A RELATIVE SPANNEDNESS FOR LOG CANONICAL PAIRS AND QUASI-LOG CANONICAL PAIRS

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Abstract. We establish a relative spannedness for log canonical pairs, which is a generalization of the basepoint-freeness for varieties with log-terminal singularities by Andreatta–Wiśniewski. Moreover, we establish a generalization for quasi-log canonical pairs.

Contents

1. Introduction

The main purpose of this paper is to establish the following relative spannedness for log canonical pairs.

Theorem 1.1 (Relative spannedness for log canonical pairs, see [1, Theorem, Remark 3.1.2, and Theorem 5.1])**.** *Let (X, Δ) be a log canonical pair and let $f\colon X \to Y$ be a projective surjective morphism onto a variety Y such that $-(K_X + \Delta)$ is f-ample. Let L be a Cartier divisor on X. Assume that $K_X + \Delta + rL$ is relatively numerically trivial over Y for some positive real number r. Let F be a fiber of f. Then the dimension of every positive-dimensional irreducible component of F is $\geq r - 1$. We further assume that $\dim F < r + 1$. Then $f^*f_*\mathcal{O}_X(L) \to \mathcal{O}_X(L)$ is surjective at every point of F.*

As an easy consequence of Theorem 1.1, we have:

Corollary 1.2 (see [11, Theorem 1])**.** *Let (X, Δ) be a log canonical pair with $\dim X = n$ and let $f\colon X \to Y$ be a projective morphism onto a variety Y. Let L be an f-ample Cartier divisor on X. Then $K_X + \Delta + (n+1)L$ is f-nef. Moreover, if $\dim Y \geq 1$, then $K_X + \Delta + nL$ is f-nef.*

By Theorem 1.1, we can quickly recover the basepoint-freeness obtained by Andreatta and Wiśniewski in [1].

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Corollary 1.3 (Relative spannedness for kawamata log terminal pairs, see [1, Theorem, Remark 3.1.2, and Theorem 5.1])**.** *In Theorem 1.1, we further assume that* (X, Δ) *is kawamata log terminal and that* $\dim X = \dim Y$*. Then the dimension of every positive-dimensional irreducible component of* F *is* $\geq \lfloor r \rfloor$*. Moreover, if* $\dim F \leq r+1$*, then* $f^*f_*\mathcal{O}_X(L) \to \mathcal{O}_X(L)$ *is surjective at every point of* F*.*

The following easy example shows that the estimates on the lower bound of the dimension of the fiber are sharp.

Example 1.4. Let Y be a smooth variety with $\dim Y = n$. Let $f\colon X \to Y$ be a blow-up at a smooth point of Y and let $E \simeq \mathbb{P}^{n-1}$ be the f-exceptional divisor on X. In this situation, $L := -E$ is an f-ample Cartier divisor on X. We put $\Delta = E$. Then we obtain that (X, Δ) is log canonical and is not kawamata log terminal and that $K_X + \Delta + nL = f^*K_Y$ holds.

The following example shows that the assumption $\dim F < r+1$ in Theorem 1.1 is sharp.

Example 1.5. Let S be a Del Pezzo surface of degree one, that is, $(-K_S)^2 = 1$. We can easily check that $\dim_{\mathbb{C}} H^0(S, \mathcal{O}_S(-K_S)) = 2$ and that $\mathrm{Bs}|-K_S|$ is a point. In particular, $|-K_S|$ is not basepoint-free. We take a positive integer m such that

$$\Phi_{|-mK_S|}\colon S \hookrightarrow \mathbb{P}^N$$

is a projectively normal embedding. Let $Y \subset \mathbb{A}^{N+1}$ be the cone over S. Then Y has only kawamata log terminal singularities. Let $f\colon X \to Y$ be the blow-up at the vertex $P \in Y$ and let $F \simeq S$ be the exceptional divisor of f. We put $\Delta = F$. Then X is smooth, (X, Δ) is log canonical and is not kawamata log terminal, and $-(K_X + \Delta)$ is f-ample. We put $L = -(K_X + \Delta)$. Then $K_X + \Delta + rL$ with $r = 1$ is obviously relatively numerically trivial over Y. We note that $\dim F = \dim S = 2 = r + 1$. By adjunction, we have $L|_F = -K_F$. Since $F \simeq S$, $|L|_F|$ is not basepoint-free. This implies that

$$f^*f_*\mathcal{O}_X(L) \to \mathcal{O}_X(L)$$

is not surjective at some point of F.

The original proof of Theorem 1.1 and Corollary 1.3 for varieties with only kawamata log terminal singularities in [1] is based on Kollár's modified basepoint-freeness method in [12]. Although Kollár's method was already generalized for log canonical pairs and quasi-log canonical pairs (see [2] and [6]), we do not use it in this paper. Our proof of Theorem 1.1 and Corollary 1.3 heavily depends on the following basepoint-free theorem for projective quasi-log schemes.

Theorem 1.6 (Spannedness for projective quasi-log schemes)**.** *Let* $[X, \omega]$ *be a projective quasi-log scheme and let* $X_{-\infty}$ *denote the non-qlc locus of* $[X, \omega]$*. We assume* $\dim(X \setminus X_{-\infty}) = n$*. Let* $\mathcal{L}$ *be an ample line bundle on* X *such that* $\omega + r\mathcal{L}$ *is numerically trivial with* $r > n-1$*. We further assume that* $|\mathcal{L}|_{X_{-\infty}}|$ *is basepoint-free. Then the complete linear system* $|\mathcal{L}|$ *is basepoint-free.*

We prove Theorem 1.6 by the theory of quasi-log schemes with the aid of Fujita's theory of Δ-genera (see [10]). Then Theorem 1.1 will be proved with an inductive argument via Theorem 1.6.

We can further generalize Theorem 1.1 for quasi-log canonical pairs. The precise statement is as follows:

Theorem 1.7 (Relative spannedness for quasi-log canonical pairs)**.** *Let* $[X, \omega]$ *be a quasi-log canonical pair and let* $\varphi\colon X \to W$ *be a projective surjective morphism onto a scheme* W *such that* $-\omega$ *is* φ*-ample. Let* $\mathcal{L}$ *be a line bundle on* X*. Assume that* $\omega + r\mathcal{L}$ *is relatively* *numerically trivial over W for some positive real number r. Let F be a fiber of f. Then the dimension of every positive-dimensional irreducible component of F is $\geq r-1$. We further assume that* $\dim F < r+1$. *Then $\varphi^*\varphi_*\mathcal{L} \to \mathcal{L}$ is surjective at every point of F.*

As a corollary of Theorem 1.7, we have the following generalization of Corollary 1.2.

Corollary 1.8. *Let $[X, \omega]$ be a quasi-log canonical pair with* $\dim X = n$ *and let* $\varphi\colon X \to W$ *be a projective morphism onto a scheme W. Let $\mathcal{L}$ be a φ-ample line bundle on X. Then $\omega + (n+1)\mathcal{L}$ is φ-nef. We further assume that X is irreducible and* $\dim W \geq 1$. *Then $\omega + n\mathcal{L}$ is φ-nef.*

Since every quasi-projective semi-log canonical pair naturally becomes a quasi-log canonical pair by [5, Theorem 1.1], we can apply Theorem 1.7 and Corollary 1.8 to semi-log canonical pairs.

We briefly explain the organization of this paper. In Section 2, we collect some basic definitions and quickly recall Fujita's theory of Δ-genera and the theory of quasi-log schemes. In Section 3, we explain three useful lemmas for quasi-log schemes for the reader's convenience. In Section 4, we give a detailed proof of Theorem 1.6. It is a combination of Fujita's theory of Δ-genera and the theory of quasi-log schemes. In Section 5, we prove Theorem 1.1. Our proof is different from Kollár's modified basepoint-freeness method in [12] and is new. It uses the framework of quasi-log schemes. In Section 6, we treat Theorem 1.7, which is a generalization of Theorem 1.1. The idea of the proof of Theorem 1.7 is completely the same as that of the proof of Theorem 1.1. However, the proof of Theorem 1.7 is harder than that of Theorem 1.1.

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We will work over $\mathbb{C}$, the complex number field, throughout this paper. In this paper, a *scheme* means a separated scheme of finite type over $\mathbb{C}$. A *variety* means an integral scheme, that is, an integral separated scheme of finite type over $\mathbb{C}$. We will use the theory of quasi-log schemes discussed in [7, Chapter 6].

2. Preliminaries

In this section, we collect some basic definitions of the minimal model program and the theory of quasi-log schemes. For the details, see [4] and [7]. We also mention Fujita's Δ-genera (see [10]), which will play a crucial role in this paper.

2.1. **Basic definitions.** Let us recall singularities of pairs and some related definitions.

Definition 2.1. Let X be a variety and let E be a prime divisor on Y for some birational morphism $f\colon Y \to X$ from a normal variety Y. Then E is called a divisor *over* X.

Definition 2.2 (Singularities of pairs)**.** A *normal pair* (X, Δ) consists of a normal variety X and an $\mathbb{R}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f\colon Y \to X$ be a projective birational morphism from a normal variety Y. Then we can write

$$K_Y = f^*(K_X + \Delta) + \sum_E a(E, X, \Delta)E$$

with

$$f_*\left(\sum_E a(E,X,\Delta)E\right) = -\Delta,$$

where E runs over prime divisors on Y. We call $a(E,X,\Delta)$ the *discrepancy* of E with respect to (X,Δ). Note that we can define the discrepancy $a(E,X,\Delta)$ for any prime divisor E over X by taking a suitable resolution of singularities of X. If $a(E,X,\Delta) \geq -1$ (resp. > -1) for every prime divisor E over X, then (X,Δ) is called *sub log canonical* (resp. *sub kawamata log terminal*). We further assume that Δ is effective. Then (X,Δ) is called *log canonical* and *kawamata log terminal* if it is sub log canonical and sub kawamata log terminal, respectively. We simply say that X has only *kawamata log terminal singularities* when $(X,0)$ is a kawamata log terminal pair.

