Introduction to the log minimal model program for log canonical pairs

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Abstract

We describe the foundation of the log minimal model program for log canonical pairs according to Ambro's idea. We generalize Kollár's vanishing and torsion-free theorems for embedded simple normal crossing pairs. Then we prove the cone and contraction theorems for quasi-log varieties, especially, for log canonical pairs.

Contents

Chapter 1

Introduction

chap1

In this book, we describe the foundation of the log minimal model program (LMMP or MMP, for short) for log canonical pairs. We follow Ambro's idea in [Am1]. First, we generalize Kollár's vanishing and torsion-free theorems (cf. [Ko1]) for embedded normal crossing pairs. Next, we introduce the notion of quasi-log varieties. The key points of the theory of quasi-log varieties are adjunction and the vanishing theorem, which directly follow from Kollár's vanishing and torsion-free theorems for embedded normal crossing pairs. Finally, we prove the cone and contraction theorems for quasi-log varieties. The proofs are more or less routine works for experts once we know adjunction and the vanishing theorem for quasi-log varieties. Chapter 2 is an expanded version of my preprint [F9] and Chapter 3 is based on the preprint [F10].

After [KM] appeared, the log minimal model program has developed drastically. Shokurov's epoch-making paper [Sh1] gave us various new ideas. The book [Book] explains some of them in details. Now, we have [BCHM], where the log minimal model program for Kawamata log terminal pairs is established on some mild assumptions. In this book, we explain nothing on the results in [BCHM]. It is because many survey articles were and will be written for [BCHM]. See, for example, [CHKLM], [Dr], and [F19]. Here, we concentrate basics of the log minimal model program for log canonical pairs.

We do not discuss the log minimal model program for toric varieties. It is because we have already established the foundation of the toric Mori theory. We recommend the reader to see [R], [M, Chapter 14], [FS], [F5], and so on. Note that we will freely use the toric geometry to construct nontrivial examples explicitly.

The main ingredient of this book is the theory of mixed Hodge struc-

tures. All the basic results for Kawamata log terminal pairs can be proved without it. I think that the classical Hodge theory and the theory of variation of Hodge structures are sufficient for Kawamata log terminal pairs. For log canonical pairs, the theory of mixed Hodge structures seems to be indispensable. In this book, we do not discuss the theory of variation of Hodge structures nor canonical bundle formulas.

Apologies. After I finished writing a preliminary version of this book, I found a more direct approach to the log minimal model program for log canonical pairs. In [F21], I obtained a correct generalization of Shokurov's non-vanishing theorem for log canonical paris. It directly implies the base point free theorem for log canonical pairs. I also proved the rationality theorem and the cone theorem for log canonical pairs without using the framework of quasi-log varieties. The vanishing and torsion-free theorems we need in [F21] are essentially contained in [EV]. The reader can learn them by [F20], where I gave a short, easy, and almost self-contained proof to them. Therefore, now we can prove some of the results in this book in a more elementary manner. However, the method developed in [F21] can be applied only to log canonical pairs. So, [F21] will not decrease the value of this book. Instead, [F21] will complement the theory of quasi-log varieties. I am sorry that I do not discuss that new approach here.

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1.1 What is a quasi-log variety ?

In this section, we informally explain why it is natural to consider *quasi-log varieties*.

Let (Z, B_Z) be a log canonical pair and let $f : V \to Z$ be a resolution with

$$K_V + S + B = f^*(K_Z + B_Z),$$

where $\mathrm{Supp}(S + B)$ is a simple normal crossing divisor, S is reduced, and $\llcorner B \lrcorner \leq 0$. It is very important to consider the *locus of log canonical singularities* W of the pair (Z, B_Z), that is, $W = f(S)$. By the Kawamata–Viehweg vanishing theorem, we can easily check that

$$\mathcal{O}_W \simeq f_*\mathcal{O}_S(\ulcorner -B_S \urcorner),$$

where $K_S + B_S = (K_V + S + B)|_S$. In our case, $B_S = B|_S$. Therefore, it is natural to introduce the following notion. Precisely speaking, a qlc pair is a quasi-log pair with only qlc singularities (see Definition 3.29).

Definition 1.1 (Qlc pairs). A qlc pair $[X, \omega]$ is a scheme X endowed with an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor ω such that there is a proper morphism $f : (Y, B_Y) \to X$ satisfying the following conditions.

(1) Y is a simple normal crossing divisor on a smooth variety M and there exists an $\mathbb{R}$-divisor D on M such that $\mathrm{Supp}(D + Y)$ is a simple normal crossing divisor, Y and D have no common irreducible components, and $B_Y = D|_Y$.

(2) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.

(3) B_Y is a subboundary, that is, $b_i \leq 1$ for any i when $B_Y = \sum b_i B_i$.

(4) $\mathcal{O}_X \simeq f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1}) \urcorner)$, where $B_Y^{<1} = \sum_{b_i<1} b_i B_i$.

It is easy to see that the pair $[W, \omega]$, where $\omega = (K_X + B)|_W$, with $f : (S, B_S) \to W$ satisfies the definition of qlc pairs. We note that the pair $[Z, K_Z + B_Z]$ with $f : (V, S + B) \to Z$ is also a qlc pair since $f_*\mathcal{O}_V(\ulcorner -B^{\urcorner}) \simeq \mathcal{O}_Z$. Therefore, we can treat log canonical pairs and loci of log canonical singularities in the same framework once we introduce the notion of qlc pairs. Ambro found that a modified version of X-method, that is, the method introduced by Kawamata and used by him to prove the foundational results

of the log minimal model program for Kawamata log terminal pairs, works for qlc pairs if we generalize Kollár's vanishing and torsion-free theorems for embedded normal crossing pairs. It is the key idea of [Amb1].

1.2 A sample computation

The following theorem must motivate the reader to study our new framework.

moti1 **Theorem 1.2 (cf. Theorem 3.39 (ii)).** *Let X be a normal projective variety and B a boundary $\mathbb{R}$-divisor on X such that (X, B) is log canonical. Let L be a Cartier divisor on X. Assume that $L - (K_X + B)$ is ample. Let $\{C_i\}$ be* any *set of lc centers of the pair (X, B). We put $W = \bigcup C_i$ with a reduced scheme structure. Then we have*

$$H^i(X, \mathcal{I}_W \otimes \mathcal{O}_X(L)) = 0$$

for any $i > 0$, where $\mathcal{I}_W$ is the defining ideal sheaf of W on X. In particular, the restriction map

$$H^0(X, \mathcal{O}_X(L)) \to H^0(W, \mathcal{O}_W(L))$$

is surjective. Therefore, if (X, B) has a zero-dimensional lc center, then the linear system $|L|$ is not empty and the base locus of $|L|$ contains no zero-dimensional lc centers of (X, B).

Let us see a simple setting to understand the difference between our new framework and the traditional one.

1.3. Let X be a smooth projective surface and let C_1 and C_2 be smooth curves on X. Assume that C_1 and C_2 intersect only at a point P transversally. Let L be a Cartier divisor on X such that $L - (K_X + B)$ is ample, where $B = C_1 + C_2$. It is obvious that (X, B) is log canonical and P is an lc center of (X, B). Then, by Theorem 1.2, we can directly obtain

$$H^i(X, \mathcal{I}_P \otimes \mathcal{O}_X(L)) = 0$$

for any $i > 0$, where $\mathcal{I}_P$ is the defining ideal sheaf of P on X.

In the classical framework, we prove it as follows. Let C be a general curve passing through P. We take small positive rational numbers ε and δ such that $(X, (1 - \varepsilon)B + \delta C)$ is log canonical at P and is Kawamata log

terminal outside P. Since ε and δ are small, $L-(K_X+(1-\varepsilon)B+\delta C)$ is still ample. By the Nadel vanishing theorem, we obtain

$$H^i(X, \mathcal{I}_P \otimes \mathcal{O}_X(L)) = 0$$

for any $i > 0$. We note that $\mathcal{I}_P$ is nothing but the *multiplier ideal sheaf* associated to the pair $(X, (1-\varepsilon)B + \delta C)$.

By our new vanishing theorem, the reader will be released from annoyance of perturbing coefficients of boundary divisors.

We give a sample computation here. It may explain the reason why Kollár's torsion-free and vanishing theorems appear in the study of log canonical pairs. The actual proof of Theorem 1.2 depends on much more sophisticated arguments on the theory of mixed Hodge structures.

Example 1.4. Let S be a normal projective surface which has only one simple elliptic Gorenstein singularity $Q \in S$. We put $X = S \times \mathbb{P}^1$ and $B = S \times \{0\}$. Then the pair (X, B) is log canonical. It is easy to see that $P = (Q, 0) \in X$ is an lc center of (X, B). Let L be a Cartier divisor on X such that $L - (K_X + B)$ is ample. We have

$$H^i(X, \mathcal{I}_P \otimes \mathcal{O}_X(L)) = 0$$

for any $i > 0$, where $\mathcal{I}_P$ is the defining ideal sheaf of P on X. We note that X is not Kawamata log terminal and that P is not an isolated lc center of (X, B).

Proof. Let $\varphi : T \to S$ be the minimal resolution. Then we can write $K_T + C = \varphi^* K_S$, where C is the φ-exceptional elliptic curve on T. We put $Y = T \times \mathbb{P}^1$ and $f = \varphi \times \mathrm{id}_{\mathbb{P}^1} : Y \to X$, where $\mathrm{id}_{\mathbb{P}^1} : \mathbb{P}^1 \to \mathbb{P}^1$ is the identity. Then f is a resolution of X and we can write

$$K_Y + B_Y + E = f^*(K_X + B),$$

where B_Y is the strict transform of B on Y and $E \simeq C \times \mathbb{P}^1$ is the exceptional divisor of f. Let $g : Z \to Y$ be the blow-up along $E \cap B_Y$. Then we can write

$$K_Z + B_Z + E_Z + F = g^*(K_Y + B_Y + E) = h^*(K_X + B),$$

where $h = f \circ g$, B_Z (resp. E_Z) is the strict transform of B_Y (resp. E) on Z, and F is the g-exceptional divisor. We note that

$$\mathcal{I}_P \simeq h_* \mathcal{O}_Z(-F) \subset h_* \mathcal{O}_Z \simeq \mathcal{O}_X.$$

Since $-F = K_Z + B_Z + E_Z - h^*(K_X + B)$, we have

$$\mathcal{I}_P \otimes \mathcal{O}_X(L) \simeq h_*\mathcal{O}_Z(K_Z + B_Z + E_Z) \otimes \mathcal{O}_X(L - (K_X + B)).$$

So, it is sufficient to prove that

$$H^i(X, h_*\mathcal{O}_Z(K_Z + B_Z + E_Z) \otimes \mathcal{L}) = 0$$

for any $i > 0$ and any ample line bundle $\mathcal{L}$ on X. We consider the short exact sequence

$$0 \to \mathcal{O}_Z(K_Z) \to \mathcal{O}_Z(K_Z + E_Z) \to \mathcal{O}_{E_Z}(K_{E_Z}) \to 0.$$

We can easily check that

$$0 \to h_*\mathcal{O}_Z(K_Z) \to h_*\mathcal{O}_Z(K_Z + E_Z) \to h_*\mathcal{O}_{E_Z}(K_{E_Z}) \to 0$$

is exact and

$$R^i h_*\mathcal{O}_Z(K_Z + E_Z) \simeq R^i h_*\mathcal{O}_{E_Z}(K_{E_Z})$$

for any $i > 0$ by the Grauert–Riemenschneider vanishing theorem. We can directly check that

$$R^1 h_*\mathcal{O}_{E_Z}(K_{E_Z}) \simeq R^1 f_*\mathcal{O}_E(K_E) \simeq \mathcal{O}_D(K_D),$$

where $D = Q \times \mathbb{P}^1 \subset X$. Therefore, $R^1 h_*\mathcal{O}_Z(K_Z + E_Z) \simeq \mathcal{O}_D(K_D)$ is a torsion sheaf on X. However, it is torsion-free as a sheaf on D. It is a generalization of Kollár's torsion-free theorem. We consider

$$0 \to \mathcal{O}_Z(K_Z + E_Z) \to \mathcal{O}_Z(K_Z + B_Z + E_Z) \to \mathcal{O}_{B_Z}(K_{B_Z}) \to 0.$$

We note that $B_Z \cap E_Z = \emptyset$. Thus, we have

$$\begin{aligned} 0 &\to h_*\mathcal{O}_Z(K_Z + E_Z) \to h_*\mathcal{O}_Z(K_Z + B_Z + E_Z) \to h_*\mathcal{O}_{B_Z}(K_{B_Z}) \\ &\xrightarrow{\delta} R^1 h_*\mathcal{O}_Z(K_Z + E_Z) \to \cdots . \end{aligned}$$

Since $\mathrm{Supp} h_*\mathcal{O}_{B_Z}(K_{B_Z}) = B$, δ is a zero map by $R^1 h_*\mathcal{O}_Z(K_Z + B_Z) \simeq \mathcal{O}_D(K_D)$. Therefore, we know that the following sequence

$$0 \to h_*\mathcal{O}_Z(K_Z + E_Z) \to h_*\mathcal{O}_Z(K_Z + B_Z + E_Z) \to h_*\mathcal{O}_{B_Z}(K_{B_Z}) \to 0$$

is exact. By Kollár's vanishing theorem on B_Z, it is sufficient to prove that $H^i(X, h_*\mathcal{O}_Z(K_Z + E_Z) \otimes \mathcal{L}) = 0$ for any $i > 0$ and any ample line bundle $\mathcal{L}$. We have

$$H^i(X, h_*\mathcal{O}_Z(K_Z) \otimes \mathcal{L}) = H^i(X, h_*\mathcal{O}_{E_Z}(K_{E_Z}) \otimes \mathcal{L}) = 0$$

for any $i > 0$ by Kollár's vanishing theorem. By the following exact sequence

$$\begin{aligned} \cdots \to H^i(X, h_*\mathcal{O}_Z(K_Z) \otimes \mathcal{L}) &\to H^i(X, h_*\mathcal{O}_Z(K_Z + E_Z)) \\ &\to H^i(X, h_*\mathcal{O}_{E_Z}(K_{E_Z})) \to \cdots , \end{aligned}$$

we obtain the desired vanishing theorem. Anyway, we have

$$H^i(X, \mathcal{I}_P \otimes \mathcal{O}_X(L)) = 0$$

for any $i > 0$. □

1.3 Overview

We summarize the contents of this book.

In the rest of Chapter 1, we collect some preliminary results and notations. Moreover, we quickly review the classical log minimal model program.

In Chapter 2, we discuss Ambro's generalizations of Kollár's injectivity, vanishing, and torsion-free theorems for embedded normal crossing pairs. These results are indispensable for the theory of quasi-log varieties. To prove them, we recall some results on the mixed Hodge structures. For the details of Chapter 2, see Section 2.1, which is the introduction of Chapter 2.

In Chapter 3, we treat the log minimal model program for log canonical pairs. In Section 3.1, we explicitly state the cone and contraction theorems for log canonical pairs and prove the log flip conjecture I for log canonical pairs in dimension four. We also discuss the length of extremal rays for log canonical pairs with the aid of the recent result by [BCHM]. Subsection 3.1.4 contains Kollár's various examples. We prove that a log canonical flop does not always exist. In Section 3.2, we introduce the notion of quasi-log varieties and prove basic results, for example, adjunction and the vanishing theorem, for quasi-log varieties. Section 3.3 is devoted to the proofs of the fundamental theorems for quasi-log varieties. First, we prove the base point free theorem for quasi-log varieties. Then, we prove the rationality theorem

and the cone theorem for quasi-log varieties. Once we understand the notion of quasi-log varieties and how to use adjunction and the vanishing theorem, there are no difficulties to prove the above fundamental theorems.

In Chapter 4, we discuss some supplementary results. Section 4.1 is devoted to the proof of the base point free theorem of Reid–Fukuda type for quasi-log varieties with only qlc singularities. In Section 4.2, we prove that the non-klt locus of a dlt pair is Cohen–Macaulay as an application of the vanishing theorem in Chapter 2. Section 4.3 is a detailed description of Alexeev's criterion for Serre's S_3 condition. It is an application of the generalized torsion-free theorem. In Section 4.4, we recall the notion of toric polyhedra. We can easily check that a toric polyhedron has a natural quasi-log structure. Section 4.5 is a short survey of the theory of non-lc ideal sheaves. In the finial section, we mention effective base point free theorems for log canonical pairs.

In the final chapter: Chapter 5, we collect various examples of toric flips.

1.4 How to read this book ?

We assume that the reader is familiar with the classical log minimal model program, at the level of Chapters 2 and 3 in [KM]. It is not a good idea to read this book without studying the classical results discussed in [KM], [KMM], or [M]. We will quickly review the classical log minimal model program in Section 1.6 for the reader's convenience. If the reader understands [KM, Chapters 2 and 3], then it is not difficult to read [F16], which is a gentle introduction to the log minimal model program for lc pairs and written in the same style as [KM]. After these preparations, the reader can read Chapter 3 in this book without any difficulties. We note that Chapter 3 can be read before Chapter 2. The hardest part of this book is Chapter 2. It is very technical. So, the reader should have strong motivations before attacking Chapter 2.

1.5 Notation and Preliminaries

We will work over the complex number field $\mathbb{C}$ throughout this book. But we note that by using Lefschetz principle, we can extend almost everything to the case where the base field is an algebraically closed field of characteristic

zero. Note that every scheme in this book is assumed to be separated. We deal not only with the usual divisors but also with the divisors with rational and real coefficients, which turn out to be fruitful and natural.

1.5 (Divisors, $\mathbb{Q}$-divisors, and $\mathbb{R}$-divisors). For an $\mathbb{R}$-Weil divisor $D = \sum_{j=1}^{r} d_j D_j$ such that $D_i \neq D_j$ for $i \neq j$, we define the *round-up* $\ulcorner D \urcorner = \sum_{j=1}^{r} \ulcorner d_j \urcorner D_j$ (resp. the *round-down* $\llcorner D \lrcorner = \sum_{j=1}^{r} \llcorner d_j \lrcorner D_j$), where for any real number x, $\ulcorner x \urcorner$ (resp. $\llcorner x \lrcorner$) is the integer defined by $x \leq \ulcorner x \urcorner < x+1$ (resp. $x - 1 < \llcorner x \lrcorner \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \llcorner D \lrcorner$. We define

$$D^{=1} = \sum_{d_j=1} D_j, \quad D^{\leq 1} = \sum_{d_j \leq 1} d_j D_j,$$
$$D^{<1} = \sum_{d_j<1} d_j D_j, \quad \text{and} \quad D^{>1} = \sum_{d_j>1} d_j D_j.$$

The *support* of $D = \sum_{j=1}^{r} d_j D_j$, denoted by $\mathrm{Supp} D$, is the subscheme $\bigcup_{d_j \neq 0} D_j$. We call D a *boundary* (resp. *subboundary*) $\mathbb{R}$-divisor if $0 \leq d_j \leq 1$ (resp. $d_j \leq 1$) for any j. $\mathbb{Q}$-*linear equivalence* (resp. $\mathbb{R}$-*linear equivalence*) of two $\mathbb{Q}$-divisors (resp. $\mathbb{R}$-*divisors*) B_1 and B_2 is denoted by $B_1 \sim_{\mathbb{Q}} B_2$ (resp. $B_1 \sim_{\mathbb{R}} B_2$). Let $f : X \to Y$ be a morphism and B_1 and B_2 two $\mathbb{R}$-divisors on X. We say that they are *linearly* f*-equivalent* (denoted by $B_1 \sim_f B_2$) if and only if there is a Cartier divisor B on Y such that $B_1 \sim B_2 + f^*B$. We can define $\mathbb{Q}$-*linear* (resp. $\mathbb{R}$-*linear*) f*-equivalence* (denoted by $B_1 \sim_{\mathbb{Q},f} B_2$ (resp. $B_1 \sim_{\mathbb{R},f} B_2$)) similarly.

Let X be a normal variety. Then X is called $\mathbb{Q}$-*factorial* if every $\mathbb{Q}$-divisor is $\mathbb{Q}$-Cartier.

We quickly review the notion of singularities of pairs. For the details, see [KM, §2.3], [Ko4], and [F7]. See also the subsection 1.6.1.

1.6 (Singularities of pairs). For a proper birational morphism $f : X \to Y$, the *exceptional locus* $\mathrm{Exc}(f) \subset X$ is the locus where f is not an isomorphism. Let X be a normal variety and let B be an $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a resolution such that $\mathrm{Exc}(f) \cup f_*^{-1}B$ has a simple normal crossing support, where $f_*^{-1}B$ is the strict transform of B on Y. We write $K_Y = f^*(K_X + B) + \sum_i a_i E_i$ and $a(E_i, X, B) = a_i$. We say that (X, B) is *sub log canonical* (resp. *sub Kawamata log terminal*) (*sub lc* (resp. *sub klt*), for short) if and only if $a_i \geq -1$ (resp. $a_i > -1$) for any i. If

(X, B) is sub lc (resp. sub klt) and B is effective, then (X, B) is called *log canonical* (resp. *Kawamata log terminal*) (lc (resp. klt), for short). Note that the *discrepancy* $a(E, X, B) \in \mathbb{R}$ can be defined for any prime divisor E *over* X. Let (X, B) be a sub lc pair. If E is a prime divisor over X such that $a(E, X, B) = -1$, then the center $c_X(E)$ is called an *lc center* of (X, B).

dlt-def **Definition 1.7 (Divisorial log terminal pairs).** Let X be a normal variety and B a boundary $\mathbb{R}$-divisor such that $K_X + B$ is $\mathbb{R}$-Cartier. If there exists a resolution $f : Y \to X$ such that

(i) both $\mathrm{Exc}(f)$ and $\mathrm{Exc}(f) \cup \mathrm{Supp}(f_*^{-1}B)$ are simple normal crossing divisors on Y, and

(ii) $a(E, X, B) > -1$ for every exceptional divisor $E \subset Y$,

then (X, B) is called *divisorial log terminal* (*dlt*, for short).

For the details of dlt pairs, see Section 4.2. The assumption that $\mathrm{Exc}(f)$ is a divisor in Definition 1.7 (i) is very important. See Example 4.16 below.

We often use resolution of singularities. We need the following strong statement. We sometimes call it Szabó's resolution lemma (see [Sz] and [F7]).

15-resol **1.8 (Resolution lemma).** Let X be a smooth variety and D a reduced divisor on X. Then there exists a proper birational morphism $f : Y \to X$ with the following properties:

(1) f is a composition of blow-ups of smooth subvarieties,

(2) Y is smooth,

(3) $f_*^{-1}D \cup \mathrm{Exc}(f)$ is a simple normal crossing divisor, where $f_*^{-1}D$ is the strict transform of D on Y, and

(4) f is an isomorphism over U, where U is the largest open set of X such that the restriction $D|_U$ is a simple normal crossing divisor on U.

Note that f is projective and the exceptional locus $\mathrm{Exc}(f)$ is of pure codimension one in Y since f is a composition of blowing-ups.

The Kleiman–Mori cone is the basic object to study in the log minimal model program.

1.9 (Kleiman–Mori cone). Let X be an algebraic scheme over $\mathbb{C}$ and let $\pi : X \to S$ be a proper morphism to an algebraic scheme S. Let $\mathrm{Pic}(X)$ be the group of line bundles on X. Take a complete curve on X which is mapped to a point by π. For $\mathcal{L} \in \mathrm{Pic}(X)$, we define the intersection number $\mathcal{L} \cdot C = \deg_{\overline{C}} f^*\mathcal{L}$, where $f : \overline{C} \to C$ is the normalization of C. Via this intersection pairing, we introduce a bilinear form

$$\cdot : \mathrm{Pic}(X) \times Z_1(X/S) \to \mathbb{Z},$$

where $Z_1(X/S)$ is the free abelian group generated by integral curves which are mapped to points on S by π.

Now we have the notion of numerical equivalence both in $Z_1(X/S)$ and in $\mathrm{Pic}(X)$, which is denoted by $\equiv$, and we obtain a perfect pairing

$$N^1(X/S) \times N_1(X/S) \to \mathbb{R},$$

where

$$N^1(X/S) = \{\mathrm{Pic}(X)/\equiv\} \otimes \mathbb{R} \quad \text{and} \quad N_1(X/S) = \{Z_1(X/S)/\equiv\} \otimes \mathbb{R},$$

namely $N^1(X/S)$ and $N_1(X/S)$ are dual to each other through this intersection pairing. It is well known that $\dim_{\mathbb{R}} N^1(X/S) = \dim_{\mathbb{R}} N_1(X/S) < \infty$. We write $\rho(X/S) = \dim_{\mathbb{R}} N^1(X/S) = \dim_{\mathbb{R}} N_1(X/S)$. We define the Kleiman–Mori cone $\overline{NE}(X/S)$ as the closed convex cone in $N_1(X/S)$ generated by integral curves on X which are mapped to points on S by π. When $S = \mathrm{Spec}\mathbb{C}$, we drop $/\mathrm{Spec}\mathbb{C}$ from the notation, e.g., we simply write $N_1(X)$ in stead of $N_1(X/\mathrm{Spec}\mathbb{C})$.

Definition 1.10. An element $D \in N^1(X/S)$ is called *π-nef* (or *relatively nef for π*), if $D \geq 0$ on $\overline{NE}(X/S)$. When $S = \mathrm{Spec}\mathbb{C}$, we simply say that D is *nef*.

klei-th **Theorem 1.11 (Kleiman's criterion for ampleness).** *Let $\pi : X \to S$ be a* projective *morphism between algebraic schemes. Then $\mathcal{L} \in \mathrm{Pic}(X)$ is π-ample if and only if the numerical class of $\mathcal{L}$ in $N^1(X/S)$ gives a positive function on $\overline{NE}(X/S) \setminus \{0\}$.*

In Theorem 1.11, we have to assume that $\pi : X \to S$ is *projective* since there are complete non-projective algebraic varieties for which Kleiman's criterion does not hold. We recall the explicit example given in [F6] for the reader's convenience. For the details of this example, see [F6, Section 3].

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Example 1.12 (cf. [F6, Section 3]). We fix a lattice $N = \mathbb{Z}^3$. We take lattice points

$$\begin{aligned} &v_1 = (1, 0, 1), && v_2 = (0, 1, 1), && v_3 = (-1, -1, 1), \\ &v_4 = (1, 0, -1), && v_5 = (0, 1, -1), && v_6 = (-1, -1, -1). \end{aligned}$$

We consider the following fan

$$\Delta = \left\{ \begin{array}{ccc} \langle v_1, v_2, v_4\rangle, & \langle v_2, v_4, v_5\rangle, & \langle v_2, v_3, v_5, v_6\rangle, \\ \langle v_1, v_3, v_4, v_6\rangle, & \langle v_1, v_2, v_3\rangle, & \langle v_4, v_5, v_6\rangle, \\ \text{and their faces} & & \end{array} \right\}.$$

Then the toric variety $X = X(\Delta)$ has the following properties.

(i) X is a non-projective complete toric variety with $\rho(X) = 1$.

(ii) There exists a Cartier divisor D on X such that D is positive on $\overline{NE}(X) \setminus \{0\}$. In particular, $\overline{NE}(X)$ is a half line.

Therefore, Kleiman's criterion for ampleness does not hold for this X. We note that X is not $\mathbb{Q}$-factorial and that there is a torus invariant curve $C \simeq \mathbb{P}^1$ on X such that C is numerically equivalent to zero.

If X has only mild singularities, for example, X is $\mathbb{Q}$-factorial, then it is known that Theorem klei-th 1.11 holds even when $\pi : X \to S$ is *proper*. However, the Kleiman–Mori cone may not have enough informations when π is only proper.

fuji-pa
Example 1.13 (cf. [FP]). There exists a smooth complete toric threefold X such that $\overline{NE}(X) = N_1(X)$.

The description below helps the reader understand examples in fuji-pa [FP].

fp-example
Example 1.14. Let Δ be the fan in $\mathbb{R}^3$ whose rays are generated by $v_1 = (1, 0, 0), v_2 = (0, 1, 0), v_3 = (0, 0, 1), v_5 = (-1, 0, -1), v_6 = (-2, -1, 0)$ and whose maximal cones are

$$\langle v_1, v_2, v_3\rangle, \langle v_1, v_3, v_6\rangle, \langle v_1, v_2, v_5\rangle, \langle v_1, v_5, v_6\rangle, \langle v_2, v_3, v_5\rangle, \langle v_3, v_5, v_6\rangle.$$

Then the associated toric variety $X_1 = X(\Delta)$ is $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2))$. We take a sequence of blow-ups

$$Y \xrightarrow{f_3} X_3 \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1,$$

where f_1 is the blow-up along the ray $v_4 = (0, -1, -1) = 3v_1 + v_5 + v_6$, f_2 is along

$$v_7 = (-1, -1, -1) = \frac{1}{3}(2v_4 + v_5 + v_6),$$

and the final blow-up f_3 is along the ray

$$v_8 = (-2, -1, -1) = \frac{1}{2}(v_5 + v_6 + v_7).$$

Then we can directly check that Y is a smooth projective toric variety with $\rho(Y) = 5$.

Finally, we remove the wall $\langle v_1, v_5 \rangle$ and add the new wall $\langle v_2, v_4 \rangle$. Then we obtain a flop $\phi : Y \dashrightarrow X$. We note that $v_2 + v_4 - v_1 - v_5 = 0$. The toric variety X is nothing but $X(\Sigma)$ given in [FP, Example 1]. Thus, X is a smooth complete toric variety with $\rho(X) = 5$ and $\overline{NE}(X) = N_1(X)$. Therefore, a simple flop $\phi : Y \dashrightarrow X$ completely destroys the projectivity of Y.

We use the following convention throughout this book.

1.15. $\mathbb{R}_{>0}$ (resp. $\mathbb{R}_{\geq 0}$) denotes the set of positive (resp. non-negative) real numbers. $\mathbb{Z}_{>0}$ denotes the set of positive integers.

1.6 Quick review of the classical MMP

se-quiqui

In this section, we quickly review the classical MMP, at the level of [KM, Chapters 2 and 3], for the reader's convenience. For the details, see [KM, Chapters 2 and 3] or [KMM]. Almost all the results explained here will be described in more general settings in subsequent chapters.

1.6.1 Singularities of pairs

161-ss

We quickly review singularities of pairs in the log minimal model program. Basically, we will only use the notion of log canonical pairs in this book. So, the reader does not have to worry about the various notions of *log terminal*.

Definition 1.16 (Discrepancy). Let (X, Δ) be a pair where X is a normal variety and Δ an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Suppose $f : Y \to X$ is a resolution. Then, we can write

$$K_Y = f^*(K_X + \Delta) + \sum_i a(E_i, X, \Delta)E_i.$$

This formula means that

$$f_*(\sum_i a(E_i, X, \Delta)E_i) = -\Delta$$

and that $\sum_i a(E_i, X, \Delta)E_i$ is numerically equivalent to K_Y over X. The real number $a(E, X, \Delta)$ is called *discrepancy* of E with respect to (X, Δ). The *discrepancy* of (X, Δ) is given by

$$\mathrm{discrep}(X, \Delta) = \inf_E \{a(E, X, \Delta) \mid E \text{ is an exceptional divisor over } X\}.$$

We note that it is indispensable to understand how to calculate discrepancies for the study of the log minimal model program.

Definition 1.17 (Singularities of pairs). Let (X, Δ) be a pair where X is a normal variety and Δ an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. We say that (X, Δ) is

$$\begin{cases} \text{terminal} \\ \text{canonical} \\ \text{klt} \\ \text{plt} \\ \text{lc} \end{cases} \quad \text{if} \quad \mathrm{discrep}(X, \Delta) \quad \begin{cases} > 0, \\ \geq 0, \\ > -1 \quad \text{and} \quad \llcorner \Delta \lrcorner = 0, \\ > -1, \\ \geq -1. \end{cases}$$

Here, plt is short for *purely log terminal*.

The basic references on this topic are [km] [KM, 2.3], [ko-sing] [Ko4], and [fujino0] [F7].

1.6.2 Basic results for klt pairs

In this subsection, we assume that X is a projective variety and Δ is an effective $\mathbb{Q}$-divisor for simplicity. Let us recall the basic results for klt pairs. A starting point is the following vanishing theorem.

z-vani **Theorem 1.18 (Vanishing theorem).** *Let X be a smooth projective variety, D a $\mathbb{Q}$-divisor such that* $\mathrm{Supp}\{D\}$ *is a simple normal crossing divisor on X. Assume that D is ample. Then*

$$H^i(X, \mathcal{O}_X(K_X + \ulcorner D \urcorner)) = 0$$

for $i > 0$.

It is a special case of the Kawamata–Viehweg vanishing theorem. It easily follows from the Kodaira vanishing theorem by using the covering trick (see [KM, Theorem 2.64]). In Chapter 2, we will prove more general vanishing theorems. See, for example, Theorem 2.39. The next theorem is Shokurov's non-vanishing theorem.

z-non **Theorem 1.19 (Non-vanishig theorem).** *Let X be a projective variety, D a nef Cartier divisor and G a $\mathbb{Q}$-divisor. Suppose*

(1) *$aD + G - K_X$ is $\mathbb{Q}$-Cartier, ample for some $a > 0$, and*

(2) *$(X, -G)$ is sub klt.*

Then, for all $m \gg 0$, $H^0(X, \mathcal{O}_X(mD + \ulcorner G \urcorner)) \neq 0$.

It plays important roles in the proof of the base point free and rationality theorems below. In the theory of quasi-log varieties described in Chapter 3, the non-vanishing theorem will be absorbed into the proof of the base point free theorem for quasi-log varieties. The following two fundamental theorems for klt pairs will be generalized for quasi-log varieties in Chapter 3. See Theorems 3.66, 3.68, and 4.1 in Chapter 4.

z-bpf **Theorem 1.20 (Base point free theorem).** *Let (X, Δ) be a projective klt pair. Let D be a nef Cartier divisor such that $aD - (K_X + \Delta)$ is ample for some $a > 0$. Then $|bD|$ has no base points for all $b \gg 0$.*

Theorem 1.21 (Rationality theorem). *Let (X, Δ) be a projective klt pair such that $K_X + \Delta$ is not nef. Let $a > 0$ be an integer such that $a(K_X + \Delta)$ is Cartier. Let H be an ample Cartier divisor, and define*

$$r = \max\{\, t \in \mathbb{R} \mid H + t(K_X + \Delta) \text{ is nef} \,\}.$$

Then r is a rational number of the form u/v $(u, v \in \mathbb{Z})$ where

$$0 < v \leq a(\dim X + 1).$$

The final theorem is the cone theorem. It easily follows from the base point free and rationality theorems.

Theorem 1.22 (Cone theorem). *Let (X, Δ) be a projective klt pair. Then we have the following properties.*

(1) *There are (countably many) rational curves $C_j \subset X$ such that*

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\Delta)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_j].$$

(2) *Let $R \subset \overline{NE}(X)$ be a $(K_X + \Delta)$-negative extremal ray. Then there is a unique morphism $\varphi_R : X \to Z$ to a projective variety such that $(\varphi_R)_*\mathcal{O}_X \simeq \mathcal{O}_Z$ and an irreducible curve $C \subset X$ is mapped to a point by φ_R if and only if $[C] \in R$.*

We note that the cone theorem can be proved for dlt pairs in the relative setting. See, for example, [KMM]. We omit it here. It is because we will give a complete generalization of the cone theorem for quasi-log varieties in Theorem 3.75.

1.6.3 X-method

subsec-X

In this subsection, we give a proof to the base point free theorem (see Theorem 1.20) by assuming the non-vanishing theorem (see Theorem 1.19). The following proof is taken almost verbatim from [KM, 3.2 Basepoint-free Theorem]. This type of argument is sometimes called X-method. It has various applications in many different contexts. So, the reader should understand X-method.

Proof of the base point free theorem. We prove the base point free theorem: Theorem 1.20.

step-z1

Step 1. In this step, we establish that $|mD| \neq \emptyset$ for every $m \gg 0$. We can construct a resolution $f : Y \to X$ such that

(1) $K_Y = f^*(K_X + \Delta) + \sum a_j F_j$ with all $a_j > -1$,

(2) $f^*(aD - (K_X + \Delta)) - \sum p_j F_j$ is ample for some $a > 0$ and for suitable $0 < p_j \ll 1$, and

(3) $\sum F_j (\supset \mathrm{Exc}(f) \cup \mathrm{Supp} f_*^{-1}\Delta)$ is a simple normal crossing divisor on Y.

We note that the F_j is not necessarily f-exceptional. On Y, we write

$$\begin{aligned} &f^*(aD - (K_X + \Delta)) - \sum p_j F_j \\ &= af^*D + \sum (a_j - p_j)F_j - (f^*(K_X + \Delta) + \sum a_j F_j) \\ &= af^*D + G - K_Y, \end{aligned}$$

where $G = \sum(a_j - p_j)F_j$. By the assumption, $\ulcorner G \urcorner$ is an effective f-exceptional divisor, $af^*D + G - K_Y$ is ample, and

$$H^0(Y, \mathcal{O}_Y(mf^*D + \ulcorner G \urcorner)) \simeq H^0(X, \mathcal{O}_X(mD)).$$

We can now apply the non-vanishing theorem (see Theorem 1.19) to get that $H^0(X, \mathcal{O}_X(mD)) \neq 0$ for all $m \gg 0$.

step-z2 **Step 2.** For a positive integer s, let $B(s)$ denote the reduced base locus of $|sD|$. Clearly, we have $B(s^u) \subset B(s^v)$ for any positive integers $u > v$. Noetherian induction implies that the sequence $B(s^u)$ stabilizes, and we call the limit B_s. So either B_s is non-empty for some s or B_s and $B_{s'}$ are empty for two relatively prime integers s and s'. In the latter case, take u and v such that $B(s^u)$ and $B(s'^v)$ are empty, and use the fact that every sufficiently large integer is a linear combination of s^u and s'^v with non-negative coefficients to conclude that $|mD|$ is base point free for all $m \gg 0$. So, we must show that the assumption that some B_s is non-empty leads to a contradiction. We let $m = s^u$ such that $B_s = B(m)$ and assume that this set is non-empty.

