On canonical rings

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History and Moriwaki's early works

History

In 1996:

- Moriwaki's seminar
  Text Book: D. Mumford: Algebraic Geometry I. Complex Projective Varieties
  Students: O. Fujino, K. Yamaki, and 4 others (Ono, Tane, Higuchi, Yuyama)
- Moriwaki's lecture
  Topic: elementary proof of Modell–Faltings
  The lecture note was published in 2017 by A. Moriwaki, S. Kawaguchi, and H. Ikoma

Moriwaki's early works

- Zariski decomposition (1986, 1988)
  There are many variants.
  Some generalization of the Zariski decomposition still plays a crucial role in MMP.
- Torsion-freeness (1987)
  His result was already generalized completely.
- Openness of nefness (1992)
  This is related to my recent work on MMP.
  This problem is still widely open.

Zariski decomposition, 1

Definition 1.1 (Zariski decomposition in C-K-M's sense)

Let $f\colon X \to Y$ be a proper surjective morphism of normal varieties. An expression $D = P + N$ of $\mathbb{R}$-Cartier divisors D, P and N is called the *Zariski decomposition in C-K-M's sense* if the following conditions are satisfied:

(1) P is f-nef,

(2) N is effective, and

(3) the natural homomorphisms $f_*\mathcal{O}_X(\lfloor mP \rfloor) \to f_*\mathcal{O}_X(\lfloor mD \rfloor)$ are bijective for all $m \in \mathbb{N}$.

In his master thesis, Moriwaki proved that P is semiample in some special case. Zariski decomposition is closely related to the finite generation of canonical rings.

Zariski decomposition, 2

Zariski decomposition still plays an important role in MMP.

Theorem 1.2 (V. Lazić and N. Tsakanikas)

Assume that any smooth projective variety V such that K_V is pseudo-effective has an NQC weak Zariski decomposition. Then every projective log canonical pair (X, Δ) such that $K_X + \Delta$ is pseudo-effective has a minimal model.

The existence problem of minimal models for log canonical pairs can be reduced to the existence problem of NQC weak Zariski decompositions for smooth projective varieties.

Torsion-freeness, 1

Theorem 1.3 (J. Kollár, 1986)

Let $f\colon X \to Y$ be a surjective morphism between complex projective varieties such that X is smooth. Then $R^i f_*\omega_X$ is torsion-free for every i.

This theorem can be proved easily by using Hodge theory.

Theorem 1.4 (A. Moriwaki, 1987)

Let $f\colon X \to Y$ be a projective surjective morphism between complex analytic varieties such that X is smooth. Then $R^i f_*\omega_X$ is torsion-free for every i.

Here, X is not necessarily compact. Hence we can not directly use the Hodge structure on X.

Torsion-freeness, 2

Takegoshi completely generalized Kollár's torsion-freeness.

Theorem 1.5 (K. Takegoshi, 1995)

Let $f\colon X \to Y$ be a proper surjective morphism between complex analytic varieties such that X is a Kähler manifold. Then $R^i f_*\omega_X$ is torsion-free for every i.

We skip my contributions on this topic. Then we finally get:

Theorem 1.6 (S. Matsumura)

Let $f\colon X \to Y$ be a proper surjective morphism between complex analytic varieties such that X is a Kähler manifold. Let (L, h) be a pseudo-effective line bundle on X. Then $R^i f_* (\omega_X \otimes L \otimes \mathcal{J}(h))$ is torsion-free for every i.

Openness of nefness, 1

- $f : X \to S$: projective morphism of algebraic varieties
- L: line bundle on X

We put:

$$\mathcal{A} := \{s \in S \mid L_s \text{ is ample on } X_s\}$$

$$\mathcal{N} := \{s \in S \mid L_s \text{ is nef on } X_s\}$$

Theorem 1.7

$\mathcal{A}$ is Zariski open in S

This theorem is well known. It is natural to ask:

Question 1.8

Is $\mathcal{N}$ Zariski open?

Openness of nefness, 2

Theorem 1.9 (Moriwaki?)

$\mathcal{N}$ is not Zariski open

Moriwaki constructed an explicit example in positive characteristic. He used Frobenius pull-backs of semistable rank 2 vector bundles on curves.

Question 1.10

Is $\mathcal{N}$ Zariski open in characteristic zero?

I think that everyone believes that $\mathcal{N}$ is not open. However, no example has been founded yet. When L is an $\mathbb{R}$-Cartier divisor, Lesieutre constructed an example such that $\mathcal{N}$ is not open.

On canonical rings

On canonical rings, 1

- X: compact complex manifold
- $\Delta = \sum_i a_i \Delta_i$ is a $\mathbb{Q}$-divisor on X with $0 \leq a_i \leq 1$ for every i, $\operatorname{Supp} \Delta$: simple normal crossing divisor

We put

$$R(X, \Delta) := \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

and call it the *log canonical ring* of (X, Δ). When $\Delta = 0$, we write

$$R(X) := \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$$

and call it the *canonical ring* of X.

