Birational Geometry of Algebraic Varieties

Open Problems

THE XXIIIRD INTERNATIONAL SYMPOSIUM

DIVISION OF MATHEMATICS

THE TANIGUCHI FOUNDATION

****

August 22 - August 27, 1988

Katata

FOREWORD

The 23rd International Symposium of the Division of Mathematics of the Taniguchi Foundation was held between August 22 and August 27 1988 in Katata, Japan. The main topic was the birational geometry of algebraic varieties; ten Japanese and nine foreign participants attended the symposium. The aim of the symposium was to bring together specialists in related areas to report on recent developments in their fields, and to exchange ideas.

Special emphasis was given to possible future developments of the theory of algebraic varieties. The collection of open problems presented here represents in part the outcome of these discussions. The organisers hope that this collection will be useful in pointing out future directions of research.

We are all grateful to the Taniguchi foundation for making this conference possible through generous financial support, and finally to the participants, whose effort made possible the success of the conference.

The organisers

List of participants

H. Clemens	University of Utah	(USA)
T. Ekedahl	University of Stockholm	(Sweden)
L. Illusie	Université Paris-Sud	(France)
V. A. Iskovskikh	Moscow University	(USSR)
J. Kollár*	University of Utah	(USA)
M. Reid	University of Warwick	(United Kingdom)
V. V. Shokurov	Yaroslavl' Pedagogical Institute	(USSR)
E. Viehweg	Universität Gesamthochschule Essen	(F. R. of Germany)
G. Xiao	East China Normal University	(P. R. of China)
T. Fujita	University of Tokyo	(Japan)
M. Ishida	Tohoku University	(Japan)
Y. Kawamata	University of Tokyo	(Japan)
M. Miyanishi	Osaka University	(Japan)
Y. Miyaoka*	Tokyo Metropolitan University	(Japan)
S. Mori*	Nagoya University	(Japan)
A. Moriwaki	Kyoto University	(Japan)
S. Mukai*	Nagoya University	(Japan)
N. Nakayama	University of Tokyo	(Japan)
S. Tsunoda	Osaka University	(Japan)

*) Member of the organizing committee.

Problems on 3-folds with trivial canonical bundle

by

Herbert Clemens

1. Some open problems about curves on hypersurfaces over $\mathbb{C}$.

Problem 1. The generic quintic $X \subset \mathbb{P}^4$ is the only case in any dimension in which a constant count predicts that X contains a finite number of rational curves of each degree. For given degree d, show that X contains only a finite number of rational curves of that degree. (True for low d.) The number of lines on X is $1 \cdot 5^3 \cdot 23$ and the number of conics is $2 \cdot 5^3 \cdot 2437$. Find the number of twisted cubics.

Problem 2. If C is a rational curve on a smooth $X \subset \mathbb{P}^4$ with $\deg X \geq 5$ and if C has infinitesimal deformations in X, that is,

$$H^0(N_{C/X}) \neq 0,$$

show that the infinitesimal Abel-Jacobi map

$$H^0(N_{C/X}) \otimes H^1(\Omega_X^2) \to H^1(\Omega_C^1),$$

induced by contracting normal vector fields against forms, is non-trivial. (This result would solve Problem 1.)

Problem 3. Decide whether, for given n and $X \subset \mathbb{P}^n$ a generic hypersurface of degree sufficiently large, every subcanonical curve on X is the intersection of a surface with X. ("Subcanonical" means that the curve's canonical bundle is induced from a line bundle on X. C. Voisin recently showed that the hypothesis of being subcanonical is necessary.)

Problem 4. Find a continuous family of lines on a <u>smooth</u>

quintic threefold which is not a cone over a quintic curve (or show that no such family exists).

Problem 5. Find a <u>smooth</u> $X \subset \mathbb{P}^4$, deg $X \geq 6$, with infinitely many rigid and smooth rational curves (or show that no such X exists).

2. Problems about 2-connected compact complex manifolds X with trivial canonical bundle.

A 2-connected compact complex manifold X is topologically the connected sum of g copies of $S^3 \times S^3$. Call g the *genus* of X. Deformations of Calabi-Eckmann manifolds were the only known examples before 1984. Those manifolds had genus one and further had the property that $h^1(\Theta_X) \neq 0$, so that they admitted non-trivial line bundles.

In 1984, R. Friedman constructed examples of X with genus > 1. Furthermore Friedman's examples have the property that $h^1(\Theta_X) = 0$ so that they admit no non-trivial line bundles. In particular, K_X is trivial. Also Friedman's examples occur in families which specialize to small contractions of projective threefolds with trivial canonical bundles. So the genera in Friedman's examples are bounded until K_X-trivial threefolds with arbitrarily high third Betti number are shown to exist.

Problem 6. Decide whether 2-connected X with K_X trivial can have arbitrarily high genus, arbitrarily low genus.

Problem 7. Construct a nodal degeneration for a family of X of genus one. A small resolution of the central fiber would give a complex structure on S^6.

Problem 8. Construct continuous (non-compact) families of curves on 2-connected X with K_X trivial. Construct a non-trivial rank 2 vector bundle on it.

3. Problems about $\mathfrak{D}$-modules and deformations of submanifolds

Let Y be a compact complex submanifold of a (possibly open) complex manifold X. Suppose Y is *locally subcanonical*, that is, there is a topological neighborhood X' of Y in X and a line bundle L on X' whose restriction to Y is K_Y. (All curves are locally subcanonical.)

Define

$\mathfrak{D}(L)$ = algebra of (right) differential endomorphisms of L.

One then defines the left $\mathfrak{D}(L)$-module

$$\Phi(Y,L) = (\mathfrak{D}(L)\otimes_{\mathcal{O}_X}\mathcal{O}_Y)\otimes_{\mathfrak{D}_Y}\mathcal{O}_Y .$$

It is a theorem in the theory of $\mathfrak{D}$-modules that

$$H^i(\Phi(Y,L))^* = \varprojlim H^{\dim Y-i}(L/\mathcal{I}_Y^n L).$$

We denote the right-hand group as $H^{q-i}(L_{\hat{X}})$. So, if Y contracts to a point under

$$f : X' \to X_0,$$

then $H^0(\Phi(Y,L))$ is finite dimensional.

Conversely, suppose this last group

$$N(Y;L) = H^0(\Phi(Y,L))$$

is finite-dimensional. Define $\mathfrak{M}$ to be the (left) $\mathcal{O}_X$-submodule in $\mathfrak{D}(L)\otimes_{\mathcal{O}_X}\mathcal{O}_Y$ which is the preimage of $\mathcal{O}_X\otimes_{\mathbb{C}}N(Y;L)$ under the natural map

$$\begin{array}{ccc}\mathfrak{D}(L)\otimes_{\mathcal{O}_X}\mathcal{O}_Y & \longrightarrow & \Phi(L) \\ \delta & \longmapsto & \delta\otimes 1\end{array} .$$

Let $L(\mathfrak{M})$ be the $\mathcal{O}_X$-submodule of L annihilated by $\mathfrak{M}$. Then, in this case,

$$\mathcal{A} = \{f \in \mathcal{O}_X \mid fL \subset L(\mathfrak{M})\}$$

is an ideal of definition for $Y \subset X$.

Problem 9. Is it true that for any coherent $\mathfrak{F}$

$$H^1(X; \mathfrak{F} \otimes \mathcal{A}^r/\mathcal{A}^{r+1}) = 0$$

for $r \gg 0$, if Y contracts? (It may not be too hard to find a counterexample.)

4. One hard miscellaneous problem

Problem 10. Find a smooth threefold X with trivial canonical bundle and first Betti number 0 which contains no rational curves.

Open problems and questions

by

Torsten Ekedahl

Problem 1. How does one compute the fundamental group of the complement of a curve in $\mathbb{P}_{\mathbb{C}}^{2}$?

The fundamental group of the complement of a plane curve has appeared in some recent (and not so recent) work. The abelianized fundamental group is easy to compute (and depends only on the number of components and the degree of the curve) and this calculation has been used by Hirzebruch to construct interesting surfaces as covers of the plane. On the other hand, the work of Fulton-Hansen and Deligne has shown that the fundamental group of a curve with only nodes is always abelian.

In the general case not much is known. Except for some special curves which occur as hyperplane sections of naturally occuring discriminant loci, etc., the only known method to "compute" the fundamental group is the one given by Zariski: Take a point not on the curve and consider the pencil of lines through that point. The fundamental group G of a general line in this pencil minus its intersection with the curve generates the fundamental group of the whole complement and the relations are obtained by following each geometric generator of G when the general line in the pencil is moved around the lines which do not intersect the curve transversally. This move gives the homotopy between the generator and another element of G and all the relations are obtained this way. The problem is that this procedure is not easy to perform in explicit cases and one would like to have either a way to follow Zariski's method or to find other methods. One test case for which no one seems to be able to compute the fundamental group is when the curve consists of a nodal curve and two flex tangents.

Problem 2. Is the fundamental group of a smooth and projective surface residually finite?

Any finitely generated group can be the fundamental group of a compact real 4-manifold. On the other hand, Thurston has proved that the fundamental group of a compact 3-manifold is always residually finite (i.e. the intersection of all subgroups of finite index is the identity element). The fundamental group of a smooth and proper surface is very special. Essentially nothing seems to be known about this problem.

Problem 3. Does there exist a minimal surface X of general type with $(K^2) = 1$, $\chi(\mathcal{O}_X) = 1$ and $P_2(X) = 3$?

Under the first two conditions the third is equivalent to $h^1(X,\omega_X^{-1}) = 1$. In [Ekedahl:Canonical models of surfaces ..., to appear in Publ. IHES] it was proved that this implies that the characteristic of the base field is two and that there is a double inseparable cover of X which is (birationally) a K3-surface or rational. This implies that $1 \leq (K_X^2) \leq 9$ and in [loc. cit.] examples with $h^1(X,\omega_X^{-1}) > 0$ were given for $2 \leq (K_X^2) \leq 9$. While probably not important in itself the question could be seen as a test case for whether one can make the same detailed analysis of surfaces of general type with small (K_X^2) as one has done in characteristic zero. In this analysis the linear system $|2K_X|$ should probably play an important role (cf. [loc.cit:II,1,10]).

Problem 4. Can an Enriques surface in characteristic 2 of geometric genus 0 have non-trivial vector fields?

As usual, a negative answer should have pleasant consequences for the deformation theory of Enriques surfaces. I can show (unpublished) that an Enriques surface of geometric genus 0 with vector fields can not have an elliptic pencil (and thus must have a quasi-elliptic one).

Problem 5. What is the correct moduli problem for Enriques surfaces in characteristic 2?

