

NAKAI–MOISHEZON AMPLENESS CRITERION FOR REAL LINE BUNDLES

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ABSTRACT. We show that the Nakai–Moishezon ampleness criterion holds for real line bundles on complete schemes. As applications, we treat the relative Nakai–Moishezon ampleness criterion for real line bundles and the Nakai–Moishezon ampleness criterion for real line bundles on complete algebraic spaces. The main ingredient of this paper is Birkar’s characterization of augmented base loci of real divisors on projective schemes.

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1. INTRODUCTION

Throughout this paper, a *scheme* means a separated scheme of finite type over an algebraically closed field k of any characteristic. We call such a scheme a *variety* if it is reduced and irreducible. Let us start with the definition of \mathbb{R} -line bundles.

Definition 1.1 (\mathbb{R} -line bundles). Let X be a scheme (or an algebraic space). An \mathbb{R} -*line bundle* (resp. a \mathbb{Q} -*line bundle*) is an element of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$) where $\mathrm{Pic}(X)$ is the Picard group of X .

Similarly, we can define \mathbb{R} -Cartier divisors.

Definition 1.2 (\mathbb{R} -Cartier divisors). Let X be a scheme. An \mathbb{R} -*Cartier divisor* (resp. a \mathbb{Q} -*Cartier divisor*) is an element of $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$) where $\mathrm{Div}(X)$ denotes the group of Cartier divisors on X .

We prove the Nakai–Moishezon ampleness criterion for \mathbb{R} -line bundles. The following theorem is the main result of this paper.

Theorem 1.3 (Nakai–Moishezon ampleness criterion for real line bundles on complete schemes). *Let X be a complete scheme over an algebraically closed field and let \mathcal{L} be an \mathbb{R} -line bundle on X . Then \mathcal{L} is ample if and only if $\mathcal{L}^{\dim Z} \cdot Z > 0$ for every positive-dimensional closed integral subscheme $Z \subset X$.*

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When X is projective, Theorem 1.3 is well known. It was first proved by Campana and Peternell (see [CP, 1.3. Theorem]). Then a somewhat simpler proof was given by Lazarsfeld in [La, Theorem 2.3.18]. Unfortunately, their arguments do not work for complete nonprojective schemes because they need an ample line bundle. Moreover, Kleiman's ampleness criterion does not always hold for complete nonprojective schemes (see [F1, Section 3] and [F4, Example 12.1]). Hence we need some new idea to prove Theorem 1.3. By the standard reduction argument, it is sufficient to treat the case where X is a complete normal variety. Therefore, all we have to do is to establish the following theorem.

Theorem 1.4 (Nakai–Moishezon ampleness criterion for real Cartier divisors on complete normal varieties). *Let X be a complete normal variety over an algebraically closed field and let L be an \mathbb{R} -Cartier divisor on X . Then L is ample if and only if $L^{\dim Z} \cdot Z > 0$ for every positive-dimensional closed subvariety $Z \subset X$.*

For the proof of Theorem 1.4, we use Birkar's characterization of augmented base loci of \mathbb{R} -divisors on projective schemes (see Theorem 3.4). Hence our approach is different from those of [CP] and [La]. Although we can not directly apply geometric arguments to \mathbb{R} -line bundles, we can generalize Theorem 1.3 for proper morphisms.

Theorem 1.5 (Relative Nakai–Moishezon ampleness criterion for real line bundles). *Let $\pi: X \rightarrow S$ be a proper morphism between schemes and let \mathcal{L} be an \mathbb{R} -line bundle on X . Then \mathcal{L} is π -ample if and only if $\mathcal{L}^{\dim Z} \cdot Z > 0$ for every positive-dimensional closed integral subscheme $Z \subset X$ such that $\pi(Z)$ is a point.*

For the details of the Nakai–Moishezon ampleness criterion and Kleiman's ampleness criterion for line bundles, see [Kl]. The reader can find many nontrivial examples of complete nonprojective varieties in [F1], [F4, Section 12], [FP], and so on. Finally, we prove the following theorem as an application of Theorem 1.3 by using some basic properties of algebraic spaces.

