ADJUNCTION FOR PURELY LOG TERMINAL PAIRS

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1. On dlt-blow-ups

The following theorem supplements [F1, Theorem 1.21 (Dlt blow-ups, I)] and [F1, Theorem 1.27 (Dlt blow-ups, II)]. It is well known for algebraic varieties and may be useful for some geometric applications. Since we are working in the complex analytic setting, the formulation is slightly complicated. We note that (X, Δ) is not assumed to be log canonical in [F1, Theorem 1.27]. On the other hand, (X, Δ) is log canonical in Theorem 1.1.

Theorem 1.1 (Dlt blow-ups, III)**.** *Let (X, Δ) be a log canonical pair. Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a Stein compact subset of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian. Let $f\colon Z \to X$ be a projective bimeromorphic morphism from a smooth variety Z such that $\mathrm{Exc}(f)$, the exceptional locus of f, is a simple normal crossing divisor on Z. We further assume that the support of $\mathrm{Exc}(f) + f^{-1}_*\Delta$ is a simple normal crossing divisor on Z. Let $\mathcal{E}$ be a subset of the f-exceptional divisors $\{E_j\}$ satisfying the following two conditions.*

(i) *If $a(E_j, X, \Delta) = -1$, then $E_j \in \mathcal{E}$.*
(ii) *If $E_j \in \mathcal{E}$, then $a(E_j, X, \Delta) \leq 0$.*

Then, after shrinking Y around W suitably, we can construct the commutative diagram

$$\begin{array}{ccc} Z & \overset{\phi}{\dashrightarrow} & Z' \\ & {}_{f}\searrow \quad \swarrow_{f'} & \\ & X & \\ & \downarrow{\scriptstyle \pi} & \\ & Y & \end{array}$$

such that

(1) *f' is a projective bimeromorphic morphism,*
(2) *ϕ extracts no divisors,*
(3) *ϕ is an isomorphism at general points of E_j with $E_j \in \mathcal{E}$, and*
(4) *ϕ contracts every f-exceptional divisor E_j with $E_j \notin \mathcal{E}$.*

Furthermore, we put

$$\Delta_{Z'} := f'^{-1}_*\Delta - \sum_{E_j \in \mathcal{E}} a(E_j, X, \Delta)\phi_* E_j.$$

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This note will be contained in [F1].

Then we have:

(5) *Z' is $\mathbb{Q}$-factorial over W and $(Z', \Delta_{Z'})$ is divisorial log terminal such that $K_{Z'} + \Delta_{Z'} = f'^*(K_X + \Delta)$ holds.*

The proof of Theorem 1.1 is an easy application of the minimal model program for projective morphisms of complex analytic spaces established in [F1]. We prove Theorem 1.1 here for the sake of completeness.

Proof of Theorem 1.1. By [F1, Lemma 2.16], we can always take an open neighborhood U of W and a Stein compact subset W' of Y such that $U \subset W'$ and that $\Gamma(W', \mathcal{O}_Y)$ is noetherian. As usual, throughout this proof, we will freely shrink Y suitably without mentioning it explicitly. Let A be a general π-ample $\mathbb{Q}$-divisor on X with $A \cdot C > 2 \dim X$ for every projective curve C on X such that $\pi(C)$ is a point. Let ε be a sufficiently small positive number. We put

$$d(E_j) = \begin{cases} -a(E_j, X, \Delta) & \text{if} \quad E_j \in \mathcal{E}, \\ \max\{-a(E_j, X, \Delta) + \varepsilon, 0\} & \text{if} \quad E_j \notin \mathcal{E}. \end{cases}$$

Then we set

$$\Theta := f_*^{-1}\Delta + \sum d(E_j) E_j.$$

By definition, we have

$$K_Z + \Theta = f^*(K_X + \Delta) + \sum_{E_j \notin \mathcal{E}} \left(d(E_j) + a(E_j, X, \Delta) \right) E_j.$$

Note that $F := \sum_{E_j \notin \mathcal{E}} \left(d(E_j) + a(E_j, X, \Delta) \right) E_j$ is effective and f-exceptional by construction. Since the support of Θ is a simple normal crossing divisor and the coefficients of Θ are in $[0, 1]$, (Z, Θ) is a divisorial log terminal pair. We take a general $(\pi \circ f)$-ample $\mathbb{Q}$-divisor H on Z such that $K_Z + \Theta + f^*A + H$ is nef over Y. We run a $(K_Z + \Theta + f^*A)$-minimal model program over Y around W' with scaling of H. We note that by [F1, Lemma 9.4] this minimal model program can be seen as a $(K_Z + \Theta)$-minimal model program over X. Then we obtain a sequence of flips and divisorial contractions over X starting from $(Z_0, \Theta_0) := (Z, \Theta)$:

$$(Z_0, \Theta_0) \overset{\phi_0}{\dashrightarrow} (Z_1, \Theta_1) \overset{\phi_1}{\dashrightarrow} \cdots \overset{\phi_{i-1}}{\dashrightarrow} (Z_i, \Theta_i) \overset{\phi_i}{\dashrightarrow},$$

where $\Theta_{i+1} := (\phi_i)_*\Theta_i$, $H_{i+1} := (\phi_i)_*H_i$, and $F_{i+1} := (\phi_i)_*F_i$, for every i, and a sequence of real numbers

$$1 \geq \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_i \geq \cdots \geq 0$$

such that $K_{Z_i} + \Theta_i + f_i^*A + \lambda_i H_i$ is nef over W', where $f_i \colon Z_i \to X$ for every i. It is obvious that (Z_i, Θ_i) is divisorial log terminal and Z_i is $\mathbb{Q}$-factorial over W'. If ϕ_i is a divisorial contraction, then ϕ_i contracts an irreducible component of F_i since F_i is effective. By [F1, Lemma 13.7] and its proof, we can check that $K_{Z_m} + \Theta_m$ is in $\mathrm{Mov}(Z_m/X; \pi^{-1}(W'))$ for some m. By applying the negativity lemma (see [F1, Lemma 4.6]) to $f_m \colon Z_m \to X$, we can check that F_m is zero on $(\pi \circ f_m)^{-1}(U)$. Although Z_m is $\mathbb{Q}$-factorial over W', it is not necessarily $\mathbb{Q}$-factorial over W. If Z_m is $\mathbb{Q}$-factorial over W, then we put $(Z', \Delta_{Z'}) := (Z_m, \Theta_m)$. If Z_m is not $\mathbb{Q}$-factorial over W, then we take a small projective $\mathbb{Q}$-factorialization $\psi \colon Z' \to Z_m$ by [F1, Theorem 1.24]. We put $(Z', \Delta_{Z'}) := (Z', \psi^*\Theta_m)$. Then the pair $(Z', \Delta_{Z'})$ is a divisorial log terminal pair such that Z' is $\mathbb{Q}$-factorial over W. By construction, we see that the induced bimeromorphic map

$\phi\colon Z \dashrightarrow Z'$ contracts F and is an isomorphism in codimension one outside the support of F. Hence, $f'\colon Z' \to X$ satisfies all the desired properties. □

2. Adjunction for purely log terminal pairs

In this section, we will see that a precise version of adjunction holds for purely log terminal pairs. It is an application of Theorem 1.1. This result is also well known for algebraic varieties.

Let X be a normal complex variety and let $S+B$ be an effective $\mathbb{R}$-divisor on X such that K_X+S+B is $\mathbb{R}$-Cartier, S is reduced and irreducible, and S and B have no common irreducible components. Let $\nu\colon S^\nu \to S$ be the normalization with $K_{S^\nu}+B_{S^\nu} = \nu^*(K_X+S+B)$. By the inversion of adjunction for log canonicity, we know that (S^ν, B_{S^ν}) is log canonical if and only if $(X, S+B)$ is log canonical in a neighborhood of S. For the details, see [F2]. By the connectedness lemma of Shokurov–Kollár, we know that (S^ν, B_{S^ν}) is kawamata log terminal if and only if $(X, S+B)$ is purely log terminal in a neighborhood of S. We note that the connectedness lemma of Shokurov–Kollár is an easy consequence of the Kawamata–Viehweg vanishing theorem for projective bimeromorphic morphisms of complex analytic spaces. We also note that if $(X, S+B)$ is purely log terminal in a neighborhood of S then S is always normal. We do not prove these results here since the proof for algebraic varieties works with only suitable modifications.

From now on, we assume that $(X, S+B)$ is purely log terminal and put $K_S+B_S := (K_X+S+B)|_S$ by adjunction. Let W be a compact subset of X. Let E be a divisor over some open neighborhood U_E of W. Then we put

$$a(E, X, S+B)_W := a(E, U_E, S|_{U_E}+B|_{U_E}).$$

In this situation,

$$\mathrm{discrep}(\mathrm{center}\cap S\neq\emptyset, X, S+B)_W$$

denotes the infimum of $a(E, X, S+B)_W$, where E runs through all divisors over some open neighborhood of W which is exceptional and whose center has non-empty intersection with $S\cap W$. Similarly,

$$\mathrm{totaldicrep}(S, B_S)_{W\cap S}$$

denotes the infimum of $a(F, S, B_S)_{W\cap S}$, where F runs through all divisors over some open neighborhood of $W\cap S$.

