Vanishing theorems for projective morphisms between complex analytic spaces

O. Fujino 2024.3.14 in Osaka

§Introduction

Theorem 1 X : smooth projective variety $/\mathbb{C}$

Y : projective variety

$f : X \to Y$: surjective morphism

$\Rightarrow$ (i) (Torsion-freeness)

$R^i f_* \omega_X$: torsion-free for $^\forall i$

(ii) (Vanishing theorem)

$H^i(Y, R^j f_* \omega_X \otimes L) = 0$ for $^\forall i > 0$, $^\forall j$

L : ample line bundle on Y

Cor 2 f : birational $\Rightarrow R^i f_* \omega_X = 0$ for $^\forall i > 0$

Cor 3 $H^i(X, \omega_X \otimes H) = 0$ for $^\forall i > 0$

$\uparrow$

Kodaira vanishing

H : ample line bundle on X

Def 4 (Simple normal crossing pairs)

(X, D) : a simple normal crossing pair

$\overset{\text{def}}{\Longleftrightarrow}$ locally, $X \hookrightarrow M$ smooth variety

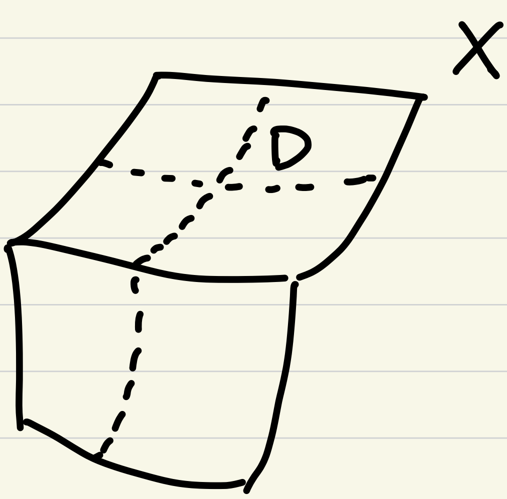
s.n.c div

B : reduced s.n.c div on M s.t

$\mathrm{Supp}(B+X)$: s.n.c div on M

B and X have no common components

with $D = B|_X$



In this talk, B (hence D) is always assumed to be reduced.

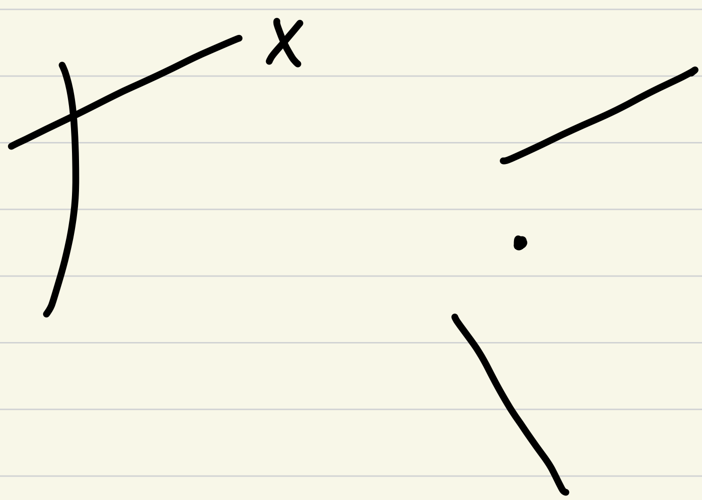
Def 5 (Stratum)

(X, D): s.n.c pair

W: closed subset of X

W is a <u>stratum</u> of (X, D)

$\overset{}{\underset{\text{def}}{\Longleftrightarrow}}$ W is an irreducible component of the intersection of some irreducible components of X and D.



<u>Theorem 6</u> (X, D) : s.n.c pair

$f : X \to Y$: projective

$\pi : Y \to Z$: projective

X, Y, Z : <u>algebraic</u>

$\Rightarrow$ (i) (Strict support condition)

every associated prime of $R^i f_* \omega_X(D)$

is the f-image of some stratum of (X, D).

(ii) (Vanishing theorem)

$$R^i \pi_* (R^j f_* \omega_X(D) \otimes L) = 0$$

for $\forall i > 0$, $\forall j$

L : π-ample line bundle on Y

<u>Remark 7</u> X : smooth irreducible, $Z = pt$

f : surjective in Theorem 6

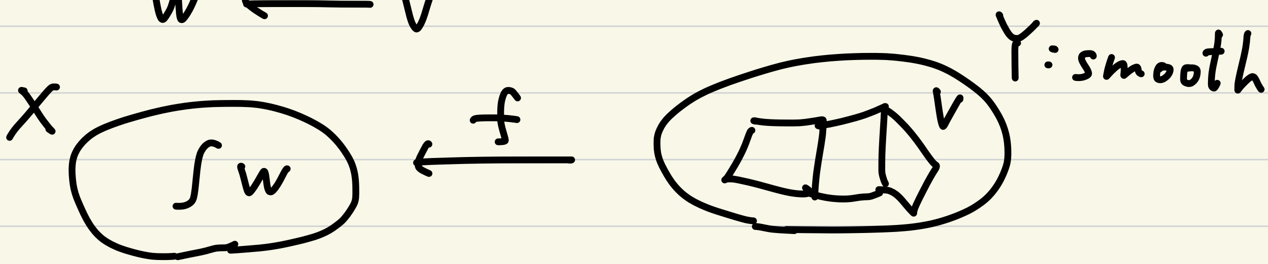
$\Rightarrow$ We can recover Kollár's original ones.