Let (X,Δ) be a normal pair. If there exist a projective birational morphism $f\colon Y \to X$ from a normal variety Y and a prime divisor E on Y such that (X,Δ) is sub log canonical in a neighborhood of the generic point of $f(E)$ and that $a(E,X,\Delta) = -1$, then $f(E)$ is called a *log canonical center* of (X,Δ).

Definition 2.3 (Operations for $\mathbb{R}$-divisors)**.** Let V be an equidimensional reduced scheme. An $\mathbb{R}$-*divisor* D on V is a finite formal sum

$$\sum_{i=1}^{l} d_i D_i,$$

where D_i is an irreducible reduced closed subscheme of V of pure codimension one with $D_i \neq D_j$ for $i \neq j$ and d_i is a real number for every i. We put

$$D^{<1} = \sum_{d_i<1} d_i D_i, \quad D^{=1} = \sum_{d_i=1} D_i, \quad \text{and} \quad D^{>1} = \sum_{d_i>1} d_i D_i.$$

For every real number x, $\lceil x \rceil$ is the integer defined by $x \leq \lceil x \rceil < x+1$. Then we put

$$\lceil D \rceil = \sum_{i=1}^{l} \lceil d_i \rceil D_i \quad \text{and} \quad \lfloor D \rfloor = -\lceil -D \rceil.$$

Definition 2.4 (Non-lc ideals and non-lc loci, see [3] and [4, Section 7])**.** Let (X,Δ) be a normal pair such that Δ is effective and let $f\colon Y \to X$ be a resolution of singularities with

$$K_Y + \Delta_Y = f^*(K_X + \Delta)$$

such that $\operatorname{Supp}\Delta_Y$ is a simple normal crossing divisor on Y. We put

$$\begin{aligned} \mathcal{J}_{\mathrm{NLC}}(X,\Delta) &:= f_*\mathcal{O}_Y(-\lfloor \Delta_Y \rfloor + \Delta_Y^{=1}) \\ &= f_*\mathcal{O}_Y(\lceil -(\Delta_Y^{<1}) \rceil - \lfloor \Delta_Y^{>1} \rfloor) \end{aligned}$$

and call it the *non-lc ideal sheaf* associated to the pair (X,Δ). We can check that $\mathcal{J}_{\mathrm{NLC}}(X,\Delta)$ is a well-defined ideal sheaf on X. The closed subscheme $\mathrm{Nlc}(X,\Delta)$ defined by $\mathcal{J}_{\mathrm{NLC}}(X,\Delta)$ is called the *non-lc locus* of (X,Δ). Note that (X,Δ) is log canonical if and only if $\mathcal{J}_{\mathrm{NLC}}(X,\Delta) = \mathcal{O}_X$.

Definition 2.5 ($\sim_{\mathbb{R}}$ and $\equiv$)**.** Let B_1 and B_2 be $\mathbb{R}$-Cartier divisors on a scheme X. Then $B_1 \sim_{\mathbb{R}} B_2$ means that B_1 is $\mathbb{R}$-*linearly equivalent* to B_2, that is, $B_1 - B_2$ is a finite $\mathbb{R}$-linear combination of principal Cartier divisors. Let $f\colon X \to Y$ be a proper morphism between schemes. Then $B_1 \equiv_Y B_2$ means that B_1 is *relatively numerically equivalent* to B_2 *over* Y. When Y is a point, we simply write $B_1 \equiv B_2$ to denote $B_1 \equiv_Y B_2$ and say that B_1 is *numerically equivalent* to B_2.

2.2. **Fujita's Δ-genera.** Let us quickly explain Fujita's theory of Δ-genera, which will play a crucial role in this paper. We start with the definition of base loci.

Definition 2.6 (Base loci)**.** Let $f\colon X \to Y$ be a proper morphism between schemes and let L be a Cartier divisor on X. Then $\mathrm{Bs}_f|L|$ denotes the support of

$$\mathrm{Coker}\,(f^*f_*\mathcal{O}_X(L) \to \mathcal{O}_X(L))$$

and is called the *relative base locus* of $|L|$. If Y is a point, then we simply write $\mathrm{Bs}|L|$ to denote $\mathrm{Bs}_f|L|$. We can define $\mathrm{Bs}_f|\mathcal{L}|$ and $\mathrm{Bs}|\mathcal{L}|$ for every line bundle $\mathcal{L}$ on X in the same way.

Let us recall the definition of Fujita's Δ-genera. In this paper, we define $\Delta(V, L)$ only when L is ample for simplicity. For the general case, see Fujita's original definition in [10].

Definition 2.7 (Fujita's Δ-genera, see [10, Definition 1.4])**.** Let V be a projective variety and let L be an ample Cartier divisor on V. Then the Δ-*genus* of (V, L) is defined to be

$$\Delta(V, L) = \dim V + L^{\dim V} - \dim_{\mathbb{C}} H^0(V, \mathcal{O}_V(L)).$$

We can define $\Delta(V, \mathcal{L})$ for every ample line bundle $\mathcal{L}$ in the same way.

The following famous theorem by Takao Fujita is one of the main ingredients of this paper. We recommend the interested reader to see Fujita's original statement (see [10, Theorem 1.9]), which is more general than Theorem 2.8.

Theorem 2.8 (Fujita, see [10, Theorem 1.9])**.** *Let V be a projective variety and let L be an ample Cartier divisor on V. Then the following inequality*

$$\dim \mathrm{Bs}|L| < \Delta(V, L)$$

holds, where $\dim \emptyset$ *is defined to be* $-\infty$*. In particular, if* $\Delta(V, L) = 0$*, then the complete linear system* $|L|$ *is basepoint-free. Of course, the same statement holds for ample line bundles* $\mathcal{L}$*.*

2.3. **Quasi-log schemes.** The notion of quasi-log schemes was first introduced by Florin Ambro in order to establish the cone and contraction theorem for (X, Δ), where X is a normal variety and Δ is an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Here we use the formulation in [7, Chapter 6], which is slightly different from Ambro's original one. We recommend the interested reader to see [8, Appendix A] for the difference between our definition of quasi-log schemes and Ambro's one.

In order to define quasi-log schemes, we need the notion of globally embedded simple normal crossing pairs.

Definition 2.9 (Globally embedded simple normal crossing pairs, see [7, Definition 6.2.1])**.** Let Y be a simple normal crossing divisor on a smooth variety M and let D be an $\mathbb{R}$-divisor on M such that $\mathrm{Supp}(D+Y)$ is a simple normal crossing divisor on M and that D and Y have no common irreducible components. We put $B_Y = D|_Y$ and consider the pair (Y, B_Y). We call (Y, B_Y) a *globally embedded simple normal crossing pair* and M the *ambient space* of (Y, B_Y). A *stratum* of (Y, B_Y) is a log canonical center of $(M, Y + D)$ that is contained in Y.

Let us recall the definition of quasi-log schemes.

Definition 2.10 (Quasi-log schemes, see [7, Definition 6.2.2])**.** A *quasi-log scheme* is a scheme X endowed with an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) ω on X, a closed subscheme $X_{-\infty} \subsetneq X$, and a finite collection $\{C\}$ of reduced and irreducible subschemes of X such that there is a proper morphism $f\colon (Y, B_Y) \to X$ from a globally embedded simple normal crossing pair satisfying the following properties:

(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.
(2) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$ induces an isomorphism

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor),$$

where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$.
(3) The collection of reduced and irreducible subschemes $\{C\}$ coincides with the images of the strata of (Y, B_Y) that are not included in $X_{-\infty}$.

We simply write $[X, \omega]$ to denote the above data

$$\big(X, \omega, f\colon (Y, B_Y) \to X\big)$$

if there is no risk of confusion. The reduced and irreducible subschemes C are called the *qlc strata* of $[X, \omega]$, $X_{-\infty}$ is called the *non-qlc locus* of $[X, \omega]$, and $f\colon (Y, B_Y) \to X$ is called a *quasi-log resolution* of $[X, \omega]$. We sometimes use $\mathrm{Nqlc}(X, \omega)$ to denote $X_{-\infty}$. If a qlc stratum C of $[X, \omega]$ is not an irreducible component of X, then it is called a *qlc center* of $[X, \omega]$.

Remark 2.11. By restricting the isomorphism

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor)$$

in Definition 2.10 to the Zariski open set $U = X \setminus X_{-\infty}$, we have

$$\mathcal{O}_U \xrightarrow{\simeq} f_*\mathcal{O}_{f^{-1}(U)}(\lceil -(B_Y^{<1})\rceil).$$

This implies that

$$\mathcal{O}_U \xrightarrow{\simeq} f_*\mathcal{O}_{f^{-1}(U)}$$

holds since $\lceil -(B_Y^{<1})\rceil$ is effective. Hence, $f\colon f^{-1}(U) \to U$ is surjective and has connected fibers. Note that a qlc stratum C of $[X, \omega]$ is the image of some stratum of (Y, B_Y) that is not included in $X_{-\infty}$. Therefore, X is the union of $\{C\}$ and $X_{-\infty}$. In particular, any irreducible component of X that is not included in $X_{-\infty}$ is a qlc stratum of $[X, \omega]$.

Definition 2.12 (Quasi-log canonical pairs, see [7, Definition 6.2.9])**.** Let

$$(X, \omega, f\colon (Y, B_Y) \to X)$$

be a quasi-log scheme. If $X_{-\infty} = \emptyset$, then it is called a *quasi-log canonical pair*.

The most important result in the theory of quasi-log scheme is adjunction and the following vanishing theorem. We will repeatedly use Theorem 2.13 in this paper. The proof of Theorem 2.13 in [7] heavily depends on the theory of mixed Hodge structures on cohomology with compact support (see [7, Chapter 5]).