Starting with the linear system obtained from Step 1, we can blow up further to obtain a new $f : Y \to X$ for which the conditions of Step 1 hold, and, for some $m > 0$,

$$f^*|mD| = |L| \text{ (moving part)} + \sum r_j F_j \text{ (fixed part)}$$

such that $|L|$ is base point free. Therefore, $\bigcup\{f(F_j) | r_j > 0\}$ is the base locus of $|mD|$. Note that $f^{-1}\mathrm{Bs}|mD| = \mathrm{Bs}|mf^*D|$. We obtain the desired contradiction by finding some F_j with $r_j > 0$ such that, for all $b \gg 0$, F_j is not contained in the base locus of $|bf^*D|$.

step-z3 **Step 3.** For an integer $b > 0$ and a rational number $c > 0$ such that $b \geq cm + a$, we define divisors:

$$\begin{aligned} N(b,c) &= bf^*D - K_Y + \sum(-cr_j + a_j - p_j)F_j \\ &= (b - cm - a)f^*D \qquad \text{(nef)} \\ &\quad + c(mf^*D - \sum r_j F_j) \qquad \text{(base point free)} \\ &\quad + f^*(aD - (K_X + \Delta)) - \sum p_j F_j \qquad \text{(ample)}. \end{aligned}$$

Thus, $N(b,c)$ is ample for $b \geq cm + a$. If that is the case then, by Theorem 1.18, $H^1(Y, \mathcal{O}_Y(\ulcorner N(b,c) \urcorner + K_Y)) = 0$, and

$$\ulcorner N(b,c) \urcorner = bf^*D + \sum \ulcorner -cr_j + a_j - p_j \urcorner F_j - K_Y.$$

step-z4 **Step 4.** c and p_j can be chosen so that

$$\sum(-cr_j + a_j - p_j)F_j = A - F$$

for some $F = F_{j_0}$, where $\ulcorner A \urcorner$ is effective and A does not have F as a component. In fact, we choose $c > 0$ so that

$$\min_j(-cr_j + a_j - p_j) = -1.$$

If this last condition does not single out a unique j, we wiggle the p_j slightly to achieve the desired uniqueness. This j satisfies $r_j > 0$ and $\ulcorner N(b,c) \urcorner + K_Y = bf^*D + \ulcorner A \urcorner - F$. Now Step 3 implies that

$$H^0(Y, \mathcal{O}_Y(bf^*D + \ulcorner A \urcorner)) \to H^0(F, \mathcal{O}_F(bf^*D + \ulcorner A \urcorner))$$

is surjective for $b \geq cm + a$. If F_j appears in $\ulcorner A \urcorner$, then $a_j > 0$, so F_j is f-exceptional. Thus, $\ulcorner A \urcorner$ is f-exceptional.

step-z5 **Step 5.** Notice that

$$N(b,c)|_F = (bf^*D + A - F - K_Y)|_F = (bf^*D + A)|_F - K_F.$$

So we can apply the non-vanishing theorem (see Theorem 1.19) on F to get

$$H^0(F, \mathcal{O}_F(bf^*D + \ulcorner A \urcorner)) \neq 0.$$

Thus, $H^0(Y, \mathcal{O}_Y(bf^*D + \ulcorner A \urcorner))$ has a section not vanishing on F. Since $\ulcorner A \urcorner$ is f-exceptional and effective,

$$H^0(Y, \mathcal{O}_Y(bf^*D + \ulcorner A \urcorner)) \simeq H^0(X, \mathcal{O}_X(bD)).$$

Therefore, $f(F)$ is not contained in the base locus of $|bD|$ for all $b \gg 0$.

This completes the proof of the base point free theorem. □

In the subsection 3.3.1, we will prove the base point free theorem for quasi-log varieties. We recommend the reader to compare the proof of Theorem 3.66 with the arguments explained here.

1.6.4 MMP for $\mathbb{Q}$-factorial dlt pairs

sub153

In this subsection, we explain the log minimal model program for $\mathbb{Q}$-factorial dlt pairs. First, let us recall the definition of the log minimal model.

z-minimal **Definition 1.23 (Log minimal model).** Let (X, Δ) be a log canonical pair and $f : X \to S$ a proper morphism. A pair (X', Δ') sitting in a diagram

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X' \\ f \searrow & & \swarrow f' \\ & S & \end{array}$$

is called a *log minimal model* of (X, Δ) over S if

(1) f' is proper,

(2) ϕ^{-1} has no exceptional divisors,

(3) $\Delta' = \phi_*\Delta$,

(4) $K_{X'} + \Delta'$ is f'-nef, and

(5) $a(E, X, \Delta) < a(E, X', \Delta')$ for every ϕ-exceptional divisor $E \subset X$.

Next, we recall the flip theorem for dlt pairs in [BCHM] and [HM]. We need the notion of small morphisms to treat flips.

Definition 1.24 (Small morphism). Let $f : X \to Y$ be a proper birational morphism between normal varieties. If $\mathrm{Exc}(f)$ has codimension ≥ 2, then f is called *small*.

z-flip **Theorem 1.25 (Log flip for dlt pairs).** *Let $\varphi : (X, \Delta) \to W$ be an extremal flipping contraction, that is,*

(1) *(X, Δ) is dlt,*

(2) *φ is small projective and φ has only connected fibers,*

(3) *$-(K_X + \Delta)$ is φ-ample,*

(4) *$\rho(X/W) = 1$, and*

(5) *X is $\mathbb{Q}$-factorial.*

Then we have the following diagram:

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow \quad \swarrow & \\ & W & \end{array}$$

(i) X^+ *is a normal variety,*

(ii) $\varphi^+ : X^+ \to W$ *is small projective, and*

(iii) $K_{X^+} + \Delta^+$ *is* φ^+*-ample, where* Δ^+ *is the strict transform of* Δ*.*

We call $\varphi^+ : (X^+, \Delta^+) \to W$ *a* $(K_X + \Delta)$*-flip of* φ*.*

Let us explain the relative log minimal model program for $\mathbb{Q}$-factorial dlt pairs.

1.26 (MMP for $\mathbb{Q}$-factorial dlt pairs). We start with a pair $(X, \Delta) = (X_0, \Delta_0)$. Let $f_0 : X_0 \to S$ be a projective morphism. The aim is to set up a recursive procedure which creates intermediate pairs (X_i, Δ_i) and projective morphisms $f_i : X_i \to S$. After some steps, it should stop with a final pair (X', Δ') and $f' : X' \to S$.

0ste **Step 0 (Initial datum).** Assume that we already constructed (X_i, Δ_i) and $f_i : X_i \to S$ with the following properties:

(1) X_i is $\mathbb{Q}$-factorial,

(2) (X_i, Δ_i) is dlt, and

(3) f_i is projective.

1ste **Step 1 (Preparation).** If $K_{X_i} + \Delta_i$ is f_i-nef, then we go directly to Step 3ste (2). If $K_{X_i} + \Delta_i$ is not f_i-nef, then we establish two results:

(1) (Cone Theorem) We have the following equality.

$$\overline{NE}(X_i/S) = \overline{NE}(X_i/S)_{(K_{X_i}+\Delta_i)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i].$$

(2) (Contraction Theorem) Any $(K_{X_i} + \Delta_i)$-negative extremal ray $R_i \subset \overline{NE}(X_i/S)$ can be contracted. Let $\varphi_{R_i} : X_i \to Y_i$ denote the corresponding contraction. It sits in a commutative diagram.

$$\begin{array}{ccc} X_i & \xrightarrow{\varphi_{R_i}} & Y_i \\ {\scriptstyle f_i}\searrow & & \swarrow{\scriptstyle g_i} \\ & S & \end{array}$$

2ste **Step 2 (Birational transformations).** If $\varphi_{R_i} : X_i \to Y_i$ is birational, then we produce a new pair (X_{i+1}, Δ_{i+1}) as follows.

(1) (Divisorial contraction) If φ_{R_i} is a divisorial contraction, that is, φ_{R_i} contracts a divisor, then we set $X_{i+1} = Y_i$, $f_{i+1} = g_i$, and $\Delta_{i+1} = (\varphi_{R_i})_*\Delta_i$.

(2) (Flipping contraction) If φ_{R_i} is a flipping contraction, that is, φ_{R_i} is small, then we set $(X_{i+1}, \Delta_{i+1}) = (X_i^+, \Delta_i^+)$, where (X_i^+, Δ_i^+) is the flip of φ_{R_i}, and $f_{i+1} = g_i \circ \varphi_{R_i}^+$. See Theorem 1.25.

In both cases, we can prove that X_{i+1} is $\mathbb{Q}$-factorial, f_{i+1} is projective and (X_{i+1}, Δ_{i+1}) is dlt. Then we go back to Step 0 with (X_{i+1}, Δ_{i+1}) and start anew.

3ste **Step 3 (Final outcome).** We expect that eventually the procedure stops, and we get one of the following two possibilities:

(1) (Mori fiber space) If φ_{R_i} is a Fano contraction, that is, $\dim Y_i < \dim X_i$, then we set $(X', \Delta') = (X_i, \Delta_i)$ and $f' = f_i$.

(2) (Minimal model) If $K_{X_i} + \Delta_i$ is f_i-nef, then we again set $(X', \Delta') = (X_i, \Delta_i)$ and $f' = f_i$. We can easily check that (X', Δ') is a log minimal model of (X, Δ) over S in the sense of Definition 1.23.

By the results in [BCHM] and [HM], all we have to do is to prove that there are no infinite sequence of flips in the above process.

We will discuss the log minimal model program for (not necessarily $\mathbb{Q}$-factorial) *lc* pairs in Section 3.1.

Chapter 2

Vanishing and Injectivity Theorems for LMMP

chap2

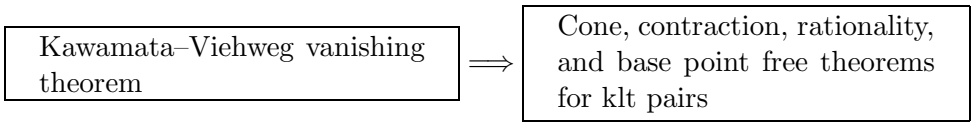
2.1 Introduction

21-sec

The following diagram is well known and described, for example, in [KM, 3.1]. See also Section 1.6.

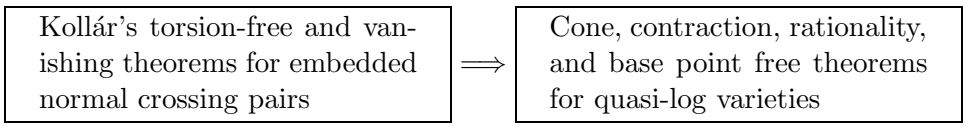
km

se-quiqui



This means that the Kawamata–Viehweg vanishing theorem produces the fundamental theorems of the log minimal model program (LMMP, for short) for klt pairs. This method is sometimes called X-method and now classical. It is sufficient for the LMMP for $\mathbb{Q}$-factorial dlt pairs. In [AmI], Ambro obtained the same diagram for *quasi-log varieties*. Note that the class of quasi-log varieties naturally contains lc pairs. Ambro introduced the notion of quasi-log varieties for the inductive treatments of lc pairs.

ambro



Namely, if we obtain Kollár's torsion-free and vanishing theorems for *embedded normal crossing pairs*, then X-method works and we obtain the fundamental theorems of the LMMP for quasi-log varieties. So, there exists an important problem for the LMMP for lc pairs.

1-pro **Problem 2.1.** *Are the injectivity, torsion-free and vanishing theorems for embedded normal crossing pairs true?*

Ambro gave an answer to Problem 2.1 in [Am1, Section 3]. Unfortunately, the proofs of injectivity, torsion-free, and vanishing theorems in [Am1, Section 3] contain various gaps. So, in this chapter, we give an affirmative answer to Problem 2.1 again.

ma-th **Theorem 2.2.** *Ambro's formulation of Kollár's injectivity, torsion-free, and vanishing theorems for embedded normal crossing pairs hold true.*

Once we have Theorem 2.2, we can obtain the fundamental theorems of the LMMP for lc pairs. The X-method for quasi-log varieties, which was explained in [Am1, Section 5] and will be described in Chapter 3, is essentially the same as the klt case. It may be more or less a routine work for the experts (see Chapter 3 and [F16]). We note that Kawamata used Kollár's injectivity, vanishing, and torsion-free theorems for *generalized normal crossing varieties* in [Ka1]. For the details, see [Ka1] or [KMM, Chapter 6]. We think that [Ka1] is the first place where X-method was used for reducible varieties.

Ambro's proofs of the injectivity, torsion-free, and vanishing theorems in [Am1] do not work even for smooth varieties. So, we need new ideas to prove the desired injectivity, torsion-free, vanishing theorems. It is the main subject of this chapter. We will explain various troubles in the proofs in [Am1, Section 3] below for the reader's convenience. Here, we give an application of Ambro's theorems to motivate the reader. It is the culmination of the works of several authors: Kawamata, Viehweg, Nadel, Reid, Fukuda, Ambro, and many others. It is the first time that the following theorem is stated explicitly in the literature.

Theorem 2.3 (cf. Theorem 2.48). *Let (X, B) be a proper lc pair such that B is a boundary $\mathbb{R}$-divisor and let L be a $\mathbb{Q}$-Cartier Weil divisor on X. Assume that $L - (K_X + B)$ is nef and log big. Then $H^q(X, \mathcal{O}_X(L)) = 0$ for any $q > 0$.*

It also contains a complete form of Kovács' Kodaira vanishing theorem for lc pairs (see Corollary 2.43). Let us explain the main trouble in [Am1, Section 3] by the following simple example.

0 **Example 2.4.** Let X be a smooth projective variety and H a Cartier divisor on X. Let A be a smooth irreducible member of $|2H|$ and S a smooth divisor on X such that S and A are disjoint. We put $B = \frac{1}{2}A+S$ and $L = H+K_X+S$. Then $L \sim_{\mathbb{Q}} K_X+B$ and $2L \sim 2(K_X+B)$. We define $\mathcal{E} = \mathcal{O}_X(-L+K_X)$ as in the proof of [ambro, Theorem 3.1]. Apply the argument in the proof of [ambro, Theorem 3.1]. Then we have a double cover $\pi : Y \to X$ corresponding to $2B \in |\mathcal{E}^{-2}|$. Then $\pi_*\Omega^p_Y(\log \pi^*B) \simeq \Omega^p_X(\log B) \oplus \Omega^p_X(\log B) \otimes \mathcal{E}(S)$. Note that $\Omega^p_X(\log B) \otimes \mathcal{E}$ is not a direct summand of $\pi_*\Omega^p_Y(\log \pi^*B)$. Theorem 3.1 in [ambro] claims that the homomorphisms $H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D))$ are injective for all q. Here, we used the notation in [ambro, Theorem 3.1]. In our case, $D = mA$ for some positive integer m. However, Ambro's argument just implies that $H^q(X, \mathcal{O}_X(L-\llcorner B \lrcorner)) \to H^q(X, \mathcal{O}_X(L-\llcorner B \lrcorner+D))$ is injective for any q. Therefore, his proof works only for the case when $\llcorner B \lrcorner = 0$ even if X is smooth.

This trouble is crucial in several applications on the LMMP. Ambro's proof is based on the mixed Hodge structure of $H^i(Y - \pi^*B, \mathbb{Z})$. It is a standard technique for vanishing theorems in the LMMP. In this chapter, we use the mixed Hodge structure of $H^i_c(Y - \pi^*S, \mathbb{Z})$, where $H^i_c(Y - \pi^*S, \mathbb{Z})$ is the cohomology group with compact support. Let us explain the main idea of this chapter. Let X be a smooth projective variety with $\dim X = n$ and D a simple normal crossing divisor on X. The main ingredient of our arguments is the decomposition

$$H^i_c(X - D, \mathbb{C}) = \bigoplus_{p+q=i} H^q(X, \Omega^p_X(\log D) \otimes \mathcal{O}_X(-D)).$$

The dual statement

$$H^{2n-i}(X - D, \mathbb{C}) = \bigoplus_{p+q=i} H^{n-q}(X, \Omega^{n-p}_X(\log D)),$$

which is well known and is commonly used for vanishing theorems, is not useful for our purposes. To solve Problem 1-pro, we have to carry out this simple idea for reducible varieties.

r-1 **Remark 2.5.** In the proof of [ambro, Theorem 3.1], if we assume that X is smooth, $B' = S$ is a reduced smooth divisor on X, and $T \sim 0$, then we need the E_1-degeneration of

$$E_1^{pq} = H^q(X, \Omega^p_X(\log S) \otimes \mathcal{O}_X(-S)) \Longrightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_X(\log S) \otimes \mathcal{O}_X(-S)).$$

However, Ambro seemed to confuse it with the E_1-degeneration of

$$E_1^{pq} = H^q(X, \Omega_X^p(\log S)) \Longrightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log S)).$$

Some problems on the Hodge theory seem to exist in the proof of [Am1, Theorem 3.1].

r0 **Remark 2.6.** In [Am2, Theorem 3.1], Ambro reproved his theorem under some extra assumptions. Here, we use the notation in [Am2, Theorem 3.1]. In the last line of the proof of [Am2, Theorem 3.1], he used the E_1-degeneration of some spectral sequence. It seems to be the E_1-degeneration of

$$E_1^{pq} = H^q(X', \widetilde{\Omega}_{X'}^p(\log \sum_{i'} E'_{i'})) \Longrightarrow \mathbb{H}^{p+q}(X', \widetilde{\Omega}_{X'}^\bullet(\log \sum_{i'} E'_{i'}))$$

since he cited [D1, Corollary 3.2.13]. Or, he applied the same type of E_1-degeneration to a desingularization of X'. However, we think that the E_1-degeneration of

$$\begin{aligned} E_1^{pq} &= H^q(X', \widetilde{\Omega}_{X'}^p(\log(\pi^* R + \sum_{i'} E'_{i'})) \otimes \mathcal{O}_{X'}(-\pi^* R)) \\ &\Longrightarrow \mathbb{H}^{p+q}(X', \widetilde{\Omega}_{X'}^\bullet(\log(\pi^* R + \sum_{i'} E'_{i'})) \otimes \mathcal{O}_{X'}(-\pi^* R)) \end{aligned}$$

is the appropriate one in his proof. If we assume that $T \sim 0$ in [Am2, Theorem 3.1], then Ambro's proof seems to imply that the E_1-degeneration of

$$E_1^{pq} = H^q(X, \Omega_X^p(\log R) \otimes \mathcal{O}_X(-R)) \Longrightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log R) \otimes \mathcal{O}_X(-R))$$

follows from the usual E_1-degeneration of

$$E_1^{pq} = H^q(X, \Omega_X^p) \Longrightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet).$$

Anyway, there are some problems in the proof of [Am2, Theorem 3.1]. In this chapter, we adopt the following spectral sequence

$$\begin{aligned} E_1^{pq} &= H^q(X', \widetilde{\Omega}_{X'}^p(\log \pi^* R) \otimes \mathcal{O}_{X'}(-\pi^* R)) \\ &\Longrightarrow \mathbb{H}^{p+q}(X', \widetilde{\Omega}_{X'}^\bullet(\log \pi^* R) \otimes \mathcal{O}_{X'}(-\pi^* R)) \end{aligned}$$

and prove its E_1-degeneration. For the details, see Sections 2.3 and 2.4.

One of the main contributions of this chapter is the rigorous proof of Proposition 2.23, which we call a fundamental injectivity theorem. Even if we prove this proposition, there are still several technical difficulties to recover Ambro's results on injectivity, torsion-free, and vanishing theorems: Theorems 2.53 and 2.54. Some important arguments are missing in [Am1]. We will discuss the other troubles on the arguments in [Am1] throughout Section 2.5. See also Section 2.9.

bunken **2.7 (Background, history, and related topics).** The standard references for vanishing, torsion-free, and injectivity theorems for the LMMP are [Ko3, Part III Vanishing Theorems] and the first half of the book [EV]. In this chapter, we closely follow the presentation of [EV] and that of [Am1]. Some special cases of Ambro's theorems were proved in [F4, Section 2]. Chapter 1 in [KMM] is still a good source for vanishing theorems for the LMMP. We note that one of the origins of Ambro's results is [Ka2, Section 4]. However, we do not treat Kawamata's generalizations of vanishing, torsion-free, and injectivity theorems for *generalized* normal crossing varieties. It is mainly because we can quickly reprove the main theorem of [Ka1] without appealing these difficult vanishing and injectivity theorems once we know a generalized version of Kodaira's canonical bundle formula. For the details, see [F11] or [F17].

We summarize the contents of this chapter. In Section 2.2, we collect basic definitions and fix some notations. In Section 2.3, we prove a fundamental cohomology injectivity theorem for simple normal crossing pairs. It is a very special case of Ambro's theorem. Our proof heavily depends on the E_1-degeneration of a certain Hodge to de Rham type spectral sequence. We postpone the proof of the E_1-degeneration in Section 2.4 since it is a purely Hodge theoretic argument. Section 2.4 consists of a short survey of mixed Hodge structures on various objects and the proof of the key E_1-degeneration. We could find no references on mixed Hodge structures which are appropriate for our purposes. So, we write it for the reader's convenience. Section 2.5 is devoted to the proofs of Ambro's theorems for embedded simple normal crossing pairs. We discuss various problems in [Am1, Section 3] and give the first rigorous proofs to [Am1, Theorems 3.1, 3.2] for embedded simple normal crossing pairs. We think that several indispensable arguments such as Lemmas 2.33, 2.34, and 2.36 are missing in [Am1, Section 3]. We treat some further generalizations of vanishing and torsion-free theorems in Section 2.6. In Section 2.7, we recover Ambro's theorems in full generality. We

recommend the reader to compare this chapter with [AmI]. We note that Section 2.7 seems to be unnecessary for applications. Section 2.8 is devoted to describe some examples. In Section 2.9, we will quickly review the structure of our proofs of the injectivity, torsion-free, and vanishing theorems. It may help the reader to understand the reason why our proofs are much longer than the original proofs in [AmI, Section 3]. In Chapter 3, we will treat the fundamental theorems of the LMMP for lc pairs as an application of our vanishing and torsion-free theorems. The reader can find various other applications of our new cohomological results in [F13], [F14], and [F15]. See also Sections 4.4, 4.5, and 4.6.

We note that we will work over $\mathbb{C}$, the complex number field, throughout this chapter.

2.2 Preliminaries

sec-pre

We explain basic notion according to [AmI, Section 2].

01 **Definition 2.8 (Normal and simple normal crossing varieties).** A variety X has *normal crossing* singularities if, for every closed point $x \in X$,

$$\widehat{\mathcal{O}}_{X,x} \simeq \frac{\mathbb{C}[[x_0, \cdots, x_N]]}{(x_0 \cdots x_k)}$$

for some $0 \leq k \leq N$, where $N = \dim X$. Furthermore, if each irreducible component of X is smooth, X is called a *simple normal crossing* variety. If X is a normal crossing variety, then X has only Gorenstein singularities. Thus, it has an invertible dualizing sheaf ω_X. So, we can define the *canonical divisor* K_X such that $\omega_X \simeq \mathcal{O}_X(K_X)$. It is a Cartier divisor on X and is well defined up to linear equivalence.

011 **Definition 2.9 (Mayer–Vietoris simplicial resolution).** Let X be a simple normal crossing variety with the irreducible decomposition $X = \bigcup_{i \in I} X_i$. Let I_n be the set of strictly increasing sequences $(i_0, \cdots, i_n)$ in I and $X^n = \coprod_{I_n} X_{i_0} \cap \cdots \cap X_{i_n}$ the disjoint union of the intersections of X_i. Let $\varepsilon_n : X^n \to X$ be the disjoint union of the natural inclusions. Then $\{X^n, \varepsilon_n\}_n$ has a natural semi-simplicial scheme structure. The face operator is induced by $\lambda_{j,n}$, where $\lambda_{j,n} : X_{i_0} \cap \cdots \cap X_{i_n} \to X_{i_0} \cap \cdots \cap X_{i_{j-1}} \cap X_{i_{j+1}} \cap \cdots \cap X_{i_n}$ is the natural closed embedding for $j \leq n$ (cf. [E2, 3.5.5]). We denote it by

$\varepsilon : X^\bullet \to X$ and call it the *Mayer–Vietoris simplicial resolution* of X. The complex

$$0 \to \varepsilon_{0*}\mathcal{O}_{X^0} \to \varepsilon_{1*}\mathcal{O}_{X^1} \to \cdots \to \varepsilon_{k*}\mathcal{O}_{X^k} \to \cdots,$$

where the differential $d_k : \varepsilon_{k*}\mathcal{O}_{X^k} \to \varepsilon_{k+1*}\mathcal{O}_{X^{k+1}}$ is $\sum_{j=0}^{k+1}(-1)^j\lambda_{j,k+1}^*$ for any $k \geq 0$, is denoted by $\mathcal{O}_{X^\bullet}$. It is easy to see that $\mathcal{O}_{X^\bullet}$ is quasi-isomorphic to $\mathcal{O}_X$. By tensoring $\mathcal{L}$, any line bundle on X, to $\mathcal{O}_{X^\bullet}$, we obtain a complex

$$0 \to \varepsilon_{0*}\mathcal{L}^0 \to \varepsilon_{1*}\mathcal{L}^1 \to \cdots \to \varepsilon_{k*}\mathcal{L}^k \to \cdots,$$

where $\mathcal{L}^n = \varepsilon_n^*\mathcal{L}$. It is denoted by $\mathcal{L}^\bullet$. Of course, $\mathcal{L}^\bullet$ is quasi-isomorphic to $\mathcal{L}$. We note that $H^q(X^\bullet, \mathcal{L}^\bullet)$ is $\mathbb{H}^q(X, \mathcal{L}^\bullet)$ by the definition and it is obviously isomorphic to $H^q(X, \mathcal{L})$ for any $q \geq 0$ because $\mathcal{L}^\bullet$ is quasi-isomorphic to $\mathcal{L}$.

0111 **Definition 2.10.** Let X be a simple normal crossing variety. A *stratum* of X is the image on X of some irreducible component of $X^\bullet$. Note that an irreducible component of X is a stratum of X.

02 **Definition 2.11 (Permissible and normal crossing divisors).** Let X be a simple normal crossing variety. A Cartier divisor D on X is called *permissible* if it induces a Cartier divisor $D^\bullet$ on $X^\bullet$. This means that $D^n = \varepsilon_n^*D$ is a Cartier divisor on X_n for any n. It is equivalent to the condition that D contains no strata of X in its support. We say that D is a *normal crossing* divisor on X if, in the notation of Definition 2.8, we have

$$\widehat{\mathcal{O}}_{D,x} \simeq \frac{\mathbb{C}[[x_0, \cdots, x_N]]}{(x_0 \cdots x_k, x_{i_1} \cdots x_{i_l})}$$

for some $\{i_1, \cdots, i_l\} \subset \{k+1, \cdots, N\}$. It is equivalent to the condition that D^n is a normal crossing divisor on X^n for any n in the usual sense. Furthermore, let D be a normal crossing divisor on a simple normal crossing variety X. If D^n is a simple normal crossing divisor on X^n for any n, then D is called a *simple normal crossing divisor* on X.

The following lemma is easy but important. We will repeatedly use it in Sections 2.3 and 2.5.

7 **Lemma 2.12.** *Let X be a simple normal crossing variety and B a permissible $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, that is, B is an $\mathbb{R}$-linear combination of permissible Cartier divisor on X, such that $\llcorner B \lrcorner = 0$. Let A be a Cartier divisor on X. Assume that $A \sim_{\mathbb{R}} B$. Then there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor C on X such that $A \sim_{\mathbb{Q}} C$, $\llcorner C \lrcorner = 0$, and* $\mathrm{Supp}C = \mathrm{Supp}B$.

Sketch of the proof. We can write $B = A + \sum_i r_i(f_i)$, where $f_i \in \Gamma(X, \mathcal{K}_X^*)$ and $r_i \in \mathbb{R}$ for any i. Here, $\mathcal{K}_X$ is the sheaf of total quotient ring of $\mathcal{O}_X$. First, we assume that X is smooth. In this case, the claim is well known and easy to check. Perturb r_i's suitably. Then we obtain a desired $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor C on X. It is an elementary problem of the linear algebra. In the general case, we take the normalization $\varepsilon_0 : X^0 \to X$ and apply the above result to X^0, ε_0^*A, ε_0^*B, and $\varepsilon_0^*(f_i)$'s. We note that $\varepsilon_0 : X_i \to X$ is a closed embedding for any irreducible component X_i of X^0. So, we get a desired $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor C on X. □

03 **Definition 2.13 (Simple normal crossing pair).** We say that the pair (X, B) is a *simple normal crossing pair* if the following conditions are satisfied.

(1) X is a simple normal crossing variety, and

(2) B is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor whose support is a simple normal crossing divisor on X.

We say that a simple normal crossing pair (X, B) is *embedded* if there exists a closed embedding $\iota : X \to M$, where M is a smooth variety of dimension $\dim X + 1$. We put $K_{X^0} + \Theta = \varepsilon_0^*(K_X + B)$, where $\varepsilon_0 : X^0 \to X$ is the normalization of X. From now on, we assume that B is a subboundary $\mathbb{R}$-divisor. A *stratum* of (X, B) is an irreducible component of X or the image of some lc center of (X^0, Θ) on X. It is compatible with Definition 2.10 0111 when $B = 0$. A Cartier divisor D on a simple normal crossing pair (X, B) is called *permissible with respect to* (X, B) if D contains no strata of the pair (X, B).

smooth **Remark 2.14.** Let (X, B) be a simple normal crossing pair. Assume that X is smooth. Then (X, B) is embedded. It is because X is a divisor on $X \times C$, where C is a smooth curve.

We give a typical example of embedded simple normal crossing pairs.

Example 2.15. Let M be a smooth variety and X a simple normal crossing divisor on M. Let A be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on M such that $\mathrm{Supp}(X+A)$ is simple normal crossing on M and that X and A have no common irreducible components. We put $B = A|_X$. Then (X, B) is an embedded simple normal crossing pair.

The reader will find that it is very useful to introduce the notion of *global embedded simple normal crossing pairs*.

gsnc0 **Definition 2.16 (Global embedded simple normal crossing pairs).** Let Y be a simple normal crossing divisor on a smooth variety M and let D be an $\mathbb{R}$-divisor on M such that $\mathrm{Supp}(D+Y)$ is simple normal crossing and that D and Y have no common irreducible components. We put $B_Y = D|_Y$ and consider the pair (Y, B_Y). We call (Y, B_Y) a *global embedded simple normal crossing pair*.

The following lemma is obvious.

use **Lemma 2.17.** *Let $(X, S+B)$ be an embedded simple normal crossing pair such that $S+B$ is a boundary $\mathbb{R}$-divisor, S is reduced, and $\llcorner B \lrcorner = 0$. Let M be the ambient space of X and $f : N \to M$ the blow-up along a smooth irreducible component C of $\mathrm{Supp}(S+B)$. Let Y be the strict transform of X on N. Then Y is a simple normal crossing divisor on N. We can write $K_Y + S_Y + B_Y = f^*(K_X + S + B)$, where $S_Y + B_Y$ is a boundary $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y such that S_Y is reduced and $\llcorner B_Y \lrcorner = 0$. In particular, $(Y, S_Y + B_Y)$ is an embedded simple normal crossing pair. By the construction, we can easily check the following properties.*

(i) *S_Y is the strict transform of S on Y if $C \subset \mathrm{Supp} B$,*

(ii) *B_Y is the strict transform of B on Y if $C \subset \mathrm{Supp} S$,*

(iii) *the f-image of any stratum of $(Y, S_Y + B_Y)$ is a stratum of $(X, S+B)$, and*

(iv) *$R^i f_* \mathcal{O}_Y = 0$ for $i > 0$ and $f_* \mathcal{O}_Y \simeq \mathcal{O}_X$.*

As a consequence of Lemma use 2.17, we obtain a very useful lemma.

useful-lemma **Lemma 2.18.** *Let (X, B_X) be an embedded simple normal crossing pair, B_X a boundary $\mathbb{R}$-divisor, and M the ambient space of X. Then there is a projective birational morphism $f : N \to M$, which is a sequence of blow-ups as in* Lemma use *2.17, with the following properties.*

(i) *Let Y be the strict transform of X on N. We put $K_Y + B_Y = f^*(K_X + B_X)$. Then (Y, B_Y) is an embedded simple normal crossing pair. Note that B_Y is a boundary $\mathbb{R}$-divisor.*

(ii) $f : Y \to X$ *is an isomorphism at the generic point of any stratum of* Y. f*-image of any stratum of* (Y, B_Y) *is a stratum of* (X, B_X).

(iii) $R^i f_* \mathcal{O}_Y = 0$ *for any* $i > 0$ *and* $f_* \mathcal{O}_Y \simeq \mathcal{O}_X$.

(iv) *There exists an* $\mathbb{R}$*-divisor* D *on* N *such that* D *and* Y *have no common irreducible components and* $\mathrm{Supp}(D + Y)$ *is simple normal crossing on* N, *and* $B_Y = D|_Y$. *This means that the pair* (Y, B_Y) *is a global embedded simple normal crossing pair.*

The next lemma is also easy to prove.

useful-lem2 **Lemma 2.19 (cf. [ambro][Am1, p.216 embedded log transformation]).** *Let* X *be a simple normal crossing divisor on a smooth variety* M *and let* D *be an* $\mathbb{R}$*-divisor on* M *such that* $\mathrm{Supp}(D + X)$ *is simple normal crossing and that* D *and* X *have no common irreducible components. We put* $B = D|_X$. *Then* (X, B) *is a global embedded simple normal crossing pair. Let* C *be a smooth stratum of* $(X, B^{=1})$. *Let* $\sigma : N \to M$ *be the blow-up along* C. *We denote by* Y *the reduced structure of the total transform of* X *in* N. *we put* $K_Y + B_Y = f^*(K_X + B)$, *where* $f = \sigma|_Y$. *Then we have the following properties.*

(i) (Y, B_Y) *is an embedded simple normal crossing pair.*

(ii) $f_* \mathcal{O}_Y \simeq \mathcal{O}_X$ *and* $R^i f_* \mathcal{O}_Y = 0$ *for any* $i > 0$.

(iii) *The strata of* $(X, B^{=1})$ *are exactly the images of the strata of* $(Y, B_Y^{=1})$.

(iv) $\sigma^{-1}(C)$ *is a maximal* (*with respect to the inclusion*) *stratum of* $(Y, B_Y^{=1})$.

(v) *There exists an* $\mathbb{R}$*-divisor* E *on* N *such that* $\mathrm{Supp}(E+Y)$ *is simple normal crossing and that* E *and* Y *have no common irreducible components such that* $B_Y = E|_Y$.

(vi) *If* B *is a boundary* $\mathbb{R}$*-divisor, then so is* B_Y.

In general, normal crossing varieties are much more difficult than *simple* normal crossing varieties. We postpone the definition of *normal crossing pairs* in Section [sec6]2.7 to avoid unnecessary confusion. Let us recall the notion of semi-ample $\mathbb{R}$-divisors since we often use it in this book.

2.20 (Semi-ample $\mathbb{R}$-divisor). Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on a variety X and $\pi : X \to S$ a proper morphism. Then, D is π-semi-ample if $D \sim_{\mathbb{R}} f^*H$, where $f : X \to Y$ is a proper morphism over S and H a relatively ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y. It is not difficult to see that D is π-semi-ample if and only if $D \sim_{\mathbb{R}} \sum_i a_i D_i$, where a_i is a positive real number and D_i is a π-semi-ample Cartier divisor on X for any i.

In the following sections, we have to treat algebraic varieties with quotient singularities. All the V-manifolds in this book are obtained as cyclic covers of smooth varieties whose ramification loci are contained in simple normal crossing divisors. So, they also have toroidal structures. We collect basic definitions according to [St, Section 1], which is the best reference for our purposes.

v-n **2.21 (V-manifold).** A *V-manifold* of dimension N is a complex analytic space that admits an open covering $\{U_i\}$ such that each U_i is analytically isomorphic to V_i/G_i, where $V_i \subset \mathbb{C}^N$ is an open ball and G_i is a finite subgroup of $\mathrm{GL}(N, \mathbb{C})$. In this paper, G_i is always a cyclic group for any i. Let X be a V-manifold and Σ its singular locus. Then we define $\widetilde{\Omega}^{\bullet}_X = j_*\Omega^{\bullet}_{X-\Sigma}$, where $j : X - \Sigma \to X$ is the natural open immersion. A divisor D on X is called a *divisor with V-normal crossings* if locally on X we have $(X, D) \simeq (V, E)/G$ with $V \subset \mathbb{C}^N$ an open domain, $G \subset \mathrm{GL}(N, \mathbb{C})$ a small subgroup acting on V, and $E \subset V$ a G-invariant divisor with only normal crossing singularities. We define $\widetilde{\Omega}^{\bullet}_X(\log D) = j_*\Omega^{\bullet}_{X-\Sigma}(\log D)$. Furthermore, if D is Cartier, then we put $\widetilde{\Omega}^{\bullet}_X(\log D)(-D) = \widetilde{\Omega}^{\bullet}_X(\log D) \otimes \mathcal{O}_X(-D)$. This complex will play crucial roles in Sections 2.3 and 2.4.

2.3 Fundamental injectivity theorems

sec2

The following proposition is a reformulation of the well-known result by Esnault–Viehweg (cf. [EV, 3.2. Theorem. c), 5.1. b)]). Their proof in [EV] depends on the characteristic p methods obtained by Deligne and Illusie. Here, we give another proof for the later usage. Note that all we want to do in this section is to generalize the following result for simple normal crossing pairs.

1 **Proposition 2.22 (Fundamental injectivity theorem I).** *Let X be a proper smooth variety and $S + B$ a boundary $\mathbb{R}$-divisor on X such that the support*

of $S+B$ is simple normal crossing, S is reduced, and $\llcorner B \lrcorner = 0$. Let L be a Cartier divisor on X and let D be an effective Cartier divisor whose support is contained in Supp$B$. *Assume that $L \sim_{\mathbb{R}} K_X + S + B$. Then the natural homomorphisms*

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D)),$$

which are induced by the inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$, are injective for all q.