The first point is to get a moduli problem which is pro-representable and as always it is the possible nonzero vector fields which cause a problem. If we consider the case of nonzero geometric genus those with vector fields are called *supersingular* and there is only one up to a scalar. The first attempt is to rigidify $\underline{\mathrm{Pic}}^{\tau}$ which for a family of Enriques surfaces is flat of order 2. Namely consider a flat group scheme $\mathcal{G}/S$ of order 2 and for every $T \to S$ the set of isomorphism classes of data $(X \to T, \varphi)$ where $X \to T$ is a family of Enriques surfaces and φ is an isomorphism between $\mathcal{G}_T$ and $\underline{\mathrm{Pic}}^{\tau}(X/T)$. This problem is pro-representable at all Enriques surfaces except those supersingular Enriques surfaces for which the square of a nonzero vector field is 0. In order to include them one needs therefore some further rigidification.

Problem 6. Is there a surface in characteristic 0 which is not uniruled but almost all of whose reductions mod p are uniruled?

To begin with such a surface must have irregularity 0. Indeed, a uniruled surface with $B_1 \neq 0$ has its Albanese map image

1-dimensional and this lifts from almost all reductions to characteristic 0. The fibers are rational and this also lifts. Furthermore, the geometric genus is 0. To see this we may suppose that the surface is defined over a number field K. If we consider the action of the Galois group of $\bar{K}/K$ on $H^2(S_{\bar{K}},\mathbb{Q}_\ell(1))$ then, as for almost all reductions the Néron-Severi group fills out all of H^2; almost all Frobenius elements have finite order and so by Chebotarev density the image of the whole Galois group is finite. If we then apply the Hodge-Tate decomposition theorem we get that $H^{0,1}(S) = 0$.

This shows that $\chi(\mathcal{O}_S) = 1$ and as any finite étale cover of S fulfills the same conditions S is in fact algebraically simply connected.

Problem 7. Does there exist a restricted class of minimal surfaces and a function $f(p)$ such that $f(p) \to 1/3$ and such that any minimal surface of general type in charecteristic p either belongs to this class or fulfills $f(p) \leq c_2/c_1^2$?

This is inspired by the result of [Ekedahl:Foliations and inseparable morphisms, Proc. Symp. Pure Math. 46, 139-150] where it is proved that for a surface of general type either $-p/(p-1)^2 \leq c_2/c_1^2$ or it is uniruled. Examples of [Szpiro:Asterisque 64 169-202, 3.4] show that in the question the restricted class has to contain non-uniruled members.

Problems on decompositions of De Rham complexes

by

Luc Illusie (*)

Problem 1. Let A be an abelian category. An object K of $D^b(A)$ is said to be *decomposable* if K is isomorphic, in $D^b(A)$, to a complex with 0 differential [1:3.1]. Let k be a perfect field with characteristic $p > 0$ and let X be a smooth k-scheme. Let $F : X \to X'$ be the relative Frobenius, where X' is the pull-back of X by the Frobenius automorphism of k, and let $\Omega^{\cdot}_X$ be the De Rham complex of X/k. Let us say that X is DR-*decomposable* if $F_*\Omega^{\cdot}_X$ is decomposable (in $D(X',\mathcal{O}_{X'})$); it is the same to say that there exists an isomorphism in $D(X',\mathcal{O}_{X'})$

$$\oplus_i \Omega^i_{X'}[-i] \xrightarrow{\sim} F_*\Omega^{\cdot}_X,$$

inducing the Cartier isomorphism C^{-1} on $\underline{H}^i$.

If X is DR-decomposable (even if $\tau_{\leq 1}F_*\Omega^{\cdot}_X$ is decomposable), X can be lifted to $W_2(k)$, [1:3.6]. Conversely, if X can be lifted to $W_2(k)$, is X DR-decomposable?

The answer is yes if X is of dimension $\leq p$ [1:2.3]. And in general $\tau_{<p}F_*\Omega^{\cdot}_X$ is decomposable. The answer is still yes if, for any n, the canonical map $\overset{n}{\otimes}\,\Omega^1_{X'} \to \Omega^n_{X'}$ has a section, which is the case, in particular, if X is parallelizable. The answer is yes, too, if there exists a smooth lifting $\tilde{X}/W_2(k)$ so that $F : X \to X'$ can be lifted to $\tilde{X}$, which is the case, in particular, if X is affine [1:2.2(ii)]. Note also that, if X and Y are DR-decomposable, then the same is true for $X \times_k Y$ and for any Z

(*) Unité associée au CNRS n° 752

étale over X. The first unknown case is that of a quadric of dimension 3 in characteristic 2.

Problem 2. Let $S = \operatorname{Spec} A$, where A is a complete discrete valuation ring of mixed characteristic with perfect residue field k of characteristic p, and let $f : X \to S$ be flat, locally of finite type, with X regular and special fiber Y a reduced divisor with normal crossings in X. Let $j : U \to X$ be the open set where f is smooth, and consider $\omega^{\cdot}_{X/S} := j_* \Omega^{\cdot}_{U/S}$ and $\omega^{\cdot}_{Y} := \omega^{\cdot}_{X/S} \otimes_A k$. The complex $\omega^{\cdot}_{Y}$, introduced by Hyodo [2], satisfies a Cartier isomorphism

$$C^{-1} : \omega^{i}_{Y'} \xrightarrow{\sim} F_* \underline{H}^{i} \omega^{\cdot}_{Y}.$$

Assume that $\dim Y < p$. Is $F_* \omega^{\cdot}_{Y}$ decomposable?

Let e be the index of ramification of A over $W(k)$. The anser is yes if $e = 1$ and f is smooth [1:2.1]. The answer is unknown otherwise. Note that we can embed S into $T = \operatorname{Spec} W(k)[t]$ in such a way that the image of t becomes a uniformizing parameter. If there exists a subscheme $\underline{X}/T$ with semi-stable reduction over $t = 0$ and inducing X/S, the answer is still yes by [3:2.2].

Problem 3. Let k be a perfect field of characteristic $p > 0$, let X/k be a smooth and projective variety of pure dimension $d \le p$, liftable to $W_2(k)$. Let L be a line bundle on X.

3.1. Assume that L is nef and big. Is it true that $H^i(X, \Omega^d_X \otimes L) = 0$ for $i > 0$?

3.2. Same question, assuming that L is semi-ample and big.

A positive answer to 3.1 (which of course implies 3.2) would give a new proof of the Kawamata-Viehweg vanishing theorem (see [4]).

Note that under the assumptions of the problem X is DR-decomposable.

For d = 2, the answer to 3.1 is yes [1:2.8].

References

[1] Deligne, P. and Illusie, L.: Relèvements modulo p^2 et décomposition du complexe de de Rham, Invent. Math. 89 (1987) 249-270.

[2] Hyodo, O.: A note on p-adic étale cohomology on the semi-stable reduction case, Invent. Math. 91 (1988) 543-557.

[3] Illusie, L.: Réduction semistable ..., Preprint, Orsay, August 1988.

[4] Viehweg, E.: Vanishing theorems and positivity in algebraic fiber spaces, Proc. of ICM Berkeley 1986, 682-688.

Problems on Fano 3-folds

by

Vasilii Alekseevich Iskovskikh

Problem 1. Is $\mathrm{Bir}_k\mathbb{P}^2$ a simple group? (k : algebraically closed) [1,6,7].

Problem 2. Describe Aut $\mathbb{Z}[x,y]$ in terms of generators and relations.

Problem 3. Describe Aut $k[x,y,z]$ in terms of generators and relations. (k : algebraically closed) cf. [Nagata:Automorphism group of k[x,y], Lecture Note Series, Kyoto University, Kinokuniya].

Problem 4. Describe Bir V, where V is one of the following Fano 3-folds:

a) $V_8 \subset \mathbb{P}^6$: complete intersection of 3 quadrics;

b) $V_{10} \subset \mathbb{P}^7$;

c) V_3 : a smooth cubic hypersurface.

cf. [3,4,2].

Problem 5. Describe Bir $\mathbb{P}^3$ [4].

Problem 6. Prove that none of the Fano 3-folds of genus 6 is rational [4,2].

Problem 7. Study the unirationality question for the following Fano 3-folds:

a) $V \to \mathbb{P}^3$: a double cover with ramification along a sextic;

b) a smooth degree 4 hypersurface;

c) $V \to W_4 \subset \mathbb{P}^6$: a double cover of the cone over the Veronese surface, cf. [4].

Problem 8. Are Fano 3-folds stably rational?

Problem 9. Compute the intermediate Jacobian of all Fano 3-folds.

Problem 10. Study the moduli space of Fano 3-folds. Which ones are rational?

Problem 11. Let $G \subset GL(4,\mathbb{C})$ be a finite subgroup. Is $\mathbb{P}^3/G$ rational? I. Kolpakov-Miroshnichenko and Ju. Prokhorov proved this for solvable G (to appear) and in a few other cases.

Problem 12. Can a small smooth deformation of a nonrational 3-fold be rational?

Problem 13. Prove the rationality criterion for conic bundles cf. [5].

Problem 14. Find rationality criteria for fiber spaces of del Pezzo surfaces. The degree 4 case was studied by V. Alekseev: Rationality criteria for 3-folds with a pencil of degree 4 del Pezzo surfaces. Mat. Zam. 41 (1987) 724-729.

Problem 15. Find rationality criteria for conic bundles over a base of higher dimension.

Problem 16. Prove that the general cubic 4-fold is not rational. cf. [10].

Problem 17. Is the general cubic n-fold rational for $n \geq 5$?

Problem 18. Describe Bir(general quartic 4-fold) cf. [8].

Problem 19. Prove that $\mathrm{Bir}\, X = \mathrm{Aut}\, X$ for every smooth hypersurface in $\mathbb{P}^n$ of degree $n \geq 5$ cf. [8].

Problem 20. Prove the existence of a uniform upper bound for $(-K)^n$ for n-dimensional Fano varieties cf. [3].

References

[1] Gizatulin, M.H.: Defining relations for the Cremona group of the plane, Math. USSR Izv. 21 (1983) 211-268.

[2] Gushel', N.P.: Fano varieties of genus 6, Izv. A.N. SSSR 46 (1982) 1159-1174.
[3] Iskovskikh, V.A.: Lectures on 3-folds: Fano varieties. Publ. of Moscow Univ. 1988
[4] ———: Birational automorphisms of 3-dimensional algebraic varieties, J. of Soviet Math. 13 (1980) 815-868.
[5] ———: On the rationality problem for conic bundles, Duke Math. J. 54 (1987) 271-294.
[6] ———: Proof of a theorem about the generators of the Cremona group, Uspekhi Mat. Nauk. 46 (1985) 255-256.
[7] ———: A simple proof of Gizatulin's result on the generators of the Cremona group, Publ. of Steklov Inst. 17 (1988).
[8] Pukhlikov, V.V.: Birational isomorphisms of 4-dimensional quintics, Invent. Math. 87 (1987) 303-329.
[9] Sarkisov, V.G.: On the structure of conic bundles, Math. USSR Izv. 20 (1982) 355-390.
[10] Tregub, S.P.: Three constructions of rationality of a four-dimensional cubic, Moscow Univ. Math. Bull. 39 (1984) 26-28.

