ON SEMIPOSITIVITY THEOREMS

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ABSTRACT. We generalize the Fujita–Zucker–Kawamata semipositivity theorem from the analytic viewpoint.

CONTENTS

1. INTRODUCTION

The main purpose of this paper is to generalize the well-known Fujita–Zucker–Kawamata semipositivity theorem (see [Kaw1, §4. Semi-positivity], [Kaw2, Theorem 2], [FF, Section 5], [FFS, Theorem 3], and [Fuj]) from the analytic viewpoint.

Theorem 1.1. *Let* X *be a complex manifold and let* $X_0 \subset X$ *be a Zariski open set such that* $D = X \setminus X_0$ *is a normal crossing divisor on* X*. Let* V_0 *be a polarizable variation of* $\mathbb{R}$*-Hodge structure over* X_0 *with unipotent monodromies around* D*. Let* F^b *be the canonical extension of the lowest piece of the Hodge filtration. Let* $F^b \to \mathscr{L}$ *be a quotient line bundle of* F^b*. Then the Hodge metric of* F^b *induces a singular hermitian metric* h *on* $\mathscr{L}$ *such that* $\sqrt{-1}\Theta_h(\mathscr{L}) \geq 0$ *and the Lelong number of* h *is zero everywhere.*

As a direct consequence of Theorem 1.1, we have:

Corollary 1.2 (cf. [Kaw3])**.** *Let* X *be a complex manifold and let* $X_0 \subset X$ *be a Zariski open set such that* $D = X \setminus X_0$ *is a normal crossing divisor on* X*. Let* V_0 *be a polarizable variation of* $\mathbb{R}$*-Hodge structure over* X_0 *with unipotent monodromies around* D*. Let* F^b *be the canonical extension of the lowest piece of the Hodge filtration. Then* $\mathscr{O}_{\mathbb{P}_X(F^b)}(1)$ *has a singular hermitian metric* h *such that* $\sqrt{-1}\Theta_h(\mathscr{O}_{\mathbb{P}_X(F^b)}(1)) \geq 0$ *and that the Lelong number of* h *is zero everywhere. Therefore,* F^b *is nef in the usual sense when* X *is projective.*

Remark 1.3. There exists a quite short published proof of Corollary 1.2 (see the proof of [Kaw3, Theorem 1.1]). However, we have been unable to follow it. We also note that the arguments in [Kaw1, §4. Semi-positivity] contain various troubles. For the details, see [FFS, 4.6. Remarks].

Date: 2017/11/10, version 0.23.

2010 *Mathematics Subject Classification.* Primary 14D07; Secondary 14C30, 32J25, 32U25.

Key words and phrases. variation of Hodge structure, singular hermitian metric, nefness, Higgs bundle, system of Hodge bundles.

Remark 1.4. When X is projective and V_0 is geometric in Corollary 1.2, the nefness of F^b has already played important roles in the Iitaka program and the minimal model program for higher-dimensional complex algebraic varieties.

More generally, we can prove:

Theorem 1.5. *Let X be a complex manifold and let $X_0 \subset X$ be a Zariski open set such that $D = X \setminus X_0$ is a normal crossing divisor on X. Let V_0 be a polarizable variation of $\mathbb{R}$-Hodge structure over X_0 with unipotent monodromies around D. If $\mathscr{M}$ is a holomorphic line subbundle of the associated system of Hodge bundles $\operatorname{Gr}^{\bullet}_F \mathscr{V} = \bigoplus_p \operatorname{Gr}^p_F \mathscr{V}$ which is contained in the kernel of the Higgs field*

$$\theta : \operatorname{Gr}^{\bullet}_F \mathscr{V} \to \Omega^1_X(\log D) \otimes_{\mathscr{O}_X} \operatorname{Gr}^{\bullet}_F \mathscr{V},$$

then the Hodge metric induces a singular hermitian metric h on its dual $\mathscr{M}^{\vee}$ such that $\sqrt{-1}\Theta_h(\mathscr{M}^{\vee}) \geq 0$ and that the Lelong number of h is zero everywhere.

For the details of the Higgs field $\theta : \operatorname{Gr}^{\bullet}_F \mathscr{V} \to \Omega^1_X(\log D) \otimes_{\mathscr{O}_X} \operatorname{Gr}^{\bullet}_F \mathscr{V}$ in Theorem 1.5, see Definition 2.7 below.

As a direct easy consequence of Theorem 1.5, we obtain:

Corollary 1.6 ([Z] and [B1, Theorem 1.8])**.** *Let X be a complex manifold and let $X_0 \subset X$ be a Zariski open set such that $D = X \setminus X_0$ is a normal crossing divisor on X. Let V_0 be a polarizable variation of $\mathbb{R}$-Hodge structure over X_0 with unipotent monodromies around D. If A is a holomorphic subbundle of the associated system of Hodge bundles $\operatorname{Gr}^{\bullet}_F \mathscr{V} = \bigoplus_p \operatorname{Gr}^p_F \mathscr{V}$ which is contained in the kernel of the Higgs field*

$$\theta : \operatorname{Gr}^{\bullet}_F \mathscr{V} \to \Omega^1_X(\log D) \otimes \operatorname{Gr}^{\bullet}_F \mathscr{V},$$

then $\mathscr{O}_{\mathbb{P}_X(A^{\vee})}(1)$ has a singular hermitian metric h such that $\sqrt{-1}\Theta_h(\mathscr{O}_{\mathbb{P}_X(A^{\vee})}(1)) \geq 0$ and that the Lelong number of h is zero everywhere. Therefore, the dual vector bundle $A^{\vee}$ is nef in the usual sense when X is projective.

Corollary 1.6 is an analytic version of [B1, Theorem 1.8] (see also [Fuj]). For some generalizations of [B1, Theorem 1.8] from the Hodge module theoretic viewpoint, see [PoS, Theorem 18.1] and [PoW, Theorem A]. For a very recent development on semipositivity theorems from the theory of Higgs bundles, see [B2].

Remark 1.7. Let a be the integer such that $F_0^{a+1} \subsetneq F_0^a = \mathscr{V}_0$. Then, in Corollary 1.6, $\operatorname{Gr}^a_F \mathscr{V}$ is a holomorphic subbundle of $\operatorname{Gr}^{\bullet}_F \mathscr{V}$ and is contained in the kernel of θ. Therefore, we can use Corollary 1.6 for $A = \operatorname{Gr}^a_F \mathscr{V}$. By considering the dual Hodge structure in Corollary 1.6 and putting $A = \operatorname{Gr}^a_F \mathscr{V}$, Corollary 1.6 is also a generalization of the Fujita–Zucker–Kawamata semipositivity theorem (see, for example, [FF, Remark 3.15]). Of course, by considering the dual Hodge structure, Theorem 1.5 contains Theorem 1.1 as a special case.

Our proof in this paper heavily depends on [Ko], which is based on [CKS], and Demailly's approximation result for quasi-plurisubharmonic functions on complex manifolds (see [D1] and [D2]).