Theorem 1.6 (Nakai–Moishezon ampleness criterion for real line bundles on complete algebraic spaces). *Let X be a complete algebraic space over an algebraically closed field and let \mathcal{L} be an \mathbb{R} -line bundle on X . Then \mathcal{L} is ample if and only if $\mathcal{L}^{\dim Z} \cdot Z > 0$ for every positive-dimensional closed integral subspace $Z \subset X$.*

We note that we treat algebraic spaces only in the final section, where we prove Theorem 1.6. We also note that the Nakai–Moishezon ampleness criterion for line bundles on complete algebraic spaces plays an crucial role in Kollár's projectivity criterion for moduli spaces (see [Ko] and [F3]).

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2. PRELIMINARIES

For simplicity of notation, we write the group law of $\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ additively.

Definition 2.1. Let \mathcal{L} be an \mathbb{R} -line bundle on a complete scheme X .

- If $\mathcal{L} = \sum_j l_j \mathcal{L}_j$ such that l_j is a positive real number and \mathcal{L}_j is an ample line bundle on X for every j , then we say that \mathcal{L} is *ample*.
- If $\mathcal{L} = \sum_j l_j \mathcal{L}_j$ such that l_j is a positive real number and \mathcal{L}_j is a semi-ample line bundle on X for every j , then we say that \mathcal{L} is *semi-ample*.
- If $\mathcal{L} \cdot C \geq 0$ for every curve C on X , then we say that \mathcal{L} is *nef*.

- We further assume that X is a variety. If $\mathcal{L} = \sum_j l_j \mathcal{L}_j$ such that l_j is a positive real number and \mathcal{L}_j is a big line bundle on X for every j , then we say that \mathcal{L} is *big*.

In the theory of minimal models, we usually use \mathbb{R} -Cartier divisors. In this paper, we do not use \mathbb{R} -Weil divisors. We only use \mathbb{R} -Cartier divisors.

Definition 2.2. Let X be a complete scheme. We consider the following natural homomorphism

$$\psi: \operatorname{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$

We note that ψ is not necessarily surjective. Let D be an \mathbb{R} -Cartier divisor on X . If the image of D by ψ is ample, semi-ample, nef, and big, then D is said to be *ample*, *semi-ample*, *nef*, and *big*, respectively. We also note that ψ is surjective when X is a variety.

For the basic properties of bigness and semi-ampleness, see [F2, Section 2.1]. Here we only explain the following important characterization of nef and big \mathbb{R} -divisors.

Lemma 2.3. *Let L be a nef \mathbb{R} -divisor on a projective variety X . Then L is big if and only if $L^{\dim X} > 0$.*

Proof. We put $n = \dim X$.

Step 1. If L is big, then we can write $L \sim_{\mathbb{R}} A + D$ where A is an ample \mathbb{R} -divisor and D is an effective \mathbb{R} -Cartier divisor on X , where $\sim_{\mathbb{R}}$ denotes the \mathbb{R} -linear equivalence of \mathbb{R} -Cartier divisors. Then

$$L^n = (A + D) \cdot L^{n-1} \geq A \cdot L^{n-1} = A \cdot (A + D) \cdot L^{n-2} \geq \dots \geq A^n > 0.$$

Hence $L^{\dim X} > 0$ holds true when L is big.

Step 2. In this step, we will check that L is big under the assumption that $L^n > 0$ holds. We will closely follow the proof of [La, Theorem 2.3.18]. We take ample \mathbb{R} -divisors A_1 and A_2 on X such that $L + A_1$ and $A_1 + A_2$ are \mathbb{Q} -Cartier divisors on X . Since $L^n > 0$, we can assume that

$$(L + A_1)^n > n \left((L + A_1)^{n-1} \cdot (A_1 + A_2) \right)$$

holds by taking A_1 and A_2 sufficiently small. By the numerical criterion for bigness (see [La, Theorem 2.2.15]),

$$L - A_2 = (L + A_1) - (A_1 + A_2)$$

is big. Hence L is also big.

We finish the proof of Lemma 2.3. □

3. AUGMENTED BASE LOCI OF \mathbb{R} -DIVISORS

In this section, we explain some properties of augmented base loci of \mathbb{R} -divisors following [B]. Let us start with the definition of base loci and stable base loci.

Definition 3.1 (Base loci and stable base loci of \mathbb{Q} -divisors). Let X be a projective scheme and let D be a Cartier divisor on X . The *base locus* of D is defined as

$$\operatorname{Bs}|D| = \{x \in X \mid \alpha \text{ vanishes at } x \text{ for every } \alpha \in H^0(X, \mathcal{O}_X(D))\}.$$

We consider $\operatorname{Bs}|D|$ with the reduced scheme structure.