Problems connected with effective Nullstellensatz

by

János Kollár

Problem 1. (Algebraic Bézout problem.) Let M be a graded $K[x_0,\cdots,x_n]$-module. Let $M_i = \{m \in M \mid \dim \operatorname{Supp} m \le i\}$. Then

$$0 = M_{-1} \subset M_0 \subset \cdots \subset M_n = M.$$

Define $\text{graded deg } M = \sum_i \deg(M_{i+1}/M_i)$.

Now let $f_1, \ldots, f_k \in K[x_0, \ldots, x_n]$. It is clear from general nonsense that $\text{graded deg } K[x_0, \ldots, x_n]/(f_1, \ldots, f_k)$ can be bounded by some function of $\deg f_1, \ldots, \deg f_k$. What is this function? It is probably doubly exponential in n.

Problem 2. Notation as above. Find a good bound for the number of the associated primes of the ideal $(f_1, \ldots, f_k)$.

Let $H_1, \ldots, H_n$ be divisors on a projective variety X of dimension n. Let $V_1, \ldots, V_k$ be the scheme-theoretic connected components of $H_1 \cap \ldots \cap H_n$. There is a natural way (cf. [F:9.1]) to assign numbers $eq(V_i, \underline{H})$ to the components V_i such that

$$\sum_i eq(V_i, \underline{H}) = H_1 \cdot \ldots \cdot H_n.$$

The number $eq(V_i, \underline{H})$ is called the equivalence of V_i. In [F:9.1.1], there is a very beautiful formula for $eq(V_i, \underline{H})$ but it is very hard to use in computations. Here are some problems.

Problem 3. Assume that $V' = V$ + one embedded closed point. Is it true that $eq(V', \underline{H}) = eq(V, \underline{H}) + 1$?

Problem 4. Let $V \subset \mathbb{P}^n$ be a reduced surface. Is it true that

$$eq(\mathbb{P}^2, \underline{H}) \le eq(V, \underline{H})?$$

Problem 5. Assume that $(X, H_1, \ldots, H_n, V_1)$ varies in a flat family. Is it true that $eq(V_1, \underline{H})$ is locally constant?

Problem 6. The individual terms of the formula [F:9.1.1] are not deformation invariants. If the answer to 5 is yes, there should be another formula that clearly shows deformation invariance. This happens if X is Gorenstein and $\dim V_1 \leq 1$.

References

[F] W. Fulton, Intersection Theory, Springer 1986.

[K] J. Kollár, Sharp effective Nullstellensatz, J. of AMS, to appear.

Some problems about 3-dimensional extremal nbds and flips

by

János Kollár and Shigefumi Mori

The following are some problems related to our joint work (to appear sooner or later).

To fix notation, let X be a 3-fold with only terminal singularities, let $C \subset X$ be an irreducible rational curve s.t. $(C \cdot K_X) < 0$ and such that there is a contraction $f : X \to Y$ which maps C to a point and is an isomorphism outside C.

Problem 1. By [M2:6.2], X can have at most 3 singular points on C. It seems likely that in fact one can have at most 2 singular points. It would be nice to have a direct proof.

Problem 2. From our classification it follows that the exceptional index 2 points [M1:12.3,23.1,25] can not occur on C. Is there a direct proof for this? Is there a deeper reason behind this?

Problem 3. Are there examples of extremal nbds with a point of type IC [M2:Appendix A] for any odd index ≥ 5? We have examples with index 5 or 7, and it is very likely that examples exist for any odd index ≥ 5. A nice construction could help a lot to understand this exceptional case. Similarly, the existence of some other exceptional cases is not fully known (2 singular points with indices 2 and $m > 2$, nonsemistable).

Problem 4. Let $(P,H) \subset (P,Y)$ be a general hyperplane section. We prove that in almost all cases, (P,H) is a quotient singularity. Possibly the only exception occurs when X has a single singular point, which is of index 3 and is the quotient of a compound D_n-type point. This would be a very interesting result. In particular, this would imply that (P,H) is always rational. Maybe there is a direct proof of this.

Problem 5. Pushing along the lines of 4, one can ask how much of [KSB:Chapter 3] generalizes to arbitrary rational singularities. To be more specific, let (P,Z') be a rational surface singularity and let (P,Z) be the 3-dimensional total space of a 1-parameter smoothing of (P,Z').

(i) By [KSB:Chapter 3], if (P,Z') is a quotient singularity, then the canonical $\mathcal{O}_Z$-algebra

$$\Sigma_{m=0}^{\infty} H^0(Z,\mathcal{O}_Z(mK_Z))$$

is finitely generated. This result also holds for some other rational singularities. It would be interesting to see more examples or counterexamples.

(ii) If the canonical algebra is finitely generated, then one can study the deformation space of (P,Z') as in [KSB:Chapter 3]. For this one has to find all rational singularities with Q-Gorenstein smoothings and understand certain partial resolutions of (P,Z') as in [KSB:Chapter 3].

Even simple individual examples are not easy to work out.

Problem 6. Going back to the original set-up, Reid's question about $|-K_X|$ having a member with only Du Val singularities is still open. This is a very nice problem. A direct proof might be very interesting.

Problem 7. If one wants to flip in the category of projective varieties then one has to understand the simultaneous contraction of several curves. Very little is known about the possible configurations of curves and singularities.

Problem 8. The existence of 3-dimensional flips with canonical singularities is proved. However nothing is known about the local structure.

Problem 9. Do 3-dimensional flips with canonical singularities vary continuously? This should be easier than 4-dimensional flips and this would imply the deformation invariance of plurigenera for 3-folds with $\mathbb{Q}$-factorial canonical singularities. We should mention that $\mathbb{Q}$-factoriality has nothing to do with plurigenera but it is needed for the proof we have.

Problem 10. While 3-dimensional flops are much easier than flips, it is still unknown whether or not 3-dimensional flops with canonical singularities vary continuously.

Problem 11. The method of [M2] can be applied in two more cases. The first is the case when a 3-fold extremal contraction contracts a surface to a curve. All the results of [M2], except (6.2.(i)), apply in this case. One should be able to work out the local structure of such contractions using the method of [M2].

The other case is when $f : X \to Z$ is an extremal contraction s.t. $\dim Z = 2$. In this case, it is easy to see that Z has only quotient singularities. If, locally, $\tilde{Z}$ is the smooth cover and if $\tilde{X}$ is the normalization of $X \times_Z \tilde{Z}$, then $\tilde{X}$ has only terminal singularities and the natural map $\tilde{f} : \tilde{X} \to \tilde{Z}$ is an extremal contraction which is flat. It should be possible to work out the local structure of $\tilde{f}$ and thereby the local structure of f. There should be only a few cases besides those listed in [M2].

The following problems are only vaguely related to flips.

Problem 12. It would be nice to have more information about 3-dimensional canonical singularities. By Shepherd-Barron (unpublished), their algebraic fundamental group is finite. Is the fundamental group itself finite? It might be possible to say something more about the simply connected ones.

Problem 13. Is a small deformation of a (3-dimensional) canonical singularity canonical? This is the most basic open problem.

Problem 14. Is there a 3-dimensional analogue of Artin's theory of fundamental cycles? From this point of view, what is the correct 3-dimensional analogue of rational surface singularities? The class of rational 3-fold singularities might be too large.

Problem 15. Now that we have a unique canonical modification of a 3-dimensional singularity, it might be possible to start a systematic study of 3-dimensional singularities. So far very little has been done in this direction.

Problem 16. Let Y be a 3-dimensional variety. Consider the set of all $f : X \to Y$, where f is birational, X has canonical singularities and K_X is f-nef. Is this set finite? By [KM], there are only finitely many such where f is also projective.

Problem 17. Finally a fairly obvious question. What about characteristic p? It is interesting that the cone theory for smooth varieties is easier in characteristic p.

References

[KM] Y. Kawamata and K. Matsuki, The number of the minimal models for a 3-fold of general type is finite, Math. Ann. 276 (1987) 595-598.

[KSB] J. Kollár and N. Shepherd-Barron, Threefolds and deformations of

surface singularities, Invent. Math. 91 (1988) 299-338.
[M1] S. Mori, On 3-dimensional terminal singularities, Nagoya Math. J., 98 (1985) 43-66.
[M2] ———, Flip theorem and the existence of minimal models for 3-folds, J. of the A.M.S. 1 (1988) 117-253.

Flippant problems

by

Miles Reid

Problem 1. (Linear bilinear problem.) Let $f : U \times V \to W$ be a bilinear map and $F : U \otimes V \to W$ the linear map it induces. The problem is to bound rank F from below in terms of the geometry of f.

Examples. (i) If $u, v \neq 0 \Rightarrow f(u,v) \neq 0$ then

$$\operatorname{rank} F \geq \dim U + \dim V - 1.$$

(ii) If $f : V \times V \to W$ is symmetric and $f(v_1,v_2) \neq f(v_3,v_3)$ for every linearly independent v_1, v_2, v_3 then $\operatorname{rank} F \geq 3 \cdot \dim V - 3$.

Proofs. By assumption, F does not kill any primitive tensor $u \otimes v$ (resp. any rank 3 tensor $v_1 \otimes v_2 + v_3 \otimes v_3$) so

$\mathbb{P}(\operatorname{Ker} F) \cap \mathbb{P}(U) \times \mathbb{P}(V) = \emptyset \Rightarrow \operatorname{codim} \operatorname{Ker} F > \dim U + \dim V - 2$;

$\mathbb{P}(\operatorname{Ker} F) \cap \mathbb{P}(\text{rank } 3) = \emptyset \Rightarrow \operatorname{codim} \operatorname{Ker} F > 3 \cdot \dim V - 4$.

This argument applies several hundred times in the literature. For example, it plays a key role in: the free pencil trick and Clifford's theorem; Castelnuovo's theorem $\chi(\mathcal{O}_S) < 0 \Rightarrow S$ ruled; Enriques' famous argument that a linear system on an Enriques surface with positive self-intersection must contain reducible divisors. More recently, applied to cup product maps on Hodge-theoretic spaces, Green and Voisin have used it to show that surfaces of degree d in $\mathbb{P}^3$ containing nontrivial curves have codimension $\geq d-3$ in all hypersurfaces (and for $d \gg 0$, codim = $d-3$ gives only lines, codim = $2d-5$ only conics).

In the last example, cases of smallest rank have an interpretation in terms of homogeneous spaces; a similar example is Castelnuovo's hundred year old theorem that $\geq 2n+3$ linearly general points of $\mathbb{P}^n$ imposing $\leq 2n+1$ conditions on quadrics must lie on a normal rational curve. See my preprint [Quadrics through a canonical surface] for a detailed attempt on a special case of this problem.

This question is above all practical, but there may also be an answer in terms of algebraic groups.