On canonical rings, 2

We are mainly interested in:

Conjecture 2.1 (Finite generation of $R(X, \Delta)$)

$R(X, \Delta)$ is a finitely generated $\mathbb{C}$-algebra.

- If $\dim X = 1$, then X is a compact Riemann surface. In this case, the above conjecture is almost obvious.
- If $\dim X = 2$, then we can check that $R(X, \Delta)$ is always finitely generated.
  When $\dim X = 2$, $K_X + \Delta$ is big, and $a_i = 1$ for some i, the proof is much more difficult than we expected. This was carried out by Takao Fujita.

Hence we may assume that $\dim X \geq 3$.

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In dimension 3, Moriwaki established:

Theorem 2.2 (A. Moriwaki, 1988)

In dimension ≤ 3, $R(X)$ is always a finitely generated $\mathbb{C}$-algebra.

- In Theorem 2.2, it is sufficient to treat the case where $\dim X = 3$, $\kappa(X) = 2$, and X is not algebraic.
- Although I have not checked the details, I believe that we can prove that $R(X, \Delta)$ is always finitely generated in dimension ≤ 3.

On canonical rings, 4

We need the condition that X is Kähler in dimension ≥ 4.

Theorem 2.3 (P. M. H. Wilson, 1981)

There exists a compact non-Kähler manifold in dimension 4 such that $R(X)$ is not finitely generated.

Hence the correct conjecture is:

Conjecture 2.4 (Finite generation of $R(X, \Delta)$)

We further assume that X is Kähler. Then $R(X, \Delta)$ is a finitely generated $\mathbb{C}$-algebra.

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When X is projective, we have:

Theorem 2.5 (Birkar–Cascini–Hacon–M^{c}Kernan, 2010)

We assume that X is projective and that $a_i < 1$ holds for every i. Then $R(X, \Delta)$ is a finitely generated $\mathbb{C}$-algebra. In particular, $R(X)$ is a finitely generated $\mathbb{C}$-algebra.

- BCHM partially established MMP for kawamata log terminal pairs and obtained the above finite generation. It is a great work on MMP.

Idea of the proof: By Fujino–Mori's canonical bundle formula, we can reduce the problem to the case where $K_X + \Delta$ is big. In this case, by BCHM, we can prove the existence of good minimal model. Then $R(X, \Delta)$ is a finitely generated $\mathbb{C}$-algebra.

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For X, which is Kähler but is not projective, we have:

Theorem 2.6 (O. Fujino, 2015)

If X is a compact Kähler manifold and $a_i < 1$ holds for every i, then $R(X, \Delta)$ is a finitely generated $\mathbb{C}$-algebra.
In particular, $R(X)$ is always a finitely generated $\mathbb{C}$-algebra when X is Kähler.

- By this theorem, Moriwak's theorem, and Wilson's example, we completely solved the finite generation conjecture of canonical rings for compact complex manifolds

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Idea of Proof:

- We may assume that $\kappa(X, K_X + \Delta) \geq 1$.
- We take the Iitaka fibration $f\colon X \to Y$ with respect to $K_X + \Delta$.
- By a generalization of Fujino–Mori's canonical bundle formula, we can construct Δ_Y such that $R(X, \Delta)$ is finitely generated if and only if $R(Y, \Delta_Y)$ is finitely generated.
  Here, we need that X is Kähler and $a_i < 1$ for every i.
- By construction, $K_Y + \Delta_Y$ is big and Y is a smooth projective variety. Therefore, we can apply the result by BCHM.

On log canonical rings of projective varieties

On log canonical rings, 1

From now on, we always assume that X is a complex projective variety.

- X: smooth projective variety defined over $\mathbb{C}$
- $\Delta = \sum_i a_i \Delta_i$ is an $\mathbb{R}$-divisor on X with $0 \leq a_i \leq 1$ for every i, $\operatorname{Supp} \Delta$: simple normal crossing divisor
- $\lfloor \Delta \rfloor := \sum_{a_i = 1} \Delta_i$, the round-down of Δ

On log canonical rings, 2

We state our main goal again.

Conjecture 3.1 (Finite generation of $R(X, \Delta)$)

If we further assume that Δ is a $\mathbb{Q}$-divisor, then $R(X, \Delta)$ is finitely generated.

- Note that we have already known that $R(X, \Delta)$ is finitely generated when $\lfloor \Delta \rfloor = 0$ by BCHM. Hence we are only interested in the case where $\lfloor \Delta \rfloor \neq 0$.

On log canonical rings, 3

Conjecture (Conjecture A_n)

If $\dim X = n$ and Δ is a $\mathbb{Q}$-divisor, then $R(X, \Delta)$ is a finitely generated $\mathbb{C}$-algebra.

Conjecture (Conjecture B_n)

If we further assume that $\dim X = n$, Δ is a $\mathbb{Q}$-divisor, $\lfloor \Delta \rfloor$ is irreducible, and $K_X + \Delta$ is big, then $R(X, \Delta)$ is a finitely generated $\mathbb{C}$-algebra.