Theorem 2.13 (see [7, Theorem 6.3.5])**.** *Let $[X, \omega]$ be a quasi-log scheme and let X' be the union of $X_{-\infty}$ with a (possibly empty) union of some qlc strata of $[X, \omega]$. Then we have the following properties.*

(i) (Adjunction). *Assume that $X' \neq X_{-\infty}$. Then X' is a quasi-log scheme with $\omega' = \omega|_{X'}$ and $X'_{-\infty} = X_{-\infty}$. Moreover, the qlc strata of $[X', \omega']$ are exactly the qlc strata of $[X, \omega]$ that are included in X'.*
(ii) (Vanishing theorem). *Assume that $\pi\colon X \to S$ is a proper morphism between schemes. Let L be a Cartier divisor on X such that $L - \omega$ is ample over S with respect to $[X, \omega]$. Then $R^i\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)) = 0$ for every $i > 0$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X' on X.*

We quickly explain the main idea of the proof of Theorem 2.13 (i) for the reader's convenience. For the details, see [7, Theorem 6.3.5].

Idea of Proof of Theorem 2.13 (i). By definition, X' is the union of $X_{-\infty}$ with a union of some qlc strata of $[X, \omega]$ set theoretically. We assume that $X' \neq X_{-\infty}$ holds. By [7, Proposition 6.3.1], we may assume that the union of all strata of (Y, B_Y) mapped to X' by f, which is denoted by Y', is a union of some irreducible components of Y. We put $Y'' = Y - Y'$, $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$, and $K_{Y'} + B_{Y'} = (K_Y + B_Y)|_{Y'}$. We set $f'' = f|_{Y''}$ and $f' = f|_{Y'}$. Then we claim that

$$(X', \omega', f' \colon (Y', B_{Y'}) \to X')$$

becomes a quasi-log scheme satisfying the desired properties. Let us consider the following short exact sequence:

$$\begin{aligned} 0 &\to \mathcal{O}_{Y''}(\lceil -(B_{Y''}^{<1}) \rceil - \lfloor B_{Y''}^{>1} \rfloor - Y'|_{Y''}) \to \mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor) \\ &\to \mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1}) \rceil - \lfloor B_{Y'}^{>1} \rfloor) \to 0, \end{aligned}$$

which is induced by

$$0 \to \mathcal{O}_{Y''}(-Y'|_{Y''}) \to \mathcal{O}_Y \to \mathcal{O}_{Y'} \to 0.$$

We take the associated long exact sequence. Then we can check that the connecting homomorphism

$$\delta \colon f'_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1}) \rceil - \lfloor B_{Y'}^{>1} \rfloor) \to R^1 f''_*\mathcal{O}_{Y''}(\lceil -(B_{Y''}^{<1}) \rceil - \lfloor B_{Y''}^{>1} \rfloor - Y'|_{Y''})$$

is zero by using a generalization of Kollár's torsion-freeness based on the theory of mixed Hodge structures on cohomology with compact support (see [7, Chapter 5]). We put

$$\mathcal{I}_{X'} := f''_*\mathcal{O}_{Y''}(\lceil -(B_{Y''}^{<1}) \rceil - \lfloor B_{Y''}^{>1} \rfloor - Y'|_{Y''}),$$

which is an ideal sheaf on X since $\mathcal{I}_{X'} \subset \mathcal{I}_{X_{-\infty}}$, and define a scheme structure on X' by $\mathcal{I}_{X'}$. Then we obtain the following big commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & f''_*\mathcal{O}_{Y''}(\lceil -(B_{Y''}^{<1}) \rceil - \lfloor B_{Y''}^{>1} \rfloor - Y'|_{Y''}) & \xrightarrow{=} & \mathcal{I}_{X'} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor) = \mathcal{I}_{X_{-\infty}} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X_{-\infty}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & f'_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1}) \rceil - \lfloor B_{Y'}^{>1} \rfloor) = \mathcal{I}_{X'_{-\infty}} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_{X'_{-\infty}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

by the above arguments. More precisely, by the above big commutative diagram,

$$\mathcal{I}_{X'_{-\infty}} = f'_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1}) \rceil - \lfloor B_{Y'}^{>1} \rfloor)$$

is an ideal sheaf on X' such that $\mathcal{O}_X/\mathcal{I}_{X_{-\infty}} = \mathcal{O}_{X'}/\mathcal{I}_{X'_{-\infty}}$. Thus we obtain that

$$(X', \omega', f' \colon (Y', B_{Y'}) \to X')$$

is a quasi-log scheme satisfying the desired properties. □

The following example is very important. It shows that we can treat log canonical pairs as quasi-log canonical pairs.

Example 2.14 ([7, 6.4.1])**.** Let (X, Δ) be a normal pair such that Δ is effective. Let $f\colon Y \to X$ be a resolution of singularities such that

$$K_Y + B_Y = f^*(K_X + \Delta)$$

and that $\mathrm{Supp} B_Y$ is a simple normal crossing divisor on Y. We put $\omega = K_X + \Delta$. Then $K_Y + B_Y \sim_{\mathbb{R}} f^*\omega$ holds. Since Δ is effective, $\lceil -(B_Y^{<1})\rceil$ is effective and f-exceptional. Therefore, the natural map

$$\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$$

is an isomorphism. We put

$$\mathcal{I}_{X_{-\infty}} := \mathcal{J}_{\mathrm{NLC}}(X, \Delta) = f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor),$$

where $\mathcal{J}_{\mathrm{NLC}}(X, \Delta)$ is the non-lc ideal sheaf associated to (X, Δ) in Definition 2.4. We put $M = Y \times \mathbb{C}$ and $D = B_Y \times \mathbb{C}$. Then $(Y, B_Y) \simeq (Y \times \{0\}, B_Y \times \{0\})$ is a globally embedded simple normal crossing pair. Thus

$$(X, \omega, f\colon (Y, B_Y) \to X)$$

becomes a quasi-log scheme. By construction, (X, Δ) is log canonical if and only if $[X, \omega]$ is quasi-log canonical. We note that C is a log canonical center of (X, Δ) if and only if C is a qlc center of $[X, \omega]$. We also note that X itself is a qlc stratum of $[X, \omega]$.

Let X' be the union of $X_{-\infty}$ with a union of some qlc centers of $[X, \omega]$. If $X' \neq X_{-\infty}$, then $[X', \omega|_{X'}]$ naturally becomes a quasi-log scheme by adjunction (see Theorem 2.13 (i) and [7, Theorem 6.3.5 (i)]). When $X_{-\infty} = \emptyset$, equivalently, (X, Δ) is log canonical, we see that $[X', \omega|_{X'}]$ is quasi-log canonical. By construction, X' is not necessarily equidimensional and is a highly singular reducible and reduced scheme.

For the basic properties of quasi-log schemes, see [7, Chapter 6].

3. Three lemmas for quasi-log schemes

In this section, we will explain three useful lemmas for quasi-log schemes for the reader's convenience. They are essentially contained in [7, Chapter 6] or easily follow from the arguments in [7, Chapter 6].

Let us start with the following easy lemma, which is almost obvious by definition.

Lemma 3.1. *Let*

$$(X, \omega, f\colon (Y, B_Y) \to X)$$

be a quasi-log canonical pair and let B be an effective $\mathbb{R}$-Cartier divisor on X. Assume that $(Y, B_Y + f^*B)$ is a globally embedded simple normal crossing pair. Then

$$(X, \omega + B, f\colon (Y, B_Y + f^*B) \to X)$$

is a quasi-log scheme. Of course, $[X, \omega + B]$ is quasi-log canonical if and only if $B_Y + f^*B$ is a subboundary $\mathbb{R}$-divisor on Y, that is, $(B_Y + f^*B)^{>1} = 0$.

Proof. By definition, $K_Y + B_Y \sim_{\mathbb{R}} f^*\omega$. Therefore, $K_Y + B_Y + f^*B \sim_{\mathbb{R}} f^*(\omega + B)$ obviously holds true. Since $[X, \omega]$ is a quasi-log canonical pair, the natural map

$$\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$$

is an isomorphism. Since it factors through $f_*\mathcal{O}_Y$, we have

$$\mathcal{O}_X \xrightarrow{\simeq} f_*\mathcal{O}_Y \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil). \tag{3.1}$$

We note that

$$0 \leq \lceil -(B_Y + f^*B)^{<1}\rceil \leq \lceil -(B_Y^{<1})\rceil.$$

Therefore, we obtain

$$\mathcal{O}_X \xrightarrow{\simeq} f_*\mathcal{O}_Y \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y+f^*B)^{<1}\rceil) \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil).$$

Thus, we get a nonzero coherent ideal sheaf

$$\mathcal{I}_{\mathrm{Nqlc}(X,\omega+B)} := f_*\mathcal{O}_Y(\lceil -(B_Y+f^*B)^{<1}\rceil - \lfloor (B_Y+f^*B)^{>1}\rfloor),$$

which defines a closed subscheme $\mathrm{Nqlc}(X,\omega+B)$. Let W be a reduced and irreducible subscheme of X. We say that W is a qlc stratum of $[X,\omega+B]$ if W is not included in $\mathrm{Nqlc}(X,\omega+B)$ and is the f-image of some stratum of (Y, B_Y+f^*B). Then

$$(X, \omega+B, f\colon (Y, B_Y+f^*B)\to X)$$

is a quasi-log scheme. By construction, $[X,\omega+B]$ is a quasi-log canonical pair if and only if $(B_Y+f^*B)^{>1}=0$. Note that $(X,\omega+B, f\colon (Y,B_Y+f^*B)\to X)$ coincides with $(X,\omega,f\colon (Y,B_Y)\to X)$ outside $\mathrm{Supp}B$. □

The next lemma is similar to the previous one. However, the proof is not so obvious because we need the argument in the proof of adjunction (see Theorem 2.13 (i)).

