is globally $\mathbb{R}$-Cartier as in Theorems 1.1 and 1.2 (see Remark 1.3). We take an arbitrary point $y \in Y$. Then it is sufficient to prove $R^i f_* \mathcal{L} = 0$ on a small open neighborhood of $y \in Y$. Hence, by shrinking Y around y suitably, we may assume that there exist Cartier divisors L and K_X on X such that $\mathcal{L} \simeq \mathcal{O}_X(L)$ and $\omega_X \simeq \mathcal{O}_X(K_X)$ hold in the proof of Lemma 2.13.

Let us prove Lemma 2.13.

Proof of Lemma 2.13. In Step 1, we will treat the case where X is irreducible. Then, in Step 2, we will treat the general case.

Step 1. In this step, we assume that X is irreducible. If $\lfloor \Delta \rfloor = 0$, then $R^i f_* \mathcal{O}_X(L) = 0$ holds for every $i > 0$ by the Kawamata–Viehweg vanishing theorem in the complex analytic setting (see, for example, [N2, Chapter II. 5.12.Corollary] and [F16, Section 5]). From now on, we assume that $\lfloor \Delta \rfloor \neq 0$. We take an irreducible component S of $\lfloor \Delta \rfloor$. We consider the following short exact sequence:

$$0 \to \mathcal{O}_X(L - S) \to \mathcal{O}_X(L) \to \mathcal{O}_S(L) \to 0.$$

Note that

$$(L - S) - (K_X + \Delta - S) = L - (K_X + \Delta)$$

and

$$L|_S - (K_S + (\Delta - S)|_S) = (L - (K_X + \Delta))|_S.$$

We use induction on the number of irreducible components of $\lfloor \Delta \rfloor$ and on dimension of X. Thus, we have $R^i f_* \mathcal{O}_X(L - S) = 0$ for every $i > 0$ and $R^i f_* \mathcal{O}_S(L) = 0$ for every $i > 0$. Hence $R^i f_* \mathcal{O}_X(L) = 0$ for every $i > 0$.

Step 2. In this step, we use induction on the number of irreducible components of X. Let Z be an irreducible component of X. Let Z' be the union of components of X other than Z. Then we have the following short exact sequence:

$$0 \to \mathcal{O}_{Z'}(L|_{Z'} - Z|_{Z'}) \to \mathcal{O}_X(L) \to \mathcal{O}_Z(L|_Z) \to 0.$$

Note that

$$(L|_{Z'} - Z|_{Z'}) - (K_{Z'} + \Delta|_{Z'}) = (L - (K_X + \Delta))|_{Z'}$$

holds. Thus, by induction, $R^i f_* \mathcal{O}_{Z'}(L|_{Z'} - Z|_{Z'}) = 0$ for every $i > 0$. Since

$$L|_Z - (K_Z + Z'|_Z + \Delta|_Z) = (L - (K_X + \Delta))|_Z$$

and Z is irreducible, $R^i f_* \mathcal{O}_Z(L|_Z) = 0$ for every $i > 0$ by Step 1. Hence, $R^i f_* \mathcal{O}_Z(L) = 0$ holds for every $i > 0$.

We finish the proof. $\square$

As an easy application of Lemma 2.13, we obtain a useful lemma. This lemma is very useful and indispensable when we treat analytic simple normal crossing pairs.

Lemma 2.15. *Let* $g \colon X' \to X$ *be a projective bimeromorphic morphism between complex analytic spaces such that* X *and* X' *are simple normal crossing. Assume that there exists a Zariski open subset* U *of* X *such that* $g \colon U' := g^{-1}(U) \to U$ *is an isomorphism and that* U *(resp.* U'*) intersects every stratum of* X *(resp.* X'*). Then* $R^i g_* \mathcal{O}_{X'} = 0$ *for every* $i > 0$ *and* $g_* \mathcal{O}_{X'} \simeq \mathcal{O}_X$ *holds.*

Proof. By assumption, we can see that g has connected fibers. Hence we may assume that $\mathrm{codim}_X(X \setminus U) \geq 2$. Thus, we have $g_* \mathcal{O}_{X'} \simeq \mathcal{O}_X$ since X satisfies Serre's S_2 condition. Since the problem is local, we can freely shrink X throughout this proof. We can write $K_{X'} = g^* K_X + E$ for some effective g-exceptional Cartier divisor E on X', where K_X (resp. $K_{X'}$) is a Cartier divisor on X (resp. X') with $\mathcal{O}_X(K_X) \simeq \omega_X$ (resp. $\mathcal{O}_{X'}(K_{X'}) \simeq \omega_{X'}$). Therefore, we have $g_* \mathcal{O}_{X'}(K_{X'}) \simeq \mathcal{O}_X(K_X)$. Since $K_{X'} - K_{X'} = 0$, $R^i g_* \mathcal{O}_{X'}(K_{X'}) = 0$ holds for every $i > 0$ by Lemma 2.13. This implies $\mathbf{R} g_* \omega_{X'}^\bullet \simeq \omega_X^\bullet$,

where $\omega_X^\bullet$ (resp. $\omega_{X'}^\bullet$) is a dualizing complex of X (resp. X'). By Grothendieck duality (see [RRV]), we have

$$\begin{aligned}
\mathcal{O}_X &\simeq \mathbf{R}\mathcal{H}om(\omega_X^\bullet, \omega_X^\bullet) \\
&\simeq \mathbf{R}\mathcal{H}om(\mathbf{R}g_*\omega_{X'}^\bullet, \omega_X^\bullet) \\
&\simeq \mathbf{R}g_*\mathbf{R}\mathcal{H}om(\omega_{X'}^\bullet, \omega_{X'}^\bullet) \\
&\simeq \mathbf{R}g_*\mathcal{O}_{X'}.
\end{aligned}$$

This implies the desired statement. $\qquad\square$

In the proof of Theorem 1.1, we will use Kawamata's covering trick. We contain the precise statement here for the sake of completeness.