Proof. We can assume that B is a $\mathbb{Q}$-divisor and $L \sim_{\mathbb{Q}} K_X+S+B$ by Lemma 2.12. We put $\mathcal{L} = \mathcal{O}_X(L-K_X-S)$. Let ν be the smallest positive integer such that $\nu L \sim \nu(K_X+S+B)$. In particular, νB is an integral Weil divisor. We take the ν-fold cyclic cover $\pi' : Y' = \mathrm{Spec}_X \bigoplus_{i=0}^{\nu-1} \mathcal{L}^{-i} \to X$ associated to the section $\nu B \in |\mathcal{L}^\nu|$. More precisely, let $s \in H^0(X, \mathcal{L}^\nu)$ be a section whose zero divisor is νB. Then the dual of $s : \mathcal{O}_X \to \mathcal{L}^\nu$ defines a $\mathcal{O}_X$-algebra structure on $\bigoplus_{i=0}^{\nu-1} \mathcal{L}^{-i}$. For the details, see, for example, [EV, 3.5. Cyclic covers]. Let $Y \to Y'$ be the normalization and $\pi : Y \to X$ the composition morphism. Then Y has only quotient singularities because the support of νB is simple normal crossing (cf. [EV, 3.24. Lemma]). We put $T = \pi^* S$. The usual differential $d : \mathcal{O}_Y \to \widetilde{\Omega}^1_Y \subset \widetilde{\Omega}^1_Y(\log T)$ gives the differential $d : \mathcal{O}_Y(-T) \to \widetilde{\Omega}^1_Y(\log T)(-T)$. This induces a natural connection $\pi_*(d) : \pi_*\mathcal{O}_Y(-T) \to \pi_*(\widetilde{\Omega}^1_Y(\log T)(-T))$. It is easy to see that $\pi_*(d)$ decomposes into ν eigen components. One of them is $\nabla : \mathcal{L}^{-1}(-S) \to \Omega^1_X(\log(S+B)) \otimes \mathcal{L}^{-1}(-S)$ (cf. [EV, 3.2. Theorem. c)]). It is well known and easy to check that the inclusion $\Omega^\bullet_X(\log(S+B)) \otimes \mathcal{L}^{-1}(-S-D) \to \Omega^\bullet_X(\log(S+B)) \otimes \mathcal{L}^{-1}(-S)$ is a quasi-isomorphism (cf. [EV, 2.9. Properties]). On the other hand, the following spectral sequence

$$\begin{aligned} E_1^{pq} &= H^q(X, \Omega^p_X(\log(S+B)) \otimes \mathcal{L}^{-1}(-S)) \\ &\Longrightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_X(\log(S+B)) \otimes \mathcal{L}^{-1}(-S)) \end{aligned}$$

degenerates in E_1. This follows from the E_1-degeneration of

$$H^q(Y, \widetilde{\Omega}^p_Y(\log T)(-T)) \Longrightarrow \mathbb{H}^{p+q}(Y, \widetilde{\Omega}^\bullet_Y(\log T)(-T))$$

where the right hand side is isomorphic to $H^{p+q}_c(Y-T, \mathbb{C})$. We will discuss this E_1-degeneration in Section 2.4. For the details, see 2.31 in Section 2.4 below. We note that $\Omega^\bullet_X(\log(S+B)) \otimes \mathcal{L}^{-1}(-S)$ is a direct summand of

$\pi_*(\widetilde{\Omega}^\bullet_Y(\log T)(-T))$. We consider the following commutative diagram for any q.

$$\begin{array}{ccc} \mathbb{H}^q(X, \Omega^\bullet_X(\log(S+B)) \otimes \mathcal{L}^{-1}(-S)) & \xrightarrow{\alpha} & H^q(X, \mathcal{L}^{-1}(-S)) \\ \uparrow{\scriptstyle \gamma} & & \uparrow{\scriptstyle \beta} \\ \mathbb{H}^q(X, \Omega^\bullet_X(\log(S+B)) \otimes \mathcal{L}^{-1}(-S-D)) & \longrightarrow & H^q(X, \mathcal{L}^{-1}(-S-D)) \end{array}$$

Since γ is an isomorphism by the above quasi-isomorphism and α is surjective by the E_1-degeneration, we obtain that β is surjective. By the Serre duality, we obtain $H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}(S)) \to H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}(S+D))$ is injective for any q. This means that $H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D))$ is injective for any q. □

The next result is a key result of this chapter.

2 **Proposition 2.23 (Fundamental injectivity theorem II).** *Let $(X, S+B)$ be a simple normal crossing pair such that X is proper, $S+B$ is a boundary $\mathbb{R}$-divisor, S is reduced, and $\llcorner B \lrcorner = 0$. Let L be a Cartier divisor on X and let D be an effective Cartier divisor whose support is contained in* Supp$B$. *Assume that $L \sim_{\mathbb{R}} K_X + S + B$. Then the natural homomorphisms*

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D)),$$

which are induced by the inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$, are injective for all q.

Proof. We can assume that B is a $\mathbb{Q}$-divisor and $L \sim_{\mathbb{Q}} K_X + S + B$ by Lemma 2.12. Without loss of generality, we can assume that X is connected. Let $\varepsilon : X^\bullet \to X$ be the Mayer–Vietoris simplicial resolution of X. Let ν be the smallest positive integer such that $\nu L \sim \nu(K_X + S + B)$. We put $\mathcal{L} = \mathcal{O}_X(L - K_X - S)$. We take the ν-fold cyclic cover $\pi' : Y' \to X$ associated to $\nu B \in |\mathcal{L}^\nu|$ as in the proof of Proposition 2.22. Let $\widetilde{Y} \to Y'$ be the normalization of Y'. We can glue $\widetilde{Y}$ naturally along the inverse image of $\varepsilon_1(X^1) \subset X$ and then obtain a connected reducible variety Y and a finite morphism $\pi : Y \to X$. For a supplementary argument, see Remark 2.24 below. We can construct the Mayer–Vietoris simplicial resolution $\varepsilon : Y^\bullet \to Y$ and a natural morphism $\pi_\bullet : Y^\bullet \to X^\bullet$. Note that Definition 2.9 makes sense without any modifications though Y has singularities. The finite morphism $\pi_0 : Y^0 \to X^0$ is essentially the same as the finite cover constructed in Proposition 2.22. Note that the inverse image of an irreducible component

X_i of X by π_0 may be a disjoint union of copies of the finite cover constructed in the proof of Proposition 2.22. More precisely, let V be any stratum of X. Then $\pi^{-1}(V)$ is not necessarily connected and $\pi : \pi^{-1}(V) \to V$ may be a disjoint union of copies of the finite cover constructed in the proof of the Proposition 2.22. Since $H^q(X^\bullet, (\mathcal{L}^{-1}(-S-D))^\bullet) \simeq H^q(X, \mathcal{L}^{-1}(-S-D))$ and $H^q(X^\bullet, (\mathcal{L}^{-1}(-S))^\bullet) \simeq H^q(X, \mathcal{L}^{-1}(-S))$, it is sufficient to see that $H^q(X^\bullet, (\mathcal{L}^{-1}(-S-D))^\bullet) \to H^q(X^\bullet, (\mathcal{L}^{-1}(-S))^\bullet)$ is surjective. First, we note that the natural inclusion

$$\Omega^\bullet_{X^n}(\log(S^n+B^n)) \otimes (\mathcal{L}^{-1}(-S-D))^n \to \Omega^\bullet_{X^n}(\log(S^n+B^n)) \otimes (\mathcal{L}^{-1}(-S))^n$$

is a quasi-isomorphism for any $n \geq 0$ (cf. [EV, 2.9. Properties]). So,

$$\Omega^\bullet_{X^\bullet}(\log(S^\bullet+B^\bullet)) \otimes (\mathcal{L}^{-1}(-S-D))^\bullet \to \Omega^\bullet_{X^\bullet}(\log(S^\bullet+B^\bullet)) \otimes (\mathcal{L}^{-1}(-S)^\bullet)$$

is a quasi-isomorphism. We put $T = \pi^*S$. Then $\Omega^\bullet_{X^n}(\log(S^n+B^n)) \otimes (\mathcal{L}^{-1}(-S))^n$ is a direct summand of $\pi_{n*}\widetilde{\Omega}^\bullet_Y(\log T^n)(-T^n)$ for any $n \geq 0$. Next, we can check that

$$E_1^{pq} = H^q(Y^\bullet, \widetilde{\Omega}^p_{Y^\bullet}(\log T^\bullet)(-T^\bullet)) \Longrightarrow \mathbb{H}^{p+q}(Y, s(\widetilde{\Omega}^\bullet_{Y^\bullet}(\log T^\bullet)(-T^\bullet)))$$

degenerates in E_1. We will discuss this E_1-degeneration in Section 2.4. See 2.32 in Section 2.4. The right hand side is isomorphic to $H^{p+q}_c(Y-T, \mathbb{C})$. Therefore,

$$\begin{aligned} E_1^{pq} &= H^q(X^\bullet, \Omega^p_{X^\bullet}(\log(S^\bullet+B^\bullet)) \otimes (\mathcal{L}^{-1}(-S))^\bullet) \\ &\Longrightarrow \mathbb{H}^{p+q}(X, s(\Omega^\bullet_{X^\bullet}(\log(S^\bullet+B^\bullet)) \otimes (\mathcal{L}^{-1}(-S))^\bullet)) \end{aligned}$$

degenerates in E_1. Thus, we have the following commutative diagram.

$$\begin{array}{ccc} \mathbb{H}^q(X, s(\Omega^\bullet_{X^\bullet}(\log(S^\bullet+B^\bullet)) \otimes (\mathcal{L}^{-1}(-S))^\bullet)) & \xrightarrow{\alpha} & H^q(X^\bullet, (\mathcal{L}^{-1}(-S))^\bullet) \\ \uparrow \gamma & & \uparrow \beta \\ \mathbb{H}^q(X, s(\Omega^\bullet_{X^\bullet}(\log(S^\bullet+B^\bullet)) \otimes (\mathcal{L}^{-1}(-S-D))^\bullet)) & \longrightarrow & H^q(X^\bullet, (\mathcal{L}^{-1}(-S-D))^\bullet) \end{array}$$

As in the proof of Proposition 2.22, γ is an isomorphism and α is surjective. Thus, β is surjective. This implies the desired injectivity results. □

su **Remark 2.24.** For simplicity, we assume that $X = X_1 \cup X_2$, where X_1 and X_2 are smooth, and that $V = X_1 \cap X_2$ is irreducible. We consider the

natural projection $p : \widetilde{Y} \to X$. We note that $\widetilde{Y} = \widetilde{Y}_1 \coprod \widetilde{Y}_2$, where $\widetilde{Y}_i$ is the inverse image of X_i by p for $i = 1$ and 2. We put $p_i = p|_{\widetilde{Y}_i}$ for $i = 1$ and 2. It is easy to see that $p_1^{-1}(V)$ is isomorphic to $p_2^{-1}(V)$ over V. We denote it by W. We consider the following surjective $\mathcal{O}_X$-module homomorphism $\mu : p_*\mathcal{O}_{\widetilde{Y}_1} \oplus p_*\mathcal{O}_{\widetilde{Y}_2} \to p_*\mathcal{O}_W : (f, g) \mapsto f|_W - g|_W$. Let $\mathcal{A}$ be the kernel of μ. Then $\mathcal{A}$ is an $\mathcal{O}_X$-algebra and $\pi : Y \to X$ is nothing but $\mathrm{Spec}_X\mathcal{A} \to X$. We can check that $\pi^{-1}(X_i) \simeq \widetilde{Y}_i$ for $i = 1$ and 2 and that $\pi^{-1}(V) \simeq W$.

rem-1 **Remark 2.25.** As pointed out in the introduction, the proof of [Amb1, Theorem 3.1] only implies that the homomorphisms

$$H^q(X, \mathcal{O}_X(L - S)) \to H^q(X, \mathcal{O}_X(L - S + D))$$

are injective for all q. When $S = 0$, we do not need the mixed Hodge structure on the cohomology with compact support. The mixed Hodge structure on the usual singular cohomology is sufficient for the case when $S = 0$.

We close this section with an easy application of Proposition 2.23. The following vanishing theorem is the Kodaira vanishing theorem for simple normal crossing varieties.

3 **Corollary 2.26.** *Let X be a projective simple normal crossing variety and $\mathcal{L}$ an ample line bundle on X. Then $H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}) = 0$ for any $q > 0$.*

Proof. We take a general member $B \in |\mathcal{L}^l|$ for some $l \gg 0$. Then we can find a Cartier divisor M such that $M \sim_{\mathbb{Q}} K_X + \frac{1}{l}B$ and $\mathcal{O}_X(K_X) \otimes \mathcal{L} \simeq \mathcal{O}_X(M)$. By Proposition 2.23, we obtain injections $H^q(X, \mathcal{O}_X(M)) \to H^q(X, \mathcal{O}_X(M + mB))$ for any q and any positive integer m. Since B is ample, Serre's vanishing theorem implies the desired vanishing theorem. □

2.4 E_1-degenerations of Hodge to de Rham type spectral sequences

sec3 From 2.27 to 2.29, we recall some well-known results on mixed Hodge structures. We use the notations in [D2] freely. The basic references on this topic are [D2, Section 8], [E1, Part II], and [E2, Chapitres 2 and 3]. The recent book [PS] may be useful. The starting point is the pure Hodge structures on proper smooth algebraic varieties.

s1 **2.27.** (Hodge structures for proper smooth varieties). Let X be a proper smooth algebraic variety over $\mathbb{C}$. Then the triple $(\mathbb{Z}_X, (\Omega_X^\bullet, F), \alpha)$, where $\Omega_X^\bullet$ is the holomorphic de Rham complex with the filtration bête F and $\alpha : \mathbb{C}_X \to \Omega_X^\bullet$ is the inclusion, is a cohomological Hodge complex (CHC, for short) of weight zero.

The next one is also a fundamental example. For the details, see [elzein [E1, I.1.]] or [elzein2 [E2, 3.5]].

s2 **2.28.** (Mixed Hodge structures for proper simple normal crossing varieties). Let D be a proper simple normal crossing algebraic variety over $\mathbb{C}$. Let $\varepsilon : D^\bullet \to D$ be the Mayer–Vietoris simplicial resolution (cf. Definition [011 2.9]). The following complex of sheaves, denoted by $\mathbb{Q}_{D^\bullet}$,

$$0 \to \varepsilon_{0*}\mathbb{Q}_{D^0} \to \varepsilon_{1*}\mathbb{Q}_{D^1} \to \cdots \to \varepsilon_{k*}\mathbb{Q}_{D^k} \to \cdots ,$$

is a resolution of $\mathbb{Q}_D$. More explicitly, the differential $d_k : \varepsilon_{k*}\mathbb{Q}_{D^k} \to \varepsilon_{k+1*}\mathbb{Q}_{D^{k+1}}$ is $\sum_{j=0}^{k+1}(-1)^j\lambda_{j,k+1}^*$ for any $k \geq 0$. For the details, see [elzein [E1, I.1.]] or [elzein2 [E2, 3.5.3]]. We obtain the resolution $\Omega_{D^\bullet}^\bullet$ of $\mathbb{C}_D$ as follows,

$$0 \to \varepsilon_{0*}\Omega_{D^0}^\bullet \to \varepsilon_{1*}\Omega_{D^1}^\bullet \to \cdots \to \varepsilon_{k*}\Omega_{D^k}^\bullet \to \cdots .$$

Of course, $d_k : \varepsilon_{k*}\Omega_{D^k}^\bullet \to \varepsilon_{k+1*}\Omega_{D^{k+1}}^\bullet$ is $\sum_{j=0}^{k+1}(-1)^j\lambda_{j,k+1}^*$. Let $s(\Omega_{D^\bullet}^\bullet)$ be the simple complex associated to the double complex $\Omega_{D^\bullet}^\bullet$. The Hodge filtration F on $s(\Omega_{D^\bullet}^\bullet)$ is defined by $F^p = s(0 \to \cdots \to 0 \to \varepsilon_*\Omega_{D^\bullet}^p \to \varepsilon_*\Omega_{D^\bullet}^{p+1} \to \cdots)$. We note that $\varepsilon_*\Omega_{D^\bullet}^p = (0 \to \varepsilon_{0*}\Omega_{D^0}^p \to \varepsilon_{1*}\Omega_{D^1}^p \to \cdots \to \varepsilon_{k*}\Omega_{D^k}^p \to \cdots)$. There exist natural weight filtrations W's on $\mathbb{Q}_{D^\bullet}$ and $s(\Omega_{D^\bullet}^\bullet)$. We omit the definition of the weight filtrations W's on $\mathbb{Q}_{D^\bullet}$ and $s(\Omega_{D^\bullet}^\bullet)$ since we do not need their explicit descriptions. See [elzein [E1, I.1.]] or [elzein2 [E2, 3.5.6]]. Then $(\mathbb{Z}_D, (\mathbb{Q}_{D^\bullet}, W), (s(\Omega_{D^\bullet}^\bullet), W, F))$ is a cohomological mixed Hodge complex (CMHC, for short). This CMHC induces a natural mixed Hodge structure on $H^\bullet(D, \mathbb{Z})$.

For the precise definitions of CHC and CMHC (CHMC, in French), see [deligne [D2, Section 8]] or [elzein2 [E2, Chapitre 3]]. The third example is not so standard but is indispensable for our injectivity theorems.

s3 **2.29.** (Mixed Hodge structure on the cohomology with compact support). Let X be a proper smooth algebraic variety over $\mathbb{C}$ and D a simple normal crossing divisor on X. We consider the mixed cone of $\mathbb{Q}_X \to \mathbb{Q}_{D^\bullet}$ with

suitable shifts of complexes and weight filtrations (for the details, see [E1, I.3.] or [E2, 3.7.14]). We obtain a complex $\mathbb{Q}_{X-D^\bullet}$, which is quasi-isomorphic to $j_!\mathbb{Q}_{X-D}$, where $j : X-D \to X$ is the natural open immersion, and a weight filtration W on $\mathbb{Q}_{X-D^\bullet}$. We define in the same way, that is, by taking a cone of a morphism of complexes $\Omega^\bullet_X \to \Omega^\bullet_{D^\bullet}$, a complex $\Omega^\bullet_{X-D^\bullet}$ with filtrations W and F. Then we obtain that the triple $(j_!\mathbb{Z}_{X-D}, (\mathbb{Q}_{X-D^\bullet}, W), (\Omega^\bullet_{X-D^\bullet}, W, F))$ is a CMHC. It defines a natural mixed Hodge structure on $H^\bullet_c(X-D, \mathbb{Z})$. Since we can check that the complex

$$\begin{aligned} 0 \to \Omega^\bullet_X(\log D)(-D) \to \Omega^\bullet_X \to \varepsilon_{0*}\Omega^\bullet_{D^0} \\ \to \varepsilon_{1*}\Omega^\bullet_{D^1} \to \cdots \to \varepsilon_{k*}\Omega^\bullet_{D^k} \to \cdots \end{aligned}$$

is exact by direct local calculations, we see that $(\Omega^\bullet_{X-D^\bullet}, F)$ is quasi-isomorphic to $(\Omega^\bullet_X(\log D)(-D), F)$ in $D^+F(X, \mathbb{C})$, where

$$\begin{aligned} &F^p\Omega^\bullet_X(\log D)(-D) \\ &\quad = (0 \to \cdots \to 0 \to \Omega^p_X(\log D)(-D) \to \Omega^{p+1}_X(\log D)(-D) \to \cdots). \end{aligned}$$

Therefore, the spectral sequence

$$E_1^{pq} = H^q(X, \Omega^p_X(\log D)(-D)) \Longrightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_X(\log D)(-D))$$

degenerates in E_1 and the right hand side is isomorphic to $H^{p+q}_c(X-D, \mathbb{C})$.

From here, we treat mixed Hodge structures on much more complicated algebraic varieties.

s4 **2.30.** (Mixed Hodge structures for proper simple normal crossing pairs). Let (X, D) be a proper simple normal crossing pair over $\mathbb{C}$ such that D is reduced. Let $\varepsilon : X^\bullet \to X$ be the Mayer–Vietoris simplicial resolution of X. As we saw in the previous step, we have a CHMC

$$(j_{n!}\mathbb{Z}_{X^n-D^n}, (\mathbb{Q}_{X^n-(D^n)^\bullet}, W), (\Omega^\bullet_{X^n-(D^n)^\bullet}, W, F))$$

on X^n, where $j_n : X^n - D^n \to X^n$ is the natural open immersion, and that $(\Omega^\bullet_{X^n-(D^n)^\bullet}, F)$ is quasi-isomorphic to $(\Omega^\bullet_{X^n}(\log D^n)(-D^n), F)$ in $D^+F(X^n, \mathbb{C})$ for any $n \geq 0$. Therefore, by using the Mayer–Vietoris simplicial resolution $\varepsilon : X^\bullet \to X$, we can construct a CMHC $(j_!\mathbb{Z}_{X-D}, (K_\mathbb{Q}, W), (K_\mathbb{C}, W, F))$ on X that induces a natural mixed Hodge structure on $H^\bullet_c(X-D, \mathbb{Z})$. We

can see that $(K_{\mathbb{C}}, F)$ is quasi-isomorphic to $(s(\Omega^{\bullet}_{X\bullet}(\log D^{\bullet})(-D^{\bullet})), F)$ in $D^{+}F(X, \mathbb{C})$, where

$$\begin{aligned} F^p = s(0 \to \cdots \to 0 \to \varepsilon_*\Omega^p_{X\bullet}(\log D^{\bullet})(-D^{\bullet}) \\ \to \varepsilon_*\Omega^{p+1}_{X\bullet}(\log D^{\bullet})(-D^{\bullet}) \to \cdots). \end{aligned}$$

We note that $\Omega^{\bullet}_{X\bullet}(\log D^{\bullet})(-D^{\bullet})$ is the double complex

$$\begin{aligned} 0 \to \varepsilon_{0*}\Omega^{\bullet}_{X^0}(\log D^0)(-D^0) \to \varepsilon_{1*}\Omega^{\bullet}_{X^1}(\log D^1)(-D^1) \to \cdots \\ \to \varepsilon_{k*}\Omega^{\bullet}_{X^k}(\log D^k)(-D^k) \to \cdots . \end{aligned}$$

Therefore, the spectral sequence

$$E_1^{pq} = H^q(X^{\bullet}, \Omega^p_{X\bullet}(\log D^{\bullet})(-D^{\bullet})) \Longrightarrow \mathbb{H}^{p+q}(X, s(\Omega^{\bullet}_{X\bullet}(\log D^{\bullet})(-D^{\bullet})))$$

degenerates in E_1 and the right hand side is isomorphic to $H^{p+q}_c(X - D, \mathbb{C})$.

Let us go to the proof of the E_1-degeneration that we already used in the proof of Proposition 2.22.

s5 **2.31 (E_1-degeneration for Proposition 2.22).** Here, we use the notation in the proof of Proposition 2.22. In this case, Y has only quotient singularities. Then $(\mathbb{Z}_Y, (\widetilde{\Omega}^{\bullet}_Y, F), \alpha)$ is a CHC, where F is the filtration bête and $\alpha : \mathbb{C}_Y \to \widetilde{\Omega}^{\bullet}_Y$ is the inclusion. For the details, see [St, (1.6)]. It is easy to see that T is a divisor with V-normal crossings on Y (see 2.21 or [St, (1.16) Definition]). We can easily check that Y is singular only over the singular locus of Supp$B$. Let $\varepsilon : T^{\bullet} \to T$ be the Mayer–Vietoris simplicial resolution. Though T has singularities, Definition 2.9 makes sense without any modifications. We note that T^n has only quotient singularities for any $n \geq 0$ by the construction of $\pi : Y \to X$. We can also check that the same construction in 2.28 works with minor modifications and we have a CMHC $(\mathbb{Z}_T, (\mathbb{Q}_{T^{\bullet}}, W), (s(\widetilde{\Omega}^{\bullet}_{T^{\bullet}}), W, F))$ that induces a natural mixed Hodge structure on $H^{\bullet}(T, \mathbb{Z})$. By the same arguments as in 2.29, we can construct a triple $(j_!\mathbb{Z}_{Y-T}, (\mathbb{Q}_{Y-T^{\bullet}}, W), (K_{\mathbb{C}}, W, F))$, where $j : Y - T \to Y$ is the natural open immersion. It is a CHMC that induces a natural mixed Hodge structure on $H^{\bullet}_c(Y - T, \mathbb{Z})$ and $(K_{\mathbb{C}}, F)$ is quasi-isomorphic to $(\widetilde{\Omega}^{\bullet}_Y(\log T)(-T), F)$ in $D^{+}F(Y, \mathbb{C})$, where

$$\begin{aligned} &F^p\widetilde{\Omega}^{\bullet}_Y(\log T)(-T) \\ &\quad = (0 \to \cdots \to 0 \to \widetilde{\Omega}^p_Y(\log T)(-T) \to \widetilde{\Omega}^{p+1}_Y(\log T)(-T) \to \cdots). \end{aligned}$$

Therefore, the spectral sequence

$$E_1^{pq} = H^q(Y, \widetilde{\Omega}_Y^p(\log T)(-T)) \Longrightarrow \mathbb{H}^{p+q}(Y, \Omega_Y^\bullet(\log T)(-T))$$

degenerates in E_1 and the right hand side is isomorphic to $H_c^{p+q}(Y - T, \mathbb{C})$.

The final one is the E_1-degeneration that we used in the proof of Proposition 2.23. It may be one of the main contributions of this chapter.

s6 **2.32 (E_1-degeneration for Proposition 2.23).** We use the notation in the proof of Proposition 2.23. Let $\varepsilon : Y^\bullet \to Y$ be the Mayer–Vietoris simplicial resolution. By the previous step, we can obtain a CHMC

$$(j_{n!}\mathbb{Z}_{Y^n - T^n}, (\mathbb{Q}_{Y^n - (T^n)^\bullet}, W), (K_{\mathbb{C}}, W, F))$$

for each $n \geq 0$. Of course, $j_n : Y^n - T^n \to Y^n$ is the natural open immersion for any $n \geq 0$. Therefore, we can construct a CMHC

$$(j_!\mathbb{Z}_{Y-T}, (K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F))$$

on Y. It induces a natural mixed Hodge structure on $H_c^\bullet(Y - T, \mathbb{Z})$. We note that $(K_{\mathbb{C}}, F)$ is quasi-isomorphic to $(s(\widetilde{\Omega}_{Y^\bullet}^\bullet(\log T^\bullet)(-T^\bullet)), F)$ in $D^+F(Y, \mathbb{C})$, where

$$\begin{aligned} F^p = s(0 \to \cdots \to 0 \to \varepsilon_* \widetilde{\Omega}_{Y^\bullet}^p(\log T^\bullet)(-T^\bullet) \\ \to \varepsilon_* \widetilde{\Omega}_{Y^\bullet}^{p+1}(\log T^\bullet)(-T^\bullet) \to \cdots). \end{aligned}$$

See s4 2.30 above. Thus, the desired spectral sequence

$$E_1^{pq} = H^q(Y^\bullet, \widetilde{\Omega}_{Y^\bullet}^p(\log T^\bullet)(-T^\bullet)) \Longrightarrow \mathbb{H}^{p+q}(Y, s(\widetilde{\Omega}_{Y^\bullet}^\bullet(\log T^\bullet)(-T^\bullet)))$$

degenerates in E_1. It is what we need in the proof of Proposition 2.23. Note that $\mathbb{H}^{p+q}(Y, s(\widetilde{\Omega}_{Y^\bullet}^\bullet(\log T^\bullet)(-T^\bullet))) \simeq H_c^{p+q}(Y - T, \mathbb{C})$.

2.5 Vanishing and injectivity theorems

sec4 The main purpose of this section is to prove Ambro's theorems (cf. [Am1, Theorems 3.1 and 3.2]) for embedded simple normal crossing pairs. The next lemma (cf. [F4, Proposition 1.11]) is missing in the proof of [Am1, Theorem 3.1]. It justifies the first three lines in the proof of [Am1, Theorem 3.1].

re-vani-lem **Lemma 2.33 (Relative vanishing lemma).** *Let $f : Y \to X$ be a proper morphism from a simple normal crossing pair $(Y, T + D)$ such that $T + D$ is a boundary $\mathbb{R}$-divisor, T is reduced, and $\llcorner D \lrcorner = 0$. We assume that f is an isomorphism at the generic point of any stratum of the pair $(Y, T+D)$. Let L be a Cartier divisor on Y such that $L \sim_{\mathbb{R}} K_Y + T + D$. Then $R^q f_* \mathcal{O}_Y(L) = 0$ for $q > 0$.*

Proof. By Lemma 7 2.12, we can assume that D is a $\mathbb{Q}$-divisor and $L \sim_{\mathbb{Q}} K_Y + T + D$. We divide the proof into two steps.

1ne **Step 1.** We assume that Y is irreducible. In this case, $L - (K_Y + T + D)$ is nef and log big over X with respect to the pair $(Y, T + D)$ (see Definition 2-46 2.46). Therefore, $R^q f_* \mathcal{O}_Y(L) = 0$ for any $q > 0$ by the vanishing theorem (see, for example, Lemma vani-rf-le 4.10).

Step 2. Let Y_1 be an irreducible component of Y and Y_2 the union of the other irreducible components of Y. Then we have a short exact sequence $0 \to i_* \mathcal{O}_{Y_1}(-Y_2|_{Y_1}) \to \mathcal{O}_Y \to \mathcal{O}_{Y_2} \to 0$, where $i : Y_1 \to Y$ is the natural closed immersion (cf. ambro [Am1, Remark 2.6]). We put $L' = L|_{Y_1} - Y_2|_{Y_1}$. Then we have a short exact sequence $0 \to i_* \mathcal{O}_{Y_1}(L') \to \mathcal{O}_Y(L) \to \mathcal{O}_{Y_2}(L|_{Y_2}) \to 0$ and $L' \sim_{\mathbb{Q}} K_{Y_1} + T|_{Y_1} + D|_{Y_1}$. On the other hand, we can check that $L|_{Y_2} \sim_{\mathbb{Q}} K_{Y_2} + Y_1|_{Y_2} + T|_{Y_2} + D|_{Y_2}$. We have already known that $R^q f_* \mathcal{O}_{Y_1}(L') = 0$ for any $q > 0$ by Step 1ne 1. By the induction on the number of the irreducible components of Y, we have $R^q f_* \mathcal{O}_{Y_2}(L|_{Y_2}) = 0$ for any $q > 0$. Therefore, $R^q f_* \mathcal{O}_Y(L) = 0$ for any $q > 0$ by the exact sequence:

$$\cdots \to R^q f_* \mathcal{O}_{Y_1}(L') \to R^q f_* \mathcal{O}_Y(L) \to R^q f_* \mathcal{O}_{Y_2}(L|_{Y_2}) \to \cdots .$$

So, we finish the proof of Lemma re-vani-lem 2.33. □

The following lemma is a variant of Szabó's resolution lemma (see the resolution lemma in 15-resol 1.8).

6 **Lemma 2.34.** *Let (X, B) be an embedded simple normal crossing pair and D a permissible Cartier divisor on X. Let M be an ambient space of X. Assume that there exists an $\mathbb{R}$-divisor A on M such that $\mathrm{Supp}(A+X)$ is simple normal crossing on M and that $B = A|_X$. Then there exists a projective birational morphism $g : N \to M$ from a smooth variety N with the following properties. Let Y be the strict transform of X on N and $f = g|_Y : Y \to X$. Then we have*

(i) $g^{-1}(D)$ *is a divisor on* N. $\mathrm{Exc}(g) \cup g_*^{-1}(A+X)$ *is simple normal crossing on* N, *where* $\mathrm{Exc}(g)$ *is the exceptional locus of* g. *In particular,* Y *is a simple normal crossing divisor on* N.

(ii) g *and* f *are isomorphisms outside* D, *in particular,* $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$.

(iii) $f^*(D+B)$ *has a simple normal crossing support on* Y. *More precisely, there exists an* $\mathbb{R}$*-divisor* A' *on* N *such that* $\mathrm{Supp}(A'+Y)$ *is simple normal crossing on* N, A' *and* Y *have no common irreducible components, and that* $A'|_Y = f^*(D+B)$.

Proof. First, we take a blow-up $M_1 \to M$ along D. Apply Hironaka's desingularization theorem to M_1 and obtain a projective birational morphism $M_2 \to M_1$ from a smooth variety M_2. Let F be the reduced divisor that coincides with the support of the inverse image of D on M_2. Apply Szabó's resolution lemma to $\mathrm{Supp}\sigma^*(A+X) \cup F$ on M_2 (see, for example, 15-resol 1.8 or fujino0 [F7, 3.5. Resolution lemma]), where $\sigma : M_2 \to M$. Then, we obtain desired projective birational morphisms $g : N \to M$ from a smooth variety N, and $f = g|_Y : Y \to X$, where Y is the strict transform of X on N, such that Y is a simple normal crossing divisor on N, g and f are isomorphisms outside D, and $f^*(D+B)$ has a simple normal crossing support on Y. Since f is an isomorphism outside D and D is permissible on X, f is an isomorphism at the generic point of any stratum of Y. Therefore, every fiber of f is connected and then $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$. □

Remark 2.35. In Lemma 6 2.34, we can directly check that $f_*\mathcal{O}_Y(K_Y) \simeq \mathcal{O}_X(K_X)$. By Lemma 5.1, $R^q f_*\mathcal{O}_Y(K_Y) = 0$ for $q > 0$. Therefore, we obtain $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$ and $R^q f_*\mathcal{O}_Y = 0$ for any $q > 0$ by the Grothendieck duality.

Here, we treat the compactification problem. It is because we can use the same technique as in the proof of Lemma 6 2.34. This lemma plays important roles in this section.

comp **Lemma 2.36.** *Let* $f : Z \to X$ *be a proper morphism from an embedded simple normal crossing pair* (Z, B). *Let* M *be the ambient space of* Z. *Assume that there is an* $\mathbb{R}$*-divisor* A *on* M *such that* $\mathrm{Supp}(A+Z)$ *is simple normal crossing on* M *and that* $B = A|_Z$. *Let* $\overline{X}$ *be a projective variety such that* $\overline{X}$ *contains* X *as a Zariski open set. Then there exist a proper embedded simple normal crossing pair* $(\overline{Z}, \overline{B})$ *that is a compactification of* (Z, B) *and a proper morphism* $\overline{f} : \overline{Z} \to \overline{X}$ *that compactifies* $f : Z \to X$.

Moreover, $\mathrm{Supp}\overline{B} \cup \mathrm{Supp}(\overline{Z} \setminus Z)$ *is a simple normal crossing divisor on* $\overline{Z}$*, and* $\overline{Z} \setminus Z$ *has no common irreducible components with* $\overline{B}$*. We note that* $\overline{B}$ *is* $\mathbb{R}$*-Cartier. Let* $\overline{M}$*, which is a compactification of* M*, be the ambient space of* $(\overline{Z}, \overline{B})$*. Then, by the construction, we can find an* $\mathbb{R}$*-divisor* $\overline{A}$ *on* $\overline{M}$ *such that* $\mathrm{Supp}(\overline{A} + \overline{Z})$ *is simple normal crossing on* $\overline{M}$ *and that* $\overline{B} = \overline{A}|_{\overline{Z}}$*.*

Proof. Let $\overline{Z}, \overline{A} \subset \overline{M}$ be any compactification. By blowing up $\overline{M}$ inside $\overline{Z} \setminus Z$, we can assume that $f : Z \to X$ extends to $\overline{f} : \overline{Z} \to \overline{X}$. By Hironaka's desingularization and the resolution lemma, we can assume that $\overline{M}$ is smooth and $\mathrm{Supp}(\overline{Z} + \overline{A}) \cup \mathrm{Supp}(\overline{M} \setminus M)$ is a simple normal crossing divisor on $\overline{M}$. It is not difficult to see that the above compactification has the desired properties. □

rem-2 **Remark 2.37.** There exists a big trouble to compactify normal crossing varieties. When we treat normal crossing varieties, we can not directly compactify them. For the details, see [F7, 3.6. Whitney umbrella], especially, Corollary 3.6.10 and Remark 3.6.11 in [F7]. Therefore, the first two lines in the proof of [Am1, Theorem 3.2] is nonsense.

It is the time to state the main injectivity theorem (cf. [Am1, Theorem 3.1]) for embedded *simple* normal crossing pairs. For applications, this formulation seems to be sufficient. We note that we will recover [Am1, Theorem 3.1] in full generality in Section 2.7 (see Theorem 2.53).

5.1 **Theorem 2.38 (cf. [Am1, Theorem 3.1]).** *Let* $(X, S + B)$ *be an embedded simple normal crossing pair such that* X *is proper,* $S + B$ *is a boundary* $\mathbb{R}$*-divisor,* S *is reduced, and* $\llcorner B \lrcorner = 0$*. Let* L *be a Cartier divisor on* X *and* D *an effective Cartier divisor that is permissible with respect to* $(X, S + B)$*. Assume the following conditions.*

(i) $L \sim_{\mathbb{R}} K_X + S + B + H$*,*

(ii) H *is a semi-ample* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor, and*

(iii) $tH \sim_{\mathbb{R}} D + D'$ *for some positive real number* t*, where* D' *is an effective* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor that is permissible with respect to* $(X, S + B)$*.*

Then the homomorphisms

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D)),$$

which are induced by the natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$*, are injective for all* q*.*

Proof. First, we use Lemma 2.18. Thus, we can assume that there exists a divisor A on M, where M is the ambient space of X, such that $\mathrm{Supp}(A+X)$ is simple normal crossing on M and that $A|_X = S$. Apply Lemma 2.34 to an embedded simple normal crossing pair (X, S) and a divisor $\mathrm{Supp}(D+D'+B)$ on X. Then we obtain a projective birational morphism $f : Y \to X$ from an embedded simple normal crossing variety Y such that f is an isomorphism outside $\mathrm{Supp}(D + D' + B)$, and that the union of the support of $f^*(S + B + D + D')$ and the exceptional locus of f has a simple normal crossing support on Y. Let B' be the strict transform of B on Y. We can assume that $\mathrm{Supp}B'$ is disjoint from any strata of Y that are not irreducible components of Y by taking blow-ups. We write $K_Y + S' + B' = f^*(K_X + S + B) + E$, where S' is the strict transform of S, and E is f-exceptional. By the construction of $f : Y \to X$, S' is Cartier and B' is $\mathbb{R}$-Cartier. Therefore, E is also $\mathbb{R}$-Cartier. It is easy to see that $E_+ = \ulcorner E \urcorner \geq 0$. We put $L' = f^*L + E_+$ and $E_- = E_+ - E \geq 0$. We note that E_+ is Cartier and E_- is $\mathbb{R}$-Cartier because $\mathrm{Supp}E$ is simple normal crossing on Y. Since f^*H is an $\mathbb{R}_{>0}$-linear combination of semi-ample Cartier divisors, we can write $f^*H \sim_{\mathbb{R}} \sum_i a_i H_i$, where $0 < a_i < 1$ and H_i is a general Cartier divisor on Y for any i. We put $B'' = B' + E_- + \frac{\varepsilon}{t} f^*(D + D') + (1 - \varepsilon) \sum_i a_i H_i$ for some $0 < \varepsilon \ll 1$. Then $L' \sim_{\mathbb{R}} K_Y + S' + B''$. By the construction, $\llcorner B'' \lrcorner = 0$, the support of $S' + B''$ is simple normal crossing on Y, and $\mathrm{Supp}B'' \supset \mathrm{Supp}f^*D$. So, Proposition 2.23 implies that the homomorphisms $H^q(Y, \mathcal{O}_Y(L')) \to H^q(Y, \mathcal{O}_Y(L' + f^*D))$ are injective for all q. By Lemma 2.33, $R^q f_* \mathcal{O}_Y(L') = 0$ for any $q > 0$ and it is easy to see that $f_* \mathcal{O}_Y(L') \simeq \mathcal{O}_X(L)$. By the Leray spectral sequence, the homomorphisms $H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D))$ are injective for all q. □

The following theorem is another main theorem of this section. It is essentially the same as [Am1, Theorem 3.2]. We note that we assume that $(Y, S + B)$ is a *simple* normal crossing pair. It is a small but technically important difference. For the full statement, see Theorem 2.54 below.