Lemma 3.2. *Let*

$$(X,\omega,f\colon (Y,B_Y)\to X)$$

be a quasi-log scheme and let B be an effective $\mathbb{R}$-Cartier divisor on X. Let X' be the union of $\mathrm{Nqlc}(X,\omega)$ and all qlc centers of $[X,\omega]$ contained in $\mathrm{Supp}B$. Assume that the union of all strata of (Y,B_Y) mapped to X' by f, which is denoted by Y', is a union of some irreducible components of Y. We put $Y''=Y-Y'$, $K_{Y''}+B_{Y''}=(K_Y+B_Y)|_{Y''}$, and $f''=f|_{Y''}$. We further assume that

$$(Y'', B_{Y''}+(f'')^*B)$$

is a globally embedded simple normal crossing pair. Then

$$(X,\omega+B, f''\colon (Y'', B_{Y''}+(f'')^*B)\to X)$$

is a quasi-log scheme.

Proof. Since $K_Y+B_Y\sim_{\mathbb{R}} f^*\omega$, we have $K_{Y''}+B_{Y''}\sim_{\mathbb{R}}(f'')^*\omega$. Therefore, $K_{Y''}+B_{Y''}+(f'')^*B\sim_{\mathbb{R}}(f'')^*(\omega+B)$ holds true. By the proof of adjunction (see Theorem 2.13 (i) and [7, Theorem 6.3.5 (i)]), we have

$$\mathcal{I}_{X'} = f''_*\mathcal{O}_{X''}(\lceil -(B_{Y''}^{<1})\rceil - \lfloor B_{Y''}^{>1}\rfloor - Y'|_{Y''}),$$

where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X' on X. Note that the following key inequality

$$\lceil -(B_{Y''}+(f'')^*B)^{<1}\rceil - \lfloor (B_{Y''}+(f'')^*B)^{>1}\rfloor \leq \lceil -(B_{Y''}^{<1})\rceil - \lfloor B_{Y''}^{>1}\rfloor - Y'|_{Y''}$$

holds. Therefore, we put

$$\mathcal{I}_{\mathrm{Nqlc}(X,\omega+B)} := f''_*\mathcal{O}_{Y''}(\lceil -(B_{Y''}+(f'')^*B)^{<1}\rceil - \lfloor (B_{Y''}+(f'')^*B)^{>1}\rfloor) \subset \mathcal{I}_{X'}\subset \mathcal{O}_X$$

and define a closed subscheme $\mathrm{Nqlc}(X,\omega+B)$ of X by $\mathcal{I}_{\mathrm{Nqlc}(X,\omega+B)}$. Then

$$(X,\omega+B, f''\colon (Y'', B_{Y''}+(f'')^*B)\to X)$$

is a quasi-log scheme. Let W be a reduced and irreducible subscheme of X. As usual, we say that W is a qlc stratum of $[X,\omega+B]$ when W is not contained in $\mathrm{Nqlc}(X,\omega+B)$ and is the f''-image of some stratum of $(Y'', B_{Y''}+(f'')^*B)$. By construction, we have $X'\subset \mathrm{Nqlc}(X,\omega+B)$. We note that $(X,\omega+B,f''\colon (Y'',B_{Y''}+(f'')^*B)\to X)$ coincides with $(X,\omega,f\colon (Y,B_Y)\to X)$ outside $\mathrm{Supp}B$. □

The final lemma is easy but very useful. We often use it without mentioning it explicitly.

Lemma 3.3 (Bertini-type theorem)**.** *Let $[X, \omega]$ be a quasi-log scheme and let Λ be a free linear system on X. If D is a general member of Λ, then $[X, \omega + cD]$ becomes a quasi-log scheme with $\mathrm{Nqlc}(X, \omega + cD) = \mathrm{Nqlc}(X, \omega)$ for every $0 \leq c \leq 1$.*

More precisely, there exists a proper morphism $f\colon (Y, B_Y) \to X$ from a globally embedded simple normal crossing pair (Y, B_Y) such that $(Y, B_Y + f^*D)$ is a globally embedded simple normal crossing pair and that

$$(X, \omega + cD, f\colon (Y, B_Y + f^*cD) \to X)$$

is a quasi-log scheme with $\mathrm{Nqlc}(X, \omega + cD) = \mathrm{Nqlc}(X, \omega)$ for every $0 \leq c \leq 1$.

When $c = 1$, every irreducible component $D^\dagger$ of D is a qlc center of

$$(X, \omega + D, f\colon (Y, B_Y + f^*D) \to X)\,.$$

Therefore, by adjunction, $[D', (\omega+D)|_{D'}]$ is a quasi-log scheme, where $D' = D^\dagger \cup \mathrm{Nqlc}(X, \omega)$.

Proof. Let $f\colon (Y, B_Y) \to X$ be a quasi-log resolution of $[X, \omega]$. Let $\nu\colon Y^\nu \to Y$ be the normalization of Y with $K_{Y^\nu} + \Theta = \nu^*(K_Y + B_Y)$ as usual. If D is a general member of Λ, then $\nu^* f^* D$ is smooth, $\nu^* f^* D$ and Θ have no common components, and $\mathrm{Supp}(\nu^* f^* D + \Theta)$ is a simple normal crossing divisor on Y^ν. By taking some blow-ups along irreducible components of f^*D repeatedly (see [7, Lemma 5.8.8]), we may further assume that $(Y, B_Y + f^*D)$ is a globally embedded simple normal crossing pair (see [7, Proposition 6.3.1]). Since

$$\lfloor (B_Y + f^*cD)^{>1} \rfloor = \lfloor B_Y^{>1} \rfloor \quad \text{and} \quad 0 \leq \lceil -(B_Y + f^*cD)^{<1} \rceil = \lceil -(B_Y^{<1}) \rceil$$

hold for every $0 \leq c \leq 1$, we obtain that the following equality

$$f_*\mathcal{O}_Y(\lceil -(B_Y + f^*cD)^{<1} \rceil - \lfloor (B_Y + f^*cD)^{>1} \rfloor) = f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor).$$

holds true for every $0 \leq c \leq 1$. Therefore, we obtain that

$$(X, \omega + cD, f\colon (Y, B_Y + f^*cD) \to X)$$

is a quasi-log scheme with $\mathrm{Nqlc}(X, \omega + cD) = \mathrm{Nqlc}(X, \omega)$ for every $0 \leq c \leq 1$. By construction, the quasi-log scheme structure of $[X, \omega + cD]$ is independent of c outside $\mathrm{Supp} D$. It is obvious that every irreducible component $D^\dagger$ of D is a qlc center of $[X, \omega + D]$. Therefore, by adjunction (see Theorem 2.13 (i)), we obtain the desired statement. □

In order to explain how to make new quasi-log scheme structures, let us treat the following proposition.

Proposition 3.4. *Let $[X, \omega]$ be a quasi-log scheme and let L be a Cartier divisor on X such that $\mathrm{Bs}|L|$ contains no qlc strata of $[X, \omega]$ and that $\mathrm{Bs}|L|$ is disjoint from $X_{-\infty}$. If D is a general member of $|L|$. Then there exists $0 < c \leq 1$ such that $[X, \omega + cD]$ becomes a quasi-log scheme with $\mathrm{Nqlc}(X, \omega + cD) = \mathrm{Nqlc}(X, \omega)$ and that there exists a qlc center C of $[X, \omega + cD]$ with $C \cap \mathrm{Bs}|L| \neq \emptyset$.*

Proof. Let $f\colon (Y, B_Y) \to X$ be a quasi-log resolution of $[X, \omega]$. Since D is a general member of $|L|$, $\mathrm{Bs}|L|$ contains no qlc strata of $[X, \omega]$, and $\mathrm{Bs}|L| \cap X_{-\infty} = \emptyset$, f^*D is a well-defined Cartier divisor on Y. We note that $[X, \omega + cD]$ becomes a quasi-log scheme with $\mathrm{Nqlc}(X, \omega + cD) = \mathrm{Nqlc}(X, \omega)$ outside $\mathrm{Bs}|L|$ for every $0 \leq c \leq 1$ by Lemma 3.3.

By taking a suitable birational modification of the ambient space M of (Y, B_Y) (see [7, Proposition 6.3.1]), we may assume that

$$(Y, f^*D + \mathrm{Supp} B_Y)$$

is a globally embedded simple normal crossing pair. We may further assume that f^*D and $\mathrm{Supp} B_Y$ have no common components outside $f^{-1}\,\mathrm{Bs}|L|$ and that f^*D is reduced outside $f^{-1}\,\mathrm{Bs}|L|$.

We put

$$c = \sup\{t \in \mathbb{R} \,|\, (tf^*D + B_Y)^{>1} = 0 \text{ holds over } X \setminus X_{-\infty}\}.$$

Then we have:

Claim. *We have* $0 < c \leq 1$.

Proof of Claim. By replacing X with $X \setminus X_{-\infty}$, we may assume that $X_{-\infty} = \emptyset$. Therefore, the natural map

$$\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil)$$

is an isomorphism. Since $B_Y^{>1} = 0$ by $X_{-\infty} = \emptyset$, the inequality $0 < c$ is obvious because D is a general member of $|L|$ and $\mathrm{Bs}|L|$ contains no qlc strata of $[X, \omega]$. We assume that the inequality $c > 1$ holds. Then the natural map

$$\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil)$$

factors through $\mathcal{O}_X(D)$, that is, we have:

$$\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D) \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil).$$

This is a contradiction. Hence we get the desired inequality $c \leq 1$. □

We consider

$$(X, \omega + cD, f\colon (Y, B_Y + cf^*D) \to X)\,.$$

It is obvious that $f^*(\omega + cD) \sim_{\mathbb{R}} K_Y + B_Y + cf^*D$ holds since $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$. We note that

$$0 \leq \lceil -(B_Y + cf^*D)^{<1} \rceil \leq \lceil -(B_Y^{<1}) \rceil$$

obviously holds and that

$$\lceil -(B_Y + cf^*D)^{<1} \rceil - \lfloor (B_Y + cf^*D)^{>1} \rfloor = \lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor$$

holds over a neighborhood of $X_{-\infty}$. Therefore,

$$(X, \omega + cD, f\colon (Y, B_Y + cf^*D) \to X)\,.$$

is a quasi-log scheme with $\mathrm{Nqlc}(X, \omega + cD) = \mathrm{Nqlc}(X, \omega)$.