Lemma 2.16 (Kawamata's covering trick). *Let $f\colon X \to Y$ be a projective morphism from a smooth complex analytic space X onto a Stein space Y. Let Σ be a reduced simple normal crossing divisor on X and let N be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $\mathrm{Supp}\{N\}$ and Σ have no common irreducible components and that the support of $\{N\} + \Sigma$ is a simple normal crossing divisor on X. Then, after replacing Y with any relatively compact open subset of Y, we can construct a finite cover $p\colon V \to X$ such that V is smooth, $\Sigma_V := p^*\Sigma$ is a reduced simple normal crossing divisor on V, $N_V := p^*N$ is a Cartier divisor on V, and $\mathcal{O}_X(K_X + \Sigma + \lceil N \rceil)$ is a direct summand of $p_*\mathcal{O}_V(K_V + \Sigma_V + N_V)$.*

Proof. The usual proof of Kawamata's covering trick (see, for example, [EV, 3.19. Lemma]) can work in the above complex analytic setting with only minor modifications. $\qquad\square$

In the proof of Theorem 1.2, we need the following elementary lemma. We describe it here for the sake of completeness.

Lemma 2.17. *Let (X, Δ) be an analytic globally embedded simple normal crossing pair and let M be the ambient space of (X, Δ) such that Δ is a boundary $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor). Let C be a stratum of (X, Δ), which is not an irreducible component of X. Let $\sigma\colon M' \to M$ be the blow-up along C and let X' denote the reduced structure of the total transform of X on M', that is, $X' = \sigma^{-1}(X)$. We put*

$$K_{X'} + \Delta' := g^*(K_X + \Delta),$$

where $g := \sigma|_{X'}$. Then we have the following properties:

(i) *(X', Δ') is an analytic globally embedded simple normal crossing pair such that Δ' is a boundary $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor),*

(ii) *M' is the ambient space of (X', Δ'),*

(iii) *$g_*\mathcal{O}_{X'} \simeq \mathcal{O}_X$ holds and $R^i g_*\mathcal{O}_{X'} = 0$ for every $i > 0$,*

(iv) *the strata of (X, Δ) are exactly the images of the strata of (X', Δ'), and*

(v) *$\sigma^{-1}(C)$ is a maximal (with respect to the inclusion) stratum of (X', Δ').*

Proof. We can write $\Delta = B|_X$ by definition. Then we have

$$K_{M'} + X' + B' = \sigma^*(K_M + X + B),$$

where B' is the strict transform of B on M'. We put $c = \mathrm{codim}_M C \geq 2$. Then we obtain $\sigma^*X = X' + (k-1)E$ with $1 \leq k \leq c$ and

$$K_{M'} = \sigma^*K_M + (c-1)E,$$

where E is the σ-exceptional divisor on M'. We consider the following short exact sequence:

$$0 \to \mathcal{O}_{M'}(-X') \to \mathcal{O}_{M'} \to \mathcal{O}_{X'} \to 0.$$

Since M is smooth, we have $R^i\sigma_*\mathcal{O}_{M'} = 0$ for every $i > 0$. Since

$$-X' - K_{M'} = -\sigma^*(K_M + X) - (c - k)E,$$

we have $R^i\sigma_*\mathcal{O}_{M'}(-X') = 0$ for every $i > 0$ by the Kawamata–Viehweg vanishing theorem in the complex analytic setting (see, for example, [N2, Chapter II. 5.12.Corollary] and [F16, Section 5]). Thus, we have $R^i g_*\mathcal{O}_{X'} = 0$ for every $i > 0$ and the following short exact sequence:

$$0 \to \sigma_*\mathcal{O}_{M'}(-X') = \mathcal{I}_X \to \mathcal{O}_M \to g_*\mathcal{O}_{X'} \to 0,$$

where $\mathcal{I}_X$ is the defining ideal sheaf of X on M. This implies that $g_*\mathcal{O}_{X'} \simeq \mathcal{O}_X$ holds. By construction, $\Delta' = B'|_{X'}$ holds. Therefore, (X', Δ') is an analytic globally embedded simple normal crossing pair such that M' is the ambient space of (X', Δ'). We can easily check that (X', Δ') satisfies all the desired properties. $\square$

3. Standard setting

In this section, we will establish the following theorem, which obviously generalizes Kollár's famous torsion-freeness, vanishing theorem, and injectivity theorem. Although our result in this section depends on Saito's theory of mixed Hodge modules through Theorem 2.6 (see [Sa1], [Sa2], [Sa3], [FFS], and [Sa5]), we do not directly use Saito's vanishing theorem in [Sa2].

Theorem 3.1. *Let (X, D) be an analytic simple normal crossing pair such that D is reduced and let $f\colon X \to Y$ be a proper morphism between complex analytic spaces. We assume that f is Kähler on each irreducible component of X. Then we have the following properties.*

(i) (*Strict support condition*). *Every associated subvariety of $R^q f_*\omega_X(D)$ is the f-image of some stratum of (X, D) for every q.*

(ii) (*Vanishing theorem*). *Let $\pi\colon Y \to Z$ be a projective morphism between complex analytic spaces and let $\mathcal{A}$ be a π-ample line bundle on Y. Then*

$$R^p\pi_*\left(\mathcal{A} \otimes R^q f_*\omega_X(D)\right) = 0$$

holds for every $p > 0$ and every q.

(iii) (*Injectivity theorem*). *Let $\mathcal{L}$ be an f-semiample line bundle on X. Let s be a nonzero element of $H^0(X, \mathcal{L}^{\otimes k})$ for some nonnegative integer k such that the zero locus of s does not contain any strata of (X, D). Then, for every q, the map*

$$\times s\colon R^q f_*\left(\omega_X(D) \otimes \mathcal{L}^{\otimes l}\right) \to R^q f_*\left(\omega_X(D) \otimes \mathcal{L}^{\otimes k+l}\right)$$

induced by $\otimes s$ is injective for every positive integer l.