Conjecture (Conjecture C_n)

If $\dim X = n$ and $K_X + \Delta$ is pseudo-effective, then (X, Δ) has a good minimal model.

On log canonical rings, 4

Conjecture (Conjecture A_n)

If $\dim X = n$ and Δ is a $\mathbb{Q}$-divisor, then $R(X, \Delta)$ is a finitely generated $\mathbb{C}$-algebra.

- This conjecture is the main goal when X is a projective variety.
- In $n \leq 4$, Conjecture A_n holds true. We note: Conjecture $C_{\leq 3} \Longrightarrow$ Conjecture A_4
- If $\lfloor \Delta \rfloor = 0$, then $R(X, \Delta)$ is finitely generated by BCHM

On log canonical rings, 5

Conjecture (Conjecture B_n)

If we further assume that $\dim X = n$, Δ is a $\mathbb{Q}$-divisor, $\lfloor \Delta \rfloor$ is irreducible, and $K_X + \Delta$ is big, then $R(X, \Delta)$ is a finitely generated $\mathbb{C}$-algebra.

- This conjecture is a very special case of Conjecture A_n.
- We can run a $(K_X + \Delta)$-MMP with scaling. However, we have not known whether it always terminates or not. If $\lfloor \Delta \rfloor = 0$, then it terminates at a good minimal model. Then $R(X, \Delta)$ is finitely generated.

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Conjecture (Conjecture C$_n$)

If $\dim X = n$ and $K_X + \Delta$ is pseudo-effective, then (X, Δ) has a good minimal model.

- This conjecture says that after finitely many flips and divisorial contractions we can always obtain a birational model (Y, Δ_Y) such that $K_Y + \Delta_Y$ is semiample.
- This conjecture is the main goal of the theory of minimal models.

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The main result is:

Theorem 3.2 (O. Fujino and Y. Gongyo, 2017)

Conjectures A_n, B_n, and $\mathrm{C}_{\leq n-1}$ are all equivalent.

- This theorem shows that Conjecture A_n is a much more difficult conjecture than we expected. If Conjecture A_n holds true for every n, then we can prove that almost all conjectures on the theory of minimal models hold true.

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Idea of Proof:

- Conjecture $\mathrm{A}_n \Longrightarrow$ Conjecture B_n
  Conjecture B_n looks much easier to treat than Conjecture A_n...
  This step is obvious

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- Conjecture $\mathrm{B}_n \Longrightarrow$ Conjecture $\mathrm{C}_{\leq n-1}$
  We consider (X, Δ) with $\dim X = n - 1$
  Y: a cone over X
  $Z \to Y$: blow-up at the vertex
  We apply Conjecture B_n to (Z, Δ_Z) for some Δ_Z. We can prove Conjecture $\mathrm{C}_{\leq n-1}$.

Key point: we can prove the nonvanishing conjecture for smooth projective varieties in dimension $n - 1$ by Conjecture B_n. Then we can freely run the minimal model program with ample scaling in dimension $n - 1$ by Hashizume's result. The main problem is the semiampleness of $K + \Delta$.

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- Conjecture $C_{\leq n-1} \Longrightarrow$ Conjecture A_n
  We consider (X, Δ) with $\dim X = n$ and $\kappa(X, K_X + \Delta) \geq 1$. Then we can prove that (X, Δ) has a good minimal model.
  More precisely, by running the minimal model program with scaling, we can prove that (X, Δ) has a minimal model. The difficult part is to prove the semiampleness of $K + \Delta$.
  Fortunately, this part was already carried out by Fujino–Gongyo.

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One of the most difficult conjectures in MMP is:

Conjecture 3.3 (Abundance conjecture)

Let (X, Δ) be a projective log canonical pair such that $K_X + \Delta$ is nef. Then $K_X + \Delta$ is semiample.

- $\dim X \leq 3$, the abundance conjecture holds true in full generality.
- $\dim X \geq 4$, it is still widely open.

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Key result is:

Theorem 3.4 (O. Fujino and Y. Gongyo, 2014)

Let (X, Δ) be a projective semi-log canonical pair such that Δ is a $\mathbb{Q}$-divisor and let $\nu\colon X^\nu \to X$ be the normalization. We put $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$. Suppose that $K_{X^\nu} + \Theta$ is semiample. Then $K_X + \Delta$ is also semiample.

- This theorem plays a crucial role in various inductive arguments.

On Kähler manifolds

The following conjecture is still widely open.

Conjecture 3.5

When X is a nonprojective compact Kähler manifold and Δ is a $\mathbb{Q}$-divisor, $R(X, \Delta)$ is a finitely generated $\mathbb{C}$-algebra.

- This conjecture holds true when $\lfloor \Delta \rfloor = 0$. Therefore, we may assume that $\lfloor \Delta \rfloor \neq 0$.
- When $\lfloor \Delta \rfloor \neq 0$, we can not use canonical bundle formula to reduce the problem to the case where X is projective.
- Since X is not projective, we can not use the framework of MMP.

Thank you

Thank you very much!