Remark 1.8 (Singular hermitian metrics on vector bundles)**.** We note that our results explained above are local analytic. Therefore, we can easily see that the Hodge metric of F^b in Theorem 1.1 is a semipositively curved singular hermitian metric in the sense of Păun–Takayama (see [PăT, Definition 2.3.1] and [HPS, Lemma 18.2]). Moreover, in Corollary 1.6, the induced metric on A is a seminegatively curved singular hermitian metric in the sense of Păun–Takayama (see [PăT, Definition 2.3.1] and [HPS, Lemma 18.2]). For the details of singular hermitian metrics on vector bundles and some related topics, see [PăT] (see also [HPS] and [B1]).

Acknowledgments. The first author was partially supported by JSPS KAKENHI Grant Numbers JP2468002, JP16H03925, JP16H06337. He thanks Professor Dano Kim whose question made him consider the problems discussed in this paper. Moreover, he pointed out an ambiguity in a preliminary version of this paper. The authors thank Professor Shin-ichi Matsumura for answering their questions and giving them some useful comments on singular hermitian metrics on vector bundles. Finally, they also thank the referee for comments.

2. Preliminaries

In this section, we collect some basic definitions and results.

2.1 (Singular hermitian metrics, multiplier ideal sheaves, and so on)**.** Let us recall some basic definitions and facts about singular hermitian metrics and plurisubharmonic functions. For the details, see [D2, (1.4), (3.12), (5.4), and so on].

Definition 2.2 (Singular hermitian metrics and curvatures)**.** Let $\mathscr{L}$ be a holomorphic line bundle on a complex manifold X. A *singular hermitian metric* h on $\mathscr{L}$ is a metric which is given in every trivialization $\theta : \mathscr{L}|_U \simeq U \times \mathbb{C}$ by

$$\|\xi\|_h = |\theta(\xi)|e^{-\varphi(x)}, \quad x \in U, \ \xi \in \mathscr{L}_x,$$

where $\varphi \in L^1_{\mathrm{loc}}(U)$ is an arbitrary function, called the *weight* of the metric with respect to the trivialization θ. Note that $L^1_{\mathrm{loc}}(U)$ is the space of locally integrable functions on U. The *curvature* $\Theta_h(\mathscr{L})$ of a singular hermitian metric h on $\mathscr{L}$ is defined by

$$\Theta_h(\mathscr{L}) := 2\partial\overline{\partial}\varphi,$$

where φ is a weight function and $\partial\overline{\partial}\varphi$ is taken in the sense of currents. It is easy to see that the right hand side does not depend on the choice of trivializations. Therefore, we get a global closed $(1,1)$-current $\Theta_h(\mathscr{L})$ on X. In this paper, $\sqrt{-1}\Theta_h(\mathscr{L}) \geq 0$ means that $\sqrt{-1}\Theta_h(\mathscr{L})$ is positive in the sense of currents.

Let $\mathscr{L}$ be a holomorphic line bundle on a smooth projective variety X. Then it is well known that there exists a singular hermitian metric h on $\mathscr{L}$ with $\sqrt{-1}\Theta_h(\mathscr{L}) \geq 0$ if and only if $\mathscr{L}$ is pseudoeffective (see [D2, (6.17) Theorem (c)]).

Definition 2.3 ((Quasi-)plurisubharmonic functions)**.** A function $\varphi : U \to [-\infty, \infty)$ defined on an open set $U \subset \mathbb{C}^n$ is called *plurisubharmonic* if

(i) φ is upper semicontinuous, and
(ii) for every complex line $L \subset \mathbb{C}^n$, $\varphi|_{U\cap L}$ is subharmonic on $U \cap L$, that is, for every $a \in U$ and $\xi \in \mathbb{C}^n$ satisfying $|\xi| < d(a, U^c) = \inf\{|a-x| \,|\, x \in U^c\}$, the function φ satisfies the mean inequality

$$\varphi(a) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi(a + e^{i\theta}\xi)d\theta.$$

Let X be an n-dimensional complex manifold. A function $\varphi : X \to [-\infty, \infty)$ is said to be *plurisubharmonic* if there exists an open cover $X = \bigcup_{i\in I} U_i$ such that $\varphi|_{U_i}$ is plurisubharmonic on U_i $(\subset \mathbb{C}^n)$ for every i. A *quasi-plurisubharmonic* function is a function φ which is locally equal to the sum of a plurisubharmonic function and of a smooth function.

Let φ be a quasi-plurisubharmonic function on a complex manifold X. Then the *multiplier ideal sheaf* $\mathscr{J}(\varphi) \subset \mathscr{O}_X$ is defined by

$$\Gamma(U, \mathscr{J}(\varphi)) = \{f \in \mathscr{O}_X(U) \,|\, |f|^2 e^{-2\varphi} \in L^1_{\mathrm{loc}}(U)\}$$

for every open set $U \subset X$. It is well known that $\mathscr{J}(\varphi)$ is a coherent ideal sheaf on X.

Definition 2.4 (Lelong numbers)**.** Let φ be a quasi-plurisubharmonic function on U ($\subset \mathbb{C}^n$). The Lelong number $\nu(\varphi, x)$ of φ at $x \in U$ is defined as follows:

$$\nu(\varphi, x) = \liminf_{z \to x} \frac{\varphi(z)}{\log |z - x|}.$$

It is well known that $\nu(\varphi, x) \geq 0$.

In this paper, we will implicitly use the following easy lemma repeatedly.

Lemma 2.5. *Let $\mathscr{L}$ be a holomorphic line bundle on a complex manifold X. Let $h = ge^{-2\varphi}$ be a singular hermitian metric on $\mathscr{L}$, where g is a smooth hermitian metric on $\mathscr{L}$ and φ is a locally integrable function on X. We assume that $\sqrt{-1}\Theta_h(\mathscr{L}) \geq 0$. Then there exists a quasi-plurisubharmonic function ψ on X such that φ coincides with ψ almost everywhere. In this situation, we put $\mathscr{J}(h) = \mathscr{J}(\psi)$. Moreover, we simply say the Lelong number of h to denote the Lelong number of ψ if there is no risk of confusion.*

2.6 (Systems of Hodge bundles, Higgs fields, curvatures, and so on)**.** Let us recall the definition of systems of Hodge bundles.

Definition 2.7 (Systems of Hodge bundles)**.** Let $V_0 = (\mathbb{V}_0, F_0)$ be a polarizable variation of $\mathbb{R}$-Hodge structure on a complex manifold X_0, where $\mathbb{V}_0$ is a local system of finite-dimensional $\mathbb{R}$-vector spaces on X_0 and $\{F_0^p\}$ is the Hodge filtration. Then we obtain a Higgs bundle (E_0, θ_0) on X_0 by setting

$$E_0 = \operatorname{Gr}^\bullet_{F_0} \mathscr{V}_0 = \bigoplus_p F_0^p / F_0^{p+1}$$

where $\mathscr{V}_0 = \mathbb{V}_0 \otimes \mathscr{O}_{X_0}$. Note that θ_0 is induced by the Griffiths transversality

$$\nabla : F_0^p \to \Omega^1_{X_0} \otimes_{\mathscr{O}_{X_0}} F_0^{p-1}.$$

More precisely, ∇ induces

$$\theta_0^p : F_0^p / F_0^{p+1} \to \Omega^1_{X_0} \otimes_{\mathscr{O}_{X_0}} \left(F_0^{p-1} / F_0^p \right)$$

for every p. Then

$$\theta_0 = \bigoplus_p \theta_0^p : E_0 \to \Omega^1_{X_0} \otimes_{\mathscr{O}_{X_0}} E_0.$$

The pair (E_0, θ_0) is usually called the *system of Hodge bundles* associated to $V_0 = (\mathbb{V}_0, F_0)$ and θ_0 is called the *Higgs field* of (E_0, θ_0).