Theorem 2.39 (cf. [Am1, Theorem 3.2]). *Let $(Y, S+B)$ be an embedded simple normal crossing pair such that $S + B$ is a boundary $\mathbb{R}$-divisor, S is reduced, and $\llcorner B \lrcorner = 0$. Let $f : Y \to X$ be a proper morphism and L a Cartier divisor on Y such that $H \sim_{\mathbb{R}} L - (K_Y + S + B)$ is f-semi-ample.*

(i) *every non-zero local section of $R^q f_* \mathcal{O}_Y(L)$ contains in its support the f-image of some strata of $(Y, S + B)$.*

(ii) *let $\pi : X \to V$ be a projective morphism and assume that $H \sim_{\mathbb{R}} f^*H'$ for some π-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor H' on X. Then $R^qf_*\mathcal{O}_Y(L)$ is π_*-acyclic, that is, $R^p\pi_*R^qf_*\mathcal{O}_Y(L) = 0$ for any $p > 0$.*

Proof. Let M be the ambient space of Y. Then, by Lemma 2.18, we can assume that there exists an $\mathbb{R}$-divisor D on M such that $\mathrm{Supp}(D + Y)$ is simple normal crossing on M and that $D|_Y = S + B$. Therefore, we can use Lemma 2.36 in Step 2 of (i) and Step 3 of (ii) below.

(i) We have already proved a very spacial case in Lemma 2.33. The argument in Step 1 is not new and it is well known.

Step 1. First, we assume that X is projective. We can assume that H is semi-ample by replacing L (resp. H) with $L+f^*A'$ (resp. $H+f^*A'$), where A' is a very ample Cartier divisor. Assume that $R^qf_*\mathcal{O}_Y(L)$ has a local section whose support does not contain the image of any $(Y, S + B)$-stratum. Then we can find a very ample Cartier divisor A with the following properties.

(a) f^*A is permissible with respect to $(Y, S + B)$, and

(b) $R^qf_*\mathcal{O}_Y(L) \to R^qf_*\mathcal{O}_Y(L) \otimes \mathcal{O}_X(A)$ is not injective.

We can assume that $H - f^*A$ is semi-ample by replacing L (resp. H) with $L+f^*A$ (resp. $H+f^*A$). If necessary, we replace L (resp. H) with $L+f^*A''$ (resp. $H + f^*A''$), where A'' is a very ample Cartier divisor. Then, we have $H^0(X, R^qf_*\mathcal{O}_Y(L)) \simeq H^q(Y, \mathcal{O}_Y(L))$ and $H^0(X, R^qf_*\mathcal{O}_Y(L) \otimes \mathcal{O}_X(A)) \simeq H^q(Y, \mathcal{O}_Y(L + f^*A))$. We obtain that

$$H^0(X, R^qf_*\mathcal{O}_Y(L)) \to H^0(X, R^qf_*\mathcal{O}_Y(L) \otimes \mathcal{O}_X(A))$$

is not injective by (b) if A'' is sufficiently ample. So, $H^q(Y, \mathcal{O}_Y(L)) \to H^q(Y, \mathcal{O}_Y(L + f^*A))$ is not injective. It contradicts Theorem 2.38. We finish the proof when X is projective.

Step 2. Next, we assume that X is not projective. Note that the problem is local. So, we can shrink X and assume that X is affine. By the argument similar to the one in Step 1 in the proof of (ii) below, we can assume that H is a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. We compactify X and apply Lemma 2.36. Then we obtain a compactification $\overline{f} : \overline{Y} \to \overline{X}$ of $f : Y \to X$. Let $\overline{H}$ be the closure of H on $\overline{Y}$. If $\overline{H}$ is not a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, then we take blowing-ups of $\overline{Y}$ inside $\overline{Y} \setminus Y$ and obtain a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\overline{H}$ on $\overline{Y}$ such that $\overline{H}|_Y = H$. Let $\overline{L}$ (resp. $\overline{B}$, $\overline{S}$) be the closure of L (resp. B,

S) on $\overline{Y}$. We note that $\overline{H} \sim_{\mathbb{R}} \overline{L} - (K_{\overline{Y}} + \overline{S} + \overline{B})$ does not necessarily hold. We can write $H + \sum_i a_i(f_i) = L - (K_Y + S + B)$, where a_i is a real number and $f_i \in \Gamma(Y, \mathcal{K}_Y^*)$ for any i. We put $E = \overline{H} + \sum_i a_i(f_i) - (\overline{L} - (K_{\overline{Y}} + \overline{S} + \overline{B}))$. We replace $\overline{L}$ (resp. $\overline{B}$) with $\overline{L} + \ulcorner E \urcorner$ (resp. $\overline{B} + \{-E\}$). Then we obtain the desired property of $R^q\overline{f}_*\mathcal{O}_{\overline{Y}}(\overline{L})$ since $\overline{X}$ is projective. We note that $\mathrm{Supp} E$ is in $\overline{Y} \setminus Y$. So, we finish the whole proof.

(ii) We divide the proof into three steps.

Step 1. We assume that $\dim V = 0$. The following arguments are well known and standard. We describe them for the reader's convenience. In this case, we can write $H' \sim_{\mathbb{R}} H'_1 + H'_2$, where H'_1 (resp. H'_2) is a π-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor (resp. π-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor) on X. So, we can write $H'_2 \sim_{\mathbb{R}} \sum_i a_i H_i$, where $0 < a_i < 1$ and H_i is a general very ample Cartier divisor on X for any i. Replacing B (resp. H') with $B + \sum_i a_i f^* H_i$ (resp. H'_1), we can assume that H' is a π-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. We take a general member $A \in |mH'|$, where m is a sufficiently large and divisible integer, such that $A' = f^*A$ and $R^q f_* \mathcal{O}_Y(L + A')$ is π_*-acyclic for all q. By (i), we have the following short exact sequences,

$$0 \to R^q f_* \mathcal{O}_Y(L) \to R^q f_* \mathcal{O}_Y(L + A') \to R^q f_* \mathcal{O}_{A'}(L + A') \to 0.$$

for any q. Note that $R^q f_* \mathcal{O}_{A'}(L + A')$ is π_*-acyclic by induction on $\dim X$ and $R^q f_* \mathcal{O}_Y(L + A')$ is also π_*-acyclic by the above assumption. Thus, $E_2^{pq} = 0$ for $p \geq 2$ in the following commutative diagram of spectral sequences.

$$\begin{array}{ccc} E_2^{pq} = R^p\pi_* R^q f_* \mathcal{O}_Y(L) & \Longrightarrow & R^{p+q}(\pi \circ f)_* \mathcal{O}_Y(L) \\ \downarrow{\scriptstyle \varphi^{pq}} & & \downarrow{\scriptstyle \varphi^{p+q}} \\ \overline{E}_2^{pq} = R^p\pi_* R^q f_* \mathcal{O}_Y(L + A') & \Longrightarrow & R^{p+q}(\pi \circ f)_* \mathcal{O}_Y(L + A') \end{array}$$

Since φ^{1+q} is injective by Theorem 2.38, $E_2^{1q} \to R^{1+q}(\pi \circ f)_* \mathcal{O}_Y(L)$ is injective by the fact that $E_2^{pq} = 0$ for $p \geq 2$, and $\overline{E}_2^{1q} = 0$ by the above assumption, we have $E_2^{1q} = 0$. This implies that $R^p\pi_* R^q f_* \mathcal{O}_Y(L) = 0$ for any $p > 0$.

o2 **Step 2.** We assume that V is projective. By replacing H' (resp. L) with $H' + \pi^* G$ (resp. $L + (\pi \circ f)^* G$), where G is a very ample Cartier divisor on V, we can assume that H' is an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. By the same argument as in Step 1, we can assume that H' is ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and $H \sim_{\mathbb{Q}} f^* H'$. If G is a sufficiently ample Cartier divisor on V,

$H^k(V, R^p\pi_*R^qf_*\mathcal{O}_Y(L)\otimes G) = 0$ for any $k \geq 1$, $H^0(V, R^p\pi_*R^qf_*\mathcal{O}_Y(L)\otimes G) \simeq H^p(X, R^qf_*\mathcal{O}_Y(L)\otimes\pi^*G)$, and $R^p\pi_*R^qf_*\mathcal{O}_Y(L)\otimes G$ is generated by its global sections. Since $H + f^*\pi^*G \sim_{\mathbb{R}} L + f^*\pi^*G - (K_Y + S + B)$, $H + f^*\pi^*G \sim_{\mathbb{Q}} f^*(H' + \pi^*G)$, and $H' + \pi^*G$ is ample, we can apply Step 1 and obtain $H^p(X, R^qf_*\mathcal{O}_Y(L + f^*\pi^*G)) = 0$ for any $p > 0$. Thus, $R^p\pi_*R^qf_*\mathcal{O}_Y(L) = 0$ for any $p > 0$ by the above arguments.

o3 **Step 3.** When V is not projective, we shrink V and assume that V is affine. By the same argument as in Step 1 above, we can assume that H' is $\mathbb{Q}$-Cartier. We compactify V and X, and can assume that V and X are projective. By Lemma comp 2.36, we can reduce it to the case when V is projective. This step is essentially the same as Step 2 in the proof of (i). So, we omit the details here.

We finish the whole proof of (ii). □

smooth-ne **Remark 2.40.** In Theorem 5.1 2.38, if X is smooth, then Proposition 1 2.22 is enough for the proof of Theorem 5.1 2.38. In the proof of Theorem 8 2.39, if Y is smooth, then Theorem 5.1 2.38 for a smooth X is sufficient. Lemmas re-vani-lem 2.33, 6 2.34, and comp 2.36 are easy and well known for smooth varieties. Therefore, the reader can find that our proof of Theorem 8 2.39 becomes much easier if we assume that Y is smooth. Ambro's original proof of ambro [Am1, Theorem 3.2 (ii)] used embedded simple normal crossing pairs even when Y is smooth (see (b) in the proof of ambro [Am1, Theorem 3.2 (ii)]). It may be a technically important difference. I could not follow Ambro's original argument in (a) in the proof of ambro [Am1, Theorem 3.2 (ii)].

9-1 **Remark 2.41.** It is easy to see that Theorem 5.1 2.38 is a generalization of Kollár's injectivity theorem. Theorem 8 2.39 (i) (resp. (ii)) is a generalization of Kollár's torsion-free (resp. vanishing) theorem.

We treat an easy vanishing theorem for lc pairs as an application of Theorem 8 2.39 (ii). It seems to be buried in ambro [Am1]. We note that we do not need the notion of embedded simple normal crossing pairs to prove Theorem lc 2.42. See Remark smooth-ne 2.40.

lc **Theorem 2.42 (Kodaira vanishing theorem for lc pairs).** *Let (X, B) be an lc pair such that B is a boundary $\mathbb{R}$-divisor. Let L be a $\mathbb{Q}$-Cartier Weil divisor on X such that $L - (K_X + B)$ is π-ample, where $\pi : X \to V$ is a projective morphism. Then $R^q\pi_*\mathcal{O}_X(L) = 0$ for any $q > 0$.*

Proof. Let $f : Y \to X$ be a log resolution of (X, B) such that $K_Y = f^*(K_X + B) + \sum_i a_i E_i$ with $a_i \geq -1$ for any i. We can assume that $\sum_i E_i \cup \mathrm{Supp} f^*L$ is a simple normal crossing divisor on Y. We put $E = \sum_i a_i E_i$ and $F = \sum_{a_j=-1}(1-b_j)E_j$, where $b_j = \mathrm{mult}_{E_j}\{f^*L\}$. We note that $A = L-(K_X+B)$ is π-ample by the assumption. So, we have $f^*A = f^*L - f^*(K_X + B) = \ulcorner f^*L + E + F \urcorner - (K_Y + F + \{-(f^*L + E + F)\})$. We can easily check that $f_*\mathcal{O}_Y(\ulcorner f^*L+E+F\urcorner) \simeq \mathcal{O}_X(L)$ and that $F+\{-(f^*L+E+F)\}$ has a simple normal crossing support and is a boundary $\mathbb{R}$-divisor on Y. By Theorem 2.39 (ii), we obtain that $\mathcal{O}_X(L)$ is π_*-acyclic. Thus, we have $R^q\pi_*\mathcal{O}_X(L) = 0$ for any $q > 0$. □

We note that Theorem 2.42 contains a complete form of [Kv2, Theorem 0.3] as a corollary. For the related topics, see [KSS, Corollary 1.3].

lcvar **Corollary 2.43 (Kodaira vanishing theorem for lc varieties).** *Let X be a projective lc variety and L an ample Cartier divisor on X. Then*

$$H^q(X, \mathcal{O}_X(K_X + L)) = 0$$

for any $q > 0$. Furthermore, if we assume that X is Cohen–Macaulay, then $H^q(X, \mathcal{O}_X(-L)) = 0$ for any $q < \dim X$.

Remark 2.44. We can see that Corollary 2.43 is contained in [F4, Theorem 2.6], which is a very special case of Theorem 2.39 (ii). I forgot to state Corollary 2.43 explicitly in [F4]. There, we do not need embedded simple normal crossing pairs. We note that there are typos in the proof of [F4, Theorem 2.6]. In the commutative diagram, $R^i f_*\omega_X(D)$'s should be replaced by $R^j f_*\omega_X(D)$'s.

We close this section with an easy example.

Example 2.45. Let X be a projective lc threefold which has the following properties: (i) there exists a projective birational morphism $f : Y \to X$ from a smooth projective threefold, and (ii) the exceptional locus E of f is an Abelian surface with $K_Y = f^*K_X - E$. For example, X is a cone over a normally projective Abelian surface in $\mathbb{P}^N$ and $f : Y \to X$ is the blow-up at the vertex of X. Let L be an ample Cartier divisor on X. By the Leray spectral sequence, we have

$$\begin{aligned} 0 \to H^1(X, f_*f^*\mathcal{O}_X(-L)) \to H^1(Y, f^*\mathcal{O}_X(-L)) \to H^0(X, R^1f_*f^*\mathcal{O}_X(-L)) \\ \to H^2(X, f_*f^*\mathcal{O}_X(-L)) \to H^2(Y, f^*\mathcal{O}_X(-L)) \to \cdots . \end{aligned}$$

Therefore, we obtain

$$H^2(X, \mathcal{O}_X(-L)) \simeq H^0(X, \mathcal{O}_X(-L) \otimes R^1 f_* \mathcal{O}_Y),$$

because $H^1(Y, f^*\mathcal{O}_X(-L)) = H^2(Y, f^*\mathcal{O}_X(-L)) = 0$ by the Kawamata–Viehweg vanishing theorem. On the other hand, we have $R^q f_* \mathcal{O}_Y \simeq H^q(E, \mathcal{O}_E)$ for any $q > 0$ since $R^q f_* \mathcal{O}_Y(-E) = 0$ for every $q > 0$. Thus, $H^2(X, \mathcal{O}_X(-L)) \simeq \mathbb{C}^2$. In particular, $H^2(X, \mathcal{O}_X(-L)) \neq 0$. We note that X is not Cohen–Macaulay. In the above example, if we assume that E is a $K3$-surface, then $H^q(X, \mathcal{O}_X(-L)) = 0$ for $q < 3$ and X is Cohen–Macaulay. For the details, see the subsection 4.3.1, especially, Lemma 4.37.

2.6 Some further generalizations

genera

Here, we treat some generalizations of Theorem 2.39. First, we introduce the notion of nef and log big (resp. nef and log abundant) divisors. See also Definition 3.37.

2-46 **Definition 2.46.** Let $f : (Y, B) \to X$ be a proper morphism from a simple normal crossing pair (Y, B) such that B is a subboundary. Let $\pi : X \to V$ be a proper morphism and H an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. We say that H is *nef and log big* (resp. *nef and log abundant*) over V if and only if $H|_C$ is nef and big (resp. nef and abundant) over V for any C, where C is the image of a stratum of (Y, B). When (X, B_X) is an lc pair, we choose a log resolution of (X, B_X) to be $f : (Y, B) \to X$, where $K_Y + B = f^*(K_X + B_X)$.

We can generalize Theorem 2.39 as follows. It is [Am1, Theorem 7.4] for embedded *simple* normal crossing pairs. His idea of the proof is very clever.

74 **Theorem 2.47 (cf. [Am1, Theorem 7.4]).** *Let $f : (Y, S + B) \to X$ be a proper morphism from an embedded simple normal crossing pair such that $S + B$ is a boundary $\mathbb{R}$-divisor, S is reduced, and $\llcorner B \lrcorner = 0$. Let L be a Cartier divisor on Y and $\pi : X \to V$ a proper morphism. Assume that $f^*H \sim_{\mathbb{R}} L - (K_Y + S + B)$, where H is nef and log big over V. Then*

(i) *every non-zero local section of $R^q f_* \mathcal{O}_Y(L)$ contains in its support the f-image of some strata of $(Y, S + B)$, and*

(ii) *$R^q f_* \mathcal{O}_Y(L)$ is π_*-acyclic.*

Proof. We note that we can assume that V is affine without loss of generality. By using Lemma 2.18, we can assume that there exists a divisor D on M, where M is the ambient space of Y, such that $\mathrm{Supp}(D+Y)$ is simple normal crossing on M and that $D|_Y = S + B$.

so-no-1 **Step 1.** We assume that each stratum of $(Y, S+B)$ dominates some irreducible component of X. By taking the Stein factorization, we can assume that f has connected fibers. Then we can assume that X is irreducible and each stratum of $(Y, S+B)$ dominates X. By Chow's lemma, there exists a projective birational morphism $\mu : X' \to X$ such that $\pi' : X' \to V$ is projective. By taking blow-ups $\varphi : Y' \to Y$ that is an isomorphism over the generic point of any stratum of $(Y, S+B)$, we have the following commutative diagram.

$$\begin{array}{ccc} Y' & \xrightarrow{\varphi} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[\mu]{} & X \end{array}$$

Then, we can write

$$K_{Y'} + S' + B' = \varphi^*(K_Y + S + B) + E,$$

where

(1) $(Y', S' + B')$ is a global embedded simple normal crossing pair such that $S' + B'$ is a boundary $\mathbb{R}$-divisor, S' is reduced, and $\llcorner B' \lrcorner = 0$.

(2) E is an effective φ-exceptional Cartier divisor.

(3) Each stratum of $(Y', S' + B')$ dominates X'.

We note that each stratum of $(Y, S+B)$ dominates X. Therefore,

$$\varphi^*L + E \sim_{\mathbb{R}} K_{Y'} + S' + B' + \varphi^*f^*H.$$

We note that $\varphi_*\mathcal{O}_{Y'}(\varphi^*L + E) \simeq \mathcal{O}_Y(L)$ and $R^i\varphi_*\mathcal{O}_{Y'}(\varphi^*L + E) = 0$ for any $i > 0$ by Theorem 2.39 (i). Thus, we can assume that $\varphi : Y' \to Y$ is an identity, that is, we have

$$\begin{array}{ccc} Y & = & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[\mu]{} & X. \end{array}$$

We put $\mathcal{F} = R^q g_* \mathcal{O}_Y(L)$. Since $\mu^* H$ is nef and big over V and $\pi' : X' \to V$ is projective, we can write $\mu^* H = E + A$, where A is a π'-ample $\mathbb{R}$-divisor on X' and E is an effective $\mathbb{R}$-divisor. By the same arguments as above, we take some blow-ups and can assume that $(Y, S + B + g^* E)$ is a global embedded simple normal crossing pair. If $k \gg 1$, then $\llcorner B + \frac{1}{k} g^* E \lrcorner = 0$,

$$\mu^* H = \frac{1}{k} E + \frac{1}{k} A + \frac{k-1}{k} \mu^* H,$$

and

$$\frac{1}{k} A + \frac{k-1}{k} \mu^* H$$

is π'-ample. Thus, $\mathcal{F}$ is μ_*-acyclic and $(\pi \circ \mu)_* = \pi'_*$-acyclic by Theorem 2.39 (ii). We note that

$$L - \Big(K_Y + S + B + \frac{1}{k} g^* E \Big) \sim_{\mathbb{R}} g^* \Big(\frac{1}{k} A + \frac{k-1}{k} \mu^* H \Big).$$

So, we have $R^q f_* \mathcal{O}_Y(L) \simeq \mu_* \mathcal{F}$ and $R^q f_* \mathcal{O}_Y(L)$ is π_*-acyclic. It is easy to see that $\mathcal{F}$ is torsion-free by Theorem 2.39 (i). Therefore, $R^q f_* \mathcal{O}_Y(L)$ is also torsion-free. Thus, we finish the proof when each stratum of $(Y, S + B)$ dominates some irreducible component of X.

so-no-2 **Step 2.** We treat the general case by induction on $\dim f(Y)$. By taking embedded log transformation (see Lemma 2.19), we can decompose $Y = Y' \cup Y''$ as follows: Y' is the union of all strata of $(Y, S + B)$ that are not mapped to irreducible components of X and $Y'' = Y - Y'$. We put $K_{Y''} + B_{Y''} = (K_Y + S + B)|_{Y''} - Y'|_{Y''}$. Then $f : (Y'', B_{Y''}) \to X$ and $L'' = L|_{Y''} - Y'|_{Y''}$ satisfy the assumption in Step 1. We consider the following short exact sequence

$$0 \to \mathcal{O}_{Y''}(L'') \to \mathcal{O}_Y(L) \to \mathcal{O}_{Y'}(L) \to 0.$$

By taking $R^q f_*$, we have short exact sequence

$$0 \to R^q f_* \mathcal{O}_{Y''}(L'') \to R^q f_* \mathcal{O}_Y(L) \to R^q f_* \mathcal{O}_{Y'}(L) \to 0$$

for any q by Step 1. It is because the connecting homomorphisms $R^q f_* \mathcal{O}_{Y'}(L) \to R^{q+1} f_* \mathcal{O}_{Y''}(L'')$ are zero maps by Step 1. Since (i) and (ii) hold for the first and third members by Step 1 and by induction on dimension, respectively, they also hold for $R^q f_* \mathcal{O}_Y(L)$.

So, we finish the proof. □

In Step 2 in the proof of Theorem 2.47, we used the embedded log transformation and the dévissage (see [Am1, Remark 2.6]). So, we need the notion of embedded simple normal crossing pairs to prove Theorem 2.47 even when Y is smooth. It is a key point.

As a corollary of Theorem 2.47, we can prove the following vanishing theorem, which is stated implicitly in the introduction of [Am1]. It is the culmination of the works of several authors: Kawamata, Viehweg, Nadel, Reid, Fukuda, Ambro, and many others (cf. [KMM, Theorem 1-2-5]).

kvn **Theorem 2.48.** *Let (X, B) be an lc pair such that B is a boundary $\mathbb{R}$-divisor and let L be a $\mathbb{Q}$-Cartier Weil divisor on X. Assume that $L-(K_X+B)$ is nef and log big over V, where $\pi : X \to V$ is a proper morphism. Then $R^q\pi_*\mathcal{O}_X(L) = 0$ for any $q > 0$.*

As a special case, we have the Kawamata–Viehweg vanishing theorem.

kv-vani **Corollary 2.49 (Kawamata–Viehweg vanishing theorem).** *Let (X, B) be a klt pair and let L be a $\mathbb{Q}$-Cartier Weil divisor on X. Assume that $L-(K_X+B)$ is nef and big over V, where $\pi : X \to V$ is a proper morphism. Then $R^q\pi_*\mathcal{O}_X(L) = 0$ for any $q > 0$.*

The proof of Theorem 2.42 works for Theorem 2.48 without any changes if we adopt Theorem 2.47. We add one example.

ex111 **Example 2.50.** Let Y be a projective surface which has the following properties: (i) there exists a projective birational morphism $f : X \to Y$ from a smooth projective surface X, and (ii) the exceptional locus E of f is an elliptic curve with $K_X + E = f^*K_Y$. For example, Y is a cone over a smooth plane cubic curve and $f : X \to Y$ is the blow-up at the vertex of Y. We note that (X, E) is a plt pair. Let H be an ample Cartier divisor on Y. We consider a Cartier divisor $L = f^*H + K_X + E$ on X. Then $L-(K_X+E)$ is nef and big, but not log big. By the short exact sequence

$$0 \to \mathcal{O}_X(f^*H + K_X) \to \mathcal{O}_X(f^*H + K_X + E) \to \mathcal{O}_E(K_E) \to 0,$$

we obtain

$$R^1f_*\mathcal{O}_X(f^*H + K_X + E) \simeq H^1(E, \mathcal{O}_E(K_E)) \simeq \mathbb{C}(P),$$

where $P = f(E)$. By the Leray spectral sequence, we have

$$\begin{aligned} 0 \to H^1(Y, f_*\mathcal{O}_X(K_X + E) \otimes \mathcal{O}_Y(H)) \to H^1(X, \mathcal{O}_X(L)) \to H^0(Y, \mathbb{C}(P)) \\ \to H^2(Y, f_*\mathcal{O}_X(K_X + E) \otimes \mathcal{O}_Y(H)) \to \cdots . \end{aligned}$$

If H is sufficiently ample, then $H^1(X, \mathcal{O}_X(L)) \simeq H^0(Y, \mathbb{C}(P)) \simeq \mathbb{C}(P)$. In particular, $H^1(X, \mathcal{O}_X(L)) \neq 0$.

Remark 2.51. In Example 2.50, there exists an effective $\mathbb{Q}$-divisor B on X such that $L - \frac{1}{k}B$ is ample for any $k > 0$ by Kodaira's lemma. Since $L \cdot E = 0$, we have $B \cdot E < 0$. In particular, $(X, E + \frac{1}{k}B)$ is not lc for any $k > 0$. This is the main reason why $H^1(X, \mathcal{O}_X(L)) \neq 0$. If $(X, E + \frac{1}{k}B)$ were lc, then the ampleness of $L - (K_X + E + \frac{1}{k}B)$ would imply $H^1(X, \mathcal{O}_X(L)) = 0$.

We modify the proof of Theorem 2.47. Then we can easily obtain the following generalization of Theorem 2.39 (i). We leave the details for the reader's exercise.

nef-labun-th **Theorem 2.52.** *Let $f : (Y, S + B) \to X$ be a proper morphism from an embedded simple normal crossing pair such that $S + B$ is a boundary, S is reduced, and $\llcorner B \lrcorner = 0$. Let L be a Cartier divisor on Y and $\pi : X \to V$ a proper morphism. Assume that $f^*H \sim_{\mathbb{R}} L - (K_Y + S + B)$, where H is nef and log abundant over V. Then, every non-zero local section of $R^q f_* \mathcal{O}_Y(L)$ contains in its support the f-image of some strata of $(Y, S + B)$.*

Sketch of the proof. In Step 1 in the proof of Theorem 2.47, we can write $\mu^*H = E + A$, where E is an effective $\mathbb{R}$-divisor such that $k\mu^*H - E$ is π'-semi-ample for any positive integer k (cf. [EV, 5.11. Lemma]). Therefore, Theorem 2.52 holds when each stratum of $(Y, S + B)$ dominates some irreducible component of X. Step 2 in the proof of Theorem 2.47 works without any changes. □

2.7 From SNC pairs to NC pairs

sec6 In this section, we recover Ambro's theorems from Theorems 2.38 and 2.39. We repeat Ambro's statements for the reader's convenience.

61 **Theorem 2.53 (cf. [Am1, Theorem 3.1]).** *Let $(X, S + B)$ be an embedded normal crossing pair such that X is proper, $S+B$ is a boundary $\mathbb{R}$-divisor, S is reduced, and $\llcorner B \lrcorner = 0$. Let L be a Cartier divisor on X and D an effective Cartier divisor that is permissible with respect to $(X, S + B)$. Assume the following conditions.*

(i) $L \sim_{\mathbb{R}} K_X + S + B + H$,

(ii) *H is a semi-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, and*

(iii) *$tH \sim_{\mathbb{R}} D + D'$ for some positive real number t, where D' is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor that is permissible with respect to $(X, S + B)$.*

Then the homomorphisms

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D)),$$

which are induced by the natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$, are injective for all q.

62 **Theorem 2.54 (cf. [Am1, Theorem 3.2]).** *Let $(Y, S+B)$ be an embedded normal crossing pair such that $S + B$ is a boundary $\mathbb{R}$-divisor, S is reduced, and $\llcorner B \lrcorner = 0$. Let $f : Y \to X$ be a proper morphism and L a Cartier divisor on Y such that $H \sim_{\mathbb{R}} L - (K_Y + S + B)$ is f-semi-ample.*

(i) *every non-zero local section of $R^q f_* \mathcal{O}_Y(L)$ contains in its support the f-image of some strata of $(Y, S + B)$.*

(ii) *let $\pi : X \to V$ be a projective morphism and assume that $H \sim_{\mathbb{R}} f^*H'$ for some π-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor H' on X. Then $R^q f_* \mathcal{O}_Y(L)$ is π_*-acyclic, that is, $R^p \pi_* R^q f_* \mathcal{O}_Y(L) = 0$ for any $p > 0$.*

Before we go to the proof, let us recall the definition of *normal crossing pairs*, which is a slight generalization of Definition 2.13. The following definition is the same as [Am1, Definition 2.3] though it may look different.

625 **Definition 2.55 (Normal crossing pair).** Let X be a normal crossing variety. We say that a reduced divisor D on X is *normal crossing* if, in the notation of Definition 2.8, we have

$$\widehat{\mathcal{O}}_{D,x} \simeq \frac{\mathbb{C}[[x_0, \cdots, x_N]]}{(x_0 \cdots x_k, x_{i_1} \cdots x_{i_l})}$$

for some $\{i_1, \cdots, i_l\} \subset \{k + 1, \cdots, N\}$. We say that the pair (X, B) is a *normal crossing pair* if the following conditions are satisfied.

(1) X is a normal crossing variety, and

(2) B is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor whose support is normal crossing on X.

We say that a normal crossing pair (X, B) is *embedded* if there exists a closed embedding $\iota : X \to M$, where M is a smooth variety of dimension $\dim X + 1$. We put $K_{X^0} + \Theta = \eta^*(K_X + B)$, where $\eta : X^0 \to X$ is the normalization of X. From now on, we assume that B is a subboundary $\mathbb{R}$-divisor. A *stratum* of (X, B) is an irreducible component of X or the image of some lc center of (X^0, Θ) on X. A Cartier divisor D on a normal crossing pair (X, B) is called *permissible with respect to* (X, B) if D contains no strata of the pair (X, B).

The following three lemmas are easy to check. So, we omit the proofs.

63 **Lemma 2.56.** *Let X be a normal crossing divisor on a smooth variety M. Then there exists a sequence of blow-ups $M_k \to M_{k-1} \to \cdots \to M_0 = M$ with the following properties.*

(i) *$\sigma_{i+1} : M_{i+1} \to M_i$ is the blow-up along a smooth stratum of X_i for any $i \geq 0$,*

(ii) *$X_0 = X$ and X_{i+1} is the inverse image of X_i with the reduced structure for any $i \geq 0$, and*

(iii) *X_k is a simple normal crossing divisor on M_k.*

For each step σ_{i+1}, we can directly check that $\sigma_{i+1*}\mathcal{O}_{X_{i+1}} \simeq \mathcal{O}_{X_i}$ and $R^q\sigma_{i+1*}\mathcal{O}_{X_{i+1}} = 0$ for any $i \geq 0$ and $q \geq 1$. Let B be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that Supp$B$ *is normal crossing. We put $B_0 = B$ and $K_{X_{i+1}} + B_{i+1} = \sigma_{i+1}^*(K_{X_i} + B_i)$ for all $i \geq 0$. Then it is obvious that B_i is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor and* SuppB_i *is normal crossing on X_i for any $i \geq 0$. We can also check that B_i is a boundary $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor) for any $i \geq 0$ if so is B. If B is a boundary, then the σ_{i+1}-image of any stratum of (X_{i+1}, B_{i+1}) is a stratum of (X_i, B_i).*

635 **Remark 2.57.** Each step in Lemma 2.56 is called *embedded log transformation* in [Am1, Section 2]. See also Lemma 2.19.

64 **Lemma 2.58.** *Let X be a simple normal crossing divisor on a smooth variety M. Let $S+B$ be a boundary $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that* Supp$(S+B)$ *is normal crossing, S is reduced, and $\llcorner B \lrcorner = 0$. Then there exists a sequence of blow-ups $M_k \to M_{k-1} \to \cdots \to M_0 = M$ with the following properties.*

(i) *$\sigma_{i+1} : M_{i+1} \to M_i$ is the blow-up along a smooth stratum of (X_i, S_i) that is contained in S_i for any $i \geq 0$,*

(ii) *we put* $X_0 = X$, $S_0 = S$, *and* $B_0 = B$, *and* X_{i+1} *is the strict transform of* X_i *for any* $i \geq 0$,

(iii) *we define* $K_{X_{i+1}} + S_{i+1} + B_{i+1} = \sigma_{i+1}^*(K_{X_i} + S_i + B_i)$ *for any* $i \geq 0$, *where* B_{i+1} *is the strict transform of* B_i *on* X_{i+1},

(iv) *the* σ_{i+1}*-image of any stratum of* $(X_{i+1}, S_{i+1} + B_{i+1})$ *is a stratum of* $(X_i, S_i + B_i)$, *and*

(v) S_k *is a simple normal crossing divisor on* X_k.

For each step σ_{i+1}, *we can easily check that* $\sigma_{i+1*}\mathcal{O}_{X_{i+1}} \simeq \mathcal{O}_{X_i}$ *and* $R^q\sigma_{i+1*}\mathcal{O}_{X_{i+1}} = 0$ *for any* $i \geq 0$ *and* $q \geq 1$. *We note that* X_i *is simple normal crossing,* $\mathrm{Supp}(S_i + B_i)$ *is normal crossing on* X_i, *and* S_i *is reduced for any* $i \geq 0$.

65 **Lemma 2.59.** *Let* X *be a simple normal crossing divisor on a smooth variety* M. *Let* $S+B$ *be a boundary* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor on* X *such that* $\mathrm{Supp}(S+B)$ *is normal crossing,* S *is reduced and simple normal crossing, and* $\llcorner B \lrcorner = 0$. *Then there exists a sequence of blow-ups* $M_k \to M_{k-1} \to \cdots \to M_0 = M$ *with the following properties.*

(i) $\sigma_{i+1} : M_{i+1} \to M_i$ *is the blow-up along a smooth stratum of* $(X_i, \mathrm{Supp}B_i)$ *that is contained in* $\mathrm{Supp}B_i$ *for any* $i \geq 0$,

(ii) *we put* $X_0 = X$, $S_0 = S$, *and* $B_0 = B$, *and* X_{i+1} *is the strict transform of* X_i *for any* $i \geq 0$,

(iii) *we define* $K_{X_{i+1}} + S_{i+1} + B_{i+1} = \sigma_{i+1}^*(K_{X_i} + S_i + B_i)$ *for any* $i \geq 0$, *where* S_{i+1} *is the strict transform of* S_i *on* X_{i+1}, *and*

(iv) $\mathrm{Supp}(S_k + B_k)$ *is a simple normal crossing divisor on* X_k.

We note that X_i *is simple normal crossing on* M_i *and* $\mathrm{Supp}(S_i + B_i)$ *is normal crossing on* X_i *for any* $i \geq 0$. *We can easily check that* $\llcorner B_i \lrcorner \leq 0$ *for any* $i \geq 0$. *The composition morphism* $M_k \to M$ *is denoted by* σ. *Let* L *be any Cartier divisor on* X. *We put* $E = \ulcorner -B_k \urcorner$. *Then* E *is an effective* σ*-exceptional Cartier divisor on* X_k *and we obtain* $\sigma_*\mathcal{O}_{X_k}(\sigma^*L+E) \simeq \mathcal{O}_X(L)$ *and* $R^q\sigma_*\mathcal{O}_{X_k}(\sigma^*L + E) = 0$ *for any* $q \geq 1$ *by* Theorem 2.39 (i). *We note that* $\sigma^*L + E - (K_{X_k} + S_k + \{B_k\}) = \sigma^*L - \sigma^*(K_X + S + B)$ *is* $\mathbb{R}$*-linearly trivial over* X *and* σ *is an isomorphism at the generic point of any stratum of* $(X_k, S_k + \{B_k\})$.

Let us go to the proof of Theorems 2.53 and 2.54.