The *stable base locus* of a \mathbb{Q} -Cartier divisor L on X is defined as

$$\mathbf{B}(L) = \bigcap_m \operatorname{Bs}|mL|$$

where m runs over all positive integers such that mL is Cartier. Note that $\mathbf{B}(L)$ is considered with the reduced scheme structure.

The notion of augmented base loci plays a crucial role in the theory of minimal models.

Definition 3.2 (Augmented base loci of \mathbb{R} -divisors). Let X be a projective scheme and let L be an \mathbb{R} -Cartier divisor on X . We put

$$\mathbf{B}_+(L) = \bigcap_H \mathbf{B}(L - H)$$

where H runs over all ample \mathbb{R} -divisors such that $L - H$ is \mathbb{Q} -Cartier. As usual, we consider $\mathbf{B}_+(L)$ with the reduced scheme structure. We call $\mathbf{B}_+(L)$ the *augmented base locus* of L .

Birkar defined $\mathbf{B}_+(L)$ differently (see [B, Definition 1,2]). Then he proved that his definition coincides with the usual one (see Definition 3.2). For the details, see [B, Lemma 3.1 (3)].

In order to explain Birkar's theorem (see Theorem 3.4), it is convenient to introduce the notion of exceptional loci of \mathbb{R} -divisors.

Definition 3.3 (Exceptional loci of \mathbb{R} -divisors). Let X be a projective scheme and let L be an \mathbb{R} -Cartier divisor on X . The *exceptional locus* of L is defined as

$$\mathbb{E}(L) = \bigcup_{\substack{\dim V > 0, \\ L|_V \text{ is not big}}} V,$$

that is, the union runs over the positive-dimensional subvarieties $V \subset X$ such that $L|_V$ is not big.

Note that $\mathbb{E}(L)$ is sometimes called the *null locus* of L when L is nef.

Theorem 3.4 ([B, Theorem 1.4]). *Let X be a projective scheme. Assume that L is a nef \mathbb{R} -divisor on X . Then*

$$\mathbf{B}_+(L) = \mathbb{E}(L)$$

holds.

For the details of Theorem 3.4, we strongly recommend the reader to see Birkar's original statement in [B, Theorem 1.4]. We will use Theorem 3.4 when X is a normal projective variety in the proof of Theorem 1.4.

4. PROOF OF THEOREM 1.4

In this section, we prove Theorem 1.4. The main ingredient of the proof of Theorem 1.4 below is Birkar's theorem (see Theorem 3.4).

Proof of Theorem 1.4. Let

$$X = \bigcup_{i=1}^k U_i$$

be a finite affine Zariski open cover of X . Let \bar{U}_i be the closure of U_i in \mathbb{P}^{N_i} . By [Lü, Lemma 2.2], which is an easy application of the flattening theorem (see [RG, Théorème (5.2.2)]), we can take an ideal sheaf \mathcal{I} on \bar{U}_i with $\text{Supp } \mathcal{O}_{\bar{U}_i}/\mathcal{I} \subset \bar{U}_i \setminus U_i$ such that the blow-up of \bar{U}_i along \mathcal{I} eliminates the indeterminacy of $\bar{U}_i \dashrightarrow X$. Therefore, by taking the normalization of the blow-up of \bar{U}_i along \mathcal{I} , we get a projective birational morphism $\pi_i: X_i \rightarrow X$ from a normal projective variety X_i such that $\pi_i: \pi_i^{-1}(U_i) \rightarrow U_i$ is an isomorphism.

$$\begin{array}{ccc} & X_i & \\ & \swarrow & \searrow \pi_i \\ \bar{U}_i & \dashrightarrow & X \end{array}$$

By Theorem 3.4, there exists an ample \mathbb{R} -divisor H_i on X_i such that $\pi_i^*L - H_i$ is \mathbb{Q} -Cartier and that

$$\mathbf{B}(\pi_i^*L - H_i) = \mathbf{B}_+(\pi_i^*L) = \mathbb{E}(\pi_i^*L)$$

holds. Let $\text{Exc}(\pi_i)$ be the exceptional locus of π_i . Then

$$\mathbb{E}(\pi_i^*L) = \text{Exc}(\pi_i)$$

holds by Lemma 2.3 and the assumption that $L^{\dim Z} \cdot Z > 0$ for every positive-dimensional closed subvariety $Z \subset X$. Since L is \mathbb{R} -Cartier, we can write