We further assume that X_0 is a Zariski open set of a complex manifold X such that $D = X \setminus X_0$ is a normal crossing divisor on X and that the local monodromy of $\mathbb{V}_0$ around D is unipotent. Then, by [S, (4.12)], we can extend (E_0, θ_0) to (E, θ) on X, where

$$E = \operatorname{Gr}^\bullet_F \mathscr{V} = \bigoplus_p F^p / F^{p+1}$$

and

$$\theta : E \to \Omega^1_X(\log D) \otimes_{\mathscr{O}_X} E.$$

Note that $\mathscr{V}$ is the canonical extension of $\mathscr{V}_0$ and F^p is the canonical extension of F_0^p, that is,

$$F^p = j_* F_0^p \cap \mathscr{V},$$

where $j : X_0 \hookrightarrow X$ is the natural open immersion, for every p.

We need the following important calculations of curvatures by Griffiths. For the basic definitions and properties of the induced metrics and curvatures for subbundles and quotient bundles of a vector bundle, see [GT, §1 and §2].

Lemma 2.8. *We use the same notation as in Definition 2.7. Let F^b_0 be the lowest piece of the Hodge filtration. Let q_0 be the metric of F^b_0 induced by the Hodge metric. Let $\Theta_{q_0}(F^b_0)$ be the curvature form of (F^b_0, q_0). Then we have*

$$\Theta_{q_0}(F^b_0) + (\theta^b_0)^* \wedge \theta^b_0 = 0$$

where $(\theta^b_0)^*$ is the adjoint of θ^b_0 with respect to the Hodge metric (see, for example, [GT] *and* [S, (7.18) Lemma]*). Let $\mathscr{L}_0$ be a quotient line bundle of F^b_0. Then we have the following short exact sequence of locally free sheaves:*

$$0 \to \mathscr{S}_0 \to F^b_0 \to \mathscr{L}_0 \to 0.$$

Let A be the second fundamental form of the subbundle $\mathscr{S}_0 \subset F^b_0$. Let h_0 be the induced metric of $\mathscr{L}_0$. Then we obtain

$$\begin{aligned}\sqrt{-1}\Theta_{h_0}(\mathscr{L}_0) &= \sqrt{-1}\Theta_{q_0}(F^b_0)|_{\mathscr{L}_0} + \sqrt{-1}A \wedge A^* \\ &= -\sqrt{-1}(\theta^b_0)^* \wedge \theta^b_0|_{\mathscr{L}_0} + \sqrt{-1}A \wedge A^*.\end{aligned}$$

Note that A^* is the adjoint of A with respect to q_0. Therefore, the curvature form of $(\mathscr{L}_0, h_0)$ is a semipositive smooth $(1,1)$-form on X_0.

In the proof of Theorem 1.1 in Section 4, we will investigate asymptotic behaviors of $\log h_0$, $\partial \log h_0$, $\partial\overline{\partial} \log h_0$ near the normal crossing divisor D and see that the largest lower semicontinuous extension h of h_0 on X has desired properties.

Lemma 2.9. *We use the same notation as in Definition 2.7. Let q_0 be the Hodge metric on the system of Hodge bundles (E_0, θ_0) induced by the original Hodge metric. Let $\Theta_{q_0}(E_0)$ be the curvature form of (E_0, q_0). Then we have*

$$\Theta_{q_0}(E_0) + \theta_0 \wedge \theta^*_0 + \theta^*_0 \wedge \theta_0 = 0$$

where θ^*_0 is the adjoint of θ_0 with respect to q_0 (see, for example, [GT] *and* [S, (7.18) Lemma]*). Therefore, we have*

$$\sqrt{-1}\Theta_{q_0}(E_0) = -\sqrt{-1}\theta_0 \wedge \theta^*_0 - \sqrt{-1}\theta^*_0 \wedge \theta_0.$$

Let $\mathscr{M}_0$ be a line subbundle of E_0 which is contained in the kernel of θ_0 and let $h^\dagger_0$ be the induced metric on $\mathscr{M}_0$. Then

$$\begin{aligned}\sqrt{-1}\Theta_{h^\dagger_0}(\mathscr{M}_0) &= \sqrt{-1}\Theta_{q_0}(E_0)|_{\mathscr{M}_0} + \sqrt{-1}A^* \wedge A \\ &= -\sqrt{-1}\theta_0 \wedge \theta^*_0|_{\mathscr{M}_0} - \sqrt{-1}\theta^*_0 \wedge \theta_0|_{\mathscr{M}_0} + \sqrt{-1}A^* \wedge A \\ &= -\sqrt{-1}\theta_0 \wedge \theta^*_0|_{\mathscr{M}_0} + \sqrt{-1}A^* \wedge A\end{aligned}$$

where A is the second fundamental form of the line subbundle $\mathscr{M}_0 \subset E_0$ and A^* is the adjoint of A with respect to q_0. Therefore, the curvature of $(\mathscr{M}_0, h^\dagger_0)$ is a seminegative smooth $(1,1)$-form on X_0.

3. NEFNESS

Let us start with the definition of nef line bundles on projective varieties.

Definition 3.1 (Nef line bundles)**.** A line bundle $\mathscr{L}$ on a projective variety X is *nef* if $\mathscr{L} \cdot C \geq 0$ for every curve C on X.

In this paper, we need the notion of nef locally free sheaves (or vector bundles) on projective varieties, which is a generalization of Definition 3.1.

Definition 3.2 (Nef locally free sheaves)**.** A locally free sheaf (or vector bundle) $\mathscr{E}$ of finite rank on a projective variety X is *nef* if the following equivalent conditions are satisfied:

(i) $\mathscr{E} = 0$ or $\mathscr{O}_{\mathbb{P}_X(\mathscr{E})}(1)$ is nef on $\mathbb{P}_X(\mathscr{E})$.

(ii) For every map from a smooth projective curve $f : C \to X$, every quotient line bundle of $f^*\mathscr{E}$ has nonnegative degree.

A nef locally free sheaf in Definition 3.2 was originally called a (*numerically*) *semipositive* sheaf in the literature.

Let us recall the definition of nef line bundles in the sense of Demailly (see [D2, (6.11) Definition]).

Definition 3.3 (Nef line bundles in the sense of Demailly)**.** A holomorphic line bundle $\mathscr{L}$ on a compact complex manifold X is said to be *nef* if for every $\varepsilon > 0$ there is a smooth hermitian metric h_ε on $\mathscr{L}$ such that $\sqrt{-1}\Theta_{h_\varepsilon}(\mathscr{L}) \geq -\varepsilon\omega$, where ω is a fixed hermitian metric on X.

We can easily check:

Lemma 3.4. *If X is projective in Definition 3.3, then $\mathscr{L}$ is nef in the sense of Demailly if and only if $\mathscr{L}$ is nef in the usual sense.*

Proof. It is an easy exercise. For the details, see [D2, (6.10) Proposition]. □

The following proposition is more or less well-known to the experts. We write the proof for the reader's convenience.