Proof of Theorems 2.53 and 2.54. We take a sequence of blow-ups and obtain a projective morphism $\sigma : X' \to X$ (resp. $\sigma : Y' \to Y$) from an embedded simple normal crossing variety X' (resp. Y') in Theorem 2.53 (resp. Theorem 2.54) by Lemma 2.56. We can replace X (resp. Y) and L with X' (resp. Y') and σ^*L by Leray's spectral sequence. So, we can assume that X (resp. Y) is simple normal crossing. Similarly, we can assume that S is simple normal crossing on X (resp. Y) by applying Lemma 2.58. Finally, we use Lemma 2.59 and obtain a birational morphism $\sigma : (X', S' + B') \to (X, S + B)$ (resp. $(Y', S' + B') \to (Y, S + B)$) from an embedded simple normal crossing pair $(X', S'+B')$ (resp. $(Y', S'+B')$) such that $K_{X'}+S'+B' = \sigma^*(K_X+S+B)$ (resp. $K_{Y'} + S' + B' = \sigma^*(K_Y + S + B)$) as in Lemma 2.59. By Lemma 2.59, we can replace $(X, S + B)$ (resp. $(Y, S + B)$) and L with $(X', S' + \{B'\})$ (resp. $(Y', S' + \{B'\})$) and $\sigma^*L + \ulcorner -B' \urcorner$ by Leray's spectral sequence. Then we apply Theorem 5.1 2.38 (resp. Theorem 8 2.39). Thus, we obtain Theorems 2.53 and 2.54. □

2.8 Examples

28-sec

In this section, we treat various supplementary examples.

2.60 (Examples for Section sec2 2.3). Let X be a smooth projective variety and let M be a Cartier divisor on X such that $N \sim mM$, where N is a simple normal crossing divisor on X and $m \geq 2$. We put $B = \frac{1}{m}N$ and $L = K_X + M$. In this setting, we can apply Proposition 1 2.22. If M is semi-ample, then the existence of such N and m is obvious by Bertini. Here, we give explicit examples where M is not nef.

Example 2.61. We consider the $\mathbb{P}^1$-bundle $\pi : X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \to \mathbb{P}^1$. Let E and G be the sections of π such that $E^2 = -2$ and $G^2 = 2$. We note that $E + 2F \sim G$, where F is a fiber of π. We consider $M = E + F$. Then $2M = 2E + 2F \sim E + G$. In this case, $M \cdot E = -1$. In particular, M is not nef. Furthermore, we can easily check that $H^i(X, \mathcal{O}_X(K_X + M)) = 0$ for any i. So, it is not interesting to apply Proposition 1 2.22.

Example 2.62. We consider the $\mathbb{P}^1$-bundle $\pi : Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(4)) \to \mathbb{P}^1$. Let G (resp. E) be the positive (resp. negative) section of π, that is, the section corresponding to the projection $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(4) \to \mathcal{O}_{\mathbb{P}^1}(4)$ (resp. $\mathcal{O}_{\mathbb{P}^1} \oplus$

$\mathcal{O}_{\mathbb{P}^1}(4) \to \mathcal{O}_{\mathbb{P}^1})$. We put $M' = -F + 2G$, where F is a fiber of π. Then M' is not nef and $2M' \sim G + E + F_1 + F_2 + H$, where F_1 and F_2 are distinct fibers of π, and H is a general member of the free linear system $|2G|$. Note that $G + E + F_1 + F_2 + H$ is a reduced simple normal crossing divisor on Y. We put $X = Y \times C$, where C is an elliptic curve, and $M = p^*M'$, where $p : X \to Y$ is the projection. Then X is a smooth projective variety and M is a Cartier divisor on X. We note that M is not nef and that we can find a reduced simple normal crossing divisor such that $N \sim 2M$. By the Künneth formula, we have

$$H^1(X, \mathcal{O}_X(K_X + M)) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \simeq \mathbb{C}^2.$$

Therefore, X with $L = K_X + M$ satisfies the conditions in Proposition 2.22 and we have $H^1(X, \mathcal{O}_X(L)) \neq 0$.

2.63 (Kodaira vanishing theorem for singular varieties). The following example is due to Sommese (cf. [So, (0.2.4) Example]). It shows that the Kodaira vanishing theorem does not necessarily hold for varieties with non-lc singularities. Therefore, Corollary 2.43 is sharp.

Proposition 2.64 (Sommese). *We consider the $\mathbb{P}^3$-bundle*

$$\pi : Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3}) \to \mathbb{P}^1$$

over $\mathbb{P}^1$. Let $\mathcal{M} = \mathcal{O}_Y(1)$ be the tautological line bundle of $\pi : Y \to \mathbb{P}^1$. We take a general member X of the linear system $|(\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1))^{\otimes 4}|$. Then X is a normal projective Gorenstein threefold and X is not lc. We put $\mathcal{L} = \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$. Then $\mathcal{L}$ is ample. In this case, we can check that $H^2(X, \mathcal{L}^{-1}) = \mathbb{C}$. By the Serre duality, $H^1(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}) = \mathbb{C}$. Therefore, the Kodaira vanishing theorem does not hold for X.

Proof. We consider the following short exact sequence

$$0 \to \mathcal{L}^{-1}(-X) \to \mathcal{L}^{-1} \to \mathcal{L}^{-1}|_X \to 0.$$

Then we have the long exact sequence

$$\begin{aligned} \cdots \to H^i(Y, \mathcal{L}^{-1}(-X)) \to H^i(Y, \mathcal{L}^{-1}) \to H^i(X, \mathcal{L}^{-1}) \\ \to H^{i+1}(Y, \mathcal{L}^{-1}(-X)) \to \cdots . \end{aligned}$$

Since $H^i(Y, \mathcal{L}^{-1}) = 0$ for $i < 4$ by the original Kodaira vanishing theorem, we obtain that $H^2(X, \mathcal{L}^{-1}) = H^3(Y, \mathcal{L}^{-1}(-X))$. Therefore, it is sufficient to prove that $H^3(Y, \mathcal{L}^{-1}(-X)) = \mathbb{C}$.

We have

$$\mathcal{L}^{-1}(-X) = \mathcal{M}^{-1} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{M}^{-4} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(4) = \mathcal{M}^{-5} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(3).$$

We note that $R^i\pi_*\mathcal{M}^{-5} = 0$ for $i \neq 3$ because $\mathcal{M} = \mathcal{O}_Y(1)$. By the Grothendieck duality,

$$R\mathcal{H}om(R\pi_*\mathcal{M}^{-5}, \mathcal{O}_{\mathbb{P}^1}(K_{\mathbb{P}^1})[1]) = R\pi_*R\mathcal{H}om(\mathcal{M}^{-5}, \mathcal{O}_Y(K_Y)[4]).$$

By the Grothendieck duality again,

$$\begin{aligned} R\pi_*\mathcal{M}^{-5} &= R\mathcal{H}om(R\pi_*R\mathcal{H}om(\mathcal{M}^{-5}, \mathcal{O}_Y(K_Y)[4]), \mathcal{O}_{\mathbb{P}^1}(K_{\mathbb{P}^1})[1]) \\ &= R\mathcal{H}om(R\pi_*(\mathcal{O}_Y(K_Y) \otimes \mathcal{M}^5), \mathcal{O}_{\mathbb{P}^1}(K_{\mathbb{P}^1}))[-3] = (*). \end{aligned}$$

By the definition, we have

$$\mathcal{O}_Y(K_Y) = \pi^*(\mathcal{O}_{\mathbb{P}^1}(K_{\mathbb{P}^1}) \otimes \det(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3})) \otimes \mathcal{M}^{-4} = \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{M}^{-4}.$$

By this formula, we obtain

$$\mathcal{O}_Y(K_Y) \otimes \mathcal{M}^5 = \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{M}.$$

Thus, $R^i\pi_*(\mathcal{O}_Y(K_Y) \otimes \mathcal{M}^5) = 0$ for any $i > 0$. We note that

$$\begin{aligned} \pi_*(\mathcal{O}_Y(K_Y) \otimes \mathcal{M}^5) &= \mathcal{O}_{\mathbb{P}^1}(1) \otimes \pi_*\mathcal{M} \\ &= \mathcal{O}_{\mathbb{P}^1}(1) \otimes (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3}) = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 3}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} (*) &= R\mathcal{H}om(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 3}, \mathcal{O}_{\mathbb{P}^1}(-2))[-3] \\ &= (\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(-4)^{\oplus 3})[-3]. \end{aligned}$$

So, we obtain $R^3\pi_*\mathcal{M}^{-5} = \mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(-4)^{\oplus 3}$. Thus, $R^3\pi_*\mathcal{M}^{-5} \otimes \mathcal{O}_{\mathbb{P}^1}(3) = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}$.

By the spectral sequence, we have

$$\begin{aligned} H^3(Y, \mathcal{L}^{-1}(-X)) &= H^3(Y, \mathcal{M}^{-5} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(3)) \\ &= H^0(\mathbb{P}^1, R^3\pi_*(\mathcal{M}^{-5} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(3))) \\ &= H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}) = \mathbb{C}. \end{aligned}$$

Therefore, $H^2(X, \mathcal{L}^{-1}) = \mathbb{C}$.

Let us recall that X is a general member of the linear system $|(\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1))^{\otimes 4}|$. Let C be the negative section of $\pi : Y \to \mathbb{P}^1$, that is, the section corresponding to the projection

$$\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^1} \to 0.$$

From now, we will check that $|\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)|$ is free outside C. Once we checked it, we know that $|(\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1))^{\otimes 4}|$ is free outside C. Then X is smooth in codimension one. Since Y is smooth, X is normal and Gorenstein by adjunction.

We take $Z \in |\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)| \neq \emptyset$. Since $H^0(Y, \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)) = 0$, Z can not have a fiber of π as an irreducible component, that is, any irreducible component of Z is mapped onto $\mathbb{P}^1$ by $\pi : Y \to \mathbb{P}^1$. On the other hand, let l be a line in a fiber of $\pi : Y \to \mathbb{P}^1$. Then $Z \cdot l = 1$. Therefore, Z is irreducible. Let $F = \mathbb{P}^3$ be a fiber of $\pi : Y \to \mathbb{P}^1$. We consider

$$\begin{aligned} 0 = H^0(Y, \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_Y(-F)) \to H^0(Y, \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)) \\ \to H^0(F, \mathcal{O}_F(1)) \to H^1(Y, \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_Y(-F)) \to \cdots . \end{aligned}$$

Since $(\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)) \cdot C = -1$, every member of $|\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)|$ contains C. We put $P = F \cap C$. Then the image of

$$\alpha : H^0(Y, \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)) \to H^0(F, \mathcal{O}_F(1))$$

is $H^0(F, m_P \otimes \mathcal{O}_F(1))$, where m_P is the maximal ideal of P. It is because the dimension of $H^0(Y, \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1))$ is three. Thus, we know that $|\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)|$ is free outside C. In particular, $|(\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1))^{\otimes 4}|$ is free outside C.

More explicitly, the image of the injection

$$\alpha : H^0(Y, \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)) \to H^0(F, \mathcal{O}_F(1))$$

is $H^0(F, m_P \otimes \mathcal{O}_F(1))$. We note that

$$H^0(Y, \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 3}) = \mathbb{C}^3,$$

and

$$H^0(Y, (\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1))^{\otimes 4}) = H^0(\mathbb{P}^1, \mathrm{Sym}^4(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 3})) = \mathbb{C}^{15}.$$

We can check that the restriction of $H^0(Y, (\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1))^{\otimes 4})$ to F is $\mathrm{Sym}^4 H^0(F, m_P \otimes \mathcal{O}_F(1))$. Thus, the general fiber f of $\pi : X \to \mathbb{P}^1$ is a cone in $\mathbb{P}^3$ on a smooth plane curve of degree 4 with the vertex $P = f \cap C$. Therefore, (Y, X) is not lc because the multiplicity of X along C is four. Thus, X is not lc by the inversion of adjunction (cf. Corollary 4.47). Anyway, X is the required variety. □

Remark 2.65. We consider the $\mathbb{P}^{k+1}$-bundle

$$\pi : Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(k+1)}) \to \mathbb{P}^1$$

over $\mathbb{P}^1$ for $k \geq 2$. We put $\mathcal{M} = \mathcal{O}_Y(1)$ and $\mathcal{L} = \mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$. Then $\mathcal{L}$ is ample. We take a general member X of the linear system $|(\mathcal{M} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1))^{\otimes(k+2)}|$. Then we can check the following properties.

(1) X is a normal projective Gorenstein $(k+1)$-fold.

(2) X is not lc.

(3) We can check that $R^{k+1}\pi_*\mathcal{M}^{-(k+3)} = \mathcal{O}_{\mathbb{P}^1}(-1-k) \oplus \mathcal{O}_{\mathbb{P}^1}(-2-k)^{\oplus(k+1)}$ and that $R^i\pi_*\mathcal{M}^{-(k+3)} = 0$ for $i \neq k+1$.

(4) Since $\mathcal{L}^{-1}(-X) = \mathcal{M}^{-(k+3)} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(k+1)$, we have

$$\begin{aligned} H^{k+1}(Y, \mathcal{L}^{-1}(-X)) &= H^0(\mathbb{P}^1, R^{k+1}\pi_*\mathcal{M}^{-(k+3)} \otimes \mathcal{O}_{\mathbb{P}^1}(k+1)) \\ &= H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(k+1)}) = \mathbb{C}. \end{aligned}$$

Thus, $H^k(X, \mathcal{L}^{-1}) = H^{k+1}(Y, \mathcal{L}^{-1}(-X)) = \mathbb{C}$.

We note that the first cohomology group of an anti-ample line bundle on a normal variety with $\dim \geq 2$ always vanishes by the following Mumford vanishing theorem.

Theorem 2.66 (Mumford). *Let V be a normal complete algebraic variety and $\mathcal{L}$ be a semi-ample line bundle on V. Assume that $\kappa(V, \mathcal{L}) \geq 2$. Then $H^1(V, \mathcal{L}^{-1}) = 0$.*

Proof. Let $f : W \to V$ be a resolution. By Leray's spectral sequence,

$$0 \to H^1(V, f_*f^*\mathcal{L}^{-1}) \to H^1(W, f^*\mathcal{L}^{-1}) \to \cdots .$$

By the Kawamata–Viehweg vanishing theorem, $H^1(W, f^*\mathcal{L}^{-1}) = 0$. Thus, $H^1(V, \mathcal{L}^{-1}) = H^1(V, f_*f^*\mathcal{L}^{-1}) = 0$. □

2.67 (On the Kawamata–Viehweg vanishing theorem). The next example shows that a naive generalization of the Kawamata–Viehweg vanishing theorem does not necessarily hold for varieties with lc singularities.

ex4 **Example 2.68.** We put $V = \mathbb{P}^2 \times \mathbb{P}^2$. Let $p_i : V \to \mathbb{P}^2$ be the i-th projection for $i = 1$ and 2. We define $\mathcal{L} = p_1^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(1)$ and consider the $\mathbb{P}^1$-bundle $\pi : W = \mathbb{P}_V(\mathcal{L} \oplus \mathcal{O}_V) \to V$. Let $F = \mathbb{P}^2 \times \mathbb{P}^2$ be the negative section of $\pi : W \to V$, that is, the section of π corresponding to $\mathcal{L} \oplus \mathcal{O}_V \to \mathcal{O}_V \to 0$. By using the linear system $|\mathcal{O}_W(1) \otimes \pi^* p_1^*\mathcal{O}_{\mathbb{P}^2}(1)|$, we can contract $F = \mathbb{P}^2 \times \mathbb{P}^2$ to $\mathbb{P}^2 \times \{\text{point}\}$.

Next, we consider an elliptic curve $C \subset \mathbb{P}^2$ and put $Z = C \times C \subset V = \mathbb{P}^2 \times \mathbb{P}^2$. Let $\pi : Y \to Z$ be the restriction of $\pi : W \to V$ to Z. The restriction of the above contraction morphism $\Phi_{|\mathcal{O}_W(1) \otimes \pi^* p_1^*\mathcal{O}_{\mathbb{P}^2}(1)|} : W \to U$ to Y is denoted by $f : Y \to X$. Then, the exceptional locus of $f : Y \to X$ is $E = F|_Y = C \times C$ and f contracts E to $C \times \{\text{point}\}$.

Let $\mathcal{O}_W(1)$ be the tautological line bundle of the $\mathbb{P}^1$-bundle $\pi : W \to V$. By the construction, $\mathcal{O}_W(1) = \mathcal{O}_W(D)$, where D is the positive section of π, that is, the section corresponding to $\mathcal{L} \oplus \mathcal{O}_W \to \mathcal{L} \to 0$. By the definition,

$$\mathcal{O}_W(K_W) = \pi^*(\mathcal{O}_V(K_V) \otimes \mathcal{L}) \otimes \mathcal{O}_W(-2).$$

By adjunction,

$$\mathcal{O}_Y(K_Y) = \pi^*(\mathcal{O}_Z(K_Z) \otimes \mathcal{L}|_Z) \otimes \mathcal{O}_Y(-2) = \pi^*(\mathcal{L}|_Z) \otimes \mathcal{O}_Y(-2).$$

Therefore,

$$\mathcal{O}_Y(K_Y + E) = \pi^*(\mathcal{L}|_Z) \otimes \mathcal{O}_Y(-2) \otimes \mathcal{O}_Y(E).$$

We note that $E = F|_Y$. Since $\mathcal{O}_Y(E) \otimes \pi^*(\mathcal{L}|_Z) \simeq \mathcal{O}_Y(D)$, we have $\mathcal{O}_Y(-(K_Y + E)) = \mathcal{O}_Y(1)$ because $\mathcal{O}_Y(1) = \mathcal{O}_Y(D)$. Thus, $-(K_Y + E)$ is nef and big.

On the other hand, it is not difficult to see that X is a normal projective Gorenstein threefold, X is lc but not klt along $G = f(E)$, and that X is smooth outside G. Since we can check that $f^*K_X = K_Y + E$, $-K_X$ is nef and big.

Finally, we consider the short exact sequence

$$0 \to \mathcal{J} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J} \to 0,$$

where $\mathcal{J}$ is the multiplier ideal sheaf of X. In our case, we can easily check that $\mathcal{J} = f_*\mathcal{O}_Y(-E) = \mathcal{I}_G$, where $\mathcal{I}_G$ is the defining ideal sheaf of G on X.

Since $-K_X$ is nef and big, $H^i(X, \mathcal{J}) = 0$ for any $i > 0$ by Nadel's vanishing theorem. Therefore, $H^i(X, \mathcal{O}_X) = H^i(G, \mathcal{O}_G)$ for any $i > 0$. Since G is an elliptic curve, $H^1(X, \mathcal{O}_X) = H^1(G, \mathcal{O}_G) = \mathbb{C}$. We note that $-K_X$ is nef and big but $-K_X$ is not log big with respect to X.

2.69 (On the injectivity theorem). The final example in this section supplements Theorem 5.1 2.38.

Example 2.70. We consider the $\mathbb{P}^1$-bundle $\pi : X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$. Let S (resp. H) be the negative (resp. positive) section of π, that is, the section corresponding to the projection $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1}$ (resp. $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1}(1)$). Then H is semi-ample and $S + F \sim H$, where F is a fiber of π.

Claim. *The homomorphism*

$$H^1(X, \mathcal{O}_X(K_X + S + H)) \to H^1(X, \mathcal{O}_X(K_X + S + H + S + F))$$

induced by the natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(S + F)$ is not injective.

Proof of Claim. It is sufficient to see that the homomorphism

$$H^1(X, \mathcal{O}_X(K_X + S + H)) \to H^1(X, \mathcal{O}_X(K_X + S + H + F))$$

induced by the natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(F)$ is not injective. We consider the short exact sequence

$$\begin{aligned} 0 \to \mathcal{O}_X(K_X + S + H) &\to \mathcal{O}_X(K_X + S + H + F) \\ &\to \mathcal{O}_F(K_F + (S + H)|_F) \to 0. \end{aligned}$$

We note that $F \simeq \mathbb{P}^1$ and $\mathcal{O}_F(K_F + (S + H)|_F) \simeq \mathcal{O}_{\mathbb{P}^1}$. Therefore, we obtain the following exact sequence

$$0 \to \mathbb{C} \to H^1(X, \mathcal{O}_X(K_X + S + H)) \to H^1(X, \mathcal{O}_X(K_X + S + H + F)) \to 0.$$

Thus, $H^1(X, \mathcal{O}_X(K_X + S + H)) \to H^1(X, \mathcal{O}_X(K_X + S + H + F))$ is not injective. We note that $S + F$ is not *permissible* with respect to (X, S). □

2.9 Review of the proofs

29-sec

We close this chapter with the review of our proofs of Theorems 2.53 and 2.54. It may help the reader to compare this chapter with [AmI, Section 3]. We think that our proofs are not so long. Ambro's proofs seem to be too short.

680 **2.71 (Review).** We review our proofs of the injectivity, torsion-free, and vanishing theorems.

step1 **Step 1.** (E_1-degeneration of a certain Hodge to de Rham type spectral sequence). We discuss this E_1-degeneration in 2.32. As we pointed out in the introduction, the appropriate spectral sequence was not chosen in [AmI]. It is one of the crucial technical problems in [AmI, Section 3]. This step is purely Hodge theoretic. We describe it in Section 2.4.

step2 **Step 2.** (Fundamental injectivity theorem: Proposition 2.23). This is a very special case of [AmI, Theorem 3.1] and follows from the E_1-degeneration in Step 1 by using covering arguments. This step is in Section 2.3.

step3 **Step 3.** (Relative vanishing lemma: Lemma 2.33). This step is missing in [AmI]. It is a very special case of [AmI, Theorem 3.2 (ii)]. However, we can not use [AmI, Theorem 3.2 (ii)] in this stage. Our proof of this lemma does *not* work directly for normal crossing pairs. So, we need to assume that the varieties are *simple* normal crossing pairs.

step4 **Step 4.** (Injectivity theorem for embedded simple normal crossing pairs: Theorem 2.38). It is [AmI, Theorem 3.1] for embedded *simple* normal crossing pairs. It follows easily from Step 2 since we already have the relative vanishing lemma in Step 3. A key point in this step is Lemma 2.34, which is missing in [AmI] and works only for embedded *simple* normal crossing pairs.

step5 **Step 5.** (Torsion-free and vanishing theorems for embedded simple normal crossing pairs: Theorem 2.39). It is [AmI, Theorem 3.2] for embedded *simple* normal crossing pairs. The proof uses the lemmas on desingularization and compactification (see Lemmas 2.34 and 2.36), which hold only for embedded *simple* normal crossing pairs, and the injectivity theorem proved for embedded *simple* normal crossing pairs in Step 4. Therefore, this step also works only for embedded *simple* normal crossing pairs. Our proof of the vanishing theorem is slightly different from Ambro's one. Compare Steps 2 and 3 in the proof of Theorem 2.39 with (a) and (b) in the proof of [AmI, Theorem 3.2 (ii)]. See Remark 2.40.

step6 **Step 6.** (Ambro's theorems: Theorems 2.53 and 2.54). In this final step, we recover Ambro's theorems, that is, [Am1, Theorems 3.1 and 3.2], in full generality. Since we have already proved [Am1, Theorem 3.2 (i)] for embedded *simple* normal crossing pairs in Step 5, we can reduce the problems to the case when the varieties are embedded *simple* normal crossing pairs by blow-ups and Leray's spectral sequences. This step is described in Section 2.7.

Chapter 3

Log Minimal Model Program for lc pairs

chap3

In this chapter, we discuss the log minimal model program (LMMP, for short) for log canonical pairs.

In Section 3.1, we will explicitly state the LMMP for lc pairs. We state the cone and contraction theorems explicitly for lc pairs with the additional estimate of lengths of extremal rays. We also write the flip conjectures for lc pairs. We note that the flip conjecture I (existence of an lc flip) is still open and that the flip conjecture II (termination of a sequence of lc flips) follows from the termination of klt flips. We give a proof of the flip conjecture I in dimension four.

Theorem 3.1 (cf. Theorem 3.13). *Log canonical flips exist in dimension four.*

In Section 3.2, we introduce the notion of quasi-log varieties. We think that the notion of quasi-log varieties is indispensable for investigating lc pairs. The reader can find that the key points of the theory of quasi-log varieties in [AmI] are adjunction and the vanishing theorem (see [AmI, Theorem 4.4] and Theorem 3.39). Adjunction and the vanishing theorems for quasi-log varieties follow from [AmI, 3. Vanishing Theorems]. However, Section 3 of [AmI] contains various troubles. Now Chapter 2 gives us sufficiently powerful vanishing and torsion-free theorems for the theory of quasi-log varieties. We succeed in removing all the troublesome problems for the foundation of the theory of quasi-log varieties. It is one of the main contributions of this chapter and [F16]. We slightly change Ambro's formulation. By this change,

the theory of quasi-log varieties becomes more accessible. As a byproduct, we have the following definition of quasi-log varieties.

new-def **Definition 3.2 (Quasi-log varieties).** A *quasi-log variety* is a scheme X endowed with an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor ω, a proper closed subscheme $X_{-\infty} \subset X$, and a finite collection $\{C\}$ of reduced and irreducible subvarieties of X such that there is a proper morphism $f : Y \to X$ from a simple normal crossing divisor Y on a smooth variety M satisfying the following properties:

(0) there exists an $\mathbb{R}$-divisor D on M such that $\mathrm{Supp}(D + Y)$ is simple normal crossing on M and that D and Y have no common irreducible components.

(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$, where $B_Y = D|_Y$.

(2) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner)$ induces an isomorphism

$$\mathcal{I}_{X_{-\infty}} \to f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1}\lrcorner),$$

where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$.

(3) The collection of subvarieties $\{C\}$ coincides with the image of (Y, B_Y)-strata that are not included in $X_{-\infty}$.

Definition new-def 3.2 is equivalent to Ambro's original definition (see ambro [Am1, Definition 4.1] and Definition glog-def-ambro 3.55). For the details, see the subsection 3-2-6 3.2.6. However, we think Definition new-def 3.2 is much better than Ambro's. Once we adopt Definition new-def 3.2, we do not need the notion of *normal crossing pairs* to define quasi-log varieties and get flexibility in the choice of *quasi-log resolutions* $f : Y \to X$ by Proposition taisetsu3 3.52.

In Section 33-sec 3.3, we will prove the fundamental theorems for the theory of quasi-log varieties such as cone, contraction, rationality, and base point free theorems.

The paper fuj-lec [F16] is a gentle introduction to the log minimal model program for lc pairs. It may be better to see fuj-lec [F16] before reading this chapter.

3.1 LMMP for log canonical pairs

31-sec

3.1.1 Log minimal model program

In this subsection, we explicitly state the log minimal model program (LMMP, for short) for log canonical pairs. It is known to some experts but we can

not find it in the standard literature. The following cone theorem is a consequence of Ambro's cone theorem for quasi-log varieties (see Theorem 5.10 in [Am1], Theorems 3.74 and 3.75 below) except for the existence of C_j with $0 < -(K_X + B) \cdot C_j \leq 2 \dim X$ in Theorem 3.3 (1). We will discuss the estimate of lengths of extremal rays in the subsection 3.1.3.

coco-th **Theorem 3.3 (Cone and contraction theorems).** *Let (X, B) be an lc pair, B an $\mathbb{R}$-divisor, and $f : X \to Y$ a projective morphism between algebraic varieties. Then we have*

(i) *There are (countably many) rational curves $C_j \subset X$ such that $f(C_j)$ is a point, $0 < -(K_X + B) \cdot C_j \leq 2 \dim X$, and*

$$\overline{NE}(X/Y) = \overline{NE}(X/Y)_{(K_X+B)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_j].$$

(ii) *For any $\varepsilon > 0$ and f-ample $\mathbb{R}$-divisor H,*

$$\overline{NE}(X/Y) = \overline{NE}(X/Y)_{(K_X+B+\varepsilon H)\geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_j].$$

(iii) *Let $F \subset \overline{NE}(X/Y)$ be a $(K_X + B)$-negative extremal face. Then there is a unique morphism $\varphi_F : X \to Z$ over Y such that $(\varphi_F)_* \mathcal{O}_X \simeq \mathcal{O}_Z$, Z is projective over Y, and an irreducible curve $C \subset X$ is mapped to a point by φ_F if and only if $[C] \in F$. The map φ_F is called the contraction of F.*

(iv) *Let F and φ_F be as in* (iii). *Let L be a line bundle on X such that $L \cdot C = 0$ for every curve C with $[C] \in F$. Then there is a line bundle L_Z on Z such that $L \simeq \varphi_F^* L_Z$.*

est-re **Remark 3.4 (Lengths of extremal rays).** In Theorem 3.3 (i), the estimate $-(K_X + B) \cdot C_j \leq 2 \dim X$ should be replaced by $-(K_X + B) \cdot C_j \leq \dim X + 1$. For toric varieties, this conjectural estimate and some generalizations were obtained in [F3] and [F5].

The following proposition is obvious. See, for example, [KM, Proposition 3.36].

di-fa-prop **Proposition 3.5.** *Let (X, B) be a $\mathbb{Q}$-factorial lc pair and let $\pi : X \to S$ be a projective morphism. Let $\varphi_R : X \to Y$ be the contraction of a $(K_X + B)$-negative extremal ray $R \subset \overline{NE}(X/S)$. Assume that φ_R is either a divisorial contraction (that is, φ_R contracts a divisor on X) or a Fano contraction (that is,* $\dim Y < \dim X$*). Then*

(1) *Y is $\mathbb{Q}$-factorial, and*

(2) *$\rho(Y/S) = \rho(X/S) - 1$.*

By the above cone and contraction theorems, we can easily see that the LMMP, that is, a recursive procedure explained in [KM, 3.31] (see also the subsection 1.6.4), works for $\mathbb{Q}$-factorial log canonical pairs if the flip conjectures (Flip Conjectures I and II) hold.

fI-conj **Conjecture 3.6.** ((Log) Flip Conjecture I: The existence of a (log) flip). *Let $\varphi\colon (X, B) \to W$ be an extremal flipping contraction of an n-dimensional pair, that is,*

(1) *(X, B) is lc, B is an $\mathbb{R}$-divisor,*

(2) *φ is small projective and φ has only connected fibers,*

(3) *$-(K_X + B)$ is φ-ample,*

(4) *$\rho(X/W) = 1$, and*

(5) *X is $\mathbb{Q}$-factorial.*

Then there should be a diagram:

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow \quad \swarrow & \\ & W & \end{array}$$

which satisfies the following conditions:

(i) *X^+ is a normal variety,*

(ii) *$\varphi^+\colon X^+ \to W$ is small projective, and*

(iii) *$K_{X^+} + B^+$ is φ^+-ample, where B^+ is the strict transform of B.*

We call $\varphi^+ : (X^+, B^+) \to W$ *a* $(K_X + B)$*-*flip *of* φ*.*

We note the following proposition. See, for example, [KM, Proposition 3.37].

fli-prop **Proposition 3.7.** *Let* (X, B) *be a* $\mathbb{Q}$*-factorial lc pair and let* $\pi : X \to S$ *be a projective morphism. Let* $\varphi_R : X \to Y$ *be the contraction of a* $(K_X + B)$*-negative extremal ray* $R \subset \overline{NE}(X/S)$*. Let* $\varphi_R : X \to Y$ *be the flipping contraction of* $R \subset \overline{NE}(X/S)$ *with flip* $\varphi_R^+ : X^+ \to Y$*. Then we have*

(1) X^+ *is* $\mathbb{Q}$*-factorial, and*

(2) $\rho(X^+/S) = \rho(X/S)$*.*

Note that to prove Conjecture 3.6 we can assume that B is a $\mathbb{Q}$-divisor, by perturbing B slightly. It is known that Conjecture 3.6 holds when $\dim X = 3$ (see [FA, Chapter 8]). Moreover, if there exists an $\mathbb{R}$-divisor B' on X such that $K_X + B'$ is klt and $-(K_X + B')$ is φ-ample, then Conjecture 3.6 is true by [BCHM]. The following famous conjecture is stronger than Conjecture 3.6. We will see it in Lemma 3.9.

fg-conj **Conjecture 3.8 (Finite generation).** *Let* X *be an* n*-dimensional smooth projective variety and* B *a boundary* $\mathbb{Q}$*-divisor on* X *such that* $\mathrm{Supp} B$ *is a simple normal crossing divisor on* X*. Assume that* $K_X + B$ *is big. Then the log canonical ring*

$$R(X, K_X + B) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\llcorner m(K_X + B) \lrcorner))$$

is a finitely generated $\mathbb{C}$*-algebra.*

Note that if there exists a $\mathbb{Q}$-divisor B' on X such that $K_X + B'$ is klt and $K_X + B' \sim_{\mathbb{Q}} K_X + B$, then Conjecture 3.8 holds by [BCHM]. See Remark 3.11.

fg-lem **Lemma 3.9.** *Let* $f : X \to S$ *be a proper surjective morphism between normal varieties with connected fibers. We assume* $\dim X = n$*. Let* B *be a* $\mathbb{Q}$*-divisor on* X *such that* (X, B) *is lc. Assume that* $K_X + B$ *is* f*-big. Then the relative log canonical ring*

$$R(X/S, K_X + B) = \bigoplus_{m \geq 0} f_* \mathcal{O}_X(\llcorner m(K_X + B) \lrcorner)$$

is a finitely generated $\mathcal{O}_S$*-algebra if* Conjecture 3.8 *holds. In particular,* Conjecture 3.8 *implies* Conjecture 3.6.

The following conjecture is the most general one.

fg-conj2 **Conjecture 3.10 (Finite Generation Conjecture).** *Let $f : X \to S$ be a proper surjective morphism between normal varieties. Let B be a $\mathbb{Q}$-divisor on X such that (X, B) is lc. Then the relative log canonical ring*

$$R(X/S, K_X + B) = \bigoplus_{m \geq 0} f_*\mathcal{O}_X(\llcorner m(K_X + B) \lrcorner)$$

is a finitely generated $\mathcal{O}_S$-algebra.

When (X, B) is klt, we can reduce Conjecture 3.10 to the case when $K_X + B$ is f-big by using a canonical bundle formula (see [FM]). Thus, Conjecture 3.10 holds for klt pairs by [BCHM]. When (X, B) is lc but not klt, we do not know if we can reduce it to the case when $K_X + B$ is f-big or not.

Before we go to the proof of Lemma 3.9, we note one easy remark.

trun-re **Remark 3.11.** For a graded integral domain $R = \bigoplus_{m \geq 0} R_m$ and a positive integer k, the truncated ring $R^{(k)}$ is defined by $R^{(k)} = \bigoplus_{m \geq 0} R_{km}$. Then R is finitely generated if and only if so is $R^{(k)}$. We consider $\mathrm{Proj}R$ when R is finitely generated. We note that $\mathrm{Proj}R^{(k)} = \mathrm{Proj}R$.

The following argument is well known to the experts.

Proof of Lemma 3.9. Since the problem is local, we can shrink S and assume that S is affine. By compactifying X and S and by the desingularization theorem, we can further assume that X and S are projective, X is smooth, B is effective, and $\mathrm{Supp}B$ is a simple normal crossing divisor. Let A be a very ample divisor on S and $H \in |rA|$ a general member for $r \gg 0$. Note that $K_X + B + (r-1)f^*A$ is big for $r \gg 0$ (cf. [KMM, Corollary 0-3-4]). Let m_0 be a positive integer such that $m_0(K_X + B + f^*H)$ is Cartier. By Conjecture 3.8, $\bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mm_0(K_X + B + f^*H)))$ is finitely generated. Thus, the relative log canonical model X' over S exists. Indeed, by assuming that m_0 is sufficiently large and divisible, $R(X, K_X + B + f^*H)^{(m_0)}$ is generated by $R(X, K_X + B + f^*H)_{m_0}$ and $|m_0(K_X + B + (r-1)f^*A)| \neq \emptyset$. Then $X' = \mathrm{Proj} \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mm_0(K_X + B + f^*H)))$ and X' is the closure of the image of X by the rational map defined by the complete linear system

$|m_0(K_X + B + rf^*A)|$. More precisely, let $g : X'' \to X$ be the elimination of the indeterminacy of the rational map defined by $|m_0(K_X + B + rf^*A)|$. Let $g' : X'' \to X'$ be the induced morphism and $h : X'' \to S$ the morphism defined by the complete linear system $|m_0 g^* f^* A|$. Then it is not difficult to see that h factors through X'.

Therefore, $\bigoplus_{m\geq 0} f_*\mathcal{O}_X(mm_0(K_X + B))$ is a finitely generated $\mathcal{O}_S$-algebra by the existence of the relative log canonical model X' over S. We finish the proof. □

The next theorem is an easy consequence of [BCHM], [AHK], [F1], and [F2].

Theorem 3.12. *Let (X, B) be a proper four-dimensional lc pair such that B is a $\mathbb{Q}$-divisor and $K_X + B$ is big. Then the log canonical ring*

$$\bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(\llcorner m(K_X + B)\lrcorner))$$

is finitely generated.

Proof. Without loss of generality, we can assume that X is smooth projective and Supp$B$ is simple normal crossing. Run a $(K_X + B)$-LMMP. Then we obtain a log minimal model (X', B') by [Sh1], [HM] and [AHK] with the aid of the special termination theorem (cf. [F8, Theorem 4.2.1]). By [F2, Theorem 3.1], which is a consequence of the main theorem in [F1], $K_{X'} + B'$ is semi-ample. In particular, $\bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(\llcorner m(K_X + B)\lrcorner)) \simeq \bigoplus_{m\geq 0} H^0(X', \mathcal{O}_{X'}(\llcorner m(K_{X'} + B')\lrcorner))$ is finitely generated. □

As a corollary, we obtain the next theorem by Lemma 3.9.

lcflip-th **Theorem 3.13.** Conjecture 3.6 *is true if* $\dim X \leq 4$.

More generally, we have the following theorem.

Theorem 3.14. Conjecture 3.10 *is true if* $\dim X \leq 4$.

For the proof, see [B], [F18], and [Fk2]. Let us go to the flip conjecture II.

fII-conj **Conjecture 3.15.** ((Log) Flip Conjecture II: Termination of a sequence of (log) flips). *A sequence of (log) flips*

$$(X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow (X_2, B_2) \dashrightarrow \cdots$$

terminates after finitely many steps. Namely, there does not exist an infinite sequence of (log) flips.