$$L = \sum_{j \in J} l_j L_j$$

such that l_j is a real number and L_j is Cartier for every $j \in J$. If $m_j \in \mathbb{Q}$ holds for every $j \in J$ and

$$\pi_i^* \left(\sum_{j \in J} m_j L_j \right) - \pi_i^*L + H_i$$

is ample, then

$$\mathbf{B} \left(\pi_i^* \left(\sum_{j \in J} m_j L_j \right) \right) \subset \mathbf{B}(\pi_i^*L - H_i) = \text{Exc}(\pi_i)$$

holds. This implies that

$$\mathbf{B} \left(\sum_{j \in J} m_j L_j \right) \subset \pi_i(\text{Exc}(\pi_i)) \subset X \setminus U_i.$$

Hence, there exists a positive real number ε such that if $m_j \in \mathbb{Q}$ and $|l_j - m_j| < \varepsilon$ for every $j \in J$ then

$$\mathbf{B} \left(\sum_{j \in J} m_j L_j \right) \subset \bigcap_{i=1}^k (X \setminus U_i) = \emptyset$$

holds. This means that $\sum_{j \in J} m_j L_j$ is semi-ample. By this fact, we can write

$$L = \sum_p r_p M_p$$

such that r_p is a positive real number and M_p is a semi-ample \mathbb{Q} -divisor for every p . Therefore, L is a semi-ample \mathbb{R} -divisor by definition. Thus there exist a morphism $f: X \rightarrow Y$ onto a normal projective variety Y with $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$ and an ample \mathbb{R} -divisor A on Y such that L is \mathbb{R} -linearly equivalent to f^*A (see [F2, Lemma 2.1.11]). By assumption, $L \cdot C > 0$ for every curve C on X . This implies that f is an isomorphism. Thus L is an ample \mathbb{R} -divisor. \square

5. PROOF OF THEOREM 1.3

In this section, we prove Theorem 1.3. More precisely, we reduce Theorem 1.3 to a special case where X is a normal variety, which is nothing but Theorem 1.4. Let us start with the following elementary lemma.

Lemma 5.1. *Let X be a complete scheme and let \mathcal{L} be an \mathbb{R} -line bundle on X . Let $X = \bigcup_{i=1}^k X_i$ be the irreducible decomposition of X . Then \mathcal{L} is ample if and only if $\mathcal{L}|_{(X_i)_{\text{red}}}$ is ample for every i .*

Proof. This statement is well known for \mathbb{Q} -line bundles. Hence we will freely use this lemma for \mathbb{Q} -line bundles in the following argument.

Step 1. If \mathcal{L} is ample, then it is obvious that so is $\mathcal{L}|_{(X_i)_{\text{red}}}$ for every i . This is because we can write

$$\mathcal{L} = \sum_j a_j \mathcal{L}_j,$$

where \mathcal{L}_j is an ample line bundle on X and $a_j > 0$ for every j .

Step 2. In this step, we will prove that \mathcal{L} is ample under the assumption that $\mathcal{L}|_{(X_i)_{\text{red}}}$ is ample for every i . Since \mathcal{L} is an \mathbb{R} -line bundle, we can write

$$\mathcal{L} = \sum_{j=1}^m l_j \mathcal{L}_j,$$

where $\mathcal{L}_j \in \text{Pic}(X)$ and $l_j \in \mathbb{R}$ for every j . We put

$$V_i = \left\{ (p_1, \dots, p_m) \in \mathbb{R}^m \left| \sum_{j=1}^m p_j \mathcal{L}_j|_{(X_i)_{\text{red}}} \text{ is ample} \right. \right\}$$

for every i . Then V_i contains an open neighborhood of $l = (l_1, \dots, l_m)$ for every i since $\sum_{j=1}^m l_j \mathcal{L}_j|_{(X_i)_{\text{red}}}$ is ample by assumption. Hence $V = \bigcap_{i=1}^k V_i$ contains a small open neighborhood of $l \in \mathbb{R}^m$. Thus we can take positive real numbers r_1, \dots, r_p and

$$v_1 = (v_{11}, \dots, v_{1m}), \dots, v_p = (v_{p1}, \dots, v_{pm}) \in V \cap \mathbb{Q}^m$$

such that $l = \sum_{\alpha=1}^p r_\alpha v_\alpha$. Then

$$\mathcal{A}_\alpha := \sum_{j=1}^m v_{\alpha j} \mathcal{L}_j \in \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is ample for every α since $v_\alpha \in V \cap \mathbb{Q}^m$. Since we can write

$$\mathcal{L} = \sum_{\alpha=1}^p r_\alpha \mathcal{A}_\alpha,$$

\mathcal{L} is ample by definition.