Proposition 3.5. *Let X be a compact complex manifold and let $\mathscr{L}$ be a holomorphic line bundle equipped with a singular hermitian metric h. Assume that $\sqrt{-1}\Theta_h(\mathscr{L}) \geq 0$ and the Lelong number of h is zero everywhere. Then $\mathscr{L}$ is a nef line bundle in the sense of Definition 3.3.*

First, we give a quick proof of Proposition 3.5 when X is projective. It is an easy application of the Nadel vanishing theorem and the Castelnuovo–Mumford regularity.

Proof of Proposition 3.5 when X is projective. Let $\mathscr{A}$ be an ample line bundle on X such that $|\mathscr{A}|$ is basepoint-free. By Skoda's theorem (see [D2, (5.6) Lemma]), we have $\mathscr{J}(h^m) = \mathscr{O}_X$ for every positive integer m, where $\mathscr{J}(h^m)$ is the multiplier ideal sheaf of h^m. Here, we used the fact that the Lelong number of h is zero everywhere. By the Nadel vanishing theorem,

$$H^i(X, \omega_X \otimes \mathscr{L}^{\otimes m} \otimes \mathscr{A}^{\otimes n+1-i}) = 0$$

for every $0 < i \leq n = \dim X$ and every positive integer m. By the Castelnuovo–Mumford regularity, $\omega_X \otimes \mathscr{L}^{\otimes m} \otimes \mathscr{A}^{\otimes n+1}$ is generated by global sections for every positive integer m. We take a curve C on X. Then $C \cdot (\omega_X \otimes \mathscr{L}^{\otimes m} \otimes \mathscr{A}^{\otimes n+1}) \geq 0$ for every positive integer m. This means that $C \cdot \mathscr{L} \geq 0$. Therefore, $\mathscr{L}$ is nef in the usual sense. □

Next, we prove Proposition 3.5 when X is not necessarily projective. The proof depends on Demailly's approximation theorem for quasi-plurisubharmonic functions on complex manifolds (see [D1]).

Proof of Proposition 3.5: general case. Let ω be a hermitian metric on X and let ε be any positive real number. We fix a smooth hermitian metric g on $\mathscr{L}$. Then we can write $h = ge^{-2\varphi}$, where φ is an integrable function on X. Since $\sqrt{-1}\Theta_h(\mathscr{L}) \geq 0$, we see that

$$\sqrt{-1}\partial\overline{\partial}\varphi \geq -\frac{1}{2}\sqrt{-1}\Theta_g(\mathscr{L}) =: \gamma.$$

By Lemma 2.5, we may assume that φ is quasi-plurisubharmonic. Note that γ is a smooth $(1,1)$-form on X. By [D1, Proposition 3.7] (see also [D2, (13.12) Theorem] and [D3, Theorem 56]), we can construct a quasi-plurisubharmonic function ψ_ε on X with only analytic singularities (see (3.1) below) such that

$$\sqrt{-1}\partial\overline{\partial}\psi_\varepsilon \geq \gamma - \frac{1}{2}\varepsilon\omega$$

(see [D1, Proposition 3.7 (iii)], [D2, (13.12) Theorem (c)], and [D3, Theorem 56 (c)]). Since the Lelong number of h is zero everywhere by assumption, we obtain

$$0 \le \nu(\psi_\varepsilon, x) \le \nu(\varphi, x) = 0$$

for every $x \in X$ by [D1, Proposition 3.7 (ii)] (see also [D2, (13.12) Theorem (b)] and [D3, Theorem 56 (b)]). Therefore, the Lelong number of ψ_ε is zero everywhere. By construction, we can easily see that ψ_ε is smooth outside $\{x \in X \mid \psi_\varepsilon(x) = -\infty\}$. As mentioned above, ψ_ε has only analytic singularities, that is, it can be written locally near every point $x_0 \in X$ as

$$\psi_\varepsilon(z) = c \log \sum_{1 \le j \le N} |g_j(z)|^2 + O(1) \tag{3.1}$$

with a family of holomorphic functions $\{g_1, \dots, g_N\}$ defined near x_0 and a positive real number c (see [D3, Definition 52]). Since $\nu(\psi_\varepsilon, x) = 0$ for every $x \in X$, we obtain that $\psi_\varepsilon \ne -\infty$ everywhere. Therefore, ψ_ε is a smooth function on X. We put $h_\varepsilon = g e^{-2\psi_\varepsilon}$. Then h_ε is a smooth hermitian metric on $\mathscr{L}$ such that $\sqrt{-1}\Theta_{h_\varepsilon}(\mathscr{L}) \ge -\varepsilon\omega$. This means that $\mathscr{L}$ is a nef line bundle in the sense of Definition 3.3. □

4. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 and Corollary 1.2. The arguments below heavily depend on [Ko, Section 5]. Therefore, we strongly recommend the reader to see [Ko, Section 5], especially [Ko, Definition 5.3], before reading this section.

4.1. We put $\Delta_a = \{z \in \mathbb{C} \mid |z| < a\}$, $\overline{\Delta}_a = \{z \in \mathbb{C} \mid |z| \le a\}$, and $\Delta_a^* = \Delta_a \setminus \{0\}$. On Δ_a^n, we fix coordinates $z_1, \dots, z_n$.

Let us quickly recall the definition of *nearly boundedness* and *almost boundedness* due to Kollár for the reader's convenience.

Definition 4.2 (see [Ko, Definition 5.3 (vi) and (vii)]). On $(\Delta_a^*)^n$ with $0 < a < e^{-1}$, we define the *Poincaré metric* by declaring the coframe

$$\left\{ \frac{dz_i}{z_i \log |z_i|}, \frac{d\bar{z}_i}{\bar{z}_i \log |z_i|} \right\}$$

to be unitary. This defines a frame of every Ω^k which we will refer to as the *Poincaré frame.*

A function f defined on a dense Zariski open set of Δ_a^n is called *nearly bounded* on Δ_a^n if f is smooth on $(\Delta_a^*)^n$ and there are $C > 0$, $k > 0$ and $\varepsilon > 0$ such that for every ordering of the coordinate functions $z_1, \dots, z_n$ at least one of the following conditions is satisfied for every $z \in \{z \in (\Delta_a^*)^n \mid |z_1| \le \cdots \le |z_n|\}$.

(**a**): $|f| \le C$,

(**b**): $|z_1| \le \exp(-|z_m|^{-\varepsilon})$ and $|f| \le C(-\log |z_m|)^k$ for some $2 \le m \le n$.

A form η defined on a dense Zariski open set of Δ_a^n is called *nearly bounded* on Δ_a^n if the coefficient functions are nearly bounded on Δ_a^n when we write η in terms of the Poincaré frame. If η_1 and η_2 are nearly bounded on the same Δ_a^n, then $\eta_1 \wedge \eta_2$ is nearly bounded on Δ_a^n.

A form η defined on a dense Zariski open set of Δ_a^n is called *almost bounded* on Δ_a^n if there is a proper bimeromorphic map $p : W \to \Delta_a^n$ such that W is smooth and every $w \in W$ has a neighborhood where $p^*\eta$ is nearly bounded.