Theorem 2.1 (Adjunction for purely log terminal pairs)**.** *Let $(X, S+B)$ be a purely log terminal pair such that $\lfloor S+B\rfloor = S$ is irreducible. Let W be a Stein compact subset of X such that $\Gamma(W, \mathcal{O}_X)$ is noetherian. Then*

$$\mathrm{totaldicrep}(S, B_S)_{W\cap S} = \mathrm{discrep}(\mathrm{center}\cap S\neq\emptyset, X, S+B)_W$$

holds.

Before we prove Theorem 2.1, we explain the reason why we adopted the above formulation in Theorem 2.1.

Remark 2.2. We put $X := \mathbb{C}$. Let $\{P_n\}_{n\in\mathbb{Z}_{>0}}$ be a set of mutually distinct discrete points of X. We consider the following divisor

$$\Delta := \sum_{n\in\mathbb{Z}_{>0}} \frac{n-1}{n} P_n.$$

Then the pair (X, Δ) is kawamata log terminal. We note that

$$a(P_n, X, \Delta) = -\frac{n-1}{n}.$$

Hence, we have $\inf_{n \in \mathbb{Z}_{>0}} \{a(P_n, X, \Delta)\} = -1$. This implies that the *discrepancy* of (X, Δ)

$$\mathrm{discrep}(X, \Delta) := \inf_E \{a(E, X, \Delta) \mid E \text{ is an exceptional divisor over } X\}$$

and the *total discrepancy* of (X, Δ)

$$\mathrm{totaldicrep}(X, \Delta) := \inf_E \{a(E, X, \Delta) \mid E \text{ is a divisor over } X\}$$

do not work well when X is a non-compact complex analytic space.

Let us prove Theorem 2.1.

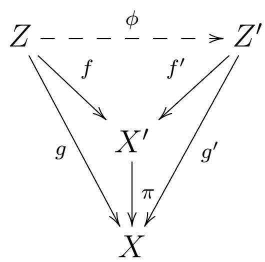
Proof of Theorem 2.1. After shrinking X around W suitably, we can take a projective bimeromorphic morphism $g\colon Z \to X$ from a smooth complex variety Z such that $\mathrm{Exc}(g)$ and the support of $\mathrm{Exc}(g) + g_*^{-1}(S+B)$ are simple normal crossing divisors on Z. By this resolution $g\colon Z \to X$ and the basic properties of discrepancy coefficients, we can easily check that the inequality

$$\mathrm{totaldicrep}(S, B_S)_{W \cap S} \geq \mathrm{discrep}(\mathrm{center} \cap S \neq \emptyset, X, S+B)_W$$

holds. Hence it is sufficient to prove the opposite inequality. Since $(X, S+B)$ is purely log terminal, there exists a small projective morphism $\pi\colon X' \to X$ such that X' is $\mathbb{Q}$-factorial over W. We put $S' := \pi_*^{-1}S$ and $B' := \pi_*^{-1}B$. We may assume that $g\colon Z \to X$ factors through X' and there exists a divisor E on Z such that

$$a(E, X, S+B)_W = \mathrm{discrep}(\mathrm{center} \cap S = \emptyset, X, S+B)_W$$

and that the center of E has non-empty intersection with $S \cap W$. We put $\mathcal{E} := \{E\}$ and apply Theorem 1.1 to $Z \to X' \to X$ and W. Then we get the following commutative diagram



satisfying the properties in Theorem 1.1. Since X' is $\mathbb{Q}$-factorial over W, the exceptional locus $\mathrm{Exc}(f')$ of f' is a divisor after shrinking X around W suitably. Hence $\mathrm{Exc}(f') = E' := \phi_* E$ holds. This implies that $E' \cap S_{Z'} \neq \emptyset$, where $S_{Z'}$ is the strict transform of S on Z'. We note that $S_{Z'}$ is normal since $(Z', S_{Z'})$ is divisorial log terminal. We also note that $E' \cap S_{Z'}$ is a divisor on $S_{Z'}$ over some open neighborhood of W since Z' is $\mathbb{Q}$-factorial over W by construction. Note that

$$K_{Z'} + S_{Z'} + B_{Z'} - a(E, X, S+B)_W E' = g'^*(K_X + S + B)$$

holds, where $B_{Z'}$ is the strict transform of B on Z'. By adjunction, we can easily see that $\mathrm{totaldicrep}(S, B_S)_{W \cap S} \leq a(E, X, S+B)_W$ holds since $E' \cap S_{Z'} \neq \emptyset$ is a divisor on $S_{Z'}$ over some open neighborhood of W. Hence we get the desired inequality. We finish the proof. □

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References

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