We finish the proof of Lemma 5.1. □

Lemma 5.2. *Let X be a complete variety and let \mathcal{L} be an \mathbb{R} -line bundle on X . Let $\pi: Y \rightarrow X$ be a finite surjective morphism between complete varieties. Then \mathcal{L} is ample if and only if $\pi^* \mathcal{L}$ is ample.*

Proof. This statement is well known for \mathbb{Q} -line bundles. Hence we will freely use this lemma for \mathbb{Q} -line bundles in this proof. Thus it is obvious that $\pi^* \mathcal{L}$ is ample when \mathcal{L} is ample. Therefore, it is sufficient to prove that \mathcal{L} is ample under the assumption that $\pi^* \mathcal{L}$ is ample. Since \mathcal{L} is an \mathbb{R} -line bundle, we can write

$$\mathcal{L} = \sum_{j=1}^m l_j \mathcal{L}_j,$$

where $\mathcal{L}_j \in \text{Pic}(X)$ and $l_j \in \mathbb{R}$ for every j . Since $\pi^* \mathcal{L}$ is ample, there exists a positive real number ε such that if $|l_j - \alpha_j| < \varepsilon$ for every j then

$$\pi^* \left(\sum_{j=1}^m \alpha_j \mathcal{L}_j \right)$$

is ample. Moreover, if we further assume $\alpha_j \in \mathbb{Q}$ for every j , then

$$\sum_{j=1}^m \alpha_j \mathcal{L}_j$$

is ample since π is a finite surjective morphism. Hence we can write

$$\mathcal{L} = \sum_i r_i \mathcal{A}_i$$

such that r_i is a positive real number and \mathcal{A}_i is an ample line bundle for every i . This means that \mathcal{L} is ample by definition. \square

Let us prove Theorem 1.3.

Proof of Theorem 1.3. By Lemma 5.1, we may assume that X is a variety. Let $\nu: X^\nu \rightarrow X$ be the normalization. Note that ν is a finite surjective morphism. Then by Lemma 5.2 it is sufficient to prove that $\nu^* \mathcal{L}$ is ample. Hence we may further assume that X is a complete normal variety. In this case, the ampleness of \mathcal{L} follows from Theorem 1.4. \square

6. PROOF OF THEOREM 1.5

In this section, we prove Theorem 1.5. The following lemma is well known for \mathbb{Q} -line bundles.

Lemma 6.1. *Let $\pi: X \rightarrow S$ be a proper surjective morphism between schemes and let \mathcal{L} be an \mathbb{R} -line bundle on X . Assume that $\mathcal{L}|_{X_s}$ is ample for every closed point $s \in S$, where $X_s = \pi^{-1}(s)$. Then \mathcal{L} is π -ample, that is, we can write*

$$\mathcal{L} = \sum_i a_i \mathcal{L}_i$$

in $\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ such that \mathcal{L}_i is a π -ample line bundle on X and a_i is a positive real number for every i .

Before we prove Lemma 6.1, we prepare the following lemma, which is also well known for \mathbb{Q} -line bundles.

Lemma 6.2. *Let $\pi: X \rightarrow S$ be a proper surjective morphism between schemes and let \mathcal{L} be an \mathbb{R} -line bundle on X . Assume that $\mathcal{L}|_{X_{s_0}}$ is ample for some closed point $s_0 \in S$, where $X_{s_0} = \pi^{-1}(s_0)$. Then there exists a Zariski open neighborhood U_{s_0} of s_0 such that $\mathcal{L}|_{\pi^{-1}(U_{s_0})}$ is ample over U_{s_0} .*

Although Lemmas 6.1 and 6.2 are more or less known to the experts, we can not find them in the standard literature. Hence we prove them here for the sake of completeness.