Remark 4.3. The definition of nearly boundedness and almost boundedness in Definition 4.2 is slightly different from Kollár's original one (see [Ko, Definition 5.3 (vii)]). We think that it is a kind of clarification. Of course, everything in [Ko, Section 5] works well for our definition.

4.4 (Proof of Theorem 1.1)**.** We fix a smooth hermitian metric g on $\mathscr{L}$. The Hodge metric induces a smooth hermitian metric h_0 on $\mathscr{L}|_{X_0}$. Then we can write

$$h_0 = ge^{-2\varphi_0}$$

for some smooth function φ_0 on X_0. We use the same notation as in Lemma 2.8. Let $\mathscr{V}$ be the canonical extension of $\mathscr{V}_0 = \mathbb{V}_0 \otimes \mathscr{O}_{X_0}$. Let q_0 be the Hodge metric on $\mathscr{V}_0$. For simplicity, we use the same notation q_0 to denote $(q_0)|_{F^b_0}$, that is, the metric on F^b_0 induced by the metric q_0 on $\mathscr{V}_0$. Let P be an arbitrary point of X. We take a suitable local coordinate $(z_1, \ldots, z_n)$ centered at P and a small positive real number a with $a < e^{-1}$. Then, by [CKS, Theorem 5.21] (see also [Kas] and [VZ, Claim 7.8]), we can write

$$\mathscr{V}|_{\Delta^n_a} \simeq \bigoplus_{i=1}^{r} \mathscr{O}_{\Delta^n_a} e_i(z),$$

where $e_i(z) \in \Gamma(\Delta^n_a, \mathscr{V})$, such that

$$q_0(e_i(z), e_i(z)) \leq C_1(-\log|z_1|)^{a_1} \cdots (-\log|z_n|)^{a_n} \tag{4.1}$$

for $z \in (\Delta^*_a)^n$, where $a_1, \ldots, a_n$ are some positive integers and C_1 is a large positive real number. By making a smaller, we may further assume that

$$\mathscr{L}|_{\Delta^n_a} \simeq \mathscr{O}_{\Delta^n_a} e(z),$$

where $e(z) \in \Gamma(\Delta^n_a, \mathscr{L})$ is a nowhere vanishing section of $\mathscr{L}$ on Δ^n_a. We take a lift $f(z) \in \Gamma(\Delta^n_a, F^b)$ of $e(z)$, that is, $p(f(z)) = e(z)$, where $p : F^b \to \mathscr{L}$. Then we can write

$$f(z) = f_1(z)e_1(z) + \cdots + f_r(z)e_r(z), \tag{4.2}$$

where $f_i(z)$ is a holomorphic function on Δ^n_a for every i. By making a smaller again, we may assume that $f_i(z)$ is holomorphic in a neighborhood of $(\overline{\Delta}_a)^n$. Of course, we may further assume that $e(z) \neq 0$ in a neighborhood of $(\overline{\Delta}_a)^n$. By (4.1) and (4.2), we obtain that there exists some large positive real number C_2 such that

$$q_0(f(z), f(z)) \leq C_2(-\log|z_1|)^{a_1} \cdots (-\log|z_n|)^{a_n}$$

holds for $z \in (\Delta^*_a)^n$. Therefore,

$$\begin{aligned} C_3 e^{-2\varphi_0(z)} &\leq g(e(z), e(z))e^{-2\varphi_0(z)} \\ &= h_0(e(z), e(z)) \\ &\leq q_0(f(z), f(z)) \leq C_2(-\log|z_1|)^{a_1} \cdots (-\log|z_n|)^{a_n} \end{aligned}$$

for $z \in (\Delta^*_a)^n$, where

$$C_3 = \min_{z \in (\overline{\Delta}_a)^n} g(e(z), e(z)) > 0.$$

Thus,

$$-\varphi_0(z) \leq \log\left(C\,(-\log|z_1|)^{a_1} \cdots (-\log|z_n|)^{a_n}\right)$$

holds for $z \in (\Delta^*_a)^n$, where C is some large positive real number. By applying similar arguments to the dual line bundle $\mathscr{L}^\vee$, we may further assume that

$$\varphi_0(z) \leq \log\left(C\,(-\log|z_1|)^{a_1} \cdots (-\log|z_n|)^{a_n}\right)$$

holds for $z \in (\Delta^*_a)^n$. Let φ be the smallest upper semicontinuous function that extends φ_0 to X. More explicitly,

$$\varphi(z) = \lim_{\varepsilon \to 0} \sup_{w \in \Delta^n_\varepsilon \cap X_0} \varphi_0(w),$$

where Δ^n_ε is a polydisc on X centered at $z \in X$. Then, by Lemma 4.6, we obtain:

Lemma 4.5. *φ is locally integrable on X.*

Proof of Lemma 4.5. Let P be an arbitrary point of X. In a small open neighborhood of P, we have

$$0 \leq \varphi_{\pm}(z) \leq \log \left(C \left(-\log |z_1|\right)^{a_1} \cdots \left(-\log |z_n|\right)^{a_n}\right)$$

where $\varphi_+ = \max\{\varphi, 0\}$ and $\varphi_- = \varphi_+ - \varphi$. By Lemma 4.6 below, we obtain that φ is locally integrable on X. $\square$

We have already used:

Lemma 4.6. *We have*

$$\int_0^a r \log(-\log r) dr < \infty$$

for $0 < a < e^{-1}$.

Proof of Lemma 4.6. We put $t = -\log r$. Then we can easily check

$$\int_0^a r \log(-\log r) dr = \int_{-\log a}^{\infty} e^{-2t}(\log t) dt \leq \int_{-\log a}^{\infty} t e^{-2t} dt \leq \int_{-\log a}^{\infty} e^{-t} dt = a < \infty$$

by direct calculations. $\square$

We put

$$h = g e^{-2\varphi}.$$

Then h is a singular hermitian metric on $\mathscr{L}$ in the sense of Definition 2.2. The following lemma is essentially contained in [Ko, Propositions 5.7 and 5.15].

Lemma 4.7. *Let P be an arbitrary point of X. Then $\partial\varphi_0$ and $\bar\partial\partial\varphi_0$ are almost bounded in a neighborhood of $P \in X$. More precisely, there exists Δ^n_a on X centered at P for some $0 < a < e^{-1}$ such that φ_0, $\partial\varphi_0$, and $\bar\partial\partial\varphi_0$ are smooth on $(\Delta^*_a)^n$ and that $\partial\varphi_0$ and $\bar\partial\partial\varphi_0$ are almost bounded on Δ^n_a.*