If $c = 1$, then we see that every irreducible component $D^\dagger$ of $\mathrm{Supp}D$ with $D^\dagger \not\subset X_{-\infty}$ is a qlc center of $[X, \omega + D]$ by the proof of Claim. Therefore, we can find a qlc center C of $[X, \omega + D]$ with $C \cap \mathrm{Bs}|L| \neq \emptyset$.

If $c < 1$, then we can find an irreducible component G of $(cf^*D + B_Y)^{=1}$ such that $f(G) \cap \mathrm{Bs}|L| \neq \emptyset$ by construction. Thus $C := f(G)$ is a desired qlc center of $[X, \omega + cD]$. □

4. Proof of Theorem 1.6

In this section, we will prove Theorem 1.6, which may look artificial but is very useful.

Let us start with an easy lemma, which follows from Fujita's theory of Δ-genera (see [10]).

Lemma 4.1. *Let $[X, \omega]$ be a projective quasi-log canonical pair such that X is irreducible with $n = \dim X \geq 1$. Let L be an ample Cartier divisor on X such that $\omega + rL \equiv 0$. Then the inequality $r \leq n + 1$ holds. We further assume that $r > n - 1$ holds. Then the complete linear system $|L|$ is basepoint-free.*

Proof. Let us consider

$$\chi(t) := \chi(X, \mathcal{O}_X(tL)) = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{C}} H^i(X, \mathcal{O}_X(tL)).$$

Since L is ample, $\chi(t)$ is a nontrivial polynomial with $\deg \chi(t) = \dim X = n$.

Step 1. In this step, we will prove that $r \le n+1$.

We assume that $r > n+1$ holds. Then

$$H^i(X, \mathcal{O}_X(tL)) = 0$$

for $i > 0$ and $t \in \mathbb{Z}$ with $t \ge -(n+1)$ since $tL - \omega \equiv (t+r)L$ is ample for $t \ge -(n+1)$ (see Theorem 2.13 (ii)). On the other hand,

$$H^0(X, \mathcal{O}_X(tL)) = 0$$

for $t < 0$ since L is ample. Therefore, we have $\chi(t) = 0$ for $t = -1, \ldots, -(n+1)$. This implies that $\chi(t) \equiv 0$ holds. This is a contradiction. Hence we obtain the desired inequality $r \le n+1$.

Step 2. In this step, we will prove that $|L|$ is basepoint-free under the assumption that $r > n-1$ holds.

As in Step 1, we have $\chi(t) = 0$ for $t = -1, \ldots, -(n-1)$ since $r > n-1$ by assumption. Therefore, we get

$$\chi(X, \mathcal{O}_X(tL)) = \frac{1}{n!}(\alpha t + \beta)(t+1)\cdots(t+n-1)$$

for some rational numbers α and β. It is well known that $\alpha = L^n$. We note that

$$\chi(X, \mathcal{O}_X) = \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X) = 1.$$

Therefore, $\beta = n$ holds. Hence we obtain

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(L)) = L^n + n.$$

This implies that

$$\Delta(X, L) = L^n + n - \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(L)) = 0$$

holds. Thus we obtain that $|L|$ is basepoint-free by Theorem 2.8 (see also [10, Corollary 1.10]).

We obtained all the desired statements. □

The following example shows that the assumption $r > n-1$ in Lemma 4.1 is sharp.

Example 4.2. Let X be a Del Pezzo surface of degree one. We put $L = -K_X$. Then $K_X + rL = 0$ with $r = 1$ holds. We note that $r = 1 = 2 - 1 = \dim X - 1$ holds. It is easy to check that $|L| = |-K_X|$ is not basepoint-free.

We can prove the following corollary.

Corollary 4.3. *Let $[X, \omega]$ be a projective quasi-log canonical pair. Note that X may be reducible. Let L be an ample Cartier divisor on X such that $\omega + rL \equiv 0$ with $r > n-1$, where $n = \dim X$. Then the complete linear system $|L|$ is basepoint-free.*

Proof. Let X_i be any irreducible component of X. Since X_i is a qlc stratum of $[X, \omega]$, $[X_i, \omega|_{X_i}]$ is a quasi-log canonical pair by adjunction (see Theorem 2.13 (i)). If $\dim X_i = 0$, then $|L|_{X_i}|$ is obviously basepoint-free. When $\dim X_i > 0$, the complete linear system $|L|_{X_i}|$ is basepoint-free by Lemma 4.1 because $\omega|_{X_i} + rL|_{X_i} \equiv 0$ with $r > \dim X_i - 1$. Since $L - \omega \equiv (r+1)L$ is ample, we have $H^1(X, \mathcal{I}_{X_i} \otimes \mathcal{O}_X(L)) = 0$ by Theorem 2.13 (ii), where $\mathcal{I}_{X_i}$ is the defining ideal sheaf of X_i on X. Therefore the restriction map

$$H^0(X, \mathcal{O}_X(L)) \to H^0(X_i, \mathcal{O}_{X_i}(L))$$

is surjective. This implies that $|L|$ is basepoint-free. □

Let us prove Theorem 1.6.

Proof of Theorem 1.6. We divide the proof into several small steps.

Step 1. If $\dim(X \setminus X_{-\infty}) = 0$, then the statement is obvious. From now on, we assume $n \geq 1$ and use induction on $\dim(X \setminus X_{-\infty})$. Therefore, we assume that the statement holds true when $\dim(X \setminus X_{-\infty}) < n$.

Step 2. Let C be a qlc stratum of $[X, \omega]$. We put $X' = C \cup X_{-\infty}$. Then, by adjunction (see Theorem 2.13 (i)), $[X', \omega|_{X'}]$ is a quasi-log scheme. Note that $\omega|_{X'} + r\mathcal{L}|_{X'} \equiv 0$ holds. Let $\mathcal{I}_{X'}$ be the defining ideal sheaf of X' on X. By Theorem 2.13 (ii), we have $H^1(X, \mathcal{I}_{X'} \otimes \mathcal{L}) = 0$ since $\mathcal{L} - \omega \equiv (r+1)\mathcal{L}$ is ample. Therefore, the natural restriction map

$$H^0(X, \mathcal{L}) \to H^0(X', \mathcal{L}|_{X'}) \tag{4.1}$$

is surjective.

Step 3. If $\dim C < n$, then $|\mathcal{L}|_{X'}|$ is basepoint-free by the induction hypothesis. By (4.1), $|\mathcal{L}|$ is basepoint-free in a neighborhood of X'.

Step 4. If $\dim C = n$ and $C \cap X_{-\infty} = \emptyset$, then $|\mathcal{L}|_C|$ is basepoint-free by Lemma 4.1 since $[C, \omega|_C]$ is an irreducible quasi-log canonical pair with

$$\omega|_C + r\mathcal{L}|_C \equiv 0$$

and

$$r > \dim(X' \setminus X'_{-\infty}) - 1 = \dim C - 1.$$

We note that $|\mathcal{L}|_{X_{-\infty}}|$ is basepoint-free by assumption. Therefore, $|\mathcal{L}|_{X'}|$ is obviously basepoint-free. Hence, by (4.1), $|\mathcal{L}|$ is basepoint-free in a neighborhood of X'.

Step 5. By Steps 3, 4, and (4.1), we may assume that $X \setminus X_{-\infty}$ is irreducible with $\dim(X \setminus X_{-\infty}) = n$ such that X is connected. Since $\mathcal{L} - \omega \equiv (r+1)\mathcal{L}$ is ample, $H^1(X, \mathcal{I}_{X_{-\infty}} \otimes \mathcal{L}) = 0$ by Theorem 2.13 (ii). Therefore, the natural restriction map

$$H^0(X, \mathcal{L}) \to H^0(X_{-\infty}, \mathcal{L}|_{X_{-\infty}})$$

is surjective. Since $|\mathcal{L}|_{X_{-\infty}}|$ is basepoint-free by assumption, the base locus $\mathrm{Bs}|\mathcal{L}|$ of $|\mathcal{L}|$ is disjoint from $X_{-\infty}$. Since $X \setminus X_{-\infty}$ is irreducible and X is connected, $\mathrm{Bs}|\mathcal{L}|$ does not contain $X \setminus X_{-\infty}$. By Step 3, $\mathrm{Bs}|\mathcal{L}|$ contains no qlc centers of $[X, \omega]$. Hence $\mathrm{Bs}|\mathcal{L}|$ contains no qlc strata of $[X, \omega]$.

We assume that $\mathrm{Bs}|\mathcal{L}| \neq \emptyset$. We take a general member D of $|\mathcal{L}|$. Then we can take $0 < c \leq 1$ such that $[X, \omega + cD]$ is a quasi-log scheme with

$$\mathrm{Nqlc}(X, \omega + cD) = \mathrm{Nqlc}(X, \omega)$$

and that there exists a qlc center C of $[X, \omega + cD]$ with $C \cap \mathrm{Bs}|\mathcal{L}| \neq \emptyset$ by construction (see Proposition 3.4). We put

$$X' = C \cup \mathrm{Nqlc}(X, \omega + cD).$$

By adjunction (see Theorem 2.13 (i)), $[X', (\omega + cD)|_{X'}]$ is a quasi-log scheme. By construction, $\dim C < n$ and

$$(\omega + cD)|_{X'} + (r - c)\mathcal{L}|_{X'} \equiv 0$$

hold. Note that

$$r - c > \dim C - 1 = \dim(X' \setminus X'_{-\infty}) - 1$$

holds. Therefore, by the induction hypothesis, $|\mathcal{L}|_{X'}|$ is basepoint-free. Since $\mathcal{L} - (\omega + cD) \equiv (r + 1 - c)\mathcal{L}$ is ample, $H^1(X, \mathcal{I}_{X'} \otimes \mathcal{L}) = 0$ by Theorem 2.13 (ii), where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X' on X. Thus, the restriction map

$$H^0(X, \mathcal{L}) \to H^0(X', \mathcal{L}|_{X'})$$

is surjective. In particular, $|\mathcal{L}|$ is basepoint-free in a neighborhood of C. This is a contradiction since $C \cap \mathrm{Bs}|\mathcal{L}| \neq \emptyset$. Hence, we obtain $\mathrm{Bs}|\mathcal{L}| = \emptyset$.