As we have already mentioned above, Theorem 3.1 generalizes Kollár's famous results. We explain it here for the reader's convenience.

Remark 3.2 (Kollár's original theorem). If X is a smooth projective variety with $D = 0$ and $f\colon X \to Y$ is a projective surjective morphism onto a projective variety Y in Theorem 3.1 (i), then the strict support condition is nothing but Kollár's torsion-freeness of $R^q f_*\omega_X$ (see [Kol1, Theorem 2.1 (i)]). We further assume that Z is a point in Theorem 3.1 (ii). Then we can recover Kollár's vanishing theorem (see [Kol1, Theorem 2.1 (iii)]). If X is a smooth projective variety, $D = 0$, and Y is a point, then Theorem 3.1 (iii) coincides with Kollár's original injectivity theorem (see [Kol1, Theorem 2.2]).

Remark 3.3. Theorem 3.1 (iii) solves [F5, Problem 1.8] completely. Note that [F11, Conjecture 2.21] is closely related to Theorem 3.1 (iii) and was recently solved in [CP] (see also [CCM]).

Let us start the proof of Theorem 3.1.

Proof of Theorem 3.1. In Step 1, we will prove (i), which is an easy consequence of Theorem 2.6 and Takegoshi's torsion-freeness (see Theorem 2.9). In Step 2, we will prove that (ii) easily follows from Takegoshi's vanishing theorem (see Theorem 2.9) with the aid of Theorem 2.6. In Step 3, we will see that (iii) is an easy consequence of (i) and (ii).

Step 1 (Strict support condition). Since the problem is local, we may assume that Y is a closed analytic subspace of a polydisc Δ^m. By replacing Y with Δ^m, we may further assume that Y itself is a polydisc (see also Proposition 2.11 (ii)). In this case, we can use Theorem 2.6. We note that $\omega_Y \simeq \mathcal{O}_Y$ holds. By Theorem 2.9, $_F E_1^{-q,i+q} \simeq \bigoplus_{k+l=n+q+1} R^i f_* \omega_{D[k,l]}$ satisfies the strict support condition, that is, every associated subvariety of $_F E_1^{-q,i+q} \simeq \bigoplus_{k+l=n+q+1} R^i f_* \omega_{D[k,l]}$ is the f-image of some stratum of (X, D). By Theorem 2.6, the associated subvariety of $_F E_2^{-q,i+q} = {}_F E_\infty^{-q,i+q}$ is the f-image of some stratum of (X, D). This implies that $R^q f_* \omega_X(D)$ satisfies the desired strict support condition.

Step 2 (Vanishing theorem). We may assume that Z is a polydisc and Y is a closed analytic subspace of $Z \times \mathbb{P}^n$ (see Proposition 2.11 (iii)). By applying Theorem 2.6 to $f \colon X \to Y \hookrightarrow Z \times \mathbb{P}^n$, we obtain the following spectral sequence

$$E_1^{-q,i+q} = \bigoplus_{k+l=n+q+1} R^i f_* \omega_{D[k,l]} \Rightarrow R^i f_* \omega_X(D)$$

which degenerates at E_2 such that its E_1-differential d_1 splits. By Theorem 2.9, we obtain $R^p \pi_* \left(\mathcal{A} \otimes E_1^{-q,i+q} \right) = 0$ for every $p > 0$. Since the $E_2^{-q,i+q} = E_\infty^{-q,i+q}$ are direct factors of $E_1^{-q,i+q}$, we have $R^p \pi_* \left(\mathcal{A} \otimes E_2^{-q,i+q} \right) = 0$ for every $p > 0$. This implies that $R^p \pi_* \left(\mathcal{A} \otimes R^q f_* \omega_X(D) \right) = 0$ holds for every $p > 0$. This is what we wanted.

Step 3 (Injectivity theorem). We take an arbitrary point $y \in Y$ and can freely shrink Y around y since the problem is local. Without loss of generality, we may assume that Y is Stein and that each irreducible component of X is Kähler (see Proposition 2.11 (ii)). By Bertini's theorem, we can take an element $u \in H^0(X, \mathcal{L}^{\otimes m}) \setminus \{0\}$ for some positive integer $m \geq 2$ such that $R := (u = 0)$ and $R + D$ are reduced and $(X, D + R)$ is an analytic simple normal crossing pair. Let $p \colon V \to X$ be the m-fold cyclic cover ramifying along R. More precisely, we define an $\mathcal{O}_X$-algebra structure of $\bigoplus_{i=0}^{m-1} \mathcal{L}^{\otimes(-i)}$ by $u \colon \mathcal{L}^{\otimes(-m)} \to \mathcal{O}_X$ and put

$$V := \mathrm{Specan}_X \bigoplus_{i=0}^{m-1} \mathcal{L}^{\otimes(-i)}.$$
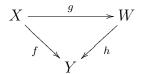
Then (V, p^*D) is an analytic simple normal crossing pair. We can check that each irreducible component of V is Kähler if we shrink Y around y suitably. By construction, we have

$$p_* \left(\omega_V(p^*D) \otimes p^* \mathcal{L}^{\otimes l} \right) = \bigoplus_{i=l}^{l+m-1} \omega_X(D) \otimes \mathcal{L}^{\otimes i}.$$