Proof of Lemma 4.7. We consider the following short exact sequence:

$$0 \to \mathscr{S} \to F^b \to \mathscr{L} \to 0.$$

We fix smooth hermitian metrics g_1, g_2 and g on $\mathscr{S}, F^b$, and $\mathscr{L}$, respectively. We assume that $g_1 = g_2|_{\mathscr{S}}$ and that g is the orthogonal complement of g_1 in g_2. Let h_1 and h_2 be the induced Hodge metrics on $\mathscr{S}_0 = \mathscr{S}|_{X_0}$ and F^b_0, respectively. By applying the calculations in [Ko, Section 5] to $\det \mathscr{S}$ and $\det F^b$, we obtain $\det h_1 = \det g_1 \cdot e^{-\varphi_1}$ and $\det h_2 = \det g_2 \cdot e^{-\varphi_2}$ on X_0 such that $\partial\varphi_1, \bar\partial\partial\varphi_1, \partial\varphi_2$, and $\bar\partial\partial\varphi_2$ are almost bounded in a neighborhood of P. More precisely, we can take a polydisc Δ^n_a centered at P for some $0 < a < e^{-1}$ and a composite of permissible blow-ups $p : W \to \Delta^n_a$ (see [Ko, 5.9] and [W, Theorem 3.5.1]) such that φ_1 and φ_2 are smooth on $(\Delta^*_a)^n$ and that every $w \in W$ has a neighborhood $\Delta^n_{a'_w}$ centered at $w \in W$ for some $0 < a'_w < e^{-1}$ where $p^*(\partial\varphi_1)$, $p^*(\overline{\partial}\partial\varphi_1)$, $p^*(\partial\varphi_2)$, and $p^*(\overline{\partial}\partial\varphi_2)$ are nearly bounded on $\Delta^n_{a'_w}$. For the details, see [Ko, Propositions 5.7 and 5.15]. By construction, $\varphi_0 = -\varphi_1 + \varphi_2$. Therefore, φ_0 is smooth on $(\Delta^*_a)^n$, and $p^*(\partial\varphi_0)$ and $p^*(\overline{\partial}\partial\varphi_0)$ are nearly bounded on $\Delta^n_{a'_w}$. This means that φ_0, $\partial\varphi_0$, and $\overline{\partial}\partial\varphi_0$ are smooth on $(\Delta^*_a)^n$ and that $\partial\varphi_0$ and $\overline{\partial}\partial\varphi_0$ are almost bounded on Δ^n_a. $\square$

We prepare an easy lemma.

Lemma 4.8. *We assume $0 < a < e^{-1}$. We have*

$$\int_0^a \frac{\log(-\log r)}{-\log r} dr < \infty.$$

Proof of Lemma 4.8. We put $t=-\log r$. Then $r=e^{-t}$. We have

$$\begin{aligned}\int_0^a \frac{\log(-\log r)}{-\log r}dr &= \int_\infty^{-\log a}\frac{\log t}{t}(-e^{-t})dt\\ &= \int_{-\log a}^\infty \frac{\log t}{t}e^{-t}dt\\ &\leq \int_{-\log a}^\infty e^{-t}dt = a<\infty.\end{aligned}$$

This is what we wanted. □

The following lemma is missing in [Ko, Section 5]. This is because it is sufficient to consider the asymptotic behaviors of $\partial\varphi_0$ and $\bar\partial\partial\varphi_0$ for the purpose of [Ko, Section 5].

Lemma 4.9. *Let* η *be a smooth* $(2n-1)$*-form on* Δ_a^n *with compact support. We put*

$$S_{\vec\varepsilon}=\{z\in\Delta_a^n\,|\,|z_i|\geq\varepsilon^i \text{ for every } i \text{ and } |z_{i_0}|=\varepsilon^{i_0} \text{ for some } i_0\}$$

where $\vec\varepsilon=(\varepsilon^1,\ldots,\varepsilon^n)$ *with* $\varepsilon^i>0$ *for every* i*. Then there is a sequence* $\{\vec\varepsilon_k\}$ *with* $\vec\varepsilon_k\searrow 0$ *such that*

$$\lim_{k\to\infty}\int_{S_{\vec\varepsilon_k}}\varphi\eta=0.$$

Proof of Lemma 4.9. We put

$$S_{\varepsilon,1}=\{z\in\Delta_a^n\,|\,|z_1|=\varepsilon\}.$$

Then it is sufficient to prove that

$$\lim_{k\to\infty}\int_{S_{\varepsilon_k,1}}\varphi\eta=0$$

for some sequence $\{\varepsilon_k\}$ with $\varepsilon_k\searrow 0$. Without loss of generality, we may assume that η is a real $(2n-1)$-form by considering $\frac{\eta+\bar\eta}{2}$ and $\frac{\eta-\bar\eta}{2\sqrt{-1}}$. Let us consider the real 1-form

$$\omega=\frac{1}{(2(-\log|z_1|)^2)^{1/2}}\left(\frac{dz_1}{z_1}+\frac{d\bar z_1}{\bar z_1}\right).$$

This form is orthogonal to the foliation $S_{\varepsilon,1}$ and has length one everywhere by the Poincaré metric. We consider the vector field

$$v=\frac{1}{(2(-\log|z_1|)^2)^{1/2}}\left(z_1(\log|z_1|)^2\frac{\partial}{\partial z_1}+\bar z_1(\log|z_1|)^2\frac{\partial}{\partial\bar z_1}\right),$$

which is dual to ω. We fix ε with $0<\varepsilon<a<e^{-1}$. We consider the flow f_t on $\Delta_a^*\times\Delta_a^{n-1}$ corresponding to $-v$. We can explicitly write

$$f_t:[0,\infty)\times S_{\varepsilon,1}\to\Delta_a^*\times\Delta_a^{n-1}$$

by

$$(t,(w,z_2,\cdots,z_n))\mapsto\left(\frac{w}{\varepsilon}\exp\left(-\exp\left(\frac{1}{\sqrt2}t+\log(-\log\varepsilon)\right)\right),z_2,\cdots,z_n\right). \tag{4.3}$$

Therefore, by using the flow f_t, we can parametrize $\{z\in\mathbb{C}\,|\,0<|z|\leq\varepsilon\}\times\Delta_a^{n-1}$ by $[0,\infty)\times S_{\varepsilon,1}$. If we write

$$\omega\wedge\varphi\eta=f(z)dV,$$

where dV is the standard volume form of $\mathbb{C}^n$, then we put

$$(\omega\wedge\varphi\eta)^+=\max\{f(z),0\}dV$$

and

$$(\omega \wedge \varphi\eta)^- = (\omega \wedge \varphi\eta)^+ - \omega \wedge \varphi\eta.$$

We can easily see that

$$\int_{\Delta_a^n} (\omega \wedge \varphi\eta)^\pm < \infty$$

by Lemmas 4.6 and 4.8. Therefore, we obtain

$$\int_{[0,\infty)\times S_{\varepsilon,1}} (\omega \wedge \varphi\eta)^\pm < \infty. \tag{4.4}$$

The image of $\{t\} \times S_{\varepsilon,1}$ in Δ_a^n is $S_{\varepsilon(t),1}$ with $0 < \varepsilon(t) \le \varepsilon$. By (4.3), we have

$$\varepsilon(t) = \exp\left(-\exp\left(\frac{1}{\sqrt{2}}t + \log(-\log\varepsilon)\right)\right).$$

We note that ω is orthogonal to $S_{\varepsilon(t),1}$ and unitary. More explicitly, we can directly check

$$f_t^*\omega = -dt.$$

Therefore, the above integral (4.4) transforms to

$$\int_{[0,\infty)} \left(\int_{S_{\varepsilon(t),1}} (\varphi\eta)^\pm\right) dt < \infty.$$

Note that $(\varphi\eta)^\pm$ is defined by

$$f_t^*(\omega \wedge \varphi\eta)^\pm = -dt \wedge (\varphi\eta)^\pm.$$

This can happen only if

$$\int_{S_{\varepsilon(t_k),1}} (\varphi\eta)^\pm \to 0$$

for some sequence $\{t_k\}$ with $t_k \nearrow \infty$. This implies that we can take a sequence $\{\varepsilon_k\}$ with $\varepsilon_k \searrow 0$ such that

$$\lim_{k\to\infty} \int_{S_{\varepsilon_k,1}} \varphi\eta = 0.$$

Therefore, we have a desired sequence $\{\vec{\varepsilon}_k\}$. $\square$

Remark 4.10. The real 1-form ω and the corresponding flow f_t in the proof of Lemma 4.9 are different from the 1-form ω and the flow v_t in the proof of [Ko, Proposition 5.16], respectively.