Problem 2. What is a general elephant? The problem is to find a common framework for the following elephantine manifestations:

a) If Y is a (weak) Fano 3-fold, the general surface $X \in |-K_Y|$ is a K3 - surface; if $P \in Y$ is a terminal 3-fold singularity then the general anticanonical surface is a Du Val singularity; the same holds for one class of flip singularities.

b) If $P \in Y$ is a flip singularity not covered in (a) then Mori proved that the double cover $g : X \to Y$ branched in a general anti-bicanonical surface has Gorenstein canonical singularities; if $f^- : Y^- \to Y$ is the flipping contraction then there is a commutative diagram

$$\begin{array}{ccc} X^- & \to & Y^- \\ h\downarrow & & \downarrow f^- \\ X & \to & Y \end{array},$$

where X^- is also a Gorenstein 3-fold with canonical singularities and h is a small partial resolution.

c) If S is a variety and $a \in |-4K|$, $b \in |-6K|$ sufficiently good sections, then a Weierstrass fibration $X \to S$ can be defined by interpreting $y^2 = x^3 + ax + b$; then $K_X = \mathcal{O}_X$ and X will have at worst canonical singularities.

In each case, we pass from a variety or singularity with K

negative, or K non-Cartier to a variety X with $K_X = 0$, and canonical singularities. The dimension can change, in (a) by -1, in (b) by 0 and in (c) by +1. Can this be generalised? Does there exist in general a reduction from a variety with K negative to X with K_X trivial and canonical singularities, possibly of larger dimension? For example, what data does one need on S to get a K3 - fibre space $X \to S$ with $K_X = 0$?

Problem 3. (Flip singularities as formal extensions of Du Val singularities.) The problem is to use the infinitesimal view to study pairs $S \subset X$, where $P \in S$ is a Du Val singularity contained as an anticanonical divisor of a 3-fold flip singularity.

Example. Let $C \subset \mathbb{P}^2$ be a nonsingular conic and $S = \operatorname{Spec} k[x,y,z]/(xz-y^2)$ the affine cone over C; as normal sheaf take the unique nontrivial divisorial sheaf (corresponding to a generator of the cone):

$$L \simeq \mathcal{O}_S u \oplus \mathcal{O}_S v/(uy-vx,\ uz-vy).$$

Take the trivial first order structure $\mathcal{O}_S \oplus L$, $L^2 = 0$, giving

$$S^{(1)} = \operatorname{Spec} k[x,y,z,u,v]/I^{(1)},$$

where

$$I^{(1)} : \operatorname{rk} \begin{bmatrix} x & y & u \\ y & z & v \end{bmatrix} \le 1,\ u^2 = uv = v^2 = 0.$$

There are just two ways of extending this to higher order:

a) Introduce a new variable w, and set

$$I_1 : \operatorname{rk} \begin{bmatrix} x & y & u \\ y & z & v \\ u & v & w \end{bmatrix} \le 1;$$

this is the cone over the Veronese surface, with $C^{(1)}$ as hyperplane section.

b) No new variables, just set

$$I_2 : \mathrm{rk}\begin{bmatrix} x & y & u \\ y & z & v \end{bmatrix} \le 1.$$

This is Francia's example, the cone over the cubic scroll $\mathbb{F}_{1,2} \subset \mathbb{P}^4$ with C as a $+1$ section.

Problem 4. (Flip singularities as toric hypersurfaces.) I believe that the 3-fold flip singularities can be classified in much the same way as terminal singularities, and that the eventual outcome will be a list of just a few series. Problem 3 was one attempt at describing the singularities "from the inside". The question here is to classify the following type of situations:

$$\begin{array}{ccc} A^- & \dashrightarrow & A^+ \\ & \varphi^- \searrow \quad \swarrow \varphi^+ & \\ & P \in X \subset A & \end{array}.$$

Here $P \in A$ is a 4-fold toric ambient space with local class group $\mathrm{CL}\, A \simeq \mathbb{Z}$, having two "$\mathbb{Q}$-factorisations" corresponding to the two ample cones in $\mathbb{Z}$ (these need not be isomorphisms outside P). Then take $X \subset A$ such that its proper transforms in A^+ and A^- give rise to a directed flip

$$\begin{array}{ccc} X^- & & X^+ \\ & \varphi^- \searrow \quad \swarrow \varphi^+ & \\ & P \in X & \end{array}.$$

For example the first series consists of the direct product $A = \mathbb{A}^1 \times A_0$ where A_0 is the toric flip studied by Danilov, and $X \subset A$ a Cartier divisor.

Problem 5. (Grandfather of flops.) The following example is contained in my pagoda paper: consider the 5-fold hypersurface

$$P \in W : (x^2 + uy^2 + vz^2 + uvt^2 = 0) \subset \mathbb{A}^6.$$

This is singular along the (u,v)-plane $\mathbb{A}^2$, and has two small resolutions

$$\begin{array}{ccc} W_1 & & W_2 \\ & \searrow \quad \swarrow & \\ & P \in W & \end{array} ,$$

where the exceptional locuses are $\mathbb{P}^1$-bundles over $\mathbb{A}^2$. The resolutions exist essentially because the discriminant of the quadric form in 4 variables

$$x^2 + uy^2 + vz^2 + vwt^2 \in k(u,v)[x,y,z,t]$$

is a square in $k(u,v)$.

Now by taking suitable sections of W by nonsingular codimension 2 subvarieties of $\mathbb{A}^6$ we obtain a whole range of 3-fold flops

$$\begin{array}{ccc} X_1 & \dashrightarrow & Y \\ & \searrow \quad \swarrow & \\ & P \in X & \end{array} ,$$

where $X_1 \to X$ contracts a $(1,-3)$-curve. This includes many examples where $P \in X$ is not a cD_4 point.

Conjecture: All 3-fold flops arise in this way.

Notes: a) This conjecture is about 7 years old, and my reason for resurrecting it is the current interest in the question of $(+1,-3)$-curves on a smooth 3-fold. If the conjecture is true it gives implicitly a list of all that can be contracted and flopped.

b) Singularity theorists have told me on one or two occasions that the conjecture is false; but I'm not certain that we were talking about the same question.

Problems about Fano varieties

by

Vyacheslav Vladimirovich Shokurov

Problem 1. Classify Fano 3-folds with (log-)terminal singularities.

a) First try to prove the existence of a good divisor in $|-K|$ or in $|-2K|$. Maybe this can be done by first using Reid's plurigenus formula to check that $|-2K|$ is not empty. Then use the existence of good member locally and try to globalize.

b) Maybe there are a few exceptions to (a). Try to classify them.

c) If there is a good divisor as above then try to bound the number of non-Gorenstein points.

d) Is the so-called Fano bound $(-K^3) \le 72$ true?

Problem 2. (Manin) Describe all pairs (S,D), where S is a possibly singular surface and D an effective reduced divisor such that there exists a finite covering $\tilde{S} \to S$ which is unramified outside D and the Kodaira dimension of $\tilde{S}$ is ≥ 0. These might be exactly those surfaces for which $S-D$ has only finitely many integral points over any given number field.

Definition. (i) The Fano index of an n-dimensional Fano variety V with log-terminal singularities is the smallest positive f such that $-K = fL$, where L is an ample Cartier divisor. It is known that $f \le n+1$ and equality holds for projective n-space only. In all other cases $f \le n$ and equality holds only for hyperquadrics.

(ii) The set

F_n = {f|f is a Fano index of an n-dimensional Fano variety}

is called the *Fano spectrum*.

(iii) We say that a subset F of $\mathbb{R}$ is semi-discontinuous from above if for every x in $\mathbb{R}$ there is an $\varepsilon > 0$ such that $[x-\varepsilon, x) \cap F = \phi$.

Problem 3. a) Is the Fano spectrum semi-discontinuous from above? This is true for $n=2$ [Alekseev, Izv. AN USSR, to appear].

b) Is it true that $F_n + m-n \subseteq F_m \cap [m+1, m-n)$ for any $m \geq n$? Maybe we even have equality.

c) Find $F_n \cap [n-1, n]$. Describe the corresponding Fano varieties.

d) The same as above for $[n-2, n-1]$.

e) Is the rank of the Néron-Severi group bounded by some function of the index on n-dimensional Fano varieties? For $n = 2$, this was proved by Nikulin.

Problem 4. A Fano variety is called *primitive* if it is not birational to a non-trivial Fano fiber space. For example by results of Iskovskih and Manin three dimensional quartics are primitive.

a) If V is a primitive Fano variety, is it true that it has only finitely many smooth (or even log-terminal) models which are also Fano?

b) Are there only finitely many deformation types of primitive Fano varieties of any given dimension?

c) Find more primitive Fano 3-folds.

Problem 5. Let V be an n-dimensional variety and assume that K_V is $\mathbb{Q}$-Cartier. Let $f : W \to V$ be a good resolution and

$K_W = f^*K_V + \sum a_iE_i$. For any point $P \in V$ we define the minimal discrepancy $md(P) = \min\{a_i | f(E_i)=P\}$. By definition V has log-terminal singularities iff $md(P) > -1$ for every P.

a) Prove that $md(P) \leq \dim V - 1$ and that equality holds iff P is smooth.

b) Prove that if P is singular then $md(P) \leq \dim V - 2$ and equality holds only for Gorenstein points.

c) Let

$$I_m = \{md(P) | P \text{ is a (log-)terminal singularity of dimension } m\}$$

be the *(log-)terminal spectrum*. One can ask the same question about the (log-)terminal spectrum that we asked about the Fano spectrum.

Problem 6. Let X be a normal projective variety, and let D_i be Weil divisors on X. Let

$$Q = \{(d_1,\ldots,d_n) | K_X+\sum d_iD_i \text{ is log-terminal and has maximal Kodaira dimension}\}.$$

We call two n-tuples in Q equivalent if they have isomorphic canonical models. Prove that the equivalence classes give a locally polyhedral and rational decomposition of Q.

Open problems on fiber spaces and moduli

by

Eckart Viehweg

Both papers, [K] and [V], are approaches to show that moduli spaces for certain classes of canonically polarized varieties are quasi-projective. Whereas J. Kollár's method works only for compact moduli spaces, [V] works only for moduli problems (over $\mathbb{C}$) where the reduced Hilbert scheme is non-singular.

Question 1. *Are there any criteria implying that the reduced Hilbert scheme for a class of polarized varieties is non-singular?*

In fact, as A. Todorov told me, he was able to prove that Question 1 has an affirmative answer for some manifolds with trivial canonical sheaf. However [V] works only for canonically polarized manifolds! Hence:

Problem 2. *Can one generalize* [V] *to manifolds with arbitrary polarizations?*

It would be much nicer to get rid of the assumption on compactness in [K] or on smoothness in [V]. The reason I failed even for canonically polarized manifolds is that I was unable to give an affirmative answer to (see [V] or the excellent survey [M] for the notations):

Problem 3. *Let* Y *be a quasi-projective variety,* $Y_0 \subset Y$ *an open subvariety and* $f_0 : X_0 \to Y_0$ *a smooth morphism. If* ω_{X_0/Y_0} *is relatively ample for* f*, can one find for some* $\nu > 1$ *a coherent extension* $\mathcal{F}$ *of* $f_{0*}\omega^{\nu}_{X_0/Y_0}$ *to* Y *such that* $\mathcal{F}$ *is weakly positive over* Y_0 *?*

Of course, the sheaf $\mathcal{F}$ should come from the direct image of the ν-th power of the dualizing sheaf of some projective morphism $f : X \to Y$ extending f_0. In some sense an affirmative answer to Problem 3 would follow from the existence of a "good" extension f of f_0. However, I do not even know what "good" is supposed to mean.