By construction again, we see that $p^*\mathcal{L}$ has a section whose zero locus is the reduced preimage of R. By iterating this process, we obtain a tower of cyclic covers:

$$V_n \to V_{n-1} \to \cdots \to V_0 := V \to X.$$

By choosing the ramification divisors suitably, we may assume that the pull-back of $\mathcal{L}$ on V_n is relatively globally generated over Y. By replacing X and $\mathcal{L}$ with V_n and the pull-back of $\mathcal{L}$, respectively, we may assume that $\mathcal{L}$ is f-free, that is, $f^*f_*\mathcal{L} \to \mathcal{L}$ is surjective. We take the contraction morphism over Y associated to the surjection $f^*f_*\mathcal{L} \to \mathcal{L}$ and take its Stein factorization. Then we obtain the following commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\quad g \quad} & W \\
& {\scriptstyle f}\searrow \quad \swarrow {\scriptstyle h} & \\
& Y &
\end{array}
$$

such that $g_*\mathcal{O}_X \simeq \mathcal{O}_W$ and that $\mathcal{L} \simeq g^*\mathcal{L}_W$ for some h-ample line bundle $\mathcal{L}_W$ on W. Since $H^0(X, \mathcal{L}^{\otimes k}) \simeq H^0(W, \mathcal{L}_W^{\otimes k})$, there exists $t \in H^0(W, \mathcal{L}_W^{\otimes k})$ such that $s = g^*t$. By (i), every associated subvariety of $R^q g_* \omega_X(D)$ is the g-image of some stratum of (X, D). By assumption, the zero locus of t does not contain any associated subvarieties of $R^q g_* \omega_X(D)$. Hence, the map

$$\times t \colon R^q g_* \omega_X(D) \otimes \mathcal{L}_W^{\otimes l} \to R^q g_* \omega_X(D) \otimes \mathcal{L}_W^{\otimes k+l}$$

induced by $\otimes t$ is injective. Thus, the map

$$\times s \colon R^q g_* \left(\omega_X(D) \otimes \mathcal{L}^{\otimes l}\right) \to R^q g_* \left(\omega_X(D) \otimes \mathcal{L}^{\otimes(k+l)}\right)$$

is injective. Therefore, by taking h_*, we obtain that

(3.1) $\qquad \times s \colon h_* \left(R^q g_* \left(\omega_X(D) \otimes \mathcal{L}^{\otimes l}\right)\right) \to h_* \left(R^q g_* \left(\omega_X(D) \otimes \mathcal{L}^{\otimes(k+l)}\right)\right)$

is injective. On the other hand,

$$R^p h_* \left(R^q g_* \left(\omega_X(D) \otimes \mathcal{L}^{\otimes n}\right)\right) \simeq R^p h_* \left(\mathcal{L}_W^{\otimes n} \otimes R^q g_* \omega_X(D)\right) = 0$$

holds for every $p > 0$ and every $n > 0$ by (ii). Thus, by using the spectral sequence, we have

$$h_* R^q g_* \left(\omega_X(D) \otimes \mathcal{L}^{\otimes n}\right) \simeq R^q (h \circ g)_* \left(\omega_X(D) \otimes \mathcal{L}^{\otimes n}\right) = R^q f_* \left(\omega_X(D) \otimes \mathcal{L}^{\otimes n}\right)$$

for every positive integer n. Therefore, (3.1) implies the desired injection

$$\times s \colon R^q f_* \left(\omega_X(D) \otimes \mathcal{L}^{\otimes l}\right) \to R^q f_* \left(\omega_X(D) \otimes \mathcal{L}^{\otimes(k+l)}\right).$$

We finish the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. PROOF OF THEOREM 1.1 WHEN X IS IRREDUCIBLE

In this section, we will prove Theorem 1.1 under the extra assumption that X is irreducible. For many geometric applications, the case where X is irreducible seems to be sufficient. When X is irreducible, we can easily reduce Theorem 1.1 to Theorem 3.1 (i) and (ii) by using some standard arguments which are repeatedly used in the theory of minimal models.

First, let us prove Theorem 1.1 (i) when X is irreducible. In the proof, we will use some covering tricks to reduce it to Theorem 3.1 (i).

Proof of Theorem 1.1 (i) when X is irreducible. We take an arbitrary point $y \in Y$. It is sufficient to prove the strict support condition on a small Stein open neighborhood of y. Therefore, we will freely shrink Y around y suitably without mentioning it explicitly throughout this proof (see also Proposition 2.11). By replacing Y with a small relatively compact Stein open neighborhood of y, we may assume that there exist Cartier divisors L and K_X such that $\mathcal{O}_X(L) \simeq \mathcal{L}$ and $\mathcal{O}_X(K_X) \simeq \omega_X$, respectively. By perturbing the coefficients of Δ suitably, we may further assume that Δ is a $\mathbb{Q}$-divisor. We put $N := L - (K_X + \Delta)$. Then N is an f-semiample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $L = K_X + \Sigma + \lceil N \rceil$, where $\Sigma := \lfloor \Delta \rfloor$, with $\{N\} = \{-\Delta\}$. By Kawamata's covering trick (see Lemma 2.16), we can take a finite cover $p \colon V \to X$ such that V is smooth, $\Sigma_V := p^*\Sigma$ is a reduced simple normal crossing divisor on V, $N_V := p^*N$ is Cartier, and $\mathcal{O}_X(L) = \mathcal{O}_X(K_X + \Sigma + \lceil N \rceil)$ is a direct summand of $p_*\mathcal{O}_V(K_V + \Sigma_V + N_V)$. Since N_V is an $(f \circ p)$-semiample Cartier divisor on V, we can take a finite cyclic cover $q \colon W \to V$ such that W is smooth, $\Sigma_W := q^*\Sigma_V = q^*p^*\Sigma$ is a reduced simple normal crossing divisor on W, and $\mathcal{O}_V(K_V + \Sigma_V + N_V)$ is a direct summand of $q_*\mathcal{O}_W(K_W + \Sigma_W)$. By Theorem 3.1 (i), every associated subvariety of $R^q(f \circ p \circ q)_*\mathcal{O}_W(K_W + \Sigma_W)$ is the $(f \circ p \circ q)$-image of some stratum of (W, Σ_W). This implies that every associated subvariety of $R^q f_*\mathcal{O}_X(L)$ is the f-image of some stratum of (X, Δ). This is what we wanted. $\square$

Next, we will use some covering tricks and Leray's spectral sequence in order to reduce Theorem 1.1 (ii) to Theorem 3.1 (ii). Although the covering tricks are the same as above, we will write all the details for the reader's convenience.