By combining the proof of [Ko, Proposition 5.16] and the proof of Lemma 4.9, we have:

Lemma 4.11. *Let η be a nearly bounded $(2n-1)$-form on Δ_a^n with compact support. Then there exists a sequence $\{\vec{\varepsilon'}_k\}$ with $\vec{\varepsilon'}_k \searrow 0$ such that*

$$\lim_{\vec{\varepsilon'}_k \searrow 0} \int_{S_{\vec{\varepsilon'}_k}} \eta = 0.$$

We leave the details of Lemma 4.11 to the reader (see the proof of [Ko, Proposition 5.16] and the proof of Lemma 4.9).

By Lemmas 4.7 and 4.9, we have the following lemma.

Lemma 4.12. *Let η be a smooth $(2n-2)$-form on Δ_a^n with compact support. We further assume that $\partial\varphi_0$ and $\bar{\partial}\partial\varphi_0$ are nearly bounded on Δ_a^n. Then*

$$\int_{\Delta_a^n} \varphi\partial\bar{\partial}\eta = \int_{\Delta_a^n} \partial\bar{\partial}\varphi_0 \wedge \eta.$$

Note that the right hand side is an improper integral. Therefore, we obtain

$$\int_{\Delta_a^n} \partial\bar{\partial}\varphi \wedge \eta = \int_{\Delta_a^n} \partial\bar{\partial}\varphi_0 \wedge \eta,$$

where we take $\partial\bar{\partial}$ of φ as a current.

Proof of Lemma 4.12. We put

$$V_{\vec{\varepsilon}_k} = \{z \in \Delta_a^n \,|\, |z_i| \geq \varepsilon_k^i \text{ for every } i\}$$

where $\vec{\varepsilon}_k = (\varepsilon_k^1, \cdots, \varepsilon_k^n)$ with $\varepsilon_k^i > 0$ for every i. Then

$$\begin{aligned}
\int_{\Delta_a^n} \varphi\partial\bar{\partial}\eta &= \lim_{\vec{\varepsilon}_k \searrow 0} \int_{V_{\vec{\varepsilon}_k}} \varphi_0 \partial\bar{\partial}\eta \\
&= \lim_{\vec{\varepsilon}_k \searrow 0} \int_{V_{\vec{\varepsilon}_k}} d(\varphi_0\bar{\partial}\eta) - \lim_{\vec{\varepsilon'}_k \searrow 0} \int_{V_{\vec{\varepsilon'}_k}} \partial\varphi_0 \wedge \bar{\partial}\eta \\
&= \lim_{\vec{\varepsilon}_k \searrow 0} \int_{S_{\vec{\varepsilon}_k}} \varphi_0\bar{\partial}\eta + \lim_{\vec{\varepsilon'}_k \searrow 0} \int_{V_{\vec{\varepsilon'}_k}} d(\partial\varphi_0 \wedge \eta) - \lim_{\vec{\varepsilon'}_k \searrow 0} \int_{V_{\vec{\varepsilon'}_k}} \bar{\partial}\partial\varphi_0 \wedge \eta \\
&= \lim_{\vec{\varepsilon'}_k \searrow 0} \int_{V_{\vec{\varepsilon'}_k}} \partial\bar{\partial}\varphi_0 \wedge \eta \\
&= \int_{\Delta_a^n} \partial\bar{\partial}\varphi_0 \wedge \eta.
\end{aligned}$$

The first equality holds since φ is locally integrable. The second one follows from integration by parts. Note that φ_0 is smooth in a neighborhood of $V_{\vec{\varepsilon}_k}$. We also note that

$$\lim_{\vec{\varepsilon}_k \searrow 0} \int_{V_{\vec{\varepsilon}_k}} \partial\varphi_0 \wedge \bar{\partial}\eta = \lim_{\vec{\varepsilon'}_k \searrow 0} \int_{V_{\vec{\varepsilon'}_k}} \partial\varphi_0 \wedge \bar{\partial}\eta$$

holds. The third one follows from Stokes' theorem and integration by parts. We obtain the fourth one by Lemmas 4.9 and 4.11. Note that

$$\int_{V_{\vec{\varepsilon'}_k}} d(\partial\varphi_0 \wedge \eta) = \int_{S_{\vec{\varepsilon'}_k}} \partial\varphi_0 \wedge \eta$$

by Stokes' theorem. The final one follows from [Ko, Proposition 5.16 (i)]. □

Lemma 4.13. *Let η be a smooth $(2n-2)$-form on Δ_a^n with compact support. We assume that $\partial\varphi_0$ and $\bar{\partial}\partial\varphi_0$ are almost bounded on Δ_a^n. Then*

$$\int_{\Delta_a^n} \varphi\partial\bar{\partial}\eta = \int_{\Delta_a^n} \partial\bar{\partial}\varphi_0 \wedge \eta.$$

Proof of Lemma 4.13. By assumption, $\partial\varphi_0$ and $\bar{\partial}\partial\varphi_0$ are almost bounded on Δ_a^n. Therefore, after taking some suitable blow-ups and a suitable partition of unity, we can apply Lemma 4.12. Then we obtain the desired equality. □

Lemma 4.14. *Let η_1 and η_2 be a smooth $(2n-2)$-form and a smooth $(2n-3)$-form on X with compact support, respectively. Then*

$$\int_X \sqrt{-1}\Theta_{h_0}(\mathscr{L}|_{X_0}) \wedge \eta_1 < \infty \tag{4.5}$$

and

$$\int_X \sqrt{-1}\Theta_{h_0}(\mathscr{L}|_{X_0}) \wedge d\eta_2 = 0. \tag{4.6}$$

Therefore, $\sqrt{-1}\Theta_{h_0}(\mathscr{L}|_{X_0})$ *can be extended to a closed positive current* T *on* X *by improper integrals. We note that* $\sqrt{-1}\Theta_{h_0}(\mathscr{L}|_{X_0})$ *is a semipositive smooth* $(1,1)$*-form on* X_0 *(see Lemma 2.8).*