Let $\tau_0 : Y_0' \to Y_0$ be a desingularization, $Y_0' \to Y'$ a smooth compactification such that $Y' - Y_0'$ is a normal crossing divisor. Let $f_0' : X_0' \to Y_0'$ be the pullback of the morphism considered in Problem 3. Let us assume that, for $k = \dim X_0 - \dim Y_0$, the monodromy of $R^k f_{0*}' \mathbb{C}_{X_0'}$ around the components of $Y' - Y_0'$ is unipotent. Then, using W. Schmid's nilpotent orbit theorem, one has a natural locally free sheaf $\mathcal{H}'$ on Y' extending $\mathcal{O}_{Y_0'} \otimes_{\mathbb{C}} R^k f_{0*}' \mathbb{C}_{X_0'}$ and $f_{0*}' \omega_{X_0'/Y_0'}$ extends to a subbundle $\mathcal{F}'$ of $\mathcal{H}'$. Using Y. Kawamata's positivity theorem and the usual techniques (§3 of [V], for example) an affirmative answer to Problem 3 would follow from an affirmative answer to:

Problem 4. *Can one construct a compactification* Y *of* Y_0 *and a locally free sheaf* $\mathcal{F}$ *(or* $\mathcal{H}$*) on* Y, *such that* $\mathcal{F}' = \tau^* \mathcal{F}$ *(or* $\mathcal{H}' = \tau^* \mathcal{H}$*) for some morphisms* $\tau : Y' \to Y$ *extending (after blowing up)* τ_0.

Finally, it would be nice to weaken the hypothesis I made in [V] on the singularities allowed for the varieties considered for the moduli problem. Also one should be able to allow certain reducible varieties (as in [K] where stable curves are considered).

References

[K] Kollár, J.: Projectivity of complete moduli, to appear.

[M] Mori, S.: Classification of higher-dimensional varieties, Proc. Symp. in Pure Math. 46 (1987) 269-331.

[V] Viehweg, E.: Weak positivity and the stability of certain Hilbert points, to appear in Invent. Math.

Problem list

by

Gang Xiao

All surfaces will be smooth projective, defined over the complex number field.

Problem 1. Characterize singular fibers of hyperelliptic fibrations in terms of the singularities of the ramification divisor, then use it to derive global invariants of the surface, just as Horikawa did for the case of genus 2 fibrations.

Reference: [Ho]

Problem 2. Find the upper limit of slope for a hyperelliptic fibration of genus g, as a function in g.

A *hyperelliptic fibration of genus* g is a fibration $f : S \to C$ where the general fiber is a hyperelliptic curve of genus g. Assume that the fibration is not locally trivial. Then the *slope* of the fibration is the ratio

$$(K_S^2-8(g-1)(g(C)-1))/(\chi(\mathcal{O}_S)-(g-1)(g(C)-1)),$$

or equivalently the unique number λ such that

$$K_S^2 = \lambda\cdot\chi(\mathcal{O}_S)+(8-\lambda)(g-1)(g(C)-1).$$

One should use the result of the preceding problem, then give bounds for the number of various possible singularities.

Problem 3. Find the largest lower bound for the slopes of surface fibrations whose general fibers are "generic" curves of genus g, as a function in g.

It seems that this bound converges towards 8 when g goes to infinity.

Reference: [X2]

Problem 4. Let $f : S \to C$ be a surface fibration of genus $g \geq 2$, not locally trivial, such that the irregularity q of S is greater than the genus b of C. It is known that the slope λ of f is at least 4 [X2, Corollary 1 to Theorem 1′]. However if $\lambda = 4$, then $q = b+1$ [X2, Theorem 3]. Presumably, the greater $q-b$ is, the bigger the lower bound of λ should be. Find a good relationship between λ, $g-b$ and g.

Conjecture 5: For almost all surfaces of general type, there are no cohomologically trivial automorphisms.

We have only to consider canonical automorphism, of order 2 or 3 [B].

Case of order 2: Try to show that there must be (-2)-curves fixed by the automorphism, then study the local behavior of $H^{1,1}$ around (-2)-curves.

Case of order 3: we conjecture that there is no infinite family of surfaces such that the canonical map is a cyclic triple cover of the image.

Reference [P1], [P2]

Problem 6. Classification of surfaces of general type whose canonical map is associated to a fibration: according to [B], the genus of the fibration is 2, 3, 4 or 5.

Case of genus 3: Find the largest coefficient c such that

$$K^2 \geq c \cdot p_g + \text{const}.$$

This c should be between $4\frac{4}{5}$ and 6 (see [X4] for 6, and an unpublished easy estimate for the other bound).

Case of genus 4: Nothing is known yet except that

$$K^2 \geq 6\frac{6}{7}p_g + \text{const.}$$

Case of genus 5: We conjecture that there is no infinite family of surfaces whose canonical map is associated to a genus 5 fibration.

It is enough to prove that the fixed part cannot be 8 times a section. The case where the fixed part is 7 times a section plus another section is already OK, due to the Miyaoka Inequality applied to the open situation which is the surface minus the section.

Problem 7. Let $f : S \to C$ be a surface fibration, F a singular fiber of f. There is a base extension $\pi : \tilde{C} \to C$ such that the minimal resolution of the pull-back of F is semi-stable. Let $\tilde{f} : \tilde{S} \to \tilde{C}$ be the pull-back fibration. The slope of $\tilde{f}$ is determined by that of f and the behavior of F. Determine the effect of F on the slope of $\tilde{f}$, and give upper and lower bounds of that effect.

Problem 8. Find a good way to classify the singular fibers of general surface fibrations.

One possible way to go through problem 7.

Problem 9. Give necessary and sufficient combinatorial conditions in terms of the irreducible components, for a curve to be realizable as a singular fiber in a surface fibration where the genus of the general fiber is ≥ 2.

Reference: [X5], [NU]

Problem 10. Find surfaces to fill in the unknown region of surface geography.

This region is $2.83c_2 \leq c_1^2 \leq 3c_2$.

Basic idea: Beginning with a fibered surface with $c_1^2 = 3c_2$, it is possible to make a series of new surfaces by making base extensions and twisting general and special fibers. If different operations give coprime Chern number changes, fill-in is possible. But experiments on some known surfaces did not give this coprimeness property.

References: [Ch], [H], [S].

Problem 11. Let $f: S \to C$ be surface fibration, g the genus of a general fiber, $b = g(C)$, q the irregularity of S.

Conjecture: $q \le \frac{g+1}{2} + b$ if f is not trivial.

Solved only for the case $b = 0$ in [X4].

Severi's conjecture 12: If $K^2 < 4\chi$, then the Albanese map is not generically finite.

A joint attempt of Igor Reider and myself to attack this problem from a local point of view around a canonical divisor has failed. Another possible approach is by proving the analogue of [X3, Theorem 1] for general surfaces.

Reference: [C1], [R]

Problem 13. Find an upper bound for the number of irreducible components of the moduli space of surfaces of general type with given Chern numbers.

Reference: [C1]

Problem 14. Find an effective bound for the number of minimal surfaces of general type which can be dominated by a given surface [DM].

Conjecture 15: The upper bound of the order of the automorphism group of a surface of general type increases proportionally with the Chern numbers of the surface.

Solved only for abelian case in [X6].

Problem 16. Let d_n be the dimension of the smallest rational variety containing the birational image of the n-canonical map of a surface of general type S. This d_n should increase towards infinity when n and the Chern invariants of S do the same. Give an estimate of d_n.

This could be useful in Problems 13 and 15.

Problem 17. Classification of surfaces with $p_g = q = 1$: Give an upper bound for the genus of the Albanese fibration as a function in K^2.

For example if $K^2 = 2$, the genus must be 2 [X2, Theorem 2].

Problem 18. Is there an algebraic surface which is algebraically simply connected but not topologically so?

This question is formulated because no such example is known.

Problem 19. Is there a surface fibration $f : S \to C$ such that $f_*\omega_{S/C}$ is not positive and $q(S') = g(C')$ for every base change $f' : S' \to C'$ of f?

References

[BPV] Barth/Peters/Van de Ven: *Compact Complex Surfaces*, Springer 1984.

[B] Beauville, A.: L'application canonique pour les surfaces de type general, Invent. Math. 55 (1979) 121-140.

[C1] Catanese, F.: Moduli of surfaces of general type, L.N.M. 997 (1983), 90-112.

[C2] ——, Canonical rings and "special" surfaces of general type, in *Algebraic Geometry I* (Proceedings of 19985 AMS Summer Institute at Bowdoin), Amer. Math. Soc. 1987, 175-194.

[Ch] Chen, Z.: On the geography of surfaces - simply connected minimal surfaces with positive index, Math. Ann. 277 (1987), 141-164.

[DM] Deschamps, M./Menegaux, R.: Surfaces de type général dominées par une varieté fixe, Bull. Soc. Math. France 110/2 (1982), 127-146.

[H] Hirzebruch, F.: Algebraic surfaces with extreme Chern numbers, preprint MPI (1984) No. 43

[Ho] Horikawa, E.: On algebraic surfaces with pencils of curves of genus 2, in *Complex Analysis and Algebraic Geometry*, Cambridge (1977) 79-90.

[M] Miranda, R.: Triple covers in algebraic geometry, Am. J. of Math.

107/5 (1985) 1123-1158.
[MT] Moishezon, B./Teicher,M.: Simply-connected algebraic surfaces of positive index, Invent. Math. 89 (1987) 601-643.
[NU] Namikawa, Y./Ueno, K.: The complete classification of fibers in pencils of curves of genus two, Manuscripta Math. 9 (1973) 143-186.
[Ps] Persson, U.: An introduction to the geography of surfaces of general type, in *Algebraic Geometry I* (Procedings of the 1985 AMS Summer Institute at Bowdoin), AMS (1987) 195-218.
[P1] Peters, C.A.M: Holomorphic automorphisms of compact Kähler surfaces and induced actions in cohomology, Invent. Math. 52 (1979) 143-148.
[P2] ——: On automorphisms of compact Kähler surfaces, in *Algebraic Geometry* (Anger 1979, Ed. A. Beauville), (1980) 249-267.
[R] Reid, M.: Quadrics through a canonical surface, manuscript.
[S] Sommese, A.: On the density of ratios of Chern numbers of algebraic surfaces, Math. Ann. 268 (1984) 207-221.
[X1] Xiao, G.: Surfaces Fibrées en Courbes de genre Deux, L.N.M. 1137 (1985)
[X2] ——: Fibered algebraic surfaces with low slope, Math. Ann. 276 (1987) 449-466.
[X3] ——: Hyperelliptic surfaces of general type with $K^2 < 4\chi$, Manuscripta Math. 57 (1987) 125-148.
[X4] ——: Irregularity of surfaces with a linear pencil, Duke Math. J. 55/3 (1987) 597-602.
[X5] ——: On the stable reduction of pencils of curves, preprint 1988.
[X6] ——: Abelian automorphism groups of surfaces of general type, manuscript 1988.