Proof of Theorem 1.1 (ii) when X is irreducible. As in the proof of Theorem 1.1 (i), we take an arbitrary point $z \in Z$ and consider a small relatively compact Stein open neighborhood of z (see also Proposition 2.11). Throughout this proof, we will freely shrink Z around z without mentioning it explicitly. By perturbing the coefficients, we may assume that H is a π-ample $\mathbb{Q}$-divisor on Y with $L - (K_X + \Delta) \sim_{\mathbb{Q}} f^*H$, where $\mathcal{L} \simeq \mathcal{O}_X(L)$ and $\omega_X \simeq \mathcal{O}_X(K_X)$. We put $N := L - (K_X + \Delta)$ and $\Sigma := \lfloor \Delta \rfloor$. By Kawamata's covering trick (see Lemma 2.16), we can construct a finite cover $p \colon V \to X$ such that V is smooth, $\Sigma_V := p^*\Sigma$ is a simple normal crossing divisor, and $N_V := p^*N$ is a Cartier divisor on V. Since $\mathcal{O}_X(L) = \mathcal{O}_X(K_X + \Sigma + \lceil N \rceil)$ is a direct summand of $p_*\mathcal{O}_V(K_V + \Sigma_V + N_V)$, it is sufficient to prove that
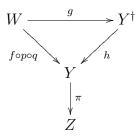
$$R^p\pi_*R^q(f \circ p)_*\mathcal{O}_V(K_V + \Sigma_V + N_V) = 0$$

for every $p > 0$. As in Step 3 in the proof of Theorem 3.1, we can take a finite cover $q \colon W \to V$ such that W is smooth, $N_W := q^*N_V$ is $(\pi \circ f \circ p \circ q)$-free, $\Sigma_W := q^*\Sigma_V$ is a reduced simple normal crossing divisor, and $\mathcal{O}_V(K_V + \Sigma_V + N_V)$ is a direct summand of $q_*\mathcal{O}_W(K_W + \Sigma_W + N_W)$. Hence it is sufficient to prove that

$$R^p\pi_*R^q(f \circ p \circ q)_*\mathcal{O}_W(K_W + \Sigma_W + N_W) = 0$$

for every $p > 0$. We take the contraction morphism over Z associated to the surjection $(\pi \circ f \circ p \circ q)^*(\pi \circ f \circ p \circ q)_*\mathcal{O}_W(N_W) \to \mathcal{O}_W(N_W)$ and take its Stein factorization. Then

we have the following commutative diagram:

$$
\begin{array}{ccc}
W & \xrightarrow{\ \ g\ \ } & Y^{\dagger} \\
{\scriptstyle f\circ p\circ q}\searrow & & \swarrow{\scriptstyle h} \\
& Y & \\
& \downarrow{\scriptstyle \pi} & \\
& Z &
\end{array}
$$

such that $g_*\mathcal{O}_W \simeq \mathcal{O}_{Y^{\dagger}}$, $\mathcal{O}_W(N_W) \simeq g^*\mathcal{A}$, where $\mathcal{A}$ is a $(\pi \circ h)$-ample line bundle on $Y^{\dagger}$, and h is finite. Then we obtain

$$
\begin{aligned}
R^p\pi_* & R^q(f \circ p \circ q)_*\mathcal{O}_W(K_W + \Sigma_W + N_W) \\
& \simeq R^p\pi_*R^q(h \circ g)_*\mathcal{O}_W(K_W + \Sigma_W + N_W) \\
& \simeq R^p\pi_*\left(h_*R^qg_*\mathcal{O}_W(K_W + \Sigma_W + N_W)\right) \\
& \simeq R^p(\pi \circ h)_*\left(\mathcal{A} \otimes R^qg_*\mathcal{O}_W(K_W + \Sigma_W)\right) \\
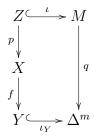& = 0
\end{aligned}
$$

by Theorem 3.1 (ii). We finish the proof. $\qquad\square$

5. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 in full generality. The proof of Theorem 1.1 given in this section is essentially the same as that of Theorem 1.1 for irreducible varieties in Section 4 although it is more complicated.

First, we prepare the following technical but very important lemma.

Lemma 5.1. *Let (X, Δ) be an analytic simple normal crossing pair such that Δ is a boundary $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor) and let $f\colon X \to Y$ be a projective morphism between complex analytic spaces. Let L be a Cartier divisor on X. We take an arbitrary point $y \in Y$. Then, after shrinking Y around y suitably, we can construct the following commutative diagram:*

$$
\begin{array}{ccc}
Z & \xhookrightarrow{\ \iota\ } & M \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\
X & & \\
{\scriptstyle f}\downarrow & & \downarrow \\
Y & \xhookrightarrow{\ \iota_Y\ } & \Delta^m
\end{array}
$$

such that

   (i) *$\iota_Y\colon Y \hookrightarrow \Delta^m$ is a closed embedding into a polydisc Δ^m,*
   (ii) *(Z, Δ_Z) is an analytic globally embedded simple normal crossing pair, where Δ_Z is a boundary $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor) on Z,*
   (iii) *M is the ambient space of (Z, Δ_Z) and is projective over Δ^m,*
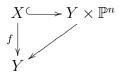   (iv) *there exists a Cartier divisor L_Z on Z satisfying*

$$
L_Z - (K_Z + \Delta_Z) = p^*(L - (K_X + \Delta)),
$$

   $p_*\mathcal{O}_Z(L_Z) \simeq \mathcal{O}_X(L)$, and $R^ip_*\mathcal{O}_Z(L_Z) = 0$ for every $i > 0$,
   (v) *$p(W)$ is a stratum of (X, Δ) for every stratum W of (Z, Δ_Z),*

(vi) *there exists a Zariski open subset U of X, which intersects every stratum of X, such that p is an isomorphism over U,*

(vii) *p maps every stratum of Z bimeromorphically onto some stratum of X, and*

(viii) *for any stratum S of (X, Δ), there exists a stratum W of (Z, Δ_Z) such that $S = p(W)$.*

Proof. We divide the proof into several small steps. The arguments below are essentially contained in [F6, Section 4], [F9, Section 5.8], and [F10, Lemmas 4.4, 4.6, and 4.8].