Note that it is sufficient to prove Conjecture 3.15 for any sequence of klt flips. The termination of dlt flips with dimension $\leq n-1$ implies the special termination in dimension n. Note that we use the formulation in [F8, Theorem 4.2.1]. The special termination and the termination of klt flips in dimension n implies the termination of dlt flips in dimension n. The termination of dlt flips in dimension n implies the termination of lc flips in dimension n. It is because we can use the LMMP for $\mathbb{Q}$-factorial dlt pairs in full generality by [BCHM] once we obtain the termination of dlt flips. The reader can find all the necessary arguments in [F8, 4.2, 4.4].

ana-re **Remark 3.16 (Analytic spaces).** The proofs of the vanishing theorems in Chapter 2 only work for algebraic varieties. Therefore, the cone, contraction, and base point free theorems stated here for lc pairs hold only for algebraic varieties. Of course, all the results should be proved for complex analytic spaces that are projective over any fixed analytic spaces.

3.1.2 Non-$\mathbb{Q}$-factorial log minimal model program

In this subsection, we explain the log minimal model program for non-$\mathbb{Q}$-factorial lc pairs. It is the most general log minimal model program. First, let us recall the definition of log canonical models.

z-log-cano **Definition 3.17 (Log canonical model).** Let (X, Δ) be a log canonical pair and $f : X \to S$ a proper morphism. A pair (X', Δ') sitting in a diagram

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X' \\ {\scriptstyle f}\searrow & & \swarrow{\scriptstyle f'} \\ & S & \end{array}$$

is called a *log canonical model* of (X, Δ) over S if

(1) f' is proper,

(2) ϕ^{-1} has no exceptional divisors,

(3) $\Delta' = \phi_*\Delta$,

(4) $K_{X'} + \Delta'$ is f'-ample, and

(5) $a(E, X, \Delta) \leq a(E, X', \Delta')$ for every ϕ-exceptional divisor $E \subset X$.

Next, we explain the minimal model program for non-$\mathbb{Q}$-factorial lc pairs (cf. [F8, 4.4]).

3.18 (MMP for non-$\mathbb{Q}$-factorial lc pairs). We start with a pair $(X, \Delta) = (X_0, \Delta_0)$. Let $f_0 : X_0 \to S$ be a projective morphism. The aim is to set up a recursive procedure which creates intermediate pairs (X_i, Δ_i) and projective morphisms $f_i : X_i \to S$. After some steps, it should stop with a final pair (X', Δ') and $f' : X' \to S$.

0ste-new **Step 0 (Initial datum).** Assume that we already constructed (X_i, Δ_i) and $f_i : X_i \to S$ with the following properties:

(1) (X_i, Δ_i) is lc,

(2) f_i is projective, and

(3) X_i is not necessarily $\mathbb{Q}$-factorial.

If X_i is $\mathbb{Q}$-factorial, then it is easy to see that X_k is also $\mathbb{Q}$-factorial for any $k \geq i$. Even when X_i is not $\mathbb{Q}$-factorial, X_{i+1} sometimes becomes $\mathbb{Q}$-factorial. See, for example, Example 5.4 below.

1ste-new **Step 1 (Preparation).** If $K_{X_i} + \Delta_i$ is f_i-nef, then we go directly to Step 3 (2). If $K_{X_i} + \Delta_i$ is not f_i-nef, then we establish two results:

(1) (Cone Theorem) We have the following equality.

$$\overline{NE}(X_i/S) = \overline{NE}(X_i/S)_{(K_{X_i}+\Delta_i)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i].$$

(2) (Contraction Theorem) Any $(K_{X_i} + \Delta_i)$-negative extremal ray $R_i \subset \overline{NE}(X_i/S)$ can be contracted. Let $\varphi_{R_i} : X_i \to Y_i$ denote the corresponding contraction. It sits in a commutative diagram.

$$\begin{array}{ccc} X_i & \xrightarrow{\varphi_{R_i}} & Y_i \\ & {\scriptstyle f_i}\searrow \quad \swarrow{\scriptstyle g_i} & \\ & S & \end{array}$$

2ste-new **Step 2 (Birational transformations).** If $\varphi_{R_i} : X_i \to Y_i$ is birational, then we can find an effective $\mathbb{Q}$-divisor B on X_i such that (X_i, B) is log canonical and $-(K_{X_i} + B)$ is φ_{R_i}-ample since $\rho(X_i/S) = 1$ (cf. Lemma sho-lem 3.20). Here, we *assume* that $\bigoplus_{m\geq 0}(\varphi_{R_i})_*\mathcal{O}_{X_i}(\llcorner m(K_{X_i} + B)\lrcorner)$ is a finitely generated $\mathcal{O}_{Y_i}$-algebra. We put

$$X_{i+1} = \mathrm{Proj}_{Y_i} \bigoplus_{m\geq 0} (\varphi_{R_i})_*\mathcal{O}_{X_i}(\llcorner m(K_{X_i} + B)\lrcorner),$$

where Δ_{i+1} is the strict transform of $(\varphi_{R_i})_*\Delta_i$ on X_{i+1}.

We note that (X_{i+1}, Δ_{i+1}) is the log canonical model of (X_i, Δ_i) over Y_i (see Definition z-log-cano 3.17). It can be checked easily that $\varphi^+_{R_i} : X_{i+1} \to Y_i$ is a small projective morphism and that (X_{i+1}, Δ_{i+1}) is log canonical. Then we go back to Step 0ste-new 0 with (X_{i+1}, Δ_{i+1}), $f_{i+1} = g_i \circ \varphi^+_{R_i}$ and start anew.

If X_i is $\mathbb{Q}$-factorial, then so is X_{i+1}. If X_i is $\mathbb{Q}$-factorial and φ_{R_i} is not small, then $\varphi^+_{R_i} : X_{i+1} \to Y_i$ is an isomorphism. It may happen that $\rho(X_i/S) < \rho(X_{i+1}/S)$ when X_i is not $\mathbb{Q}$-factorial. See, for example, Example final-ex 5.4 below.

3ste-new **Step 3 (Final outcome).** We expect that eventually the procedure stops, and we get one of the following two possibilities:

(1) (Mori fiber space) If φ_{R_i} is a Fano contraction, that is, $\dim Y_i < \dim X_i$, then we set $(X', \Delta') = (X_i, \Delta_i)$ and $f' = f_i$.

(2) (Minimal model) If $K_{X_i} + \Delta_i$ is f_i-nef, then we again set $(X', \Delta') = (X_i, \Delta_i)$ and $f' = f_i$. We can easily check that (X', Δ') is a log minimal model of (X, Δ) over S in the sense of Definition z-minimal 1.23.

Therefore, all we have to do is to prove Conjecture fg-conj2 3.10 for birational morphisms and Conjecture fll-conj 3.15.

We close this subsection with an example of a non-$\mathbb{Q}$-factorial log canonical variety.

Example 3.19. Let $C \subset \mathbb{P}^2$ be a smooth cubic curve and $Y \subset \mathbb{P}^3$ be a cone over C. Then Y is log canonical. In this case, Y is not $\mathbb{Q}$-factorial. We can check it as follows. Let $f : X = \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}) \to Y$ be a resolution such that $K_X + E = f^*K_Y$, where $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(1)|_C$ and E is the exceptional curve. We take $P, Q \in C$ such that $\mathcal{O}_C(P - Q)$ is not a torsion in $\mathrm{Pic}^0(C)$. We consider

$D = \pi^*P - \pi^*Q$, where $\pi : X = \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}) \to C$. We put $D' = f_*D$. If D' is $\mathbb{Q}$-Cartier, then $mD = f^*mD' + aE$ for some $a \in \mathbb{Z}$ and $m \in \mathbb{Z}_{>0}$. Restrict it to E. Then $\mathcal{O}_C(m(P-Q)) \simeq \mathcal{O}_E(aE) \simeq (\mathcal{L}^{-1})^{\otimes a}$. Therefore, we obtain that $a = 0$ and $m(P-Q) \sim 0$. It is a contradiction. Thus, D' is not $\mathbb{Q}$-Cartier. In particular, Y is not $\mathbb{Q}$-factorial.

3.1.3 Lengths of extremal rays

leng-ssec

In this subsection, we consider the estimate of lengths of extremal rays. Related topics are in [BCHM]. Let us recall the following easy lemma.

sho-lem

Lemma 3.20 (cf. [Sh2, Lemma 1]). *Let (X, B) be an lc pair, where B is an $\mathbb{R}$-divisor. Then there are positive real numbers r_i and effective $\mathbb{Q}$-divisors B_i for $1 \leq i \leq l$ and a positive integer m such that $\sum_{i=1}^{l} r_i = 1$, $K_X + B = \sum_{i=1}^{l} r_i(K_X + B_i)$, (X, B_i) is lc, and $m(K_X + B_i)$ is Cartier for any i.*

The next result is essentially due to [Ka2] and [Sh2, Proposition 1].

leng-prop

Proposition 3.21. *We use the notation in* Lemma 3.20. *Let (X, B) be an lc pair, B an $\mathbb{R}$-divisor, and $f : X \to Y$ a projective morphism between algebraic varieteis. Let R be a $(K_X + B)$-negative extremal ray of $\overline{NE}(X/Y)$. Then we can find a rational curve C on X such that $[C] \in R$ and $-(K_X + B_i) \cdot C \leq 2 \dim X$ for any i. In particular, $-(K_X + B) \cdot C \leq 2 \dim X$. More precisely, we can write $-(K_X + B) \cdot C = \sum_{i=1}^{l} \frac{r_i n_i}{m}$, where $n_i \in \mathbb{Z}$ and $n_i \leq 2m \dim X$ for any i.*

Proof. By replacing $f : X \to Y$ with the extremal contraction $\varphi_R : X \to W$ over Y, we can assume that the relative Picard number $\rho(X/Y) = 1$. In particular, $-(K_X + B)$ is f-ample. Therefore, we can assume that $-(K_X + B_1)$ is f-ample and $-(K_X + B_i) = -s_i(K_X + B_1)$ in $N^1(X/Y)$ with $s_i \leq 1$ for any $i \geq 2$. Thus, it is sufficient to find a rational curve C such that $f(C)$ is a point and that $-(K_X + B_1) \cdot C \leq 2 \dim X$. So, we can assume that $K_X + B$ is $\mathbb{Q}$-Cartier and lc. By [BCHM], there is a birational morphism $g : (W, B_W) \to (X, B)$ such that $K_W + B_W = g^*(K_X + B)$, W is $\mathbb{Q}$-factorial, B_W is effective, and $(W, \{B_W\})$ is klt. By [Ka2, Theorem 1], we can find a rational curve C' on W such that $-(K_W + B_W) \cdot C' \leq 2 \dim W = 2 \dim X$ and that C' spans a $(K_W + B_W)$-negative extremal ray. Note that Kawamata's proof works in the above situation with only small modifications. See the

proof of Theorem 10-2-1 in [M] and Remark 3.22 below. By the projection formula, the g-image of C' is a desired rational curve. So, we finish the proof. □

re-03 **Remark 3.22.** Let (X, D) be an lc pair, D an $\mathbb{R}$-divisor. Let $\phi : X \to Y$ be a projective morphism and H a Cartier divisor on X. Assume that $H - (K_X + D)$ is f-ample. By Theorem 2.48, $R^q\phi_*\mathcal{O}_X(H) = 0$ for any $q > 0$ if X and Y are *algebraic* varieties. If this vanishing theorem holds for analytic spaces X and Y, then Kawamata's original argument in [Ka2] works directly for lc pairs. In that case, we do not need the results in [BCHM] in the proof of Proposition 3.21.

We consider the proof of [M, Theorem 10-2-1] when (X, D) is lc such that $(X, \{D\})$ is klt. We need $R^1\phi_*\mathcal{O}_X(H) = 0$ after shrinking X and Y analytically. In our situation, $(X, D - \varepsilon\llcorner D\lrcorner)$ is klt for $0 < \varepsilon \ll 1$. Therefore, $H - (K_X + D - \varepsilon\llcorner D\lrcorner)$ is ϕ-ample and $(X, D - \varepsilon\llcorner D\lrcorner)$ is klt for $0 < \varepsilon \ll 1$. Thus, we can apply the analytic version of the relative Kawamata–Viehweg vanishing theorem. So, we do not need the analytic version of Theorem 2.48.

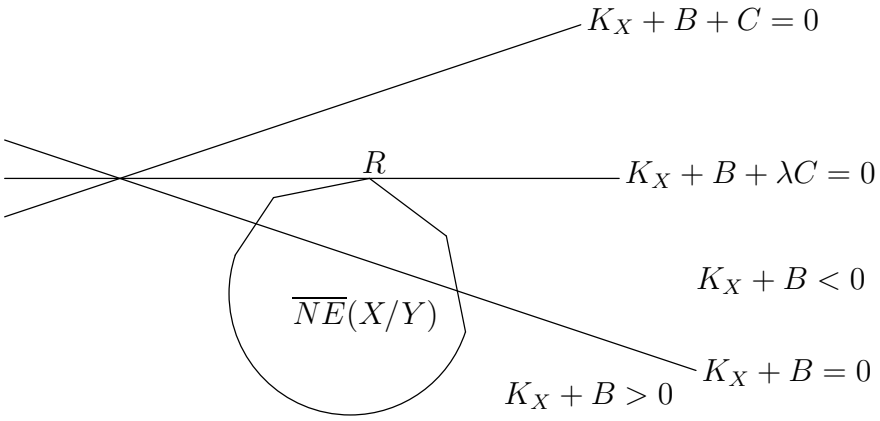
By Proposition 3.21, Lemma 2.6 in [B] holds for lc pairs. For the proof, see [B, Lemma 2.6]. It may be useful for the LMMP with scaling.

bir-prop **Proposition 3.23.** *Let (X, B) be an lc pair, B an $\mathbb{R}$-divisor, and $f : X \to Y$ a projective morphism between algebraic varieties. Let C be an effective $\mathbb{R}$-Cartier divisor on X such that $K_X + B + C$ is f-nef and $(X, B + C)$ is lc. Then, either $K_X + B$ is also f-nef or there is a $(K_X + B)$-negative extremal ray R such that $(K_X + B + \lambda C) \cdot R = 0$, where*

$$\lambda := \inf\{t \geq 0 \,|\, K_X + B + tC \text{ is } f\text{-nef}\}.$$

Of course, $K_X + B + \lambda C$ is f-nef.

The following picture helps the reader understand Proposition 3.23.



3.1.4 Log canonical flops

313ss

The following theorem is an easy consequence of [BCHM].

exe85 **Theorem 3.24.** *Let (X, Δ) be a klt pair and D a $\mathbb{Q}$-divisor on X. Then $\bigoplus_{m\geq 0} \mathcal{O}_X(\llcorner mD \lrcorner)$ is a finitely generated $\mathcal{O}_X$-algebra.*

Sketch of the proof. If D is $\mathbb{Q}$-Cartier, then the claim is obvious. So, we assume that D is not $\mathbb{Q}$-Cartier. We can also assume that X is quasi-projective. By [BCHM], we take a birational morphism $f : Y \to X$ such that Y is $\mathbb{Q}$-factorial, f is small projective, and (Y, Δ_Y) is klt, where $K_Y + \Delta_Y = f^*(K_X + \Delta)$. Then the strict transform D_Y of D on Y is $\mathbb{Q}$-Cartier. Let ε be a small positive number. By applying the MMP with scaling for the pair $(Y, \Delta_Y + \varepsilon D_Y)$ over X, we can assume that D_Y is f-nef, Therefore, by the base point free theorem, $\bigoplus_{m\geq 0} f_*\mathcal{O}_Y(\llcorner mD_Y \lrcorner) \simeq \bigoplus_{m\geq 0} \mathcal{O}_X(\llcorner mD \lrcorner)$ is finitely generated as an $\mathcal{O}_X$-algebra. □

The next example shows that Theorem 3.24 is not true for lc pairs. In other words, if (X, Δ) is lc, then $\bigoplus_{m\geq 0} \mathcal{O}_X(\llcorner mD \lrcorner)$ is not necessarily finitely generated as an $\mathcal{O}_X$-algebra.

exe88 **Example 3.25 (cf. [Ko5, Exercise 95]).** Let $E \subset \mathbb{P}^2$ be a smooth cubic curve. Let S be a surface obtained by blowing up nine general points on E and $E_S \subset S$ the strict transform of E. Let H be a very ample divisor on S giving a projectively normal embedding $S \subset \mathbb{P}^n$. Let $X \subset \mathbb{A}^{n+1}$ be the cone over S and $D \subset X$ the cone over E_S. Then (X, D) is lc since $K_S + E_S \sim 0$

(cf. Proposition 4.38). Let $P \in D \subset X$ be the vertex of the cones D and X. Since X is normal, we have

$$\begin{aligned} H^0(X, \mathcal{O}_X(mD)) &= H^0(X \setminus P, \mathcal{O}_X(mD)) \\ &\simeq \bigoplus_{r \in \mathbb{Z}} H^0(S, \mathcal{O}_S(mE_S + rH)). \end{aligned}$$

By the construction, $\mathcal{O}_S(mE_S)$ has only the obvious section which vanishes along mE_S for any $m > 0$. It can be checked by the induction on m using the following exact sequence

$$0 \to H^0(X, \mathcal{O}_S((m-1)E_S)) \to H^0(S, \mathcal{O}_S(mE_S)) \to H^0(E_S, \mathcal{O}_{E_S}(mE_S)) \to \cdots$$

since $\mathcal{O}_{E_S}(E_S)$ is not a torsion element in $\mathrm{Pic}^0(E_S)$. Therefore, $H^0(S, \mathcal{O}_S(mE_S + rH)) = 0$ for any $r < 0$. So, we have

$$\bigoplus_{m \geq 0} \mathcal{O}_X(mD) \simeq \bigoplus_{m \geq 0} \bigoplus_{r \geq 0} H^0(S, \mathcal{O}_S(mE_S + rH)).$$

Since E_S is nef, $\mathcal{O}_S(mE_S + 4H) \simeq \mathcal{O}_S(K_S + E_S + mE_S + 4H)$ is very ample for any $m \geq 0$. Therefore, by replacing H with $4H$, we can assume that $\mathcal{O}_S(mE_S + rH)$ is very ample for any $m \geq 0$ and $r > 0$. In this setting, the multiplication maps

$$\begin{aligned} \bigoplus_{a=0}^{m-1} H^0(S, \mathcal{O}_S(aE_S + H)) \otimes H^0(S, \mathcal{O}_S((m-a)E_S)) \\ \to H^0(S, \mathcal{O}_S(mE_S + H)) \end{aligned}$$

are never surjective. This implies that $\bigoplus_{m \geq 0} \mathcal{O}_X(mD)$ is not finitely generated as an $\mathcal{O}_X$-algebra.

Let us recall the definition of log canonical flops (cf. [FA, 6.8 Definition]).

lc-flop-def **Definition 3.26 (Log canonical flop).** Let (X, B) be an lc pair. Let H be a Cartier divisor on X. Let $f : X \to Z$ be a small contraction such that $K_X + B$ is numerically f-trivial and $-H$ is f-ample. The opposite of f with respect to H is called an H-flop with respect to $K_X + B$ or simply say an H-flop.

The following example shows that log canonical flops do not always exist.

exe89 **Example 3.27 (cf. [Ko5, Exercise 96]).** Let E be an elliptic curve and L a degree zero line bundle on E. We put $S = \mathbb{P}_E(\mathcal{O}_E \oplus L)$. Let C_1 and C_2 be the sections of the $\mathbb{P}^1$-bundle $p : S \to E$. We note that $K_S + C_1 + C_2 \sim 0$. As in Example 3.25, we take a sufficiently ample divisor $H = aF + bC_1$ on S giving a projectively normal embedding $S \subset \mathbb{P}^n$, where F is a fiber of the $\mathbb{P}^1$-bundle $p : S \to E$, $a > 0$, and $b > 0$. We can assume that $\mathcal{O}_S(mC_i + rH)$ is very ample for any i, $m \geq 0$, and $r > 0$. Moreover, we can assume that $\mathcal{O}_S(M + rH)$ is very ample for any nef divisor M and any $r > 0$. Let $X \subset \mathbb{A}^{n+1}$ be a cone over S and $D_i \subset X$ the cones over C_i. Since $K_S + C_1 + C_2 \sim 0$, $(X, D_1 + D_2)$ is lc and $K_X + D_1 + D_2 \sim 0$ (cf. Proposition 4.38). By the same arguments as in Example 3.25, we can prove the following statement.

exe-cl1 **Claim 1.** *If L is a non-torsion element in* $\mathrm{Pic}^0(E)$*, then* $\bigoplus_{m\geq 0} \mathcal{O}_X(mD_i)$ *is not a finitely generated sheaf of $\mathcal{O}_X$-algebra for $i = 1$ and 2.*

We note that $\mathcal{O}_S(mC_i)$ has only the obvious section which vanishes along mC_i for any $m > 0$. Let $B \subset X$ be the cone over F. Then we have the following result.

exe-cl2 **Claim 2.** *The graded $\mathcal{O}_X$-algebra $\bigoplus_{m\geq 0} \mathcal{O}_X(mB)$ is a finitely generated $\mathcal{O}_X$-algebra.*

Proof of Claim 2. By the same arguments as in Example 3.25, we have

$$\bigoplus_{m\geq 0} \mathcal{O}_X(mB) \simeq \bigoplus_{m\geq 0} \bigoplus_{r\geq 0} H^0(S, \mathcal{O}_S(mF + rH)).$$

We consider $V = \mathbb{P}_S(\mathcal{O}_S(F) \oplus \mathcal{O}_S(H))$. Then $\mathcal{O}_V(1)$ is semi-ample. Therefore,

$$\bigoplus_{n\geq 0} H^0(V, \mathcal{O}_V(n)) \simeq \bigoplus_{m\geq 0} \bigoplus_{r\geq 0} H^0(S, \mathcal{O}_S(mF + rH))$$

is finitely generated. □

Let $P \in X$ be the vertex of the cone X and let $f : Y \to X$ be the blow-up at P. Let $A \simeq S$ be the exceptional divisor of f. We consider the $\mathbb{P}^1$-bundle $\pi : \mathbb{P}_S(\mathcal{O}_S \oplus \mathcal{O}_S(H)) \to S$. Then $Y \simeq \mathbb{P}_S(\mathcal{O}_S \oplus \mathcal{O}_S(H)) \setminus G$, where G is the section of π corresponding to $\mathcal{O}_S \oplus \mathcal{O}_S(H) \to \mathcal{O}_S(H) \to 0$. We consider π^*F on Y. Then $\mathcal{O}_Y(\pi^*F)$ is f-semi-ample. So, we obtain a contraction morphism $g : Y \to Z$ over X. It is easy to see that $Z \simeq \mathrm{Proj}_X \bigoplus_{m\geq 0} \mathcal{O}_X(mB)$

and that $h : Z \to X$ is a small projective contraction. On Y, we have $-A \sim \pi^*H = a\pi^*F + b\pi^*C_1$. Therefore, we obtain $aB + bD_1 \sim 0$ on X. If L is not a torsion element, then the flop of $h : Z \to X$ with respect to D_1 does not exist since $\bigoplus_{m\geq 0} \mathcal{O}_X(mD_1)$ is not finitely generated as an $\mathcal{O}_X$-algebra.

Let C be any Cartier divisor on Z such that $-C$ is h-ample. Then the flop of $h : Z \to X$ with respect to C exists if and only if $\bigoplus_{m\geq 0} h_*\mathcal{O}_Z(mC)$ is a finitely generated $\mathcal{O}_X$-algebra. We can find a positive constant m_0 and a degree zero Cartier divisor N on E such that the finite generation of $\bigoplus_{m\geq 0} h_*\mathcal{O}_Z(mC)$ is equivalent to that of $\bigoplus_{m\geq 0} \mathcal{O}_X(m(m_0D_1 + \widetilde{N}))$, where $\widetilde{N} \subset X$ is the cone over $p^*N \subset S$.

exe-cl3 **Claim 3.** *If L is not a torsion element in $\mathrm{Pic}^0(E)$, then $\bigoplus_{m\geq 0} \mathcal{O}_X(m(m_0D_1+\widetilde{N}))$ is not finitely generated as an $\mathcal{O}_X$-algebra. In particular, the flop of $h : Z \to X$ with respect to C does not exist.*

Proof of Claim exe-cl3. By the same arguments as in Example exe88, we have

$$\begin{aligned}&\bigoplus_{m\geq 0} \mathcal{O}_X(m(m_0D_1 + \widetilde{N}))\\ &\simeq \bigoplus_{m\geq 0}\bigoplus_{r\in\mathbb{Z}} H^0(S, \mathcal{O}_S(m(m_0C_1 + p^*N) + rH)).\end{aligned}$$

Since $\dim H^0(S, \mathcal{O}_S(m(m_0C_1+p^*N))) \leq 1$ for any $m \geq 0$, we can easily check that the above $\mathcal{O}_X$-algebra is not finitely generated. See the arguments in Example exe88. We note that $\mathcal{O}_S(m(m_0C_1 + p^*N) + rH)$ is very ample for any $m \geq 0$ and $r > 0$ because $m_0C_1 + p^*N$ is nef. □

Anyway, if L is not a torsion element in $\mathrm{Pic}^0(E)$, then the flop of $h : Z \to X$ does not exist.

In the above setting, we assume that L is a torsion element in $\mathrm{Pic}^0(E)$. Then $\mathcal{O}_Y(\pi^*C_1)$ is f-semi-ample. So, we obtain a contraction morphism $g' : Y \to Z^+$ over X. It is easy to see that $\bigoplus_{m\geq 0} \mathcal{O}_X(mD_i)$ is finitely generated as an $\mathcal{O}_X$-algebra for $i = 1, 2$ (cf. Claim exe-cl2), $Z^+ \simeq \mathrm{Proj}_X \bigoplus_{m\geq 0} \mathcal{O}_X(mD_1)$, and that $Z^+ \to X$ is the flop of $Z \to X$ with respect to D_1.

Let C be any Cartier divisor on Z such that $-C$ is h-ample. If $-C \sim_{\mathbb{Q},h} cB$ for some positive rational number c, then it is obvious that the above $Z^+ \to X$ is the flop of $h : Z \to X$ with respect to C. Otherwise, the flop of $h : Z \to X$ with respect to C does not exist. As above, we can find a positive constant m_0 and a non-torsion element N in $\mathrm{Pic}^0(E)$ such that $\bigoplus_{m\geq 0} h_*\mathcal{O}_Z(mC)$

is finitely generated if and only if so is $\bigoplus_{m\geq 0}\mathcal{O}_X(m(m_0D_1+\widetilde{N}))$, where $\widetilde{N}\subset X$ is the cone over $p^*N\subset S$. By the same arguments as in the proof of Claim 3, we can easily check that $\bigoplus_{m\geq 0}\mathcal{O}_X(m(m_0D_1+\widetilde{N}))$ is not finitely generated as an $\mathcal{O}_X$-algebra. We note that $\dim H^0(S,\mathcal{O}_S(m(m_0D_1+p^*N)))=0$ for any $m>0$ since N is a non-torsion element and L is a torsion element in $\mathrm{Pic}^0(E)$.

3.2 Quasi-log varieties

qlog-sec

3.2.1 Definition of quasi-log varieties

In this subsection, we introduce the notion of *quasi-log varieties* according to [Am1]. Our definition requires slightly stronger assumptions than Ambro's original one. However, we will check that our definition is equivalent to Ambro's in the subsection 3.2.6.

Let us recall the definition of global embedded simple normal crossing pairs (see Definition 2.16).

gsnc **Definition 3.28 (Global embedded simple normal crossing pairs).** Let Y be a simple normal crossing divisor on a smooth variety M and let D be an $\mathbb{R}$-divisor on M such that $\mathrm{Supp}(D+Y)$ is simple normal crossing and that D and Y have no common irreducible components. We put $B_Y=D|_Y$ and consider the pair (Y,B_Y). We call (Y,B_Y) a *global embedded simple normal crossing pair*.

It's time for us to define *quasi-log varieties*.

quasi-log **Definition 3.29 (Quasi-log varieties).** A *quasi-log variety* is a scheme X endowed with an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor ω, a proper closed subscheme $X_{-\infty}\subset X$, and a finite collection $\{C\}$ of reduced and irreducible subvarieties of X such that there is a proper morphism $f:(Y,B_Y)\to X$ from a global embedded simple normal crossing pair satisfying the following properties:

(1) $f^*\omega\sim_{\mathbb{R}} K_Y+B_Y$.

(2) The natural map $\mathcal{O}_X\to f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner)$ induces an isomorphism

$$\mathcal{I}_{X_{-\infty}}\to f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner-\llcorner B_Y^{>1}\lrcorner),$$

where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$.

(3) The collection of subvarieties $\{C\}$ coincides with the image of (Y, B_Y)-strata that are not included in $X_{-\infty}$.

We sometimes simply say that $[X, \omega]$ is a *quasi-log pair*. We use the following terminology according to Ambro. The subvarieties C are the *qlc centers* of X, $X_{-\infty}$ is the *non-qlc locus* of X, and $f : (Y, B_Y) \to X$ is a *quasi-log resolution* of X. We say that X has *qlc singularities* if $X_{-\infty} = \emptyset$. Assume that $[X, \omega]$ is a quasi-log pair with $X_{-\infty} = \emptyset$. Then we simply say that $[X, \omega]$ is a *qlc pair*. Note that a quasi-log variety X is the union of its qlc centers and $X_{-\infty}$. A *relative quasi-log variety* X/S is a quasi-log variety X endowed with a proper morphism $\pi : X \to S$.

cano **Remark 3.30 (Quasi-log canonical class).** In Definition 3.29, we assume that ω is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. However, it may be better to see $\omega \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. It is because the *quasi-log canonical class* ω is defined up to $\mathbb{R}$-linear equivalence and we often restrict ω to a subvariety of X.

new-example **Example 3.31.** Let X be a normal variety and let B be an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. We take a resolution $f : Y \to X$ such that $K_Y + B_Y = f^*(K_X + B)$ and that $\mathrm{Supp} B_Y$ is a simple normal crossing divisor on Y. Then the pair $[X, K_X + B]$ is a quasi-log variety with a quasi-log resolution $f : (Y, B_Y) \to X$. By this quasi-log structure, $[X, K_X + B]$ is qlc if and only if (X, B) is lc. See also Corollary 3.51.

hosoku **Remark 3.32.** By Definition 3.29, X has only qlc singularities if and only if B_Y is a subboundary. In this case, $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$ since $\mathcal{O}_X \simeq f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner)$. In particular, f is surjective when X has only qlc singularities.

hosoku2 **Remark 3.33 (Semi-normality).** In general, we have

$$\mathcal{O}_{X \setminus X_{-\infty}} \simeq f_*\mathcal{O}_{f^{-1}(X \setminus X_{-\infty})}(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1} \lrcorner) = f_*\mathcal{O}_{f^{-1}(X \setminus X_{-\infty})}(\ulcorner -(B_Y^{<1})\urcorner).$$

This implies that $\mathcal{O}_{X \setminus X_{-\infty}} \simeq f_*\mathcal{O}_{f^{-1}(X \setminus X_{-\infty})}$. Therefore, $X \setminus X_{-\infty}$ is semi-normal since $f^{-1}(X \setminus X_{-\infty})$ is a simple normal crossing variety.

sing **Remark 3.34.** To prove the cone and contraction theorems for lc pairs, it is enough to treat quasi-log varieties with only qlc singularities. For the details, see [F16].

We close this subsection with an obvious lemma.

ob-lem **Lemma 3.35.** *Let $[X, \omega]$ be a quasi-log pair. Assume that $X = V \cup X_{-\infty}$ and $V \cap X_{-\infty} = \emptyset$. Then $[V, \omega']$ is a qlc pair, where $\omega' = \omega|_V$.*

3.2.2 Quick review of vanishing and torsion-free theorems

In this subsection, we quickly review Ambro's formulation of torsion-free and vanishing theorems in a simplified form. For more advanced topics and the proof, see Chapter 2.

We consider a global embedded simple normal crossing pair (Y, B). More precisely, let Y be a simple normal crossing divisor on a smooth variety M and let D be an $\mathbb{R}$-divisor on M such that $\mathrm{Supp}(D+Y)$ is simple normal crossing and that D and Y have no common irreducible components. We put $B = D|_Y$ and consider the pair (Y, B). Let $\nu : Y^\nu \to Y$ be the normalization. We put $K_{Y^\nu} + \Theta = \nu^*(K_Y + B)$. A *stratum* of (Y, B) is an irreducible component of Y or the image of some lc center of $(Y^\nu, \Theta^{=1})$.

When Y is smooth and B is an $\mathbb{R}$-divisor on Y such that $\mathrm{Supp}B$ is simple normal crossing, we put $M = Y \times \mathbb{A}^1$ and $D = B \times \mathbb{A}^1$. Then $(Y, B) \simeq (Y \times \{0\}, B \times \{0\})$ satisfies the above conditions.

The following theorem is a special case of Theorem 2.39.

ap1 **Theorem 3.36.** *Let (Y, B) be as above. Assume that B is a boundary $\mathbb{R}$-divisor. Let $f : Y \to X$ be a proper morphism and L a Cartier divisor on Y.*

(1) Assume that $H \sim_{\mathbb{R}} L - (K_Y + B)$ is f-semi-ample. Then every non-zero local section of $R^q f_* \mathcal{O}_Y(L)$ contains in its support the f-image of some strata of (Y, B).

(2) Let $\pi : X \to V$ be a proper morphism and assume that $H \sim_{\mathbb{R}} f^*H'$ for some π-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor H' on X. Then, $R^q f_* \mathcal{O}_Y(L)$ is π_*-acyclic, that is, $R^p \pi_* R^q f_* \mathcal{O}_Y(L) = 0$ for any $p > 0$.

We need a slight generalization of Theorem 3.36 in Section 4.1. Let us recall the definition of *nef and log big divisors* for the vanishing theorem.

nlb-def **Definition 3.37 (Nef and log big divisors).** Let $f : (Y, B_Y) \to X$ be a proper morphism from a simple normal crossing pair (Y, B_Y). Let $\pi : X \to V$ be a proper morphism and H an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. We say that H is *nef and log big* over V if and only if $H|_C$ is nef and big over V for any C, where

(i) C is a qlc center when X is a quasi-log variety and $f : (Y, B_Y) \to X$ is a quasi-log resolution, or

(ii) C is the image of a stratum of (Y, B_Y) when B_Y is a subboundary.

If X is a quasi-log variety with only qlc singularities and $f : (Y, B_Y) \to X$ is a quasi-log resolution, then the above two cases (i) and (ii) coincide. When (X, B_X) is an lc pair, we choose a log resolution of (X, B_X) to be $f : (Y, B_Y) \to X$, where $K_Y + B_Y = f^*(K_X + B_X)$. We note that if H is ample over V then it is obvious that H is nef and log big over V.

nlb-vani-th **Theorem 3.38 (cf. Theorem 2.47).** *Let (Y, B) be as above. Assume that B is a boundary $\mathbb{R}$-divisor. Let $f : Y \to X$ be a proper morphism and L a Cartier divisor on Y. We put $H \sim_{\mathbb{R}} L - (K_X + B)$. Let $\pi : X \to V$ be a proper morphism and assume that $H \sim_{\mathbb{R}} f^*H'$ for some π-nef and π-log big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor H' on X. Then, every non-zero local section of $R^q f_*\mathcal{O}_Y(L)$ contains in its support the f-image of some strata of (Y, B), and $R^q f_*\mathcal{O}_Y(L)$ is π_*-acyclic, that is, $R^p\pi_* R^q f_*\mathcal{O}_Y(L) = 0$ for any $p > 0$.*

For the proof, see Theorem 2.47.

3.2.3 Adjunction and Vanishing Theorem

The following theorem is one of the key results in the theory of quasi-log varieties (cf. [Amb1, Theorem 4.4]).

adj-th **Theorem 3.39 (Adjunction and vanishing theorem).** *Let $[X, \omega]$ be a quasi-log pair and X' the union of $X_{-\infty}$ with a (possibly empty) union of some qlc centers of $[X, \omega]$.*

(i) *Assume that $X' \neq X_{-\infty}$. Then X' is a quasi-log variety, with $\omega' = \omega|_{X'}$ and $X'_{-\infty} = X_{-\infty}$. Moreover, the qlc centers of $[X', \omega']$ are exactly the qlc centers of $[X, \omega]$ that are included in X'.*

(ii) *Assume that $\pi : X \to S$ is proper. Let L be a Cartier divisor on X such that $L - \omega$ is nef and log big over S. Then $\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)$ is π_*-acyclic, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X' on X.*

Theorem 3.39 is the hardest part to prove in the theory of quasi-log varieties. It is because it depends on the non-trivial vanishing and torsion-free theorems for simple normal crossing pairs. The adjunction for normal divisors on normal varieties is investigated in [F15]. See also Section 4.5.