Problems

by

Takao Fujita

1. Prove or disprove the following.

Conjecture. Let L be an ample line bundle on a smooth projective variety M whose canonical bundle is K. Then $K+tL$ is spanned for any integer t with $t \geq 1+\dim M$. Moreover $K+tL$ is very ample if $t > 1+\dim M$.

Comments. (1.1) This is obvious when $n = \dim M = 1$.

(1.2) There are several examples showing that the bounds $t \geq 1+n$, $t > 1+n$ are sharp. Also $t \geq n$ and (or $t > n$) might be enough if we assume that $L^n > 1$.

(1.3) One can prove that $K+(n+1)L$ is nef. Moreover if $\mathrm{char}(k) = 0$, then $m(K+(n+1)L)$ is spanned for some $m > 0$.

(1.4) When M is of general type and K is ample, the conjecture predicts, as a special case in which $L = K$, that tK is spanned for $t \geq 2+n$ and is very ample for $t \geq 3+n$. In case $n = 2$, this is contained in Bombieri's result.

Thus, it would be reasonable to ask the following

Question. Let M be a smooth variety of general type such that K is nef. Then, is tK spanned for $t \geq 2+n$? Is

$$H^0(M,tK)\otimes H^0(M,sK) \to H^0(M,(t+s)K)$$

surjective for any $t \geq 2+n$ and $s \geq 3+n$?

(1.5) When $n = \dim M = 2$ and $\mathrm{char}(k) = 0$, the conjecture follows from Reider's theory on adjoint systems.

It would be nice if we could generalize this approach as follows:

Supposed Lemma: *Let* A *be a nef and big line bundle on* M *with* $(A^n) > n^n$. *Then* $K+A$ *is spanned (or very ample) unless* (*).

Here (*) is a condition which satisfies the following requirements: (i) we can prove the above Lemma under this condition, (ii) the conjecture follows from the Lemma. ····· Unfortunately, I cannot even guess how this condition (*) should be formulated.

(1.6) If L is ample and spanned and if $\mathrm{char}(k) = 0$, then $K+tL$ is spanned for $t \geq 1+n$. Moreover $K+nL$ is spanned too unless $(M,L) \simeq (\mathbb{P}^n, \mathcal{O}(1))$.

(1.7) I believe that this conjecture is extremely difficult. Even a partial solution in some special case would be very nice.

2. Let L be a nef and big line bundle on a variety V. Define the sectional genus $g(V,L)$ by means of the Hilbert polynomial $\chi(V,tL)$. Then

Conjecture. The inequality $g(V,L) \geq 0$ always holds. Moreover, if $g(V,L) = 0$ and if V is normal, then $\Delta(V,L) = 0$.

Comments. (2.1) The Δ-genus is defined by $\Delta(V,L) = n + d - h^0(V,L)$, where $n = \dim V$ and $d = (L^n)$.

(2.2) If L is nef and big and if $\Delta(V,L) = 0$, then there is a birational morphism $f : V \to W$ and a polarized variety (W,H) with $\Delta(W,H) = 0$ such that $L = f^*H$. In such a case H is always very ample.

(2.3) When $\mathrm{char}(k) = 0$, the conjecture would follow from the "Flip Conjecture". In particular, it is true in case $n = \dim V \leq 3$ by virtue of Mori's result.

3. Prove (or disprove) the following

Conjecture: $h^1(M,\mathcal{O}) \leq g(M,L)$ for any polarized manifold (M,L).

Furthermore, describe the structure of (M,L) if equality holds.

Comments: (3.1) It would be safer to assume $\mathrm{char}(k) = 0$.

(3.2) The conjecture is true if $g(M,L) = 0, 1, 2$ (at least if $\mathrm{char}(k) = 0$).

(3.3) The conjecture is true if $\mathrm{char}(k) = 0$ and L is spanned.

(3.4) If $g = h^1(M,\mathcal{O}) = 1$, then (M,L) is a scroll over an elliptic curve.

(3.5) If $g = h^1 = 2$, then (M,L) is a scroll over a curve of genus 2, or numerically equivalent to a Jacobian scroll of a curve of genus two, or M is an abelian surface or one point blow-up of an abelian surface. Here by a Jacobian scroll of a curve C we mean the following: Fix a point x on C and take the product of n copies of C. Let p_i be the projections. Let $D = \sum p_i^*(x)$. Then D descends to a divisor A on the symmetric product S. A is ample and the Albanese map induces a morphism $S \to J(C)$, which is a $\mathbb{P}^{n-g}$-bundle if $n > 2g-2$. Moreover (S,A) is a scroll over $J(C)$. This is called the Jacobian scroll.

(3.6) In order to solve the conjecture we should perhaps study the relation between adjoint mappings of (M,L) and the Albanese mapping of M.

4. Let $P_1, \ldots, P_r$ be points on $\mathbb{P}^2$ in a general position, and let S be the surface obtained by blowing up these points. Determine the shape of the cone of effective curves on S.

Conjectured Answer: Let $P \subset N_1(S)$ be the cone of 1-cycles with positive self-intersection. Then the cone of effective curves is generated by the positive half of P and the (-1)-curves.

Of course the cone of effective curves is bigger if the points are in a special position.

Comments. (4.1) If $r < 9$ then S is a Del Pezzo surface and the cone of effective curves is generated by the (-1)-curves.

(4.2) If $r = 9$ then the cone of effective curves is not generated by the (-1)-curves but I am sure that the conjectured answer is true.

(4.3) Suppose that $r = 15$ and let E_i be the exceptional curve over P_i. The conjecture would imply that $4H-\sum E_i$ is ample. Even this special case is unknown.

(4.4) If the conjectured answer is true then all the possible deformation types of rational polarized surfaces enumerated by my computer program do really exist.

(4.5) I would like to ask similar questions for irrational ruled surfaces too.

A smoothing problem related to Mumford's fake projective plane

by

Masa-Nori Ishida

In [3], Mumford constructed an algebraic surface of general type with $K^2 = 9$ and $p_g = q = 0$. This surface is called Mumford's fake projective plane because it has the same Betti numbers as the complex projective plane.

Mumford's surface is given as the generic fiber of a regular $\mathbb{Z}_2$-scheme M. The closed fiber M_0 is a normal crossing divisor in M. Its normalization is isomorphic to the rational surface B obtained by blowing up the projective plane over the prime field at the seven $\mathbb{F}_2$-rational point. In particular, the automorphism group of B is equal to the simple group $PSL(3,\mathbb{F}_2)$ of order 168.

The rational surface B has seven exceptional (-1)-curves and seven (-2)-curves which are the proper transforms of the seven $\mathbb{F}_2$-rational lines in the plane. These 14 curves form 7 pairs of a (-1)-curve and a (-2)-curve and the closed fiber M_0 is obtained from B by identifying the curves in each pair. Mumford's surface is a smoothing of M_0.

In [2] we explicitly describe the identification of the curves on B. We can show that there are exactly 6 other ways (up to isomorphism) to identify the (-1)-curves and the (-2)-curves on B such that we obtain a variety with only normal crossings. We can show that we obtain projective $\mathbb{F}_2$-surfaces with $K^2 = 3c_2 = 9$ and $\chi(\mathcal{O}) = 1$.

Problem Are these surfaces smoothable over $\mathbb{Z}_2$?

These surfaces are d-semi-stable in the sense of [1].

References

[1] Friedman, R.: Global smoothings of varieties with normal crossings, Annals of Math. 118 (1983) 75-114.

[2] Ishida, M.-N.: An elliptic surface covered by Mumford's fake projective plane, Tohoku Math. J. 40 (1988) 367-396.

[3] Mumford, D.: An algebraic surface with K ample $K^2 = 9$, $p_g = q = 0$, in Contributions to Algebraic Geometry, Johns Hopkins University Press (1979) 233-244.

Open problems

by

Yujiro Kawamata

Problem 1. Flip conjectures:

(i) existence of flip (dim $\geq$ 4)

(ii) termination of flips (dim $\geq$ 5)

Problem 2. Log-flip conjectures (generalization of (1)):

(i) existence (dim $\geq$ 3), (ii) termination (dim $\geq$ 3).

Problem 3. Crepant flip conjectures (special case of (2)):

(i) existence (dim $\geq$ 4), (ii) termination (dim $\geq$ 4).

3(i) implies 1(i) in one dimension less.

Problem 4. Abundance conjecture (dim $\geq$ 3).

Problem 5. If a canonical divisor K of a nonsingular projective variety X is not pseudo-effective, does there exist a covering family of curves $\{C_\lambda\}$ on X such that $(K \cdot C_\lambda) < 0$? More generally, describe the cone of pseudo-effective divisors.

Problem 6. Inductive structures of canonical and terminal singularities: Investigate the simplification process of those singularities via (i) blowing up with small discrepancies; (ii) hyperplane cutting; (iii) coverings; (iv) deformations.

Problem 7. Are deformations of canonical (resp. terminal) singularities canonical (resp. terminal)?

Problem 8. Is an extremal ray represented by a rational curve? Is the exceptional locus of an extremal contraction covered by rational curves?

Open problems

by

Masayoshi Miyanishi

1. Topologically contractible algebraic surfaces

Let X be a smooth algebraic variety defined over $\mathbb{C}$. X is a *complex homology n-cell* if $H_i(X;\mathbb{Z}) = 0$ for every $i > 0$. When $\dim X = 2$, we call X a *homology plane.*

Problem 1.1. Suppose $\kappa(X) = 2$.

(1) Is a homology plane topologically contractible?

(2) Classify all homology planes or topologically contractible algebraic surfaces with $\kappa(X) = 2$.

Problem 1.2. (Conjecture of Petrie-tom Dieck). Let X be a homology plane with a nontrivial automorphism of finite order. Is X then isomorphic to $\mathbb{A}^2$?

Problem 1.3. Let X be a complex homology 3-cell and let V be an algebraic compactification with boundary divisor $D := V-X$ of simple normal crossings. Say something on irreducible components of D and the dual graph of D.

2. Logarithmic del Pezzo surfaces

A normal projective surface V defined over $\mathbb{C}$ is a *log del Pezzo surface* if (1) V has at worst quotient singularities and (2) $-K_V$ is ample. There are some basic works on log del Pezzo surfaces by D.-Q. Zhang. Let $V_0 := V - \mathrm{Sing}\, V$. Then $\pi_1(V_0)$ is a finite group. Let U_0 be the universal covering space of V_0 and let U be the normalization of V in the function field $\mathbb{C}(U_0)$.

The surface U with the normalization morphism $p : U \to V$ is called a *quasi-universal covering*. Then U is again a log del Pezzo surface.

Problem 2.1. Let V be a log del Pezzo surface. Does it hold that V is a compactification of $\mathbb{C}^2$ if and only if $\pi_1(V_0) = \{e\}$?

Problem 2.2. Let V be a log del Pezzo surface of rank 1. Then does it hold that $V \simeq \mathbb{P}^2/G$ with $G \subset PGL(2)$ if and only if $\pi_1(V_0) \neq \{e\}$ and U has rank 1?