Step 1. Since $f\colon X \to Y$ is projective, we have the following commutative diagram

$$\begin{array}{ccc} X & \hookrightarrow & Y \times \mathbb{P}^n \\ {\scriptstyle f}\downarrow & \swarrow & \\ Y & & \end{array}$$

after shrinking Y around y. Without loss of generality, we may further assume that Y is a closed analytic subspace of a polydisc Δ^m. Thus we get the following commutative diagram

$$\begin{array}{ccc} X & \hookrightarrow & \Delta^m \times \mathbb{P}^n \\ {\scriptstyle f}\downarrow & & \downarrow \\ Y & \underset{\iota_Y}{\hookrightarrow} & \Delta^m \end{array}$$

such that $\iota_Y(y) = 0 \in \Delta^m$.

Step 2. We put $M_0 := \Delta^m \times \mathbb{P}^n$. We take an irreducible component C of $\mathrm{Supp}\,\Delta$ and take the blow-up $p_1\colon M_1 \to M_0$ along C. Let X_1 be the strict transform of X on M_1. We put $K_{X_1} + \Delta_1 = p_1^*(K_X + \Delta)$ and $L_1 = p_1^* L$. By construction, Δ_1 is a boundary $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor). By Lemma 2.15, $R^i p_{1*} \mathcal{O}_{X_1} = 0$ for every $i > 0$ and $p_{1*} \mathcal{O}_{X_1} \simeq \mathcal{O}_X$ holds. Therefore, we obtain that $R^i p_{1*} \mathcal{O}_{X_1}(L_1) = 0$ for every $i > 0$ and $p_{1*} \mathcal{O}_{X_1}(L_1) \simeq \mathcal{O}_X(L)$. Note that M_1 is smooth and that (X_1, Δ_1) is a simple normal crossing pair. By repeating this process finitely many times, we obtain a sequence of blow-ups

$$M_k \xrightarrow{p_k} M_{k-1} \xrightarrow{p_{k-1}} \cdots \xrightarrow{p_1} M_0$$

and simple normal crossing pairs (X_i, Δ_i) and Cartier divisors L_i on X_i such that $\Delta_k = D|_{X_k}$ for some $\mathbb{R}$-Cartier $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor) D on M_k.

Step 3 (see the proof of [Kol4, Proposition 10.59]). We shrink Δ^m slightly and assume that M_k is a closed analytic subspace of $\Delta^m \times \mathbb{P}^{n'}$. We pick a finite set of points W of X_k such that each stratum of $(X_k, \mathrm{Supp}\,\Delta_k)$ contains some point of W. We take a sufficiently large positive integer d such that $\mathcal{I}_{X_k} \otimes \mathcal{O}_{M_k}(d)$ is globally generated, where $\mathcal{I}_{X_k}$ is the defining ideal sheaf of X_k on M_k and $\mathcal{O}_{M_k}(d) := (q^* \mathcal{O}_{\mathbb{P}^{n'}}(d))|_{M_k}$ with the second projection $q\colon \Delta^m \times \mathbb{P}^{n'} \to \mathbb{P}^{n'}$. We take a complete intersection of $(m + n' - \dim X_k - 1)$ general members of $|\mathcal{I}_{X_k} \otimes \mathcal{O}_{M_k}(d)|$. Then we can construct $X_k \subset V$ such that V is smooth at every point of W. We note that we used the fact that X_k has only hypersurface singularities near W. By construction, we have $V \not\subset \mathrm{Supp}\,D$.

Step 4. In this step, we will freely shrink Δ^m slightly without mentioning it explicitly. By applying the resolution of singularities to V (see [BM, Theorem 13.3]), we can construct a projective bimeromorphic morphism $\alpha\colon V' \to V$ from a smooth analytic space V'. We may assume that the exceptional locus E of α is a simple normal crossing divisor on

V' and that $\alpha\colon V' \to V$ is an isomorphism over the largest Zariski open subset of V on which V is smooth. Let X_{k+1} be the strict transform of X_k on V'. We apply the resolution of singularities to $(V', X_{k+1} + \Sigma)$ (see [BM, Theorems 13.3 and 12.4]), where Σ is the support of the union of $\alpha^*(D|_V)$ and E. Then we get a projective bimeromorphic morphism $M \to V'$ from a smooth analytic space M such that $M \to V'$ is an isomorphism over the locus where $X_{k+1} + \Sigma$ is a simple normal crossing divisor. Let Z be the strict transform of X_{k+1} on M. We put $K_Z + \Theta := \beta^*(K_{X_k} + \Delta_k)$, where $\beta\colon Z \to X_k$ is the natural induced morphism. We put $\Delta_Z := \Theta + \lceil -(\Theta^{<1}) \rceil$ and $L_Z := \beta^* L_k + \lceil -(\Theta^{<1}) \rceil$. By construction, (Z, Δ_Z) is an analytic globally embedded simple normal crossing pair and M is the ambient space of (Z, Δ_Z). We note that Δ_Z is a boundary $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor) and that

$$L_Z - (K_Z + \Delta_Z) = \beta^* L_k + \lceil -(\Theta^{<1}) \rceil - \beta^*(K_{X_k} + \Delta_k) - \lceil -(\Theta^{<1}) \rceil$$
$$= \beta^*(L_k - (K_{X_k} + \Delta_k))$$

holds. Note that $\lceil -(\Theta^{<1}) \rceil$ is an effective β-exceptional divisor on Z. By construction again, β maps every stratum W of (Z, Δ_Z) bimeromorphically onto $\beta(W)$. Therefore, we have $R^i \beta_* \mathcal{O}_Z(L_Z) = 0$ for every $i > 0$ by Lemma 2.13 and $\beta_* \mathcal{O}_Z \simeq \mathcal{O}_{X_k}$ holds (see Lemma 2.15). Hence (Z, Δ_Z) is an analytic globally embedded simple normal crossing pair with all the desired properties.