We obtained the desired statement. □

5. Proof of Theorem 1.1

Let us explain the idea of the proof of the relative spannedness in Theorem 1.1. We construct a sequence of closed subschemes

$$Z_0 \subset Z_1 \subset \cdots \subset Z_{k-1} \subset X$$

such that $Z_{k-1} = F$ holds set theoretically. For every i, there exists an $\mathbb{R}$-Cartier divisor $\omega_i|_{Z_i}$ on Z_i such that $[Z_i, \omega_i|_{Z_i}]$ is a quasi-log scheme and that

$$\omega_i|_{Z_i} + rL|_{Z_i} \equiv 0$$

holds with

$$r > \dim F - 1 \geq \dim Z_i - 1.$$

We can make $[Z_i, \omega_i|_{Z_i}]$ satisfy that $(Z_0)_{-\infty} = \emptyset$ and that $(Z_{i+1})_{-\infty} \subset Z_i$ set theoretically for every i. By Theorem 1.6, the complete linear system $|L|_{Z_0}|$ is basepoint-free. By the vanishing theorem for quasi-log schemes, we obtain that the natural restriction map

$$f_*\mathcal{O}_X(L) \to f_*\mathcal{O}_{Z_i}(L|_{Z_i})$$

is surjective for every i. Therefore, if $|L|_{Z_i}|$ is basepoint-free, then the relative base locus $\mathrm{Bs}_f|L|$ is disjoint from Z_i. This implies that $|L|_{(Z_{i+1})_{-\infty}}|$ is basepoint-free since we have $(Z_{i+1})_{-\infty} \subset Z_i$. Then, by Theorem 1.6, the complete linear system $|L|_{Z_{i+1}}|$ is basepoint-free. Hence we finally obtain that the relative base locus $\mathrm{Bs}_f|L|$ is disjoint from F. This means that

$$f^*f_*\mathcal{O}_X(L) \to \mathcal{O}_X(L)$$

is surjective at every point of F.

We start with the following easy lemma on log canonical pairs.

Lemma 5.1. *Let (X, Δ) be a log canonical pair and let B be an effective $\mathbb{R}$-Cartier divisor on X such that $(X, \Delta + B)$ is not log canonical. Then there exists an increasing sequence of real numbers*

$$0 \leq c_0 < c_1 < \cdots < c_{k-1} < c_k = 1$$

with the following properties.

(i) *c_0 is the log canonical threshold of (X, Δ) with respect to B.*
(ii) *We put $U_i = X \setminus \mathrm{Nlc}(X, \Delta + c_iB)$ for every i. Then $U_{i+1} \subsetneq U_i$ holds for every $0 \leq i \leq k-1$.*
(iii) *For every $1 \leq i \leq k$, $X \setminus \mathrm{Nlc}(X, \Delta + tB) = U_i$ holds for any $t \in (c_{i-1}, c_i]$.*

In this situation, for each i with $0 \leq i \leq k-1$, there exists a finite set of log canonical centers $\{C_j\}_{j \in I_i}$ of $(X, \Delta + c_iB)$ such that

$$U_i \setminus U_{i+1} \subset \bigcup_{j \in I_i} C_j$$

and that

$$\mathrm{Nlc}(X, \Delta + B) = \bigcup_{i=0}^{k-1} \left(\bigcup_{j \in I_i} C_j \right)$$

holds set theoretically.

Proof. We note that c_i is a kind of jumping numbers of (X, Δ) with respect to B for every i. More precisely, we consider the following Zariski open set

$$U_t := X \setminus \mathrm{Nlc}(X, \Delta + tB)$$

for every $t \in [0, 1]$ and increase t from 0 to 1. Then there exists an increasing sequence of real numbers

$$0 \leq c_0 < c_1 < \cdots < c_{k-1} < c_k = 1$$

satisfying the desired properties.

The following description may be helpful. By the above construction, c_0 is the log canonical threshold of (X, Δ) with respect to B and $c_k = 1$. Let $\mathrm{Nklt}(X, \Delta + c_{i-1}B)$ denote the non-klt locus of $(X, \Delta + c_{i-1}B)$ for $1 \leq i \leq k-1$. Equivalently, $V_{i-1} := X \setminus \mathrm{Nklt}(X, \Delta + c_{i-1}B)$ is the largest Zariski open set of X such that $(V_{i-1}, (\Delta + c_{i-1}B)|_{V_{i-1}})$ is kawamata log terminal. Then $c_i - c_{i-1}$ is the log canonical threshold of $(X, \Delta + c_{i-1}B)$ with respect to B on the Zariski open set $V_{i-1} = X \setminus \mathrm{Nklt}(X, \Delta + c_{i-1}B)$ for $1 \leq i \leq k-1$. □

We prepare one more easy lemma.

Lemma 5.2. *Let (X, Δ) be a log canonical pair and let $B_1, \ldots, B_k$ be effective Cartier divisors on X passing through a closed point P of X. If $(X, \Delta + \sum_{i=1}^{k} B_i)$ is log canonical around P, then the inequality $k \leq \dim X$ holds.*

Although Lemma 5.2 is well known, we prove it here for the reader's convenience.

Proof. By shrinking X around P, we may assume that $(X, \Delta + \sum_{i=1}^{k} B_i)$ is log canonical. If $\dim X = 1$, then the statement is obvious. We use the induction on $\dim X$. So we assume that $\dim X \geq 2$ holds. Let $\nu \colon Z \to B_k$ be the normalization of B_k. We put

$$K_Z + \Delta_Z = \nu^*(K_X + \Delta + B_k).$$

Then (Z, Δ_Z) is log canonical by adjunction since $(X, \Delta + B_k)$ is log canonical. We note that $\mathrm{Supp} B_i$ and $\mathrm{Supp} B_k$ have no common irreducible components for $1 \leq i \leq k-1$ since $(X, \Delta + \sum_{i=1}^{k} B_i)$ is log canonical. We take $Q \in \nu^{-1}(P)$. Then $(Z, \Delta_Z + \sum_{i=1}^{k-1} \nu^* B_i)$ is log canonical by adjunction and $Q \in \mathrm{Supp}\, \nu^* B_i$ for $1 \leq i \leq k-1$. Therefore, we obtain

$$k - 1 \leq \dim Z = \dim X - 1$$

by the induction hypothesis. This means that the desired inequality $k \leq \dim X$ holds. □

Let us prove Theorem 1.1 by using Theorem 1.6 and Lemma 5.1.

Proof of Theorem 1.1. Since $K_X + \Delta + rL \equiv_Y 0$, $-(K_X + \Delta)$ is f-ample, and $r > 0$, we see that L is f-ample. We put $f(F) = P$ and shrink Y around P. Then we may assume that Y is affine without loss of generality. We put $n = \dim X$ and take general hyperplane sections $B_1, \ldots, B_{n+1}$ on Y such that $P \in \mathrm{Supp} B_i$ for every i. We put

$$B = \sum_{i=1}^{n+1} f^* B_i.$$

Then $(X, \Delta + B)$ is log canonical outside F and is not log canonical at every point of F by Lemma 5.2.

Step 1. Let F' be any positive-dimensional irreducible component of F. In this step, we will prove that $\dim F' \geq r - 1$ holds.

We put

$$c = \max\{t \in \mathbb{R} \,|\, (X, \Delta + tB) \text{ is log canonical at the generic point of } F'\},$$

that is, c is the log canonical threshold of (X, Δ) with respect to B at the generic point of F'. By construction, $0 \leq c < 1$ and F' is a log canonical center of $(X, \Delta + cB)$. We now consider the natural quasi-log scheme structure of $[X, \Delta + cB]$ as in Example 2.14. We put

$$X' = F' \cup \mathrm{Nqlc}(X, \Delta + cB)$$

and consider the induced quasi-log scheme $[X', (K_X + \Delta + cB)|_{X'}]$ by adjunction (see Theorem 2.13 (i)). Note that

$$tL|_{X'} - (K_X + \Delta + cB)|_{X'} \equiv (t + r)L|_{X'}$$

is ample for $t > -r$ since $f(X') = P$. We note that

$$\deg \chi(X', \mathcal{I}_{X'_{-\infty}} \otimes \mathcal{O}_{X'}(tL)) = \dim F'$$

holds because $L|_{X'}$ is ample and the coherent ideal sheaf $\mathcal{I}_{X'_{-\infty}}$ on X' can be considered a coherent sheaf on F'. More precisely, $\mathcal{I}_{X'_{-\infty}} \subset \mathcal{O}_{F'}$ holds since $\{0\} = \mathcal{I}_{F'} \cap \mathcal{I}_{X'_{-\infty}} \subset \mathcal{O}_{X'}$, where $\mathcal{I}_{F'}$ is the defining ideal sheaf of F' on X'. By Theorem 2.13,

$$H^i(X', \mathcal{I}_{X'_{-\infty}} \otimes \mathcal{O}_{X'}(tL)) = 0$$

for $i > 0$ and $t \in \mathbb{Z}$ with $t > -r$. Since $L|_{X'}$ is ample,

$$H^0(X', \mathcal{I}_{X'_{-\infty}} \otimes \mathcal{O}_{X'}(tL)) = 0$$

for $t \in \mathbb{Z}$ with $t < 0$. Therefore, we obtain

$$\chi(X', \mathcal{I}_{X'_{-\infty}} \otimes \mathcal{O}_{X'}(tL)) = 0$$

for $t \in \mathbb{Z}$ with $-r < t \leq -1$. Hence, we obtain

$$\dim F' = \deg \chi(X', \mathcal{I}_{X'_{-\infty}} \otimes \mathcal{O}_{X'}(tL)) \geq r - 1.$$

This means that the dimension of every positive-dimensional irreducible component of F is $\geq r - 1$.