3. Affine cancellation problem in dimension 3

The said problem asks if $X \simeq \mathbb{A}^3$ provided $X \times \mathbb{A}^1 \simeq \mathbb{A}^4$. The condition implies $\kappa(X) = -\infty$. We say that X is *affine r-ruled* if X contains a Zariski open set U isomorphic to $U_0 \times \mathbb{A}^r$.

Problem 3.1. Suppose X is a smooth affine 3-fold with $\kappa(X) = -\infty$. Is X then affine 1-ruled?

For example, let X be a cubic hypersurface $x_1^3+x_2^3+x_3^3+x_4^3 = -1$ in $\mathbb{A}^4$. Then $\kappa(X) = -\infty$ and X is not affine 2-ruled, though we do not know yet if it is not affine 1-ruled either.

Problem 3.2. Let $f : X \to Y$ be a flat surjective morphism from a smooth algebraic 3-fold X to a smooth algebraic surface Y such that general fibers of f are isomorphic to $\mathbb{A}^1$ or $\mathbb{A}^1_* := \mathbb{A}^1 - \{0\}$. Let F be any degenerate fiber of f. Determine F_{red}.

Problem 3.3. Let X be a normal affine algebraic 3-fold with only isolated singularities. When X is affine 1-ruled, determine the type of singularities. Are they canonical?

Problem 3.4. Obtain any kind of algebraic characterization of the affine 3-space $\mathbb{A}^3$.

Open problems

by

Yoichi Miyaoka

Classification theory.

Problem 1. Let X be a compact complex projective manifold with trivial canonical bundle. When $\pi_1(X) = 1$, it is known that Ω_X^1 is never semi-positive (Bogomolov-Donaldson). In particular, there is a curve $C \subset X$ such that $\Omega_X^1|_C$ is not semi-positive. Is it possible to deform C in X to produce a rational curve? A partial result is known in dimension 2 (Mumford). If the answer is affirmative, study the set of rational curves on X, which would not be empty.

Problem 2. Is the set of smooth n-dimensional Fano manifolds bounded? If $\mathrm{Pic}\, X \simeq \mathbb{Z}$, two general points on a Fano mainfold can be joined by n rational curves. Can they be joined by a single rational curve? (If so, the boundedness of such Fano manifolds follows).

p-adic cohomology

Problem 3. Let X be a proper flat scheme over a discrete valuation ring R with quotient field K of characteristic 0 and with perfect residue field k of characteristic $p > 0$. When X is smooth over R, Fontaine-Messing has proved that

$$H^m(X_{\overline{K}}, \mathbb{Z}_p)(r) \simeq (B^+_{\mathbb{Z}_p} \otimes_W H^m_{crys}(Y/W))^{fil^r, p^{-r}f=1},$$

$$p > \dim X, \ r \geq m,$$

$$B^+_{\mathbb{Z}_p} = \varprojlim_n H^0_{crys}\left((\mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}})/W\right).$$

Find an alalogue of this isomorphism in the semistable case.

Problem of Zariski decomposition

by

Atsushi Moriwaki

1. Introduction

The origin of Zariski decomposition of a divisor on an algebraic variety is Zariski's famous paper "The theorem of Riemann-Roch for high multiples of an effective divisors on an algebraic surface". In his paper, he studied a certain property of a ring $R(S,D) = \oplus_{n\geq 0} H^0(S,nD)$, where S is a projective surface and D a divisor on S. In consideration of the above ring, he found some decomposition of a divisor D. This decomposition was very useful for special study of algebraic surfaces beyond Zariski's purpose, for example "Cancellation problem of affine plane". Hence, it is very important to extend this decomposition to higher dimensional varieties. The first problem of this attempt is to find a natural definition. The second is to prove the existence of a decomposition and the final one is to find applications. So far we have completed only the first and the third steps. For example, the success of this theory includes not only the minimal model theory but also its logarithmic version. The existence of the minimal model with respect to a logarithmic canonical divisor of 3-fold is still open. We expect more fascinating applications. In this note, we summarize several problems about Zariski decomposition.

2. Review of surface case

Let S be a smooth projective surface and D an $\mathbb{R}$-divisor on S, that is, an element of $\mathrm{Div}(S)\otimes\mathbb{R}$. We say that D is *pseudo-effective* if $\kappa(X,D+A) \neq -\infty$ for any ample $\mathbb{R}$-divisor A on S. We consider the following two condisions (A), (B) about a decomposition $D = P+N$. In the following context, we assume that D is pseudo-effective.

Condition (A)

(a1) P, N are $\mathbb{R}$-divisors such that P is nef and N is effective.

(a2) Letting $N = \sum_i a_i N_i$ be the irreducible decomposition of N, the intersection matrix $((N_i \cdot N_j))$ is negative definite.

(a3) $(P \cdot N_i) = 0$ for all i.

Condition (B)

(b1) same as (a1)

(b2) The natural homomorphisms $H^0(S,[nP]) \to H^0(S,[nD])$ are bijective for all n.

Then we have

Proposition 2.1. *If* D *is big, Condition*(A) *is equivalent to* (B).

Furthermore, let $D + \varepsilon A = P_\varepsilon + N_\varepsilon$ and $D = P + N$ be a decomposition satisfying Condition (A) or (B) for small positive real number ε and ample divisor A. Then,

Proposition 2.2. *The limits*

$$\lim_{\varepsilon\downarrow 0} P_\varepsilon \quad \textit{and} \quad \lim_{\varepsilon\downarrow 0} N_\varepsilon$$

exists and are equal to P *and* N *respectively.*

Hence, it is sufficient to consider Condition (B) for a big divisor. The construction of a decomposition satisfying (B) is very easy. This is done in the following manners.

Let $[nD] = M_n + F_n$ be the decomposition such that F_n is the fixed part of the complete linear system of $|[nD]|$. Then it is clear that M_n is nef. Set $N = \inf(F_n/n)$ and $P = D-N$. So we have a decomposition $D = P + N$, which is the desired decomposition.

Finally, we remark that if D is a $\mathbb{Q}$-divisor, then P and N are also $\mathbb{Q}$-divisors by Conditions (a2) and (a3).

3. Definition of Zariski decomposition

Let X be a smooth projective variety and D an $\mathbb{R}$-divisor on X. Let I_n be the ideal sheaf defined by the image of the natural homomorphism $H^0(X,[nD])\otimes\mathcal{O}_X(-[nD]) \to \mathcal{O}_X$. Let R be a discrete valuation ring of the rational function field of X. Define the integer a_n by $I_nR = t^{a_n} R$, where t is a prime element of R. Set $v_R(D) = \inf(a_n/n)$. If R is defined by a subvariety C of X, we denote v_R by v_C.

We say a decomposition $D = P + N$ is a *Zariski decomposition* (resp. *sectional decomposition*) if the following conditions are satisfied.

(1) N is an effective $\mathbb{R}$-divisor.

(2) $H^0(X,[nP]) \to H^0(X,[nD])$ is bijective for every n.

(3) $v_R(P) = 0$ for all DVR R (resp. $v_\Gamma(P) = 0$ for all prime divisors Γ).

Note that (i) if $v_C(P) = 0$ for all curves C, then P is nef, (ii) if P is nef and big, $v_R(P) = 0$ for all DVR R and (iii) rationality of a decomposition is not expected (example of Cutkosky).

4. Sectional decomposition and base curve

In this section, we assume $\dim X = 3$ and D is big. We consider the sum $N = \sum v_\Gamma(D)\Gamma$, where summation runs over all prime divisors. We can easily check that this summation is finite. Set $P = D - N$. Then $D = P + N$ is a sectional decomposition. Next we consider the formal 1-cycle $Z = \sum_C v_C(P)C$. The first fundamental problem is

PROBLEM A. Is the formal 1-cycle Z finite?

If there exists a Zariski decomposition of D, Problem A is true. With respect to this, we have

Proposition 4.1. Z *has a limit in*

$$N_1(X)\ (=(\{1\text{-cycles on } X\}/\equiv)\otimes\mathbb{R}).$$

By this proposition, for any positive ε, the set $\{C \mid v_C(P) > \varepsilon\}$ is finite.

PROBLEM B. Control Z by blowing-up process.

For this, there is a partial answer by Nakayama. His work is very local to control Z. This is closely related to find a good invariant for Z and induction step to vanishing of Z.

Finally, we propose the fundamental problem.

PROBLEM C. For an effective divisor D on a projective 3-fold X, is there a birational morphism $f : Y \to X$ such that $f^*(D)$ has a Zariski decomposition?

If $\kappa(X) < 3$, then this is true using the Zariski decomposition on a surface.

5. Paradise of Zariski decomposition

The reason why we consider a Zariski decomposition is to prove finitely generatedness of the canonical ring. With respect to this, we have the following theorem.

Theorem 5.1. *Let* (X,Δ) *be a normal projective variety with only log-terminal singularities and* $K_X + \Delta = P + N$ *a Zariski* *decomposition. Then* P, N ∈ Div(X)⊗ℚ *and there exists* m *such that* |[mP]| *is base point free.*

Problems on characterization of the complex projective space

by

Shigeru Mukai

A compact complex manifold X is a Fano manifold if its 1st Chern class $c_1(X) \in H^1(X,\mathbb{Z})$ is positive, or equivalently, the anticanonical class $-K_X$ is ample. The projective space $\mathbb{P}^n$ is the most typical example. In this note, I pose some problems on characterization of $\mathbb{P}^n$ which was conceived during my study on Fano manifolds of coindex 3 [Mu].

1. Characterization by index

For a Fano manifold X, the largest integer r which divides $c_1(X)$ in $H^2(X,\mathbb{Z})$ is called the *index* of X. The index of $\mathbb{P}^n$ is equal to $n+1$.

Theorem 1. ([K-O]). *Let* X *be a Fano manifold. Then* index $X \le \dim X + 1$. *Moreover, the equality holds if and only if* $X \simeq \mathbb{P}^n$.

If X is a Fano manifold of index r, then the vector bundle $\mathcal{O}_X(-K_X/r)^{\oplus r}$ is ample and its first Chern class is equal to $c_1(X)$. So we consider ample vector bundles E on X with $c_1(E) = c_1(X)$. How big can the rank $r(E)$ of E be? By [Mo], there exists a rational curve C on X with $(C \cdot c_1(X)) \le \dim X + 1$. Since every vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles, we have $r(E) = r(E|_C) \le \dim X + 1$.

Conjecture 1. Let X be a compact complex manifold and E an ample vector bundle on it with $c_1(E) = c_1(X)$. If $r(E) = \dim X + 1$, then $(X,E) \simeq (\mathbb{P}^n, \mathcal{O}(1)^{\oplus(n+1)})$.

2. Characterization by the tangent bundle

The following was conjectured by [Ha].

Theorem 2. ([Mo]). *A compact complex manifold* X *with ample tangent bundle* T_X *is isomorphic to* $\mathbb{P}^n$.