Proof of Lemma 6.2. Since \mathcal{L} is an \mathbb{R} -line bundle, there exist line bundles \mathcal{M}_j for $1 \leq j \leq k$ such that

$$\mathcal{L} = \sum_{j=1}^k b_j \mathcal{M}_j$$

in $\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, where $b_j \in \mathbb{R}$ for every j . We put

$$\mathcal{A} = \left\{ (c_1, \dots, c_k) \in \mathbb{R}^k \mid \sum_{j=1}^k c_j \mathcal{M}_j|_{X_{s_0}} \text{ is ample} \right\}.$$

Then \mathcal{A} contains a small open neighborhood of (b_1, \dots, b_k) . Hence we can write

$$\mathcal{L} = \sum_i a_i \mathcal{L}_i,$$

where \mathcal{L}_i is a line bundle on X such that a_i is a positive real number and $\mathcal{L}_i|_{X_{s_0}}$ is ample for every i . Since $\mathcal{L}_i|_{X_{s_0}}$ is ample for every i , there exists a Zariski open neighborhood U_{s_0} of s_0 such that $\mathcal{L}_i|_{\pi^{-1}(U_{s_0})}$ is ample over U_{s_0} for every i (see, for example, [KoM, Proposition 1.41]). Therefore, $\mathcal{L}|_{\pi^{-1}(U_{s_0})} = \sum_i a_i \mathcal{L}_i|_{\pi^{-1}(U_{s_0})}$ is ample over U_{s_0} . \square

Let us prove Lemma 6.1.

Proof of Lemma 6.1. We use the same notation as in the proof of Lemma 6.2. By Lemma 6.2, we can take $s_1, \dots, s_l \in S$ such that

$$\bigcup_{\alpha=1}^l U_{s_\alpha} = S$$

and that $\mathcal{L}|_{\pi^{-1}(U_{s_\alpha})}$ is ample over U_{s_α} for every α . We put

$$\mathcal{A}_\alpha = \left\{ (c_1, \dots, c_k) \in \mathbb{R}^k \mid \sum_{j=1}^k c_j \mathcal{M}_j|_{\pi^{-1}(U_{s_\alpha})} \text{ is } \pi\text{-ample over } U_{s_\alpha} \right\}.$$

Then \mathcal{A}_α contains a small open neighborhood of (b_1, \dots, b_k) . Therefore, $\bigcap_{\alpha=1}^l \mathcal{A}_\alpha$ also contains a small open neighborhood of (b_1, \dots, b_k) . Hence we can write

$$\mathcal{L} = \sum_i a_i \mathcal{L}_i$$

in $\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ such that \mathcal{L}_i is a π -ample line bundle on X and a_i is a positive real number for every i . \square

Finally, we prove Theorem 1.5.

Proof of Theorem 1.5. If \mathcal{L} is π -ample, then it is obvious that it satisfies the desired property. Hence, by Lemma 6.1, it is sufficient to prove that $\mathcal{L}|_{X_s}$ is ample for every closed point $s \in S$, where $X_s = \pi^{-1}(s)$, under the assumption that $\mathcal{L}^{\dim Z} \cdot Z > 0$. This follows from the Nakai–Moishezon ampleness criterion for \mathbb{R} -line bundles on complete schemes (see Theorem 1.3). \square

7. PROOF OF THEOREM 1.6

In this final section, we prove the Nakai–Moishezon ampleness criterion for \mathbb{R} -line bundles on complete algebraic spaces (Theorem 1.6) by using some basic properties of algebraic spaces and Theorem 1.3.

Proof of Theorem 1.6. It is well known that the Nakai–Moishezon ampleness criterion holds for line bundles on complete algebraic spaces (see, for example, [Ko, 3.11. Theorem] and [P, (1.4) Theorem]). It is also well known that there exists a finite surjective morphism $f: Y \rightarrow X$ from a complete scheme Y (see, for example, [Ko, 2.8. Lemma]). By Theorem 1.3, $f^*\mathcal{L}$ is an ample \mathbb{R} -line bundle on Y . We write

$$\mathcal{L} = \sum_j a_j \mathcal{L}_j,$$

where \mathcal{L}_j is a line bundle on X and a_j is a real number for every j . We put

$$\mathcal{M} = \sum_j b_j \mathcal{L}_j,$$

where b_j is a rational number for every j . If $|a_j - b_j| \ll 1$ for every j , then $f^*\mathcal{M}$ is an ample \mathbb{Q} -line bundle on Y since $f^*\mathcal{L}$ is ample. Therefore, $m\mathcal{M}$ is an ample line bundle on X for some positive integer m by the Nakai–Moishezon ampleness criterion for line bundles on complete algebraic spaces. This implies that X is projective. Thus, by Theorem 1.3 again, \mathcal{L} is an ample \mathbb{R} -line bundle on X . \square

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