Step 2. In Steps 2 and 3, we will prove that $f^* f_* \mathcal{O}_X(L) \to \mathcal{O}_X(L)$ is surjective at every point of F.

By Lemma 5.1, we have an increasing sequence of real numbers

$$0 \leq c_0 < c_1 < \cdots < c_k = 1$$

satisfying the properties in Lemma 5.1. We consider normal pairs $(X, \Delta + c_i B)$ for $0 \leq i \leq k-1$. We put $\omega_i = K_X + \Delta + c_i B$. Then $[X, \omega_i]$ is a quasi-log scheme for $0 \leq i \leq k-1$ (see Example 2.14). We put

$$Z_i = \bigcup_{j \in I_i} C_j \cup \mathrm{Nqlc}(X, \omega_i)$$

and consider the pair $[Z_i, \omega_i|_{Z_i}]$ for every i with $0 \leq i \leq k-1$. Then, by adjunction (see Theorem 2.13 (i)), $[Z_i, \omega_i|_{Z_i}]$ is a quasi-log scheme with $f(Z_i) = P$ for $0 \leq i \leq k-1$. We note that $\mathrm{Nqlc}(X, \omega_0) = \emptyset$ since $(X, \Delta + c_0 B)$ is log canonical by definition. We also note that $(Z_i)_{-\infty} = \mathrm{Nqlc}(Z_i, \omega_i|_{Z_i}) = \mathrm{Nqlc}(X, \omega_i)$ for every i by construction. Since $L - \omega_i$ is numerically equivalent to

$$L - (K_X + \Delta) \equiv_Y (r+1)L$$

over Y, $L - \omega_i$ is f-ample. Therefore, by Theorem 2.13 (ii),

$$H^1(X, \mathcal{I}_{Z_i} \otimes \mathcal{O}_X(L)) = 0,$$

where $\mathcal{I}_{Z_i}$ is the defining ideal sheaf of Z_i on X. Hence, the restriction map

$$H^0(X, \mathcal{O}_X(L)) \to H^0(Z_i, \mathcal{O}_{Z_i}(L)) \tag{5.1}$$

is surjective for every $0 \leq i \leq k-1$.

Step 3. Since $[Z_0, \omega_0|_{Z_0}]$ is a projective quasi-log canonical pair such that

$$\omega_0|_{Z_0} + rL|_{Z_0} \equiv 0$$

with $r > \dim F - 1 \geq \dim Z_0 - 1$, the complete linear system $|L|_{Z_0}|$ is basepoint-free by Corollary 4.3.

If $|L|_{Z_i}|$ is basepoint-free, then the relative base locus $\mathrm{Bs}_f|L|$ is disjoint from Z_i by (5.1). By Lemma 5.1, $\mathrm{Nqlc}(X, \omega_{i+1}) \subset Z_i$ holds set theoretically. This implies that $\mathrm{Bs}_f|L|$

does not intersect with $\mathrm{Nqlc}(X, \omega_{i+1}) = \mathrm{Nqlc}(Z_{i+1}, \omega_{i+1}|_{Z_{i+1}}) = (Z_{i+1})_{-\infty}$. Therefore, $|L|_{(Z_{i+1})_{-\infty}}|$ is basepoint-free. Since

$$\omega_{i+1}|_{Z_{i+1}} + rL|_{Z_{i+1}} \equiv 0$$

with $r > \dim F - 1 \geq \dim Z_{i+1} - 1$, $|L|_{Z_{i+1}}|$ is basepoint-free by Theorem 1.6. We repeat this process. We note that $F = \mathrm{Nlc}(X, \Delta + B) = \mathrm{Nqlc}(X, \omega_k)$ set theoretically. Hence we finally obtain that the complete linear system $|L|_{Z_{k-1}}|$ is basepoint-free and that the relative base locus $\mathrm{Bs}_f|L|$ is disjoint from $F = \mathrm{Nqlc}(X, \omega_k)$, equivalently, $f^*f_*\mathcal{O}_X(L) \to \mathcal{O}_X(L)$ is surjective at every point of F.

We obtained all the desired statements. □

Proof of Corollary 1.2. We assume that $K_X + \Delta + (n+1)L$ is not f-nef. Then, by the cone and contraction theorem for log canonical pairs (see [4, Theorem 1.1]), we get a $(K_X+\Delta+(n+1)L)$-negative extremal contraction $\varphi\colon X \to W$ over Y. Thus, by replacing $f\colon X \to Y$ with $\varphi\colon X \to W$, we may assume that the relative Picard number $\rho(X/Y) = 1$. Therefore, there exists r with $r > n+1$ such that $K_X + \Delta + rL$ is relatively numerically trivial over Y. By Theorem 1.1, we have

$$n = \dim X \geq r - 1 > n.$$

This is a contradiction. This means that $K_X + \Delta + (n+1)L$ is f-nef. Similarly, we can check that $K_X + \Delta + nL$ is f-nef when $\dim Y \geq 1$. □

We close this section with the proof of Corollary 1.3.

Proof of Corollary 1.3. Without loss of generality, we may assume that Y is affine by shrinking Y around $f(F)$. Since $\dim X = \dim Y$ and f is surjective, f is generically finite. Hence $f_*\mathcal{O}_X(-L) \neq 0$ holds. Thus we can take an effective Cartier divisor D on X such that $D \sim -L$. Since (X, Δ) is kawamata log terminal, $(X, \Delta + \varepsilon D)$ is also kawamata log terminal for $0 < \varepsilon \ll 1$. By construction,

$$K_X + \Delta + \varepsilon D + (r + \varepsilon)L$$

is relatively numerically trivial over Y. Therefore, by Theorem 1.1, the dimension of every positive-dimensional irreducible component of F is $\geq (r+\varepsilon) - 1$, that is, $\geq \lfloor r \rfloor$. If $\dim F \leq r + 1$, then $\dim F < (r + \varepsilon) + 1$ obvioully holds. Thus, by Theorem 1.1,

$$f^*f_*\mathcal{O}_X(L) \to \mathcal{O}_X(L)$$

is surjective at every point of F. □

6. Generalizations for quasi-log canonical pairs

In this section, we will prove Theorem 1.7. The following lemma is a generalization of Lemma 5.1 for quasi-log canonical pairs.

Lemma 6.1. *Let $[X, \omega]$ be an irreducible quasi-log canonical pair and let B be an effective $\mathbb{R}$-Cartier divisor on X. Then there exist an increasing sequence of real numbers*

$$c_{-1} = 0 \leq c_0 < c_1 < \cdots < c_{k-1} < c_k = 1,$$

globally embedded simple normal crossing pairs (Y_i, B_{Y_i}) for $0 \leq i \leq k$, and proper surjective morphisms $f_i\colon Y_i \to X$ for all $0 \leq i \leq k$ with the following properties.

(i) *For every $0 \leq i \leq k$,*

$$(X, \omega + c_iB, f_i\colon (Y_i, B_{Y_i}) \to X)$$

is a quasi-log scheme.

(ii) *We put*
$$U_i = X \setminus \mathrm{Nqlc}(X, \omega + c_i B)$$
for every $0 \leq i \leq k$. *Then*
$$U_k \subsetneq U_{k-1} \subsetneq \cdots \subsetneq U_0 = X$$
holds.

(iii) *For every* $0 \leq i \leq k$,
$$(X, \omega + tB, f_i \colon (Y_i, B_{Y_i} + (t - c_i) f_i^* B) \to X)$$
is a quasi-log scheme such that
$$U_i = X \setminus \mathrm{Nqlc}(X, \omega + tB)$$
holds for any $t \in (c_{i-1}, c_i]$.

(iv) *For each* $0 \leq i \leq k-1$, *there exists a finite set of qlc centers* $\{C_j\}_{j \in I_i}$ *of* $[X, \omega + c_i B]$ *such that*
$$U_i \setminus U_{i+1} \subset \bigcup_{j \in I_i} C_j$$
holds.

Before we prove Lemma 6.1, we make an important remark.

Remark 6.2. In Lemma 6.1, let $f \colon (Y, B_Y) \to X$ be a quasi-log resolution of $[X, \omega]$. Let X' be the union of all qlc centers of $[X, \omega]$ contained in $\mathrm{Supp}\, B$. Assume that the union of all strata of (Y, B_Y) mapped to X' by f, which is denoted by Y', is a union of some irreducible components of Y. We put $Y'' = Y - Y'$, $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$, and $f'' = f|_{Y''}$. By [7, Proposition 6.3.1] and [13, Theorem 3.35], we may further assume that
$$(Y'', B_{Y''} + (f'')^* B)$$
is a globally embedded simple normal crossing pair. Then, by Lemma 3.2,
$$(X, \omega + B, f'' \colon (Y'', B_{Y''} + (f'')^* B) \to X)$$
is a quasi-log scheme. By the following proof of Lemma 6.1, we see that
$$\mathrm{Nqlc}(X, \omega + B) = \bigcup_{i=0}^{k-1} \left(\bigcup_{j \in I_i} C_j \right)$$
holds set theoretically.

We give a detailed proof of Lemma 6.1 for the reader's convenience, although it is similar to the proof of Lemma 5.1.

Proof of Lemma 6.1. Let $f \colon (Y, B_Y) \to X$ be a quasi-log resolution of $[X, \omega]$.