Proof of Lemma 4.14. We note that

$$\sqrt{-1}\Theta_{h_0}(\mathscr{L}|_{X_0}) = \sqrt{-1}\Theta_g(\mathscr{L})|_{X_0} + 2\sqrt{-1}\partial\overline{\partial}\varphi_0$$

by definition and that $\sqrt{-1}\Theta_g(\mathscr{L})$ is a d-closed smooth $(1,1)$-form on X. Therefore, it is sufficient to prove that

$$\int_{\Delta^n_a} \sqrt{-1}\partial\overline{\partial}\varphi_0 \wedge \eta_1 < \infty \tag{4.7}$$

and

$$\int_{\Delta^n_a} \partial\overline{\partial}\varphi_0 \wedge d\eta_2 = 0 \tag{4.8}$$

by taking some suitable partition of unity. We see that (4.7) and (4.8) follow from [Ko, Corollary 5.17] since $\partial\overline{\partial}\varphi_0$ is almost bounded on Δ^n_a (see Lemma 4.7). More precisely, by taking some suitable blow-ups and a suitable partition of unity, we can reduce the problems to the case where $\partial\overline{\partial}\varphi_0$ is nearly bounded on some polydisc Δ^n_a. Then (4.7) follows from [Ko, Proposition 5.16 (i)]. By [Ko, Proposition 5.16 (i)], integration by parts, Stokes' theorem, and Lemma 4.11, we can directly check that

$$\int_{\Delta^n_a} \partial\overline{\partial}\varphi_0 \wedge d\eta_2 = 0$$

as in the proof of Lemma 4.12. □

By Lemma 4.13, we can see that

$$\sqrt{-1}\Theta_h(\mathscr{L}) = \sqrt{-1}\Theta_g(\mathscr{L}) + 2\sqrt{-1}\partial\overline{\partial}\varphi \tag{4.9}$$

coincides with T. Note that we took $\partial\overline{\partial}$ of φ as a current in (4.9). In particular,

$$\sqrt{-1}\Theta_h(\mathscr{L}) \geq 0,$$

that is, $\sqrt{-1}\Theta_h(\mathscr{L})$ is a closed positive current on X. By Lemma 2.5, φ is a quasi-plurisubharmonic function since φ is the smallest upper semicontinuous function that extends φ_0 to X.

Finally, we prove:

Lemma 4.15. *Let* φ *be a quasi-plurisubharmonic function on* Δ^n_a *for some* $0 < a < e^{-1}$. *Assume that there exist some positive integers* $a_1, \cdots, a_n$ *and a positive real number* C *such that*

$$-\varphi(z) \leq \log\left(C\left(-\log|z_1|\right)^{a_1} \cdots \left(-\log|z_n|\right)^{a_n}\right)$$

holds for all $z \in (\Delta^*_a)^n$. *Then the Lelong number of* φ *at* 0 *is zero.*

Proof. We denote the Lelong number of φ at x by $\nu(\varphi, x)$. We can easily see that

$$0 \leq \nu(\varphi, 0) = \liminf_{z\to 0} \frac{\varphi(z)}{\log|z|} \leq \liminf_{z\to 0} \frac{\log\left(C\left(-\log|z_1|\right)^{a_1} \cdots \left(-\log|z_n|\right)^{a_n}\right)}{-\log|z|} \leq 0$$

holds. Therefore, the Lelong number $\nu(\varphi, 0)$ of φ at 0 is zero. □

Thus we obtain Theorem 1.1 by Lemma 4.15.

Now Corollary 1.2 is almost obvious by Theorem 1.1.

Proof of Corollary 1.2. We put $\pi : Y = \mathbb{P}_X(F^b) \to X$ and $Y_0 = \pi^{-1}(X_0)$. We consider the variation of Hodge structure $\pi^* V_0$ on Y_0. Then $\pi^* F^b$ is the canonical extension of the lowest piece of the Hodge filtration. By applying Theorem 1.1 to the natural map $\pi^* F^b \to \mathscr{O}_{\mathbb{P}_X(F^b)}(1) \to 0$, we obtain a singular hermitian metric on $\mathscr{O}_{\mathbb{P}_X(F^b)}(1)$ with the desired properties. $\square$

5. Proof of Theorem 1.5

In this section, we will prove Theorem 1.5 and Corollary 1.6. We will only explain how to modify the arguments in Section 4.

5.1 (Proof of Theorem 1.5)**.** Let $\{F_0^p\}$ be the Hodge filtration of the polarizable variation of $\mathbb{R}$-Hodge structure $V_0 = (\mathbb{V}_0, F_0)$ on X_0. We put

$$0 = F_0^{b+1} \subsetneq F_0^b \subseteq \cdots \subseteq F_0^{a+1} \subsetneq F_0^a = \mathscr{V}_0 := \mathbb{V}_0 \otimes \mathscr{O}_{X_0}.$$

By assumption, $\mathscr{M}$ is a holomorphic line subbundle of $\bigoplus_p \operatorname{Gr}_F^p \mathscr{V}$. Therefore, $\mathscr{M}$ is naturally a holomorphic line subbundle of $\mathscr{Q} := \bigoplus_{p=a+1}^{b+1} \mathscr{V}/F^p$. Then we have the following big commutative diagram of holomorphic vector bundles on X.

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & \mathscr{M} \arrow[d] & \\
0 \arrow[r] & \mathscr{S}' \arrow[r] \arrow[d] & \bigoplus\limits_{\text{finite}} \mathscr{V} \arrow[r] \arrow[d, equal] & \mathscr{Q} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{S} \arrow[r] \arrow[d] & \bigoplus\limits_{\text{finite}} \mathscr{V} \arrow[r] & \mathscr{Q}' \arrow[r] \arrow[d] & 0 \\
 & \mathscr{M} \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}$$

We note that $\mathscr{S}' = \bigoplus_{p=a+1}^{b+1} F^p$ and $\mathscr{Q}' = \mathscr{Q}/\mathscr{M}$ and that $\mathscr{S}$ is the kernel of the naturally induced surjection $\bigoplus_{\text{finite}} \mathscr{V} \to \mathscr{Q}'$. By taking the dual of the above commutative diagram, $\mathscr{M}^\vee$ is a quotient bundle of $\mathscr{Q}^\vee$ and $\mathscr{Q}^\vee$ is a subbundle of $\bigoplus\limits_{\text{finite}} \mathscr{V}^\vee$. Therefore, we can apply the same arguments as in Section 4 to $\mathscr{M}^\vee$ by considering the polarizable variation of $\mathbb{R}$-Hodge structure $\bigoplus\limits_{\text{finite}} V_0^\vee$. Then we see that the Hodge metric of $\bigoplus\limits_{\text{finite}} V_0^\vee$ induces the desired singular hermitian metric h on $\mathscr{M}^\vee$ by Lemma 2.9.

Finally, we give a proof of Corollary 1.6.

Proof of Corollary 1.6. We put $\pi : Y = \mathbb{P}_X(A^\vee) \to X$ and $Y_0 = \pi^{-1}(X_0)$. We consider $\pi^* V_0$ on Y_0. Then $\pi^* A$ is contained in the kernel of the Higgs field

$$\pi^*\theta : \operatorname{Gr}_F^{\bullet} \pi^* \mathscr{V} \to \Omega_Y^1(\log \pi^* D) \otimes \operatorname{Gr}_F^{\bullet} \pi^* \mathscr{V}.$$

By applying Theorem 1.5 to the line subbundle $\mathcal{O}_{\mathbb{P}_X(A^\vee)}(-1)$ of $\pi^* A$, we obtain the desired result. $\square$

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