Proof. By blowing up the ambient space M of Y, we can assume that the union of all strata of (Y, B_Y) mapped to X', which is denoted by Y', is a union of irreducible components of Y (cf. Lemma 2.19). We will justify this reduction in a more general setting in Proposition 3.50 below. We put $K_{Y'} + B_{Y'} = (K_Y + B_Y)|_{Y'}$ and $Y'' = Y - Y'$. We claim that $[X', \omega']$ is a quasi-log pair and that $f : (Y', B_{Y'}) \to X'$ is a quasi-log resolution. By the construction, $f^*\omega' \sim_{\mathbb{R}} K_{Y'} + B_{Y'}$ on Y' is obvious. We put $A = \ulcorner -(B_Y^{<1})\urcorner$ and $N = \llcorner B_Y^{>1} \lrcorner$. We consider the following short exact sequence

$$0 \to \mathcal{O}_{Y''}(-Y') \to \mathcal{O}_Y \to \mathcal{O}_{Y'} \to 0.$$

By applying $\otimes \mathcal{O}_Y(A - N)$, we have

$$0 \to \mathcal{O}_{Y''}(A - N - Y') \to \mathcal{O}_Y(A - N) \to \mathcal{O}_{Y'}(A - N) \to 0.$$

By applying f_*, we obtain

$$\begin{aligned} 0 \to f_*\mathcal{O}_{Y''}(A - N - Y') &\to f_*\mathcal{O}_Y(A - N) \to f_*\mathcal{O}_{Y'}(A - N) \\ &\to R^1 f_*\mathcal{O}_{Y''}(A - N - Y') \to \cdots . \end{aligned}$$

By Theorem 3.36 (i), the support of any non-zero local section of $R^1 f_*\mathcal{O}_{Y''}(A - N - Y')$ can not be contained in $X' = f(Y')$. We note that

$$(A - N - Y')|_{Y''} - (K_{Y''} + \{B_{Y''}\} + B_{Y''}^{=1} - Y'|_{Y''}) = -(K_{Y''} + B_{Y''}) \sim_{\mathbb{R}} f^*\omega|_{Y''},$$

where $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$. Therefore, the connecting homomorphism $f_*\mathcal{O}_{Y'}(A - N) \to R^1 f_*\mathcal{O}_{Y''}(A - N - Y')$ is a zero map. Thus,

$$0 \to f_*\mathcal{O}_{Y''}(A - N - Y') \to \mathcal{I}_{X_{-\infty}} \to f_*\mathcal{O}_{Y'}(A - N) \to 0$$

is exact. We put $\mathcal{I}_{X'} = f_*\mathcal{O}_{Y''}(A - N - Y')$. Then $\mathcal{I}_{X'}$ defines a scheme structure on X'. We define $\mathcal{I}_{X'_{-\infty}} = \mathcal{I}_{X_{-\infty}}/\mathcal{I}_{X'}$. Then $\mathcal{I}_{X'_{-\infty}} \simeq f_*\mathcal{O}_{Y'}(A - N)$ by the above exact sequence. By the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f_*\mathcal{O}_{Y''}(A - N - Y') & \longrightarrow & f_*\mathcal{O}_Y(A - N) & \longrightarrow & f_*\mathcal{O}_{Y'}(A - N) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & f_*\mathcal{O}_{Y''}(A - Y') & \longrightarrow & f_*\mathcal{O}_Y(A) & \longrightarrow & f_*\mathcal{O}_{Y'}(A) & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{I}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & 0, \end{array}$$

we can see that $\mathcal{O}_{X'} \to f_*\mathcal{O}_{Y'}(\ulcorner -(B^{<1}_{Y'})\urcorner)$ induces an isomorphism $\mathcal{I}_{X'_{-\infty}} \to f_*\mathcal{O}_{Y'}(\ulcorner -(B^{<1}_{Y'})\urcorner - \llcorner B^{>1}_{Y'} \lrcorner)$. Therefore, $[X', \omega']$ is a quasi-log pair such that $X'_{-\infty} = X_{-\infty}$. By the construction, the property about qlc centers are obvious. So, we finish the proof of (i).

Let $f : (Y, B_Y) \to X$ be a quasi-log resolution as in the proof of (i). Then $f^*(L-\omega) \sim_{\mathbb{R}} f^*L - (K_{Y''} + B_{Y''})$ on Y'', where $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$. Note that

$$f^*L - (K_{Y''} + B_{Y''}) = (f^*L + A - N - Y')|_{Y''} - (K_{Y''} + \{B_{Y''}\} + B^{=1}_{Y''} - Y'|_{Y''})$$

and that any stratum of $(Y'', B^{=1}_{Y''} - Y'|_{Y''})$ is not mapped to $X_{-\infty} = X'_{-\infty}$. Then by Theorem 3.38 (Theorem 3.36 (ii) when $L-\omega$ is π-ample),

$$R^p\pi_*(f_*\mathcal{O}_{Y''}(f^*L + A - N - Y')) = R^p\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)) = 0$$

for any $p > 0$. Thus, we finish the proof of (ii). □

Remark 3.40. We make a few comments on Theorem 3.39 for the reader's convenience. We slightly changed the big diagram in the proof of [Amb1, Theorem 4.4] and incorporated [Amb1, Theorem 7.3] into [Amb1, Theorem 4.4]. Please compare Theorem 3.39 with the original statements in [Amb1].

irre-cor **Corollary 3.41.** *Let $[X, \omega]$ be a qlc pair and let X' be an irreducible component of X. Then $[X', \omega']$, where $\omega' = \omega|_{X'}$, is a qlc pair.*

Proof. It is because X' is a qlc center of $[X, \omega]$ by Remark 3.32. □

The next example shows that the definition of quasi-log varieties is reasonable.

311 **Example 3.42.** Let (X, B_X) be an lc pair. Let $f : Y \to (X, B_X)$ be a resolution such that $K_Y + S + B = f^*(K_X + B_X)$, where $\mathrm{Supp}(S + B)$ is simple normal crossing, S is reduced, and $\llcorner B \lrcorner \leq 0$. We put $K_S + B_S = (K_X + S + B)|_S$ and consider the short exact sequence

$$0 \to \mathcal{O}_Y(\ulcorner -B\urcorner - S) \to \mathcal{O}_Y(\ulcorner -B\urcorner) \to \mathcal{O}_S(\ulcorner -B_S\urcorner) \to 0.$$

Note that $B_S = B|_S$ since Y is smooth. By the Kawamata–Viehweg vanishing theorem, $R^1f_*\mathcal{O}_Y(\ulcorner -B\urcorner - S) = 0$. This implies that $f_*\mathcal{O}_S(\ulcorner -B_S\urcorner) \simeq \mathcal{O}_{f(S)}$ since $f_*\mathcal{O}_Y(\ulcorner -B\urcorner) \simeq \mathcal{O}_X$. This argument is well known as the proof of the connectedness lemma. We put $W = f(S)$ and $\omega = (K_X + B_X)|_W$. Then $[W, \omega]$ is a quasi-log pair with only qlc singularities and $f : (S, B_S) \to W$ is a quasi-log resolution.

Example 3.42 is a very special case of Theorem 3.39 (i), that is, adjunction from $[X, K_X + B_X]$ to $[W, \omega]$. For other examples, see [F12, §5] or Section 4.4, where we treat toric polyhedra as quasi-log varieties. In the proof of Theorem 3.39 (i), we used Theorem 3.36 (i), which is a generalization of Kollár's theorem, instead of the Kawamata–Viehweg vanishing theorem.

3.2.4 Miscellanies on qlc centers

The notion of *lcs locus* is important for X-method on quasi-log varieties.

lcs **Definition 3.43 (LCS locus).** The *LCS locus* of a quasi-log pair $[X, \omega]$, denoted by $\mathrm{LCS}(X)$ or $\mathrm{LCS}(X, \omega)$, is the union of $X_{-\infty}$ with all qlc centers of X that are not maximal with respect to the inclusion. The subscheme structure is defined in Theorem 3.39 (i), and we have a natural embedding $X_{-\infty} \subseteq \mathrm{LCS}(X)$. In this book and [F16], $\mathrm{LCS}(X, \omega)$ is denoted by $\mathrm{Nqklt}(X, \omega)$.

When X is normal and B is an effective $\mathbb{R}$-divisor such that $K_X + B$ is $\mathbb{R}$-Cartier, $\mathrm{Nqklt}(X, K_X + B)$ is denoted by $\mathrm{Nklt}(X, B)$ and is called the *non-klt locus* of the pair (X, B).

The next proposition is easy to prove. However, in some applications, it may be useful.

normal **Proposition 3.44 (cf. [Am1, Proposition 4.7]).** *Let X be a quasi-log variety whose LCS locus is empty. Then X is normal.*

Proof. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution. By the assumption, every stratum of Y dominates X. Therefore, $f : Y \to X$ passes through the normalization $X^\nu \to X$ of X. This implies that X is normal since $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$ by Remark 3.32. □

qlc-cent-prop **Theorem 3.45 (cf. [Am1, Proposition 4.8]).** *Assume that $[X, \omega]$ is a qlc pair. We have the following properties:*

(i) *The intersection of two qlc centers is a union of qlc centers.*

(ii) *For any point $P \in X$, the set of all qlc centers passing through P has a unique element W. Moreover, W is normal at P.*

Proof. Let C_1 and C_2 be two qlc centers of $[X, \omega]$. We fix $P \in C_1 \cap C_2$. It is enough to find a qlc center C such that $P \in C \subset C_1 \cap C_2$. The union $X' = C_1 \cup C_2$ with $\omega' = \omega|_{X'}$ is a qlc pair having two irreducible components. Hence, it is not normal at P. By Proposition 3.44, $P \in \mathrm{Nqklt}(X', \omega')$. Therefore, there exists a qlc center $C \subset C_1$ with $\dim C < \dim C_1$ such that $P \in C \cap C_2$. If $C \subset C_2$, we are done. Otherwise, we repeat the argument with $C_1 = C$ and reach the conclusion in a finite number of steps. So, we finish the proof of (i). The uniqueness of the minimal qlc center follows from (i) and the normality of the minimal center follows from Proposition 3.44. Thus, we have (ii). □

a1 **Theorem 3.46 (cf. [Am2, Theorem 1.1]).** *We assume that (X, B) is log canonical. Then we have the following properties.*

(1) *(X, B) has at most finitely many lc centers.*

(2) *An intersection of two lc centers is a union of lc centers.*

(3) *Any union of lc centers of (X, B) is semi-normal.*

(4) *Let $x \in X$ be a closed point such that (X, B) is log canonical but not Kawamata log terminal at x. Then there is a unique minimal lc center W_x passing through x, and W_x is normal at x.*

Proof. Let $f : (Y, B_Y) \to (X, B)$ be a resolution such that $K_Y + B_Y = f^*(K_X + B)$ and $\mathrm{Supp} B_Y$ is a simple normal crossing divisor. Then an lc center of (X, B) is the image of some stratum of a simple normal crossing variety $B_Y^{=1}$. Therefore, (X, B) has at most finitely many lc centers. This is (1). The statements (2) and (4) are obvious by Theorem 3.45. Let $\{C_i\}_{i \in I}$ be a set of lc centers of (X, B). We put $X' = \bigcup_{i \in I} C_i$ and $\omega' = (K_X + B)|_{X'}$. Then $[X', \omega']$ is a qlc pair. Therefore, X' is semi-normal by Remarks 3.32 and 3.33. This is (3). □

The following result is an easy consequence of adjunction and the vanishing theorem: Theorem 3.39.

qlc-cent-th **Theorem 3.47 (cf. [Am1, Theorem 6.6]).** *Let $[X, \omega]$ be a quasi-log pair and let $\pi : X \to S$ be a proper morphism such that $\pi_* \mathcal{O}_X \simeq \mathcal{O}_S$ and $-\omega$ is nef and log big over S. Let $P \in S$ be a closed point.*

(i) *Assume that $X_{-\infty} \cap \pi^{-1}(P) \neq \emptyset$ and C is a qlc center such that $C \cap \pi^{-1}(P) \neq \emptyset$. Then $C \cap X_{-\infty} \cap \pi^{-1}(P) \neq \emptyset$.*

(ii) *Assume that $[X, \omega]$ is a qlc pair. Then the set of all qlc centers intersecting $\pi^{-1}(P)$ has a unique minimal element with respect to inclusion.*

Proof. Let C be a qlc center of $[X, \omega]$ such that $P \in \pi(C) \cap \pi(X_{-\infty})$. Then $X' = C \cup X_{-\infty}$ with $\omega' = \omega|_{X'}$ is a quasi-log variety and the restriction map $\pi_*\mathcal{O}_X \to \pi_*\mathcal{O}_{X'}$ is surjective by Theorem 3.39. Since $\pi_*\mathcal{O}_X \simeq \mathcal{O}_S$, $X_{-\infty}$ and C intersect over a neighborhood of P. So, we have (i).

Assume that $[X, \omega]$ is a qlc pair, that is, $X_{-\infty} = \emptyset$. Let C_1 and C_2 be two qlc centers of $[X, \omega]$ such that $P \in \pi(C_1) \cap \pi(C_2)$. The union $X' = C_1 \cup C_2$ with $\omega' = \omega|_{X'}$ is a qlc pair and the restriction map $\pi_*\mathcal{O}_X \to \pi_*\mathcal{O}_{X'}$ is surjective. Therefore, C_1 and C_2 intersect over P. Furthermore, the intersection $C_1 \cap C_2$ is a union of qlc centers by Proposition 3.45. Therefore, there exists a unique qlc center C_P over a neighborhood of P such that $C_P \subset C$ for every qlc center C with $P \in \pi(C)$. So, we finish the proof of (ii). □

The following corollary is obvious by Theorem 3.47.

Corollary 3.48. *Let (X, B) be a proper lc pair. Assume that $-(K_X + B)$ is nef and log big and that (X, B) is not klt. Then there exists a unique minimal lc center C_0 such that every lc center contains C_0. In particular,* $\mathrm{Nklt}(X, B)$ *is connected.*

The next theorem easily follows from [FT, Section 2].

Theorem 3.49. *Let (X, B) be a projective lc pair. Assume that $K_X + B$ is numerically trivial. Then* $\mathrm{Nklt}(X, B)$ *has at most two connected components.*

Proof. By [BCHM], there is a birational morphism $f : (Y, B_Y) \to (X, B)$ such that $K_Y + B_Y = f^*(K_X + B)$, Y is projective and $\mathbb{Q}$-factorial, B_Y is effective, and $(Y, \{B_Y\})$ is klt. Therefore, it is sufficient to prove that $\llcorner B_Y \lrcorner$ has at most two connected components. We assume that $\llcorner B_Y \lrcorner \neq 0$. Then $K_Y + \{B_Y\}$ is $\mathbb{Q}$-factorial klt and is not pseudo-effective. Apply the arguments in [FT, Proposition 2.1] with using the LMMP with scaling (see [BCHM]). Then we obtain that $\llcorner B_Y \lrcorner$ and $\mathrm{Nklt}(X, B)$ have at most two connected components. □

3.2.5 Useful lemmas

In this subsection, we prepare some useful lemmas for making quasi-log resolutions with good properties.

taisetsu **Proposition 3.50.** *Let $f : Z \to Y$ be a proper birational morphism between smooth varieties and let B_Y be an $\mathbb{R}$-divisor on Y such that* $\mathrm{Supp}B_Y$ *is simple normal crossing. Assume that $K_Z + B_Z = f^*(K_Y + B_Y)$ and that* $\mathrm{Supp}B_Z$ *is simple normal crossing. Then we have*

$$f_*\mathcal{O}_Z(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner) \simeq \mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1}\lrcorner).$$

Furthermore, let S be a simple normal crossing divisor on Y such that $S \subset \mathrm{Supp}B_Y^{=1}$. Let T be the union of the irreducible components of $B_Z^{=1}$ that are mapped into S by f. Assume that $\mathrm{Supp}f_*^{-1}B_Y \cup \mathrm{Exc}(f)$ *is simple normal crossing on Z. Then we have*

$$f_*\mathcal{O}_T(\ulcorner -(B_T^{<1})\urcorner - \llcorner B_T^{>1}\lrcorner) \simeq \mathcal{O}_S(\ulcorner -(B_S^{<1})\urcorner - \llcorner B_S^{>1}\lrcorner),$$

where $(K_Z + B_Z)|_T = K_T + B_T$ and $(K_Y + B_Y)|_S = K_S + B_S$.

Proof. By $K_Z + B_Z = f^*(K_Y + B_Y)$, we obtain

$$\begin{aligned} K_Z =& f^*(K_Y + B_Y^{=1} + \{B_Y\}) \\ &+ f^*(\llcorner B_Y^{<1}\lrcorner + \llcorner B_Y^{>1}\lrcorner) - (\llcorner B_Z^{<1}\lrcorner + \llcorner B_Z^{>1}\lrcorner) - B_Z^{=1} - \{B_Z\}. \end{aligned}$$

If $a(\nu, Y, B_Y^{=1} + \{B_Y\}) = -1$ for a prime divisor ν over Y, then we can check that $a(\nu, Y, B_Y) = -1$ by using [KM, Lemma 2.45]. Since $f^*(\llcorner B_Y^{<1}\lrcorner + \llcorner B_Y^{>1}\lrcorner) - (\llcorner B_Z^{<1}\lrcorner + \llcorner B_Z^{>1}\lrcorner)$ is Cartier, we can easily see that $f^*(\llcorner B_Y^{<1}\lrcorner + \llcorner B_Y^{>1}\lrcorner) = \llcorner B_Z^{<1}\lrcorner + \llcorner B_Z^{>1}\lrcorner + E$, where E is an effective f-exceptional divisor. Thus, we obtain

$$f_*\mathcal{O}_Z(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner) \simeq \mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1}\lrcorner).$$

Next, we consider

$$\begin{aligned} 0 \to& \mathcal{O}_Z(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner - T) \\ &\to \mathcal{O}_Z(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner) \to \mathcal{O}_T(\ulcorner -(B_T^{<1})\urcorner - \llcorner B_T^{>1}\lrcorner) \to 0. \end{aligned}$$

Since $T = f^*S - F$, where F is an effective f-exceptional divisor, we can easily see that

$$f_*\mathcal{O}_Z(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner - T) \simeq \mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1}\lrcorner - S).$$

We note that

$$\begin{aligned} (\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner - T) - (K_Z + \{B_Z\} + B_Z^{=1} - T) \\ = -f^*(K_Y + B_Y). \end{aligned}$$

Therefore, every local section of $R^1f_*\mathcal{O}_Z(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner - T)$ contains in its support the f-image of some strata of $(Z, \{B_Z\} + B_Z^{=1} - T)$ by Theorem 3.36 (i).

Claim. *No strata of $(Z, \{B_Z\} + B_Z^{=1} - T)$ are mapped into S by f.*

Proof of Claim. Assume that there is a stratum C of $(Z, \{B_Z\} + B_Z^{=1} - T)$ such that $f(C) \subset S$. Note that $\mathrm{Supp} f^*S \subset \mathrm{Supp} f_*^{-1}B_Y \cup \mathrm{Exc}(f)$ and $\mathrm{Supp} B_Z^{=1} \subset \mathrm{Supp} f_*^{-1}B_Y \cup \mathrm{Exc}(f)$. Since C is also a stratum of $(Z, B_Z^{=1})$ and $C \subset \mathrm{Supp} f^*S$, there exists an irreducible component G of $B_Z^{=1}$ such that $C \subset G \subset \mathrm{Supp} f^*S$. Therefore, by the definition of T, G is an irreducible component of T because $f(G) \subset S$ and G is an irreducible component of $B_Z^{=1}$. So, C is not a stratum of $(Z, \{B_Z\} + B_Z^{=1} - T)$. It is a contradiction. □

On the other hand, $f(T) \subset S$. Therefore,

$$f_*\mathcal{O}_T(\ulcorner -(B_T^{<1})\urcorner - \llcorner B_T^{>1}\lrcorner) \to R^1f_*\mathcal{O}_Z(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner - T)$$

is a zero map by the assumption on the strata of $(Z, B_Z^{=1} - T)$. Thus,

$$f_*\mathcal{O}_T(\ulcorner -(B_T^{<1})\urcorner - \llcorner B_T^{>1}\lrcorner) \simeq \mathcal{O}_S(\ulcorner -(B_S^{<1})\urcorner - \llcorner B_S^{>1}\lrcorner).$$

We finish the proof. □

The following corollary is obvious by Proposition 3.50.

346346 **Corollary 3.51.** *Let X be a normal variety and let B be an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. Let $f_i : Y_i \to X$ be a resolution of (X, B) for $i = 1, 2$. We put $K_{Y_i} + B_{Y_i} = f_i^*(K_X + B)$ and assume that $\mathrm{Supp} B_{Y_i}$ is simple normal crossing. Then $f_i : (Y_i, B_{Y_i}) \to X$ defines a quasi-log structure on $[X, K_X + B]$ for $i = 1, 2$. By taking a common log resolution of (Y_1, B_{Y_1}) and (Y_2, B_{Y_2}) suitably and applying* Proposition 3.50*, we can see that these two quasi-log structures coincide. Moreover, let X' be the union of $X_{-\infty}$ with a union of some qlc centers of $[X, K_X + B]$. Then we can see that $f_1 : (Y_1, B_{Y_1}) \to X$ and $f_2 : (Y_2, B_{Y_2}) \to X$ induce the same quasi-log structure on $[X', (K_X + B)|_{X'}]$ by* Proposition 3.50.

The final results in this section are very useful and indispensable for some applications.

taisetsu3 **Proposition 3.52.** *Let $[X, \omega]$ be a quasi-log pair and let $f : (Y, B_Y) \to X$ be a quasi-log resolution. Assume that (Y, B_Y) is a global embedded simple normal crossing pair as in* Definition 3.28. *Let $\sigma : N \to M$ be a proper birational morphism from a smooth variety N. We define $K_N + D_N = \sigma^*(K_M + D + Y)$ and assume that $\mathrm{Supp}\sigma_*^{-1}(D+Y) \cup \mathrm{Exc}(\sigma)$ is simple normal crossing on N. Let Z be the union of the irreducible components of $D_N^{=1}$ that are mapped into Y by σ. Then $f \circ \sigma : (Z, B_Z) \to X$ is a quasi-log resolution of $[X, \omega]$, where $K_Z + B_Z = (K_N + D_N)|_Z$.*

The proof of Proposition 3.52 is obvious by Proposition 3.50.

Remark 3.53. In Proposition 3.52, $\sigma : (Z, B_Z) \to (Y, B_Y)$ is not necessarily a composition of *embedded log transformations* and blow-ups whose centers contain no strata of the pair $(Y, B_Y^{=1})$ (see [Am1, Section 2]). Compare Proposition 3.52 with [Am1, Remark 4.2.(iv)].

The final proposition in this subsection will play very important roles in the following sections.

tai4 **Proposition 3.54.** *Let $f : (Y, B_Y) \to X$ be a quasi-log resolution of a quasi-log pair $[X, \omega]$, where (Y, B_Y) is a global embedded simple normal crossing pair as in* Definition 3.28. *Let E be a Cartier divisor on X such that $\mathrm{Supp}E$ contains no qlc centers of $[X, \omega]$. By blowing up M, the ambient space of Y, inside $\mathrm{Supp}f^*E$, we can assume that $(Y, B_Y + f^*E)$ is a global embedded simple normal crossing pair.*

Proof. First, we take a blow-up of M along f^*E and apply Hironaka's resolution theorem to M. Then we can assume that there exists a Cartier divisor F on M such that $\mathrm{Supp}(F \cap Y) = \mathrm{Supp}f^*E$. Next, we apply Szabó's resolution lemma to $\mathrm{Supp}(D + Y + F)$ on M. Thus, we obtain the desired properties by Proposition 3.50. □

3.2.6 Ambro's original formulation

3-2-6

Let us recall Ambro's original definition of quasi-log varieties.

qlog-def-ambro **Definition 3.55 (Quasi-log varieties).** A *quasi-log variety* is a scheme X endowed with an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor ω, a proper closed subscheme $X_{-\infty} \subset X$, and a finite collection $\{C\}$ of reduced and irreducible subvarieties of X such that there is a proper morphism $f : (Y, B_Y) \to X$ from an *embedded normal crossing pair* satisfying the following properties:

(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.

(2) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner)$ induces an isomorphism

$$\mathcal{I}_{X_{-\infty}} \to f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1}\lrcorner),$$

where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$.

(3) The collection of subvarieties $\{C\}$ coincides with the image of (Y, B_Y)-strata that are not included in $X_{-\infty}$.

For the definition of normal crossing pairs, see Definition 2.55.

def-ni-tuite **Remark 3.56.** We can always construct an embedded *simple* normal crossing pair $(Y', B_{Y'})$ and a proper morphism $f' : (Y', B_{Y'}) \to X$ with the above conditions (1), (2), and (3) by blowing up M suitably, where M is the ambient space of Y (see [Am1, p.218, embedded log transformations, and Remark 4.2.(iv)]). We leave the details for the reader's exercies (see also Lemmas 2.56, 2.58, and 2.59, and the proof of Proposition 3.50). Therefore, we can assume that (Y, B_Y) is a simple normal crossing pair in Definition 3.55. We note that the proofs of the vanishing and injectivity theorems on normal crossing pairs are much harder than on *simple* normal crossing pairs (see Chapter 2). Therefore, there are no advantages to adopt *normal crossing pairs* in the definition of quasi-log varieties.

The next proposition is the main result in this section. Proposition 3.50 becomes very powerful if it is combined with Proposition 3.57. See Proposition 3.52.

taisetsu2 **Proposition 3.57.** *We assume that (Y, B_Y) is an embedded simple normal crossing pair in* Definition 3.55. *Let M be the ambient space of Y. We can assume that there exists an $\mathbb{R}$-divisor D on M such that* $\mathrm{Supp}(D + Y)$ *is simple normal crossing and $B_Y = D|_Y$.*

Proof. We can construct a sequence of blow-ups $M_k \to M_{k-1} \to \cdots \to M_0 = M$ with the following properties.

(i) $\sigma_{i+1} : M_{i+1} \to M_i$ is the blow-up along a smooth irreducible component of $\mathrm{Supp} B_{Y_i}$ for any $i \geq 0$,

(ii) we put $Y_0 = Y$, $B_{Y_0} = B_Y$, and Y_{i+1} is the strict transform of Y_i for any $i \geq 0$,

(iii) we define $K_{Y_{i+1}} + B_{Y_{i+1}} = \sigma_{i+1}^*(K_{Y_i} + B_{Y_i})$ for any $i \geq 0$,

(iv) there exists an $\mathbb{R}$-divisor D on M_k such that $\mathrm{Supp}(Y_k + D)$ is simple normal crossing on M_k and that $D|_{Y_k} = B_{Y_k}$, and

(v) $\sigma_* \mathcal{O}_{Y_k}(\ulcorner -(B_{Y_k}^{<1})\urcorner - \llcorner B_{Y_k}^{>1} \lrcorner) \simeq \mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1} \lrcorner)$, where $\sigma : M_k \to M_{k-1} \to \cdots \to M_0 = M$.

We note that we can directly check $\sigma_{i+1*}\mathcal{O}_{Y_{i+1}}(\ulcorner -(B_{Y_{i+1}}^{<1})\urcorner - \llcorner B_{Y_{i+1}}^{>1} \lrcorner) \simeq \mathcal{O}_{Y_i}(\ulcorner -(B_{Y_i}^{<1})\urcorner - \llcorner B_{Y_i}^{>1} \lrcorner)$ for any $i \geq 0$ by computations similar to the proof of Proposition 3.50. We replace M and (Y, B_Y) with M_k and (Y_k, B_{Y_k}). □

uun **Remark 3.58.** In the proof of Proposition 3.57, M_k and (Y_k, B_{Y_k}) depend on the order of blow-ups. If we change the order of blow-ups, we have another tower of blow-ups $\sigma' : M'_k \to M'_{k-1} \to \cdots \to M'_0 = M$, D', Y'_k on M'_k, and $D'|_{Y'_k} = B_{Y'_k}$ with the desired properties. The relationship between M_k, Y_k, D and M'_k, Y'_k, D' is not clear.

multi **Remark 3.59 (Multicrossing vs simple normal crossing).** In [AmI, Section 2], Ambro discussed *multicrossing singularities* and *multicrossing pairs*. However, we think that *simple normal crossing varieties* and *simple normal crossing divisors* on them are sufficient for the later arguments in [AmI]. Therefore, we did not introduce the notion of *multicrossing singularities* and their simplicial resolutions. For the theory of quasi-log varieties, we may not even need the notion of *simple normal crossing pairs*. The notion of *global embedded simple normal crossing pairs* seems to be sufficient.

3.2.7 A remark on the ambient space

In this subsection, we make a remark on the ambient space M of a quasi-log resolution $f : (Y, B_Y) \to X$ in Definition 3.29.

The following lemma is essentially the same as Proposition 3.57. We repeat it here since it is important. The proof is obvious.

se-bl **Lemma 3.60.** *Let (Y, B_Y) be a simple normal crossing pair. Let V be a smooth variety such that $Y \subset V$. Then we can construct a sequence of blow-ups*

$$V_k \to V_{k-1} \to \cdots \to V_0 = V$$

with the following properties.

(1) $\sigma_{i+1} : V_{i+1} \to V_i$ *is the blow-up along a smooth irreducible component of* $\mathrm{Supp} B_{Y_i}$ *for any* $i \geq 0$,

(2) *we put* $Y_0 = Y$, $B_{Y_0} = B_Y$, *and* Y_{i+1} *is the strict transform of* Y_i *for any* $i \geq 0$,

(3) *we define* $K_{Y_{i+1}} + B_{Y_{i+1}} = \sigma_{i+1}^*(K_{Y_i} + B_{Y_i})$ *for any* $i \geq 0$,

(4) *there exists an* $\mathbb{R}$*-divisor* D *on* V_k *such that* $D|_{Y_k} = B_{Y_k}$, *and*

(5) $\sigma_* \mathcal{O}_{Y_k}(\ulcorner -(B_{Y_k}^{<1})\urcorner - \llcorner B_{Y_k}^{\geq 1} \lrcorner) \simeq \mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{\geq 1} \lrcorner)$, *where* $\sigma : V_k \to V_{k-1} \to \cdots \to V_0 = V$.

When a simple normal crossing variety Y is quasi-projective, we can make a *singular* ambient space whose singular locus dose not contain any strata of Y.

gene-int

Lemma 3.61. *Let* Y *be a simple normal crossing variety. Let* V *be a smooth quasi-projective variety such that* $Y \subset V$. *Let* $\{P_i\}$ *be any finite set of closed points of* Y. *Then we can find a quasi-projecive variety* W *such that* $Y \subset W \subset V$, $\dim W = \dim Y + 1$, *and* W *is smooth at* P_i *for any* i.

Proof. Let $\mathcal{I}_Y$ be the defining ideal sheaf of Y on V. Let H be an ample Cartier divisor. Then $\mathcal{I}_Y \otimes \mathcal{O}_Y(dH)$ is generated by global sections for $d \gg 0$. We can further assume that

$$H^0(V, \mathcal{I}_Y \otimes \mathcal{O}_V(dH)) \to \mathcal{I}_Y \otimes \mathcal{O}_V(dH) \otimes \mathcal{O}_V/m_{P_i}^2$$

is surjective for any i, where m_{P_i} is the maximal ideal corresponding to P_i. By taking a complete intersection of $(\dim V - \dim Y - 1)$ general members in $H^0(V, \mathcal{I}_Y \otimes \mathcal{O}_V(dH))$, we obtain a desired variety W. □

Of course, we can not always make W smooth in Lemma 3.61. gene-int

Example 3.62. Let $V \subset \mathbb{P}^5$ be the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$. In this case, there are no smooth hypersurfaces of $\mathbb{P}^5$ containing V. We can check it as follows. If there exists a smooth hypersurface S such that $V \subset S \subset \mathbb{P}^5$, then $\rho(V) = \rho(S) = \rho(\mathbb{P}^5) = 1$ by the Lefschetz hyperplane theorem. It is a contradiction.

By the above lemmas, we can prove the final lemma.

le-amb **Lemma 3.63.** *Let (Y, B_Y) be a simple normal crossing pair such that Y is quasi-projective. Then there exist a global embedded simple normal crossing pair (Z, B_Z) and a morphism $\sigma : Z \to Y$ such that*

$$\sigma_*\mathcal{O}_Z(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner) \simeq \mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1}\lrcorner).$$

Proof. Let V be a smooth quasi-projective variety such that $Y \subset V$. By Lemma 3.60, we can assume that there exists an $\mathbb{R}$-divisor D on V such that $D|_Y = B_Y$. Then we apply Lemma 3.61. We can find a quasi-projectie variety W such that $Y \subset W \subset V$, $\dim W = \dim Y + 1$, and W is smooth at the generic point of any stratum of (Y, B_Y). Of course, we can make $W \not\subset \mathrm{Supp} D$ (see the proof of Lemma 3.61). We apply Hironaka's resolution to W and use Szabó's resolution lemma. Then we obtain a desired global embedded simple normal crossing pair (Z, B_Z). □

Therefore, we obtain the following statement.

goodbye **Theorem 3.64.** *In* Definition 3.29*, it is sufficient to assume that (Y, B_Y) is a simple normal crossing pair if Y is quasi-projective.*

We note that we have a natural quasi-projective ambient space M in almost all the applications of the theory of quasi-log varieties to log canonical pairs. Therefore, Definition 3.29 seems to be reasonable.

We close this subsection with a remark on Chow's lemma. Proposition 3.65 is a bottleneck to construct a good ambient space of a simple normal crossing pair.

bottle **Proposition 3.65.** *There exists a complete simple normal crossing variety Y with the following property. If $f : Z \to Y$ is a proper surjective morphism from a simple normal crossing variety Z such that f is an isomorphism at the generic point of any stratum of Z, then Z is non-projective.*

Proof. We take a smooth complete non-projective toric variety X (cf. Example 1.14). We put $V = X \times \mathbb{P}^1$. Then V is a toric variety. We consider $Y = V \setminus T$, where T is the big torus of V. We will see that Y has the desired property. By the above construction, there is an irreducible component Y' of Y that is isomorphic to X. Let Z' be the irreducible component of Z mapped onto Y' by f. So, it is sufficient to see that Z' is not projective. On $Y' \simeq X$, there is an torus invariant effective one cycle C such that C is numerically trivial. By the construction and the assumption, $g = f|_{Z'} : Z' \to Y' \simeq X$ is

birational and an isomorphism over the generic point of any torus invariant curve on $Y' \simeq X$. We note that any torus invariant curve on $Y' \simeq X$ is a stratum of Y. We assume that Z' is projective, then there is a very ample effective divisor A on Z' such that A does not contain any irreducible components of the inverse image of C. Then $B = f_*A$ is an effective Cartier divisor on $Y' \simeq X$ such that $\mathrm{Supp}B$ contains no irreducible components of C. It is a contradiction because $\mathrm{Supp}B \cap C \neq \emptyset$ and C is numerically trivial. □

The phenomenon described in Proposition 3.65 is annoying when we treat non-normal varieties.

3.3 Fundamental Theorems

33-sec

In this section, we will prove the fundamental theorems for quasi-log pairs. First, we prove the base point free theorem for quasi-log pairs in the subsection 3.3.1. The reader can find that the notion of quasi-log pairs is very useful for inductive arguments. Next, we give a proof to the rationality theorem for quasi-log pairs in the subsection 3.3.2. Our proof is essentially the same as the proof for klt pairs. In the subsection 3.3.3, we prove the cone theorem for quasi-log varieties. The cone and contraction theorems are the main results in this section.

3.3.1 Base Point Free Theorem

331-ssec

The next theorem is the main theorem of this subsection. It is [Am1, Theorem 5.1]. This formulation is useful for the inductive treatment of log canonical pairs.

bpf-th

Theorem 3.66 (Base Point Free Theorem). *Let $[X, \omega]$ be a quasi-log pair and let $\pi : X \to S$ be a projective morphism. Let L be a π-nef Cartier divisor on X. Assume that*

(i) *$qL - \omega$ is π-ample for some real number $q > 0$, and*

(ii) *$\mathcal{O}_{X_{-\infty}}(mL)$ is $\pi|_{X_{-\infty}}$-generated for $m \gg 0$.*

Then $\mathcal{O}_X(mL)$ is π-generated for $m \gg 0$, that is, there exists a positive number m_0 such that $\mathcal{O}_X(mL)$ is π-generated for any $m \geq m_0$.

Proof. Without loss of generality, we can assume that S is affine.

bpf-c1 **Claim 1.** *$\mathcal{O}_X(mL)$ is π-generated around* $\mathrm{Nqklt}(X,\omega)$ *for $m \gg 0$.*

We put $X' = \mathrm{Nqklt}(X,\omega)$. Then $[X',\omega']$, where $\omega' = \omega|_{X'}$, is a quasi-log pair by adjunction (see Theorem adj-th 3.39 (i)). If $X' = X_{-\infty}$, then $\mathcal{O}_{X'}(mL)$ is π-generated for $m \gg 0$ by the assumption (ii). If $X' \neq X_{-\infty}$, then $\mathcal{O}_{X'}(mL)$ is π-generated for $m \gg 0$ by the induction on the dimension of $X \setminus X_{-\infty}$. By the following commutative diagram:

$$\begin{array}{ccccc} \pi^*\pi_*\mathcal{O}_X(mL) & \xrightarrow{\alpha} & \pi^*\pi_*\mathcal{O}_{X'}(mL) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathcal{O}_X(mL) & \longrightarrow & \mathcal{O}_{X'}(mL) & \longrightarrow & 0, \end{array}$$

we know that $\mathcal{O}_X(mL)$ is π-generated around X' for $m \gg 0$.

bpf-c2 **Claim 2.** *$\mathcal{O}_X(mL)$ is π-generated on a non-empty Zariski open set for $m \gg 0$.*

By Claim bpf-c1 1, we can assume that $\mathrm{Nqklt}(X,\omega)$ is empty. We will see that we can also assume that X is irreducible. Let X' be an irreducible component of X. Then X' with $\omega' = \omega|_{X'}$ has a natural quasi-log structure induced by $[X,\omega]$ by adjunction (see Corollary irre-cor 3.41). By the vanishing theorem (see Theorem adj-th 3.39 (ii)), we have $R^1\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(mL)) = 0$ for any $m \geq q$. We consider the following commutative diagram.

$$\begin{array}{ccccc} \pi^*\pi_*\mathcal{O}_X(mL) & \xrightarrow{\alpha} & \pi^*\pi_*\mathcal{O}_{X'}(mL) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathcal{O}_X(mL) & \longrightarrow & \mathcal{O}_{X'}(mL) & \longrightarrow & 0 \end{array}$$

Since α is surjective for $m \geq q$, we can assume that X is irreducible when we prove this claim.

If L is π-numerically trivial, then $\pi_*\mathcal{O}_X(L)$ is not zero. It is because $h^0(X_\eta, \mathcal{O}_{X_\eta}(L)) = \chi(X_\eta, \mathcal{O}_{X_\eta}(L)) = \chi(X_\eta, \mathcal{O}_{X_\eta}) = h^0(X_\eta, \mathcal{O}_{X_\eta}) > 0$ by Theorem adj-th 3.39 (ii) and by [kleiman Kl, Chapter II §2 Theorem 1], where X_η is the generic fiber of $\pi : X \to S$. Let D be a general member of $|L|$. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution. By blowing up M, we can assume that $(Y, B_Y + f^*D)$ is a global embedded simple normal crossing pair by Proposition tai4 3.54. We note that any stratum of (Y, B_Y) is mapped onto X by the assumption. We can take a positive real number $c \leq 1$ such that

$B_Y + cf^*D$ is a subboundary and some stratum of $(Y, B_Y + cf^*D)$ does not dominate X. Note that $f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner) \simeq \mathcal{O}_X$. Then the pair $[X, \omega + cD]$ is qlc and $f : (Y, B_Y + cf^*D) \to X$ is a quasi-log resolution. We note that $qL - (\omega + cD)$ is π-ample. By Claim 1, $\mathcal{O}_X(mL)$ is π-generated around $\mathrm{Nqklt}(X, \omega + cD)$ for $m \gg 0$. So, we can assume that L is not π-numerically trivial.