§ Motivation

X : singular variety
$\cup$
W : closed subvariety

$$\begin{tikzcd} X & Y \arrow[l, "f"'] \\ W \arrow[u, phantom, "\cup" rotate=90] & V \arrow[l] \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

f : resolution



V : s.n.c div

In general, $f^{-1}W$ is not smooth.
$f^{-1}W$ is a s.n.c divisor.

We consider

$f : V \longrightarrow W$ ← highly singular variety

↑ s.n.c variety

We want to apply Theorem 6 to $f : V \to W$.

Traditionally, if X has only mild singularities (for example, X : klt), then, by the perturbation technique, we can take a smooth irreducible V.
(Kawamata, Shokurov, Reid, Kollár, …
…, BCHM, …).

If X has bad singularities (for example, X : lc), then we can not make V irreducible.

§ How to prove Theorem 6

For simplicity, we assume that

$$X : \text{irreducible}, \quad D \neq 0.$$

Since X : algebraic, we can always take a compactification. So we may assume X : complete.

By MHS, we have

$$E_1^{p,q} = H^q(X, \Omega_X^p(\log D) \otimes \mathcal{O}_X(-D))$$
$$\Rightarrow H_c^{p+q}(X \setminus D, \mathbb{C}),$$

which degenerates at E_1.

$$\omega_X(D) = \mathcal{H}om(\mathcal{O}_X(-D), \omega_X)$$

$$\mathcal{O}_X(-D) = \Omega_X^0(\log D) \otimes \mathcal{O}_X(-D)$$

Roughly speaking, by the above E_1-degeneration, we first prove a generalization of Kollár's injectivity theorem.

Then, we prove Theorem 6 (i) and (ii).

§ Analytic generalization

Theorem 8 (Main theorem)

(X, D): analytic simple normal crossing pair

$f: X \to Y$: projective

$\pi: Y \to Z$: projective

X, Y, Z : complex analytic spaces

$\Rightarrow$ (i) (Strict support condition)

every associated subvariety of $R^i f_* \omega_X(D)$ is the f-image of some stratum of (X, D) for $\forall i$

(ii) (Vanishing theorem)

$R^i \pi_* (R^j f_* \omega_X(D) \otimes L) = 0$ for $\forall i > 0$, $\forall j$

L : π-ample line bundle on Y

§ How to use Theorem 8

By Theorem 8, we can translate many results in the MMP for projective morphisms of complex analytic spaces.

Theorem 9 (Cone and contraction theorem)

X, Y : complex analytic spaces

$f: X \to Y$: projective

(X, Δ) : lc

W : compact subset of Y with some finiteness assumption

$\Longrightarrow$

$$\overline{NE}(X/Y ; W) = \overline{NE}(X/Y ; W)_{K_X+\Delta \geq 0} + \sum_j \mathbb{R}_{\geq 0}[C_j],$$

and so on.

§ Why MMP for analytic spaces?

Example 10 $P \in X$: complex analytic germ

$$\begin{tikzcd} Y \arrow[rr, dashed, "\mathrm{MMP}"] \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

projective resolution

$g : Z \to X$: partial resolution of singularities with some good properties.

We can use $g : Z \to X$ for the study of $(P \in X)$.

§ Idea of Proof of Theorem 9

We assume Y : smooth.

By using Saitô's MHM, we have:

$$\circledast \quad E_1^{-q,i+q} = \bigoplus_S R^i f_* \omega_{S/Y} \Rightarrow R^i f_* \omega_{X/Y}(D)$$

S runs over $(-q)$-dimensional strata of (X, D).

$\circledast$ degenerates at E_2 and its E_1-differential d_1 splits.

Thus, $E_2^{-q,i+q}$ is a direct summand of $E_1^{-q,i+q}$.

By this spectral sequence ⊛, we can reduce Theorem 7 to the simpler case where X : smooth, irreducible, and $D=0$.

In this case, Theorem 7 is a special case of Takegoshi's theorem.

<u>Remark 11</u> After I obtained the above proof, I found an approach without using Saito's MHM.

VMHS is sufficient.

This is a joint work with Fujisawa.