We finish the proof.   □

Let us prove Theorem 1.1 (i).

Proof of Theorem 1.1 (i). We take an arbitrary point $y \in Y$. We can freely shrink Y around y suitably. Hence we may assume that Y is Stein and there exist L and K_X such that $\mathcal{O}_X(L) \simeq \mathcal{L}$ and $\mathcal{O}_X(K_X) \simeq \omega_X$, respectively. By perturbing the coefficients of Δ, we may assume that Δ is a boundary $\mathbb{Q}$-divisor such that $N := L - (K_X + \Delta)$ is an f-semiample $\mathbb{Q}$-divisor. By Lemma 5.1, we may further assume that (X, Δ) is an analytic globally embedded simple normal crossing pair and that the ambient space M of (X, Δ) is projective over a polydisc Δ^m with $Y \hookrightarrow \Delta^m$. We put $\Sigma := \lfloor \Delta \rfloor$. By taking a suitable Kawamata covering of the ambient space M (see Lemma 2.16) and restricting it to X, we have a finite cover $p\colon V \to X$ such that $(V, \Sigma_V := p^*\Sigma)$ is an analytic globally embedded simple normal crossing pair, Σ_V is reduced, $N_V := p^*N$ is Cartier, and $\mathcal{O}_X(K_X + \Sigma + \lceil N \rceil) = \mathcal{O}_X(L)$ is a direct summand of $p_* \mathcal{O}_V(K_V + \Sigma_V + N_V)$. We note that $(K_M + X)|_X = K_X$ and (X, Δ) is an analytic globally embedded simple normal crossing pair. As before, we can take a finite cyclic cover $q\colon W \to V$ such that $(W, \Sigma_W := q^*\Sigma_V)$ is an analytic simple normal crossing pair, Σ_W is reduced, and $\mathcal{O}_V(K_V + \Sigma_V + N_V)$ is a direct summand of $q_* \mathcal{O}_W(K_W + \Sigma_W)$ (for the details, see Step 3 in the proof of Theorem 3.1 and Proof of Theorem 1.1 (i) when X is irreducible). By Theorem 3.1 (i), every associated subvariety of $R^q(f \circ p \circ q)_* \mathcal{O}_W(K_W + \Sigma_W)$ is the $(f \circ p \circ q)$-image of some stratum of (W, Σ_W). Therefore, every associated subvariety of $R^q f_* \mathcal{O}_X(L)$ is the f-image of some stratum of (X, Δ).   □

Finally, we prove Theorem 1.1 (ii).

Proof of Theorem 1.1 (ii). We take an arbitrary point $z \in Z$ and can freely shrink Z around z. So we may assume that Z is Stein. As usual, by perturbing the coefficients, we may assume that $L - (K_X + \Delta) \sim_{\mathbb{Q}} f^*H$, where $\mathcal{O}_X(L) \simeq \mathcal{L}$, $\mathcal{O}_X(K_X) \simeq \omega_X$, H is a π-ample $\mathbb{Q}$-divisor on Y, such that Δ is a boundary $\mathbb{Q}$-divisor. We put $\Sigma := \lfloor \Delta \rfloor$. By Lemma 5.1, we may further assume that (X, Δ) is an analytic globally embedded simple normal

crossing pair and the ambient space M of (X, Δ) is projective over a polydisc Δ^m with $Z \hookrightarrow \Delta^m$. By taking a suitable Kawamata covering of the ambient space M (see Lemma 2.16) and restricting it to X, we get a finite cover $p \colon V \to X$ such that $(V, \Sigma_V := p^*\Sigma)$ is an analytic globally embedded simple normal crossing pair, Σ_V is reduced, $N_V := p^*N$ is Cartier, and $\mathcal{O}_X(L)$ is a direct summand of $p_*\mathcal{O}_V(K_V + \Sigma_V + N_V)$. As in Step 3 in the proof of Theorem 3.1, by taking a tower of cyclic covers, we can construct a finite cover $q \colon W \to V$ such that $(W, \Sigma_W := q^*\Sigma_V)$ is an analytic simple normal crossing pair, Σ_W is reduced, $N_W := q^*N_V$ is free over Z, and $\mathcal{O}_V(K_V + \Sigma_V + N_V)$ is a direct summand of $q_*\mathcal{O}_W(K_W + \Sigma_W + N_W)$. Hence it is sufficient to prove

$$R^p\pi_* R^q(f \circ p \circ q)_*\mathcal{O}_W(K_W + \Sigma_W + N_W) = 0$$

for every $p > 0$. Since Theorem 3.1 (ii) holds for analytic simple normal crossing pairs, the argument in the proof of Theorem 1.1 (ii) when X is irreducible in Section 4 works without any changes. Thus we get the desired vanishing theorem. $\square$

6. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2.

Proof of Theorem 1.2. We take an arbitrary point $z \in Z$. It is sufficient to prove that $R^p\pi_* R^q f_*\mathcal{L} = 0$ holds for every $p > 0$ in a neighborhood of $z \in Z$. By Lemma 5.1, we may assume that the following commutative diagram

$$\begin{array}{ccc} X & \overset{\iota}{\hookrightarrow} & M \\ {\scriptstyle \pi \circ f}\downarrow & & \downarrow \\ Z & \underset{\iota_Z}{\hookrightarrow} & \Delta^m \end{array}$$

exists, where (X, Δ) is an analytic globally embedded simple normal crossing pair, M is the ambient space of (X, Δ) and is projective over Δ^m, and Z is a closed analytic subspace of Δ^m with $\iota_Z(z) = 0 \in \Delta^m$. Since Z is Stein and f and π are both projective, there exist Cartier divisors L and K_X such that $\mathcal{L} \simeq \mathcal{O}_X(L)$ and $\omega_X \simeq \mathcal{O}_X(K_X)$, respectively, and that $L - (K_X + \Delta) \sim_{\mathbb{R}} f^*H$ holds for some $\mathbb{R}$-Cartier divisor H, which is nef and log big over Z with respect to (X, Δ), on Y.