The tangent bundle T_X is a vector bundle on X with $r(T_X) = \dim X$ and $c_1(T_X) = c_1(X)$. The vector bundles $\mathcal{O}(1)^{\oplus(n-1)} \oplus \mathcal{O}(2)$ over $\mathbb{P}^n$ and $\mathcal{O}(1)^{\oplus n}$ over a hyperquadric $Q^n \subset \mathbb{P}^{n+1}$ also satisfy these conditions.

Conjecture 2. Let E be an ample vector bundle on X with $\mathrm{rk}\, E = \dim X$ and $c_1(E) = c_1(X)$. Then the pair (X, E) is isomorphic to $(\mathbb{P}^n, T_{\mathbb{P}})$, $(\mathbb{P}^n, \mathcal{O}(1)^{\oplus(n-1)} \oplus \mathcal{O}(2))$ or $(Q^n, \mathcal{O}(1)^{\oplus n})$.

3. The logarithmic version of Hartshorne conjecture

The "log analogue" of the tangent bundle T_X is the sheaf of vector fields with logarithmic zeroes along D, which is denoted by $T_X(-\log D)$. $T_X(-\log D)$ is characterized by the natural exact sequence

$$0 \to T_X(-\log D) \to T_X \to N_{D/X} \to 0,$$

where $N_{D/X}$ is the normal bundle $\mathcal{O}_D(D)$ of D and we regard it as a sheaf on X with support on D. If $X = \mathbb{P}^n$ and D is a hyperplane, then $T_X(-\log D)$ is isomorphic to $\mathcal{O}_{\mathbb{P}}(1)^{\oplus n}$.

Conjecture 3.(*) Let X be a compact complex manifold and D a nonzero reduced effective divisor on it. If the logarithmic tangent bundle $T_X(-\log D)$ is ample, then $(X,D) \simeq (\mathbb{P}^n, \text{hyperplane})$.

(*) In the problem session, Mori said that this would be proved by essentially the same argument as in [Mo].

The tangent bundle T_X is ample if the bisectional curvature is positive.

Problem. Find a sufficient condition on the curvature for $T_X(-\log D)$ to be ample, that is, formulate a logarithmic version of the Frankel conjecture which characterizes $\mathbb{C}^n$.

4. Relation with the classification of Fano manifolds

Let E be a rank r vector bundle on X with $c_1(E) = c_1(X)$ and put $Y := \mathbb{P}(E)$. Then $c_1(Y)$ is r times the tautological line bundle $\mathcal{O}_Y(1)$. Hence if E is ample then Y is a Fano manifold of index r. If $r = n+1$, $n = \dim X$, then Y is a Fano $2n$-fold of index $n+1$. We note $\rho(Y) = \rho(X)+1 \geq 2$, where ρ denotes the Picard number. The following is a refinement of Theorem 1.

Conjecture 4. If Y is a Fano manifold with Picard number ρ, then $\text{index } Y \leq \dim Y/\rho + 1$. Moreover, the equality holds iff $Y \simeq (\mathbb{P}^{\text{index } Y-1})^{\rho}$.

For a Fano manifold Y, we define the coindex by $\dim Y - \text{index } Y + 1$, which is nonnegative by Theorem 1. Conjecture 4 implies

Conjecture 4'. If Y is a Fano manifold with Picard number ≥ 2, then $\dim Y \leq 2\cdot\text{coindex } Y$. Moreover, the equality holds if $Y \simeq \mathbb{P}^{\text{coindex } Y} \times \mathbb{P}^{\text{coindex } Y}$.

This conjecture implies Conjecture 1. In the case coindex $Y \leq 3$, Conjecture 4' is easily obtained from the following;

Proposition. *Let* Y *be a Fano manifold of coindex* $c \leq 3$ *and* R *an extremal ray of* Y. *Let* $f : Y \to Z$ *be the contraction morphism of* R. *Then we have either* $\dim Z = \dim Y$ *or* $\dim Z \leq c$.

In the former case, f is birational and contracts a divisor to a point or to a curve.

(This proposition is also observed in [Fuj].)

Proof of Conjecture 4' in the case coindex 3:

In the case $\dim Y \geq 4$, Y has a nef extremal ray R_1. Since $\rho(Y) \geq 2$, Y has another extremal ray R_2. Let F_2 be a fiber of maximal dimension of cont_{R_2}. By the proposition, $\dim F_2 \geq \dim Y - 3$. Since the restriction of cont_{R_1} to F_1 is finite, we have $\dim Y - 3 \leq 3$. Moreover, if the equality holds, then both cont_{R_1} and cont_{R_2} are $\mathbb{P}^3$-bundles over 3-folds. Hence we have $Y \simeq \mathbb{P}^3 \times \mathbb{P}^3$.

References

[Fuj]Fujita,T.: On polarized manifolds whose adjoint bundles are not semipositive, Adv. Stud. Pure Math. 10 (1987) 167-178.

[Ha] Hartshorne, R.: Ample subvarieties, L.N., Springer.

[K-O]Kobayashi,S. and Ochiai,T.: Characterization of complex projective spaces and hyperquadrics, J. Math. Kyoto Univ. 13 (1973) 31-47.

[Mo] Mori, S.: Projective manifolds with ample tangent bundle, Annals of Math. 110 (1979) 593-606.

[Mu] Mukai, S.: New classification of Fano 3-folds and Fano manifolds of coindex 3, preprint.

Open problems

by

Noboru Nakayama

Let X be an n-dimensional projective manifold over $\mathbb{C}$ and let D be a pseudo-effective $\mathbb{R}$-divisor on X. We have two decompositions $D = P_\sigma(D) + N_\sigma(D) = P_\nu(D) + N_\nu(D)$ which have the following properties:

1a. N_σ is an effective $\mathbb{R}$-divisor and P_σ is pseudo-effective.

1b. For any ample $\mathbb{R}$-divisor A the intersection of all effective $\mathbb{R}$-divisors numerically equivalent to $D+A$ is an analytic subset of codimension at least 2.

1c. If $D = P'+N'$ satisfy conditions 1a and 1b then $N' \geq N_\sigma$.

2a. N_ν is an effective $\mathbb{R}$-divisor and P_ν is pseudo-effective.

2b. For any prime divisor G, $P_\nu|G$ is pseudo-effective.

2c. If $D = P''+N''$ satisfy conditions 2a and 2b then $N'' \geq N_\nu$.

If $\dim X = 2$ these two decompositions are the same and it is called the Zariski decomposition of D. Here P_σ is always nef and N_σ can be contracted. If $\dim X \geq 3$ then these two decompositions are different in general.

Problem 1. (Zariski decomposition problem) For a pseudo-effective $\mathbb{R}$-divisor D find a modification $g : Y \to X$ such that $P_\sigma(g^*D)$ is nef.

If P_σ is nef, then the first decomposition is called the Zariski decomposition of D. If P_ν is nef then $P_\nu = P_\sigma$.

Problem 2. In Problem 1 can one get a modification g such that

$P_\nu(g^*D)$ is nef?

It is also important to consider the relative versions of these problems.

Problem 3. Let $f : M \to S$ be a projective smooth morphism between complex manifolds and let L be a line bundle on N. Then is the set $\{s \in S \mid L|M_s \text{ is nef}\}$ open in S?

If the relative version of Problem 1 is true then the anser to Problem 3 is yes.

Problem 4. Characterize the dual of the pseudo-effective cone. Especially, we are interested in the following:

Problem 5. For a line bundle L, if $L|G$ is pseudo-effective for all prime divisors G, then is L pseudo-effective?

Relating to Problem 4 the following question is also important.

Problem 6. On a compact Kähler manifold, characterize the dual of the Kähler cone.

Two problems

by

Shuichiro Tsunoda

Problem 1. (Generalized Weierstrass points) Let V a nonsingular algebraic variety and E a vector bundle on V. Let J_k be the sheaf of k-jets. Then we have a natural map $\delta : \mathcal{O}_V \to J_k$ which is a homomorphism between sheaves of rings. We denote by $J_k(E)$ the J_k-module associated with E. If f_{ij} is a system of transition functions for E then $\delta(f_{ij})$ is a system of transition functions for $J_k(E)$. Let W be a subspace of $H^0(E)$. Then we have $\delta(W) \subset H^0(J_k(E))$. A GW-point (Generalized Weierstrass) is a "special" point on V with respect to $\delta(W)$ and $J_k(E)$.

Example 1. A point at which the quotient of $J_k(E)$ by the $\mathcal{O}_V$-subsheaf generated by $\delta(W)$ is not locally free is a GW-point.

Example 2. Assume $n \geq \text{rank } J_k := \ell$, and that E is a line bundle. If $f_1, \ldots, f_\ell \in W$, then we have a global section $\delta f_1 \wedge \ldots \delta f_\ell \in \det(J_k \otimes E)$. Consider the subspace $W' \subset H^0(\det(J_k \otimes E))$ generated by the above section. A point at which the rational map associated with W' is not an isomorphism is a GW-point.

Example 3. Assume $k = 1$. In this case $J_1 \simeq \mathcal{O}_V + \Omega_V^1$ as an $\mathcal{O}_V$-module and $\delta f = f + df$. Let f_i be a basis of $W \subset H^0(E)$. Then $f_i df_j - f_j df_i \in H^0(S^2 E \otimes \Omega_V^1)$. As in Example 2 a base point of the subspace generated by these elements is a GW-point.

Iitaka [Advances in Math. Vol. 33 (1979)] defined Λ-Weierstrass points using symmetric forms. We conjecture that Λ-Weierstrass points above are GW-points.

Problem 2. (Real analytic varieties with corners) Let D be the unit ball in $\mathbb{R}^n$ and let $f_1, \ldots, f_n$ be real analytic functions on D. We shall study the set $V = \{x \in D \mid f_i(x) \geq 0 \ \forall i\}$. These are natural generalizations of algebraic geometry and linear programming. One can easily put a sheaf structure on V and globalize in the natural way. Thus we obtain the notion of real analytic varieties with corners. A unique feature of this category is the "dissection" (a counterpart of blow-up). Two dimensional dissections are obtained as follows:

We identify two points on $\mathbb{R}^2-\{0\}$ if they are on the same half line. Then we get S^1. The rest of the construction of dissection is completely the same as that of blow-up.

Now one can ask many questions about this category e.g. resolution (should be easy), classification, and so on.

Fact Sheet on the Taniguchi Foundation

Official name and origin

Toyosaburo Taniguchi, chairman emeritus of the board of Toyobo Corporation, established the Taniguchi Foundation in 1929 with his personal funds in accordance with his late father's will.

Purpose of the Foundation

1) To promote research in the field of basic sciences thus contributing to the development and progress of industry and economy.

2) To promote mutual understanding and friendship on the international level through international exchange of ideas and research in the sciences.

Activities of the Foundation

1) Initiating and/or financing international symposia of the following character.

a) The symposium should center especially around promising young scholars giving them an opportunity to exchange and develop their ideas.

b) The number of participants shall be limited to discussion group size to allow maximum flow of communication and personal contact.

2) Supporting publications of the results of the research of foreign as well as Japanese scholars.

3) Promoting and financing the sending of young researchers abroad and also inviting young researchers to Japan.