Step 1. If there exists a qlc center C of $[X, \omega]$ such that $C \subset \mathrm{Supp} B$. Then we put $c_0 = 0$, $(Y_0, B_{Y_0}) = (Y, B_Y)$, and $f_0 = f$.

Step 2. We assume that there are no qlc centers of $[X, \omega]$ contained in $\mathrm{Supp} B$. By [7, Proposition 6.3.1] and [13, Theorem 3.35], we may assume that
$$(Y, f^* B + \mathrm{Supp} B_Y)$$
is a globally embedded simple normal crossing pair.

If $(B_Y + f^* B)^{>1} = 0$, then we put $c_0 = 1$, $(Y_0, B_{Y_0}) = (Y, B_Y + f^* B)$, $f_0 = f$, and we stop this process (see Lemma 3.1).

If $(B_Y+f^*B)^{>1}\neq 0$, then we can take $0<c_0<1$ such that $(B_Y+c_0f^*B)^{>1}=0$ and that there exists a component G of $(B_Y+c_0f^*B)^{=1}$ with $f(G)\subset \mathrm{Supp}B$. In this situation, we put $(Y_0, B_{Y_0})=(Y, B_Y+c_0f^*B)$ and $f_0=f$. Then we see that

$$(X, \omega+c_0B, f_0\colon (Y_0, B_{Y_0})\to X)$$

is the desired quasi-log canonical pair (see Lemma 3.1).

Step 3. We assume that we have already constructed

$$(X, \omega+c_iB, f_i\colon (Y_i, B_{Y_i})\to X)$$

for $i\geq 0$ with $c_i<1$.

Let X'_i be the union of $\mathrm{Nqlc}(X, \omega+c_iB)$ and all qlc centers of $[X, \omega+c_iB]$ contained in $\mathrm{Supp}B$. By [7, Proposition 6.3.1], we may assume that the union of all strata of (Y_i, B_{Y_i}) mapped to X'_i by f_i, which is denoted by Y'_i, is a union of some irreducible components of Y_i. We put $Y''_i=Y_i-Y'_i$,

$$K_{Y''_i}+B_{Y''_i}=(K_{Y_i}+B_{Y_i})|_{Y''_i},$$

and $f''_i=f_i|_{Y''_i}$. We may further assume that

$$\left(Y''_i, (f''_i)^*B+\mathrm{Supp}B_{Y''_i}\right)$$

is a globally embedded simple normal crossing pair by [7, Proposition 6.3.1] and [13, Theorem 3.35].

If

$$f''_i\left(\mathrm{Supp}\big(B_{Y''_i}+(1-c_i)(f''_i)^*B\big)^{>1}\right)\subset X'_i$$

holds, then we put $c_{i+1}=1$,

$$\left(Y_{i+1}, B_{Y_{i+1}}\right)=\left(Y''_i, B_{Y''_i}+(1-c_i)(f''_i)^*B\right),$$

$f_{i+1}=f''_i$, and we stop this process. We can see that

$$\left(X, \omega+B, f_{i+1}\colon (Y_{i+1}, B_{Y_{i+1}})\to X\right)$$

with $c_{i+1}=1$ is a quasi-log scheme with the desired properties (see Lemma 3.2).

Otherwise, we put

$$c_{i+1}=\sup\left\{s\in\mathbb{R}\;\middle|\; f''_i\left(\mathrm{Supp}\big(B_{Y''_i}+(s-c_i)(f''_i)B\big)^{>1}\right)\subset X'_i\right\}.$$

In this situation, we have $c_i<c_{i+1}<1$. Then we put

$$\left(Y_{i+1}, B_{Y_{i+1}}\right)=\left(Y''_i, B_{Y''_i}+(c_{i+1}-c_i)(f''_i)^*B\right)$$

and $f_{i+1}=f''_i$. We can see that

$$\left(X, \omega+c_{i+1}B, f_{i+1}\colon (Y_{i+1}, B_{Y_{i+1}})\to X\right)$$

is a quasi-log scheme with the desired properties (see Lemma 3.2).

Step 4. After finitely many steps, we get a finite increasing sequence of real numbers:

$$c_{-1}=0\leq c_0<c_1<\cdots<c_{k-1}<c_k=1.$$

By the above construction, we obviously have the desired properties.

Roughly speaking, c_i-c_{i-1} is the quasi-log canonical threshold of $[X, \omega+c_{i-1}B]$ with respect to B on the Zariski open set $X\setminus X'_{i-1}$ of X for $1\leq i\leq k-1$. Hence we can see this lemma as a quasi-log scheme analogue of Lemma 5.1. □

We give a sketch of the proof of Theorem 1.7 for the reader's convenience, although the proof of Theorem 1.7 is essentially the same as that of Theorem 1.1.

Sketch of Proof of Theorem 1.7. We divide the proof into several small steps.

Step 1. Since $\mathcal{L} - \omega \equiv (r+1)\mathcal{L}$ is φ-ample, we have $R^1\varphi_*(\mathcal{I}_{X_i} \otimes \mathcal{L}) = 0$, where X_i is any irreducible component of X and $\mathcal{I}_{X_i}$ is the defining ideal sheaf of X_i on X. Therefore, the restriction map

$$\varphi_*\mathcal{L} \to \varphi_*(\mathcal{L}|_{X_i})$$

is surjective. We note that $[X_i, \omega|_{X_i}]$ is a quasi-log canonical pair by adjunction (see Theorem 2.13 (i)). We also note that $\omega|_{X_i} + r\mathcal{L}|_{X_i}$ is relatively numerically trivial over W. Therefore, by replacing $[X, \omega]$ with $[X_i, \omega|_{X_i}]$, we may assume that X is irreducible. Furthermore, by replacing W with $\varphi(X)$, we may assume that W is an irreducible variety. By shrinking W around $\varphi(F)$, we may further assume that W is an affine variety.

Step 2. If $\varphi(X) = W$ is a point, then we have $X = F$ and may assume that $\dim X = \dim F \geq 1$ holds. In this case, by Lemma 4.1, $r \leq \dim F + 1$ holds. This means that $\dim F \geq r - 1$ holds true. If $\dim F < r + 1$, equivalently, $r > \dim F - 1 = \dim X - 1$, then $|\mathcal{L}|$ is basepoint-free by Lemma 4.1 again. This means that

$$\varphi^*\varphi_*\mathcal{L} \to \mathcal{L}$$

is surjective at every point of F.

Step 3. From now on, we may assume that $\dim W \geq 1$ holds. We put $n = \dim X$ and take general hyperplane sections $B_1, \ldots, B_{n+1}$ on W such that $\varphi(F) \in \mathrm{Supp} B_i$ for every i. We put

$$B = \sum_{i=1}^{n+1} \varphi^* B_i.$$

Step 4. Let F' be any positive-dimensional irreducible component of F.

If F' is a qlc center of $[X, \omega]$. Then $[F', \omega|_{F'}]$ is a quasi-log canonical pair by adjunction (see Theorem 2.13 (i)). Hence we obtain

$$\dim F' = \deg \chi(F', \mathcal{L}^{\otimes t}|_{F'}) \geq r - 1$$

by the usual application of the vanishing theorem (see Theorem 2.13 (ii) and Step 1 in the proof of Lemma 4.1).

From now on, we may assume that F' is not a qlc center of $[X, \omega]$. Let $f\colon (Y, B_Y) \to X$ be a quasi-log resolution of $[X, \omega]$. Let X' be the union of all qlc centers contained in F. By [7, Proposition 6.3.1], we may assume that the union of all strata of (Y, B_Y) mapped to X' by f, which is denoted by Y', is a union of some irreducible components of Y. We put $Y'' = Y - Y'$, $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$, and $f'' = f|_{Y''}$. We may further assume that

$$(Y'', (f'')^*B + \mathrm{Supp} B_{Y''})$$

is a globally embedded simple normal crossing pair by [7, Proposition 6.3.1] and [13, Theorem 3.35]. By [7, Lemma 6.3.13], we can take $0 < c < 1$ such that there exists an irreducible component G of $(B_{Y''} + c(f'')^*B)^{=1}$ with $f''(G) = F'$ and that $F' \not\subset f''\left(\mathrm{Supp}(B_{Y''} + c(f'')^*B)^{>1}\right)$. Then

$$(X, \omega + cB, f''\colon (Y'', B_{Y''} + c(f'')^*B) \to X)$$

is a quasi-log scheme such that F' is a qlc center of $[X, \omega + cB]$ (see Lemma 3.2). We put

$$X' = F' \cup \mathrm{Nqlc}(X, \omega + cB).$$

Then, by adjunction (see Theorem 2.13 (i)), $[X', (\omega + cB)|_{X'}]$ is a quasi-log scheme. By construction,

$$\dim F' = \deg \chi(X', \mathcal{I}_{X'_{-\infty}} \otimes \mathcal{L}^{\otimes t}) \geq r - 1$$

as in Step 1 in the proof of Theorem 1.1.

Step 5. We note that

$$(X, \omega + B, f'' \colon (Y'', B_{Y''} + (f'')^*B) \to X)$$

is a quasi-log scheme (see Lemma 3.2) such that $\mathrm{Nqlc}(X, \omega+B) = F$ holds set theoretically (see [7, Lemma 6.3.13]). By Lemma 6.1, the arguments in Steps 2 and 3 in the proof of Theorem 1.1 work with some minor modifications. Hence, we obtain that

$$\varphi^*\varphi_*\mathcal{L} \to \mathcal{L}$$

is surjective at every point of $F = \mathrm{Nqlc}(X, \omega + B)$.

We obtained all the desired statements. □

Proof of Corollary 1.8. We note that the cone and contraction theorem holds true for quasi-log canonical pairs (see [7, Theorems 6.7.3 and 6.7.4]). Therefore, the proof of Corollary 1.2 works as well in this case, using Theorem 1.7. □

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