Step 1. In this step, we will prove the desired vanishing theorem under the extra assumption that every stratum of (X, Δ) is dominant onto some irreducible component of $f(X)$.

From now on, we assume that every stratum of (X, Δ) is dominant onto some irreducible component of $f(X)$. By taking the Stein factorization, we may assume that $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$ holds. In particular, Y is reduced. Let $X^\dagger$ be any connected component of X. Since f has connected fibers, $Y^\dagger := f(X^\dagger)$ is an irreducible component of Y and every irreducible component of $X^\dagger$ is mapped to $Y^\dagger$ by f. Hence, we may further assume that X is connected and that Y is irreducible. Since H is π-big, we can write $H = E + A$, where A is a π-ample $\mathbb{R}$-divisor and E is an effective $\mathbb{R}$-Cartier divisor. We take a projective bimeromorphic morphism $\alpha \colon M' \to M$ from a smooth complex variety M', which is an isomorphism outside $\operatorname{Supp} f^*E$. Let X' be the strict transform of X on M'. We put $\varphi := \alpha|_{X'} \colon X' \to X$. By taking $\alpha \colon M' \to M$ suitably, we may assume that (X', Σ) is an analytic globally embedded simple normal crossing pair, M' is the ambient space of

(X', Σ), and Σ contains $\operatorname{Supp} \varphi^* \Delta$ and $\operatorname{Supp} \varphi^* f^* E$ (see [BM, Theorems 13.3 and 12.4]). For $k \gg 1$, we can write

$$K_{X'} + \Delta' = \varphi^* \left(K_X + \Delta + \frac{1}{k} f^* E \right) + E'$$

such that

(1) (X', Δ') is an analytic globally embedded simple normal crossing pair such that Δ' is a boundary $\mathbb{R}$-divisor,
(2) M' is the ambient space of (X', Δ'),
(3) E' is an effective φ-exceptional Cartier divisor on X', and
(4) φ maps every stratum of (X', Δ') bimeromorphically onto some stratum of (X, Δ).

We put $L' := \varphi^* L + E'$. Then we have $\varphi_* \mathcal{O}_{X'}(L') \simeq \mathcal{O}_X(L)$ and $R^i \varphi_* \mathcal{O}_{X'}(L') = 0$ for every $i > 0$ by Lemma 2.13 since $L' - (K_X' + \Delta') = \varphi^*(L - (K_X + \Delta + \frac{1}{k} f^* E))$. We note that

$$L' - (K_{X'} + \Delta') = \varphi^* L + E' - \varphi^* \left(K_X + \Delta + \frac{1}{k} f^* E \right) - E'$$

$$= \varphi^*(L - (K_X + \Delta)) - \frac{1}{k} \varphi^* f^* E$$

$$\sim_{\mathbb{R}} \varphi^* \left(f^* H - \frac{1}{k} f^* E \right)$$

$$= (f \circ \varphi)^* \left(\frac{1}{k} A + \frac{k-1}{k} H \right).$$

and that $\frac{1}{k} A + \frac{k-1}{k} H$ is π-ample. Thus, by Theorem 1.1 (ii), we obtain

$$R^p \pi_* R^q f_* \mathcal{O}_X(L) = R^p \pi_* R^q (f \circ \varphi)_* \mathcal{O}_{X'}(L') = 0$$

for every $p > 0$.

Step 2. In this step, we will treat the general case by induction on $\dim f(X)$. We note that the desired vanishing theorem holds when $\dim f(X) = 0$ by Step 1. By using Lemma 2.17 finitely many times, we can decompose $X = X' \cup X''$ as follows: X' is the union of all strata of (X, Δ) that are not mapped to irreducible components of $f(X)$ and $X'' = X - X'$. We put

$$K_{X''} + \Delta_{X''} := (K_X + \Delta)|_{X''} - X'|_{X''}.$$

Then $f : (X'', \Delta_{X''}) \to X$ and $L'' := L|_{X''} - X'|_{X''}$ satisfy the assumption in Step 1. We consider the following short exact sequence:

$$0 \to \mathcal{O}_{X''}(L'') \to \mathcal{O}_X(L) \to \mathcal{O}_{X'}(L) \to 0.$$

We note that every associated subvariety of $R^q f_* \mathcal{O}_{X''}(L'')$ is an irreducible component of $f(X)$ by Theorem 1.1 (i) and that every associated subvariety of $R^q f_* \mathcal{O}_{X'}(L)$ is contained in $f(X')$ for every q. Therefore, the connecting homomorphisms

$$\delta : R^q f_* \mathcal{O}_{X'}(L) \to R^{q+1} f_* \mathcal{O}_{X''}(L'')$$

are zero for all q. Thus we have the following short exact sequence

$$0 \to R^q f_* \mathcal{O}_{X''}(L'') \to R^q f_* \mathcal{O}_X(L) \to R^q f_* \mathcal{O}_{X'}(L) \to 0$$

for every q. By Step 1, we have $R^p \pi_* R^q f_* \mathcal{O}_{X''}(L'') = 0$ for every $p > 0$. On the other hand, $R^p \pi_* R^q f_* \mathcal{O}_{X'}(L) = 0$ for every $p > 0$ by induction on $\dim f(X)$. This implies that

$$R^p \pi_* R^q f_* \mathcal{O}_X(L) = 0$$

holds for every $p > 0$.

We finish the proof. □

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