Let $x \in X$ be a general smooth point. Then we can take an $\mathbb{R}$-divisor D such that $\mathrm{mult}_x D > \dim X$ and that $D \sim_{\mathbb{R}} (q + r)L - \omega$ for some $r > 0$ (see [KM, 3.5 Step 2]). By blowing up M, we can assume that $(Y, B_Y + f^*D)$ is a global embedded simple normal crossing pair by Proposition 3.54. By the construction of D, we can find a positive real number $c < 1$ such that $B_Y + cf^*D$ is a subboundary and some stratum of $(Y, B_Y + cf^*D)$ does not dominate X. Note that $f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner) \simeq \mathcal{O}_X$. Then the pair $[X, \omega + cD]$ is qlc and $f : (Y, B_Y + cf^*D) \to X$ is a quasi-log resolution. We note that $q'L - (\omega + cD)$ is π-ample by $c < 1$, where $q' = q + cr$. By the construction, $\mathrm{Nqklt}(X, \omega + cD)$ is non-empty. Therefore, by applying Claim 1 to $[X, \omega + cD]$, $\mathcal{O}_X(mL)$ is π-generated around $\mathrm{Nqklt}(X, \omega + cD)$ for $m \gg 0$. So, we finish the proof of Claim 2.

Let p be a prime number and let l be a large integer. Then $\pi_*\mathcal{O}_X(p^l L) \neq 0$ by Claim 2 and $\mathcal{O}_X(p^l L)$ is π-generated around $\mathrm{Nqklt}(X, \omega)$ by Claim 1.

bpf-c3 **Claim 3.** *If the relative base locus* $\mathrm{Bs}_\pi|p^l L|$ *(with reduced scheme structure) is not empty, then* $\mathrm{Bs}_\pi|p^l L|$ *is not contained in* $\mathrm{Bs}_\pi|p^{l'} L|$ *for* $l' \gg l$.

Let $f : (Y, B_Y) \to X$ be a quasi-log resolution. We take a general member $D \in |p^l L|$. We note that S is affine and $|p^l L|$ is free around $\mathrm{Nqklt}(X, \omega)$. Thus, f^*D intersects any strata of $(Y, \mathrm{Supp} B_Y)$ transversally over $X \setminus \mathrm{Bs}_\pi|p^l L|$ by Bertini and f^*D contains no strata of (Y, B_Y). By taking blow-ups of M suitably, we can assume that $(Y, B_Y + f^*D)$ is a global embedded simple normal crossing pair. See the proofs of Propositions 3.54 and 3.50. We take the maximal positive real number c such that $B_Y + cf^*D$ is a subboundary over $X \setminus X_{-\infty}$. We note that $c \leq 1$. Here, we used $\mathcal{O}_X \simeq f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner)$ over $X \setminus X_{-\infty}$. Then $f : (Y, B_Y + cf^*D) \to X$ is a quasi-log resolution of $[X, \omega' = \omega + cD]$. Note that $[X, \omega']$ has a qlc center C that intersects $\mathrm{Bs}_\pi|p^l L|$ by the construction. By the induction, $\mathcal{O}_C(mL)$ is π-generated for $m \gg 0$ since $(q + cp^l)L - (\omega + cD)$ is π-ample. We can lift the sections of $\mathcal{O}_C(mL)$ to X for $m \geq q + cp^l$ by Theorem 3.39 (ii). Then we obtain that $\mathcal{O}_X(mL)$ is π-generated around C for $m \gg 0$. Therefore, $\mathrm{Bs}_\pi|p^{l'} L|$ is strictly smaller than $\mathrm{Bs}_\pi|p^l L|$ for $l' \gg l$.

bpf-c4 **Claim 4.** *$\mathcal{O}_X(mL)$ is π-generated for $m \gg 0$.*

By Claim 3 and the noetherian induction, $\mathcal{O}_X(p^l L)$ and $\mathcal{O}_X(p'^{l'} L)$ are π-generated for large l and l', where p and p' are prime numbers and they are relatively prime. So, there exists a positive number m_0 such that $\mathcal{O}_X(mL)$ is π-generated for any $m \geq m_0$. □

The next corollary is a special case of Theorem 3.66.

bpf-co **Corollary 3.67 (Base Point Free Theorem for lc pairs).** *Let (X, B) be an lc pair and let $\pi : X \to S$ be a projective morphism. Let L be a π-nef Cartier divisor on X. Assume that $qL - (K_X + B)$ is π-ample for some positive real number q. Then $\mathcal{O}_X(mL)$ is π-generated for $m \gg 0$.*

3.3.2 Rationality Theorem

332-ssec In this subsection, we prove the following rationality theorem (cf. [AmI, Theorem 5.9]).

rat-th **Theorem 3.68 (Rationality Theorem).** *Assume that $[X, \omega]$ is a quasi-log pair such that ω is $\mathbb{Q}$-Cartier. We note that this means ω is $\mathbb{R}$-linearly equivalent to a $\mathbb{Q}$-Cartier divisor on X (see* Remark 3.30*). Let $\pi : X \to S$ be a projective morphism and let H be a π-ample Cartier divisor on X. Assume that r is a positive number such that*

(1) *$H + r\omega$ is π-nef but not π-ample, and*

(2) *$(H + r\omega)|_{X_{-\infty}}$ is $\pi|_{X_{-\infty}}$-ample.*

Then r is a rational number, and in reduced form, r has denominator at most $a(\dim X + 1)$, where $a\omega$ is $\mathbb{R}$-linearly equivalent to a Cartier divisor on X.

Before we go to the proof, we recall the following lemmas.

pol1-le **Lemma 3.69 (cf. [KM, Lemma 3.19]).** *Let $P(x, y)$ be a non-trivial polynomial of degree $\leq n$ and assume that P vanishes for all sufficiently large integral solutions of $0 < ay - rx < \varepsilon$ for some fixed positive integer a and positive ε for some $r \in \mathbb{R}$. Then r is rational, and in reduced form, r has denominator $\leq a(n + 1)/\varepsilon$.*

For the proof, see [KM, Lemma 3.19].

pol2-le **Lemma 3.70 (cf. [KM, 3.4 Step 2]).** *Let $[Y, \omega]$ be a projective qlc pair and let $\{D_i\}$ be a finite collection of Cartier divisors. Consider the Hilbert polynomial*

$$P(u_1, \cdots, u_k) = \chi(Y, \mathcal{O}_Y(\sum_{i=1}^{k} u_i D_i)).$$

Suppose that for some values of the u_i, $\sum_{i=1}^{k} u_i D_i$ is nef and $\sum_{i=1}^{k} u_i D_i - \omega$ is ample. Then $P(u_1, \cdots, u_k)$ is not identically zero by the base point free theorem for qlc pairs (see Theorem 3.66*) and the vanishing theorem (see* Theorem 3.39 (ii)*), and its degree is $\leq \dim Y$.*

Note that the arguments in [KM, 3.4 Step 2] work for our setting.

Proof of Theorem 3.68. By using mH with various large m in place of H, we can assume that H is very ample over S (cf. [KM, 3.4 Step 1]). For each $(p, q) \in \mathbb{Z}^2$, let $L(p, q)$ denote the relative base locus of the linear system $M(p, q)$ on X (with reduced scheme structure), that is,

$$L(p, q) = \mathrm{Supp}(\mathrm{Coker}(\pi^*\pi_*\mathcal{O}_X(M(p, q)) \to \mathcal{O}_X(M(p, q)))),$$

where $M(p, q) = pH + qD$, where D is a Cartier divisor such that $D \sim_{\mathbb{R}} a\omega$. By the definition, $L(p, q) = X$ if and only if $\pi_*\mathcal{O}_X(M(p, q)) = 0$.

rat-c1 **Claim 1 (cf. [KM, Claim 3.20]).** *Let ε be a positive number. For (p, q) sufficiently large and $0 < aq - rp < \varepsilon$, $L(p, q)$ is the same subset of X. We call this subset L_0. We let $I \subset \mathbb{Z}^2$ be the set of (p, q) for which $0 < aq - rp < 1$ and $L(p, q) = L_0$. We note that I contains all sufficiently large (p, q) with $0 < aq - rp < 1$.*

For the proof, see [KM, Claim 3.20]. See also the proof of Claim 2 below.

rat-c2 **Claim 2.** *We have $L_0 \cap X_{-\infty} = \emptyset$.*

Proof of Claim 2. We take $(\alpha, \beta) \in \mathbb{Q}^2$ such that $\alpha > 0$, $\beta > 0$, and $\beta a/\alpha > r$ is sufficiently close to r. Then $(\alpha H + \beta a\omega)|_{X_{-\infty}}$ is $\pi|_{X_{-\infty}}$-ample because $(H + r\omega)|_{X_{-\infty}}$ is $\pi|_{X_{-\infty}}$-ample. If $0 < aq - rp < 1$ and $(p, q) \in \mathbb{Z}^2$ is sufficiently large, then $M(p, q) = mM(\alpha, \beta) + (M(p, q) - mM(\alpha, \beta))$ such that $M(p, q) - mM(\alpha, \beta)$ is π-very ample and that $m(\alpha H + \beta D)|_{X_{-\infty}}$ is also $\pi|_{X_{-\infty}}$-very ample. Therefore, $\mathcal{O}_{X_{-\infty}}(M(p, q))$ is π-very ample. Since $\pi_*\mathcal{O}_X(M(p, q)) \to \pi_*\mathcal{O}_{X_{-\infty}}(M(p, q))$ is surjective by the vanishing theorem (see Theorem 3.39 (ii)), $L(p, q) \cap X_{-\infty} = \emptyset$. We note that $M(p, q) - \omega$ is

π-ample because (p,q) is sufficiently large and $aq-rp<1$. By Claim 1, we have $L_0\cap X_{-\infty}=\emptyset$. □

rat-c3 **Claim 3.** *We assume that r is not rational or that r is rational and has denominator $>a(n+1)$ in reduced form, where $n=\dim X$. Then, for (p,q) sufficiently large and $0<aq-rp<1$, $\mathcal{O}_X(M(p,q))$ is π-generated at the generic point of any qlc center of $[X,\omega]$.*

Proof of Claim 3. We note that $M(p,q)-\omega\sim_{\mathbb{R}} pH+(qa-1)\omega$. If $aq-rp<1$ and (p,q) is sufficiently large, then $M(p,q)-\omega$ is π-ample. Let C be a qlc center of $[X,\omega]$. We note that we can assume $C\cap X_{-\infty}=\emptyset$ by Claim 2. Then $P_{C_\eta}(p,q)=\chi(C_\eta,\mathcal{O}_{C_\eta}(M(p,q)))$ is a non-zero polynomial of degree at most $\dim C_\eta\leq\dim X$ by Lemma 3.70 (see also Lemma 3.35). Note that C_η is the generic fiber of $C\to\pi(C)$. By Lemma 3.69, there exists (p,q) such that $P_{C_\eta}(p,q)\neq 0$ and that (p,q) sufficiently large and $0<aq-rp<1$. By the π-ampleness of $M(p,q)-\omega$, $P_{C_\eta}(p,q)=\chi(C_\eta,\mathcal{O}_{C_\eta}(M(p,q)))=h^0(C_\eta,\mathcal{O}_{C_\eta}(M(p,q)))$ and $\pi_*\mathcal{O}_X(M(p,q))\to\pi_*\mathcal{O}_C(M(p,q))$ is surjective. We note that $C'=C\cup X_{-\infty}$ has a natural quasi-log structure induced by $[X,\omega]$ and that $C\cap X_{-\infty}=\emptyset$. Therefore, $\mathcal{O}_X(M(p,q))$ is π-generated at the generic point of C. By combining this with Claim 1, $\mathcal{O}_X(M(p,q))$ is π-generated at the generic point of any qlc center of $[X,\omega]$ if (p,q) is sufficiently large with $0<aq-rp<1$. So, we obtain Claim 2. □

Note that $\mathcal{O}_X(M(p,q))$ is not π-generated for $(p,q)\in I$ because $M(p,q)$ is not π-nef. Therefore, $L_0\neq\emptyset$. We shrink S to an affine open subset intersecting $\pi(L_0)$. Let $D_1,\cdots,D_{n+1}$ be general members of $\pi_*\mathcal{O}_X(M(p_0,q_0))=H^0(X,\mathcal{O}_X(M(p_0,q_0)))$ with $(p_0,q_0)\in I$. Around the generic point of any irreducible component of L_0, by taking general hyperplane cuts and applying Lemma 3.71 below, we can check that $\omega+\sum_{i=1}^{n+1}D_i$ is not qlc at the generic point of any irreducible component of L_0. Thus, $\omega+\sum_{i=1}^{n+1}D_i$ is not qlc at the generic point of any irreducible component of L_0 and is qlc outside $L_0\cup X_{-\infty}$. Let $0<c<1$ be the maximal real number such that $\omega+c\sum_{i=1}^{n+1}D_i$ is qlc outside $X_{-\infty}$. Note that $c>0$ by Claim 3. Thus, the quasi-log pair $[X,\omega+c\sum_{i=1}^{n+1}D_i]$ has some qlc centers contained in L_0. Let C be a qlc center contained in L_0. We note that $C\cap X_{-\infty}=\emptyset$. We consider

$$\omega'=\omega+c\sum_{i=1}^{n+1}D_i\sim_{\mathbb{R}} c(n+1)p_0H+(1+c(n+1)q_0a)\omega.$$

Thus we have

$$pH + qa\omega - \omega' \sim_{\mathbb{R}} (p - c(n+1)p_0)H + (qa - (1 + c(n+1)q_0a))\omega.$$

If p and q are large enough and $0 < aq - rp \leq aq_0 - rp_0$, then $pH + qa\omega - \omega'$ is π-ample. It is because

$$\begin{aligned}&(p - c(n+1)p_0)H + (qa - (1 + c(n+1)q_0a))\omega \\ &= (p - (1 + c(n+1))p_0)H + (qa - (1 + c(n+1))q_0a)\omega + p_0H + (q_0a - 1)\omega.\end{aligned}$$

Suppose that r is not rational. There must be arbitrarily large (p, q) such that $0 < aq - rp < \varepsilon = aq_0 - rp_0$ and $H^0(C_\eta, \mathcal{O}_{C_\eta}(M(p,q))) \neq 0$ by Lemma 3.69. It is because $M(p,q) - \omega'$ is π-ample by $0 < aq - rp < aq_0 - rp_0$, $P_{C_\eta}(p,q) = \chi(C_\eta, \mathcal{O}_{C_\eta}(M(p,q)))$ is a non-trivial polynomial of degree at most $\dim C_\eta$ by Lemma 3.70, and $\chi(C_\eta, \mathcal{O}_{C_\eta}(M(p,q))) = h^0(C_\eta, \mathcal{O}_{C_\eta}(M(p,q)))$ by the ampleness of $M(p,q) - \omega'$. By the vanishing theorem, $\pi_*\mathcal{O}_X(M(p,q)) \to \pi_*\mathcal{O}_C(M(p,q))$ is surjective because $M(p,q) - \omega'$ is π-ample. We note that $C' = C \cup X_{-\infty}$ has a natural quasi-log structure induced by $[X, \omega']$ and that $C \cap X_{-\infty} = \emptyset$. Thus C is not contained in $L(p,q)$. Therefore, $L(p,q)$ is a proper subset of $L(p_0, q_0) = L_0$, giving the desired contradiction. So now we know that r is rational.

We next suppose that the assertion of the theorem concerning the denominator of r is false. Choose $(p_0, q_0) \in I$ such that $aq_0 - rp_0$ is the maximum, say it is equal to d/v. If $0 < aq - rp \leq d/v$ and (p,q) is sufficiently large, then $\chi(C_\eta, \mathcal{O}_{C_\eta}(M(p,q))) = h^0(C_\eta, \mathcal{O}_{C_\eta}(M(p,q)))$ since $M(p,q) - \omega'$ is π-ample. There exists sufficiently large (p,q) in the strip $0 < aq - rp < 1$ with $\varepsilon = 1$ for which $h^0(C_\eta, \mathcal{O}_{C_\eta}(M(p,q))) \neq 0$ by Lemma 3.69. Note that $aq - rp \leq d/v = aq_0 - rp_0$ holds automatically for $(p,q) \in I$. Since $\pi_*\mathcal{O}_X(M(p,q)) \to \pi_*\mathcal{O}_C(M(p,q))$ is surjective by the π-ampleness of $M(p,q) - \omega'$, we obtain the desired contradiction by the same reason as above. So, we finish the proof of the rationality theorem. □

We used the following lemma in the proof of Theorem 3.68.

boun **Lemma 3.71.** *Let $[X, \omega]$ be a qlc pair and $x \in X$ a closed point. Let $D_1, \cdots, D_m$ be effective Cartier divisors passing through x. If $[X, \omega + \sum_{i=1}^m D_i]$ is qlc, then $m \leq \dim X$.*

Proof. First, we assume $\dim X = 1$. If $x \in X$ is a qlc center of $[X, \omega]$, then m must be zero. So, we can assume that $x \in X$ is not a qlc center of $[X, \omega]$. Let

$f : (Y, B_Y) \to X$ be a quasi-log resolution of $[X, \omega]$. By shrinking X around x, we can assume that any stratum of Y dominates X and that X is smooth by Proposition 3.44. Since $f_*\mathcal{O}_Y(\ulcorner -(B_Y^{\leq 1})\urcorner) \simeq \mathcal{O}_X$, we can easily check that $m \leq 1 = \dim X$. In general, $[X, \omega + D_1]$ is qlc. Let V be the union of qlc centers of $[X, \omega + D_1]$ contained in $\mathrm{Supp} D_1$. Then both $[V, (\omega + D_1)|_V]$ and $[V, (\omega + D_1)|_V + \sum_{i=2}^m D_i|_V]$ are qlc by adjunction. By the induction on the dimension, $m - 1 \leq \dim V$. Therefore, we obtain $m \leq \dim X$. □

3.3.3 Cone Theorem

333-ssec

The main theorem of this subsection is the cone theorem for quasi-log varieties (cf. [Am1, Theorem 5.10]). Before we state the main theorem, let us fix the notation.

Definition 3.72. Let $[X, \omega]$ be a quasi-log pair with the non-qlc locus $X_{-\infty}$. Let $\pi : X \to S$ be a projective morphism. We put

$$\overline{NE}(X/S)_{-\infty} = \mathrm{Im}(\overline{NE}(X_{-\infty}/S) \to \overline{NE}(X/S)).$$

For an $\mathbb{R}$-Cartier divisor D, we define

$$D_{\geq 0} = \{z \in N_1(X/S) \mid D \cdot z \geq 0\}.$$

Similarly, we can define $D_{>0}$, $D_{\leq 0}$, and $D_{<0}$. We also define

$$D^{\perp} = \{z \in N_1(X/S) \mid D \cdot z = 0\}.$$

We use the following notation

$$\overline{NE}(X/S)_{D \geq 0} = \overline{NE}(X/S) \cap D_{\geq 0},$$

and similarly for > 0, ≤ 0, and < 0.

Definition 3.73. An *extremal face* of $\overline{NE}(X/S)$ is a non-zero subcone $F \subset \overline{NE}(X/S)$ such that $z, z' \in F$ and $z + z' \in F$ imply that $z, z' \in F$. Equivalently, $F = \overline{NE}(X/S) \cap H^{\perp}$ for some π-nef $\mathbb{R}$-divisor H, which is called a *supporting function* of F. An *extremal ray* is a one-dimensional extremal face.

(1) An extremal face F is called *ω-negative* if $F \cap \overline{NE}(X/S)_{\geq 0} = \{0\}$.

(2) An extremal face F is called *rational* if we can choose a π-nef $\mathbb{Q}$-divisor H as a support function of F.

(3) An extremal face F is called *relatively ample at infinity* if $F \cap \overline{NE}(X/S)_{-\infty} = \{0\}$. Equivalently, $H|_{X_{-\infty}}$ is $\pi|_{X_{-\infty}}$-ample for any supporting function H of F.

(4) An extremal face F is called *contractible at infinity* if it has a rational supporting function H such that $H|_{X_{-\infty}}$ is $\pi|_{X_{-\infty}}$-semi-ample.

The following theorem is a direct consequence of Theorem 3.66.

cont-th **Theorem 3.74 (Contraction Theorem).** *Let $[X, \omega]$ be a quasi-log pair and let $\pi : X \to S$ be a projective morphism. Let H be a π-nef Cartier divisor such that $F = H^{\perp} \cap \overline{NE}(X/S)$ is ω-negative and contractible at infinity. Then there exists a projective morphism $\varphi_F : X \to Y$ over S with the following properties.*

(1) *Let C be an integral curve on X such that $\pi(C)$ is a point. Then $\varphi_F(C)$ is a point if and only if $[C] \in F$.*

(2) $\mathcal{O}_Y \simeq (\varphi_F)_* \mathcal{O}_X$.

(3) *Let L be a line bundle on X such that $L \cdot C = 0$ for every curve C with $[C] \in F$. Then there is a line bundle L_Y on Y such that $L \simeq \varphi_F^* L_Y$.*

Proof. By the assumption, $qH - \omega$ is π-ample for some positive integer q and $H|_{X_{-\infty}}$ is $\pi|_{X_{-\infty}}$-semi-ample. By Theorem 3.66, $\mathcal{O}_X(mH)$ is π-generated for $m \gg 0$. We take the Stein factorization of the associated morphism. Then, we have the contraction morphism $\varphi_F : X \to Y$ with the properties (1) and (2).

We consider $\varphi_F : X \to Y$ and $\overline{NE}(X/Y)$. Then $\overline{NE}(X/Y) = F$, L is numerically trivial over Y, and $-\omega$ is φ_F-ample. Applying the base point free theorem (cf. Theorem 3.66) over Y, both $L^{\otimes m}$ and $L^{\otimes(m+1)}$ are pull-backs of line bundles on Y. Their difference gives a line bundle L_Y such that $L \simeq \varphi_F^* L_Y$. □

cone-thm **Theorem 3.75 (Cone Theorem).** *Let $[X, \omega]$ be a quasi-log pair and let $\pi : X \to S$ be a projective morphism. Then we have the following properties.*

(1) $\overline{NE}(X/S) = \overline{NE}(X/S)_{\omega\geq 0} + \overline{NE}(X/S)_{-\infty} + \sum R_j$, *where* R_j*'s are the* ω*-negative extremal rays of* $\overline{NE}(X/S)$ *that are rational and relatively ample at infinity. In particular, each* R_j *is spanned by an integral curve* C_j *on* X *such that* $\pi(C_j)$ *is a point.*

(2) *Let* H *be a* π*-ample* $\mathbb{R}$*-divisor on* X*. Then there are only finitely many* R_j*'s included in* $(\omega + H)_{<0}$*. In particular, the* R_j*'s are discrete in the half-space* $\omega_{<0}$*.*

(3) *Let* F *be an* ω*-negative extremal face of* $\overline{NE}(X/S)$ *that is relatively ample at infinity. Then* F *is a rational face. In particular,* F *is contractible at infinity.*

Proof. First, we assume that ω is $\mathbb{Q}$-Cartier. This means that ω is $\mathbb{R}$-linearly equivalent to a $\mathbb{Q}$-Cartier divisor. We can assume that $\dim_{\mathbb{R}} N_1(X/S) \geq 2$ and ω is not π-nef. Otherwise, the theorem is obvious.

cone-st1 **Step 1.** We have

$$\overline{NE}(X/S) = \overline{\overline{NE}(X/S)_{\omega\geq 0} + \overline{NE}(X/S)_{-\infty} + \sum_F F},$$

where F's vary among all rational proper ω-negative faces that are relatively ample at infinity and $\overline{\ }$ denotes the closure with respect to the real topology.

Proof. We put

$$B = \overline{\overline{NE}(X/S)_{\omega\geq 0} + \overline{NE}(X/S)_{-\infty} + \sum_F F}.$$

It is clear that $\overline{NE}(X/S) \supset B$. We note that each F is spanned by curves on X mapped to points on S by Theorem cont-th 3.74 (1). Supposing $\overline{NE}(X/S) \neq B$, we shall derive a contradiction. There is a separating function M which is Cartier and is not a multiple of ω in $N^1(X/S)$ such that $M > 0$ on $B\setminus\{0\}$ and $M \cdot z_0 < 0$ for some $z_0 \in \overline{NE}(X/S)$. Let C be the dual cone of $\overline{NE}(X/S)_{\omega\geq 0}$, that is,

$$C = \{D \in N^1(X/S) \mid D \cdot z \geq 0 \text{ for } z \in \overline{NE}(X/S)_{\omega\geq 0}\}.$$

Then C is generated by π-nef divisors and ω. Since $M > 0$ on $\overline{NE}(X/S)_{\omega\geq 0}\setminus\{0\}$, M is in the interior of C, and hence there exists a π-ample $\mathbb{Q}$-Cartier

divisor A such that $M - A = L' + p\omega$ in $N^1(X/S)$, where L' is a π-nef $\mathbb{Q}$-Cartier divisor on X and p is a non-negative rational number. Therefore, M is expressed in the form $M = H + p\omega$ in $N^1(X/S)$, where $H = A + L'$ is a π-ample $\mathbb{Q}$-Cartier divisor. The rationality theorem (see Theorem 3.68) implies that there exists a positive rational number $r < p$ such that $L = H + r\omega$ is π-nef but not π-ample, and $L|_{X_{-\infty}}$ is $\pi|_{X_{-\infty}}$-ample. Note that $L \neq 0$ in $N^1(X/S)$, since M is not a multiple of ω. Thus the extremal face F_L associated to the supporting function L is contained in B, which implies $M > 0$ on F_L. Therefore, $p < r$. It is a contradiction. This completes the proof of our first claim. □

Step 2. In the equality of Step 1, we may take such L that has the extremal face F_L of dimension one.

Proof. Let F be a rational proper ω-negative extremal face that is relatively ample at infinity, and assume that $\dim F \geq 2$. Let $\varphi_F : X \to W$ be the associated contraction. Note that $-\omega$ is φ_F-ample. By Step 1, we obtain

$$F = \overline{NE}(X/W) = \overline{\sum_G G},$$

where the G's are the rational proper ω-negative extremal faces of $\overline{NE}(X/W)$. We note that $\overline{NE}(X/W)_{-\infty} = 0$ because φ_F embeds $X_{-\infty}$ into W. The G's are also ω-negative extremal faces of $\overline{NE}(X/S)$ that are ample at infinity, and $\dim G < \dim F$. By induction, we obtain

$$\overline{NE}(X/S) = \overline{\overline{NE}(X/S)_{\omega\geq 0} + \overline{NE}(X/S)_{-\infty} + \sum R_j}, \tag{3.1}$$ siki1

where the R_j's are ω-negative rational extremal rays. Note that each R_j does not intersect $\overline{NE}(X/S)_{-\infty}$. □

cone-st3 **Step 3.** The contraction theorem (cf. Theorem 3.74) guarantees that for each extremal ray R_j there exists a reduced irreducible curve C_j on X such that $[C_j] \in R_j$. Let $\psi_j : X \to W_j$ be the contraction morphism of R_j, and let A be a π-ample Cartier divisor. We set

$$r_j = -\frac{A \cdot C_j}{\omega \cdot C_j}.$$

Then $A + r_j\omega$ is ψ_j-nef but not ψ_j-ample, and $(A + r_j\omega)|_{X_{-\infty}}$ is $\psi_j|_{X_{-\infty}}$-ample. By the rationality theorem (see Theorem 3.68), expressing $r_j = u_j/v_j$ with $u_j, v_j \in \mathbb{Z}_{>0}$ and $(u_j, v_j) = 1$, we have the inequality $v_j \leq a(\dim X + 1)$.

Step 4. Now take π-ample Cartier divisors $H_1, H_2, \cdots, H_{\rho-1}$ such that ω and the H_i's form a basis of $N^1(X/S)$, where $\rho = \dim_{\mathbb{R}} N^1(X/S)$. By Step 3, the intersection of the extremal rays R_j with the hyperplane

$$\{z \in N_1(X/S) \mid a\omega \cdot z = -1\}$$

in $N_1(X/S)$ lie on the lattice

$$\Lambda = \{z \in N_1(X/S) \mid a\omega \cdot z = -1, H_i \cdot z \in (a(a(\dim X + 1))!)^{-1}\mathbb{Z}\}.$$

This implies that the extremal rays are discrete in the half space

$$\{z \in N_1(X/S) \mid \omega \cdot z < 0\}.$$

Thus we can omit the closure sign $\overline{\ }$ from the formula (3.1) and this completes the proof of (1) when ω is $\mathbb{Q}$-Cartier.

Step 5. Let H be a π-ample $\mathbb{R}$-divisor on X. We choose $0 < \varepsilon_i \ll 1$ for $1 \leq i \leq \rho - 1$ such that $H - \sum_{i=1}^{\rho-1} \varepsilon_i H_i$ is π-ample. Then the R_j's included in $(\omega + H)_{<0}$ correspond to some elements of the above lattice Λ for which $\sum_{i=1}^{\rho-1} \varepsilon_i H_i \cdot z < 1/a$. Therefore, we obtain (2).

Step 6. The vector space $V = F^{\perp} \subset N^1(X/S)$ is defined over $\mathbb{Q}$ because F is generated by some of the R_j's. There exists a π-ample $\mathbb{R}$-divisor H such that F is contained in $(\omega + H)_{<0}$. Let $\langle F \rangle$ be the vector space spanned by F. We put

$$W_F = \overline{NE}(X/S)_{\omega+H\geq 0} + \overline{NE}(X/S)_{-\infty} + \sum_{R_j \not\subset F} R_j.$$

Then W_F is a closed cone, $\overline{NE}(X/S) = W_F + F$, and $W_F \cap \langle F \rangle = \{0\}$. The supporting functions of F are the elements of V that are positive on $W_F \setminus \{0\}$. This is a non-empty open set and thus it contains a rational element that, after scaling, gives a π-nef Cartier divisor L such that $F = L^{\perp} \cap \overline{NE}(X/S)$. Therefore, F is rational. So, we have (3).

From now on, ω is $\mathbb{R}$-Cartier.

Step 7. Let H be a π-ample $\mathbb{R}$-divisor on X. We shall prove (2). We assume that there are infinitely many R_j's in $(\omega + H)_{<0}$ and get a contradiction. There exists an affine open subset U of S such that $\overline{NE}(\pi^{-1}(U)/U)$ has infinitely many $(\omega + H)$-negative extremal rays. So, we shrink S and can

assume that S is affine. We can write $H = E + H'$, where H' is π-ample, $[X, \omega + E]$ is a quasi-log pair with the same qlc centers and non-qlc locus as $[X, \omega]$, and $\omega + E$ is $\mathbb{Q}$-Cartier. Since $\omega + H = \omega + E + H'$, we have

$$\overline{NE}(X/S) = \overline{NE}(X/S)_{\omega+H\geq 0} + \overline{NE}(X/S)_{-\infty} + \sum_{\text{finite}} R_j.$$

It is a contradiction. Thus, we obtain (2). The statement (1) is a direct consequence of (2). Of course, (3) holds by Step 6 once we obtain (1).

So, we finish the proof of the cone theorem. □

We close this subsection with the following non-trivial example.

cone-ex **Example 3.76.** We consider the first projection $p : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. We take a blow-up $\mu : Z \to \mathbb{P}^1 \times \mathbb{P}^1$ at $(0, \infty)$. Let A_∞ (resp. A_0) be the strict transform of $\mathbb{P}^1 \times \{\infty\}$ (resp. $\mathbb{P}^1 \times \{0\}$) on Z. We define $M = \mathbb{P}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z(A_0))$ and X is the restriction of M on $(p \circ \mu)^{-1}(0)$. Then X is a simple normal crossing divisor on M. More explicitly, X is a $\mathbb{P}^1$-bundle over $(p \circ \mu)^{-1}(0)$ and is obtained by gluing $X_1 = \mathbb{P}^1 \times \mathbb{P}^1$ and $X_2 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ along a fiber. In particular, $[X, K_X]$ is a quasi-log pair with only qlc singularities. By the construction, $M \to Z$ has two sections. Let D^+ (resp. D^-) be the restriction of the section of $M \to Z$ corresponding to $\mathcal{O}_Z \oplus \mathcal{O}_Z(A_0) \to \mathcal{O}_Z(A_0) \to 0$ (resp. $\mathcal{O}_Z \oplus \mathcal{O}_Z(A_0) \to \mathcal{O}_Z \to 0$). Then it is easy to see that D^+ is a nef Cartier divisor on X and that the linear system $|mD^+|$ is free for any $m > 0$ by Remark 3.77 below. We take a general member $B_0 \in |mD^+|$ with $m \geq 2$. We consider $K_X + B$ with $B = D^- + B_0 + B_1 + B_2$, where B_1 and B_2 are general fibers of $X_1 = \mathbb{P}^1 \times \mathbb{P}^1 \subset X$. We note that B_0 does not intersect D^-. Then (X, B) is an embedded simple normal crossing pair. In particular, $[X, K_X + B]$ is a quasi-log pair with $X_{-\infty} = \emptyset$. It is easy to see that there exists only one integral curve C on $X_2 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \subset X$ such that $C \cdot (K_X + B) < 0$. Note that $(K_X + B)|_{X_1}$ is ample on X_1. By the cone theorem, we obtain

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+B)\geq 0} + \mathbb{R}_{\geq 0}[C].$$

By the contraction theorem, we have $\varphi : X \to W$ which contracts C. We can easily see that W is a simple normal crossing surface but $K_W + B_W$, where $B_W = \varphi_* B$, is not $\mathbb{Q}$-Cartier. Therefore, we can not run the LMMP for reducible varieties.

The above example implies that the cone and contraction theorems for quasi-log varieties do not directly produce the LMMP for quasi-log varieties.

free-re **Remark 3.77.** In Example cone-ex 3.76, M is a projective toric variety. Let E be the section of $M \to Z$ corresponding to $\mathcal{O}_Z \oplus \mathcal{O}_Z(A_0) \to \mathcal{O}_Z(A_0) \to 0$. Then, it is easy to see that E is a nef Cartier divisor on M. Therefore, the linear system $|E|$ is free. In particular, $|D^+|$ is free on X. Note that $D^+ = E|_X$. So, $|mD^+|$ is free for any $m \geq 0$.

Chapter 4

Related Topics

chap4

In this chapter, we treat related topics. In Section 4.1, we discuss the base point free theorem of Reid–Fukuda type. In Section 4.2, we prove that the non-klt locus of a dlt pair is Cohen–Macaulay as an application of Lemma 2.33. Section 4.3 is a description of Alexeev's criterion for Serre's S_3 condition. It is a clever application of Theorem 2.39 (i). Section 4.4 is an introduction to the theory of toric polyhedra. A toric polyhedron has a natural quasi-log structure. In Section 4.5, we quickly explain the notion of *non-lc ideal sheaves* and the restriction theorem in [F15]. It is related to the inversion of adjunction on log canonicity. In the final section, we state effective base point free theorems for log canonical pairs. We give no proofs there.

4.1 Base Point Free Theorem of Reid–Fukuda type

34-sec

One of my motivations to study [Am1] is to understand [Am1, Theorem 7.2], which is a complete generalization of [F2]. The following theorem is a special case of Theorem 7.2 in [Am1], which was stated without proof. Here, we will reduce it to Theorem 3.66 by using Kodaira's lemma.

bpf-rd-th

Theorem 4.1 (Base point free theorem of Reid–Fukuda type). *Let $[X, \omega]$ be a quasi-log pair with $X_{-\infty} = \emptyset$, $\pi : X \to S$ a* projective *morphism, and L a π-nef Cartier divisor on X such that $qL - \omega$ is nef and log big over S for some positive real number q. Then $\mathcal{O}_X(mL)$ is π-generated for $m \gg 0$.*

Remark 4.2. In [Am1, Section 7], Ambro said that the proof of [Am1, Theorem 7.2] is parallel to [Am1, Theorem 5.1]. However, I could not check it. Steps 1, 2, and 4 in the proof of [Am1, Theorem 5.1] work without any modifications. In Step 3 (see Claim 3 in the proof of Theorem 3.66), $q'L-\omega'$ is π-nef, but I think that $q'L-\omega' = qL-\omega$ is not always log big over S with respect to $[X, \omega']$, where $\omega' = \omega + cD$ and $q' = q + cp^l$. So, we can not directly apply the argument in Step 1 (see Claim 1 in the proof of Theorem 3.66) to this new quasi-log pair $[X, \omega']$.

Proof. We divide the proof into three steps.

Step 1. We take an irreducible component X' of X. Then X' has a natural quasi-log structure induced by X (see Theorem 3.39 (i)). By the vanishing theorem (see Theorem 3.39 (ii)), we have $R^1\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(mL)) = 0$ for $m \geq q$. Therefore, we obtain that $\pi_*\mathcal{O}_X(mL) \to \pi_*\mathcal{O}_{X'}(mL)$ is surjective for $m \geq q$. Thus, we can assume that X is irreducible for the proof of this theorem by the following commutative diagram.

$$\begin{array}{ccccc} \pi^*\pi_*\mathcal{O}_X(mL) & \longrightarrow & \pi^*\pi_*\mathcal{O}_{X'}(mL) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathcal{O}_X(mL) & \longrightarrow & \mathcal{O}_{X'}(mL) & \longrightarrow & 0 \end{array}$$

Step 2. Without loss of generality, we can assume that S is affine. Since $qL-\omega$ is nef and big over S, we can write $qL-\omega \sim_{\mathbb{R}} A+E$ by Kodaira's lemma, where A is a π-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X and E is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. We note that X is projective over S and that X is not necessarily normal. By Lemma 4.3 below, we have a new quasi-log structure on $[X, \widetilde{\omega}]$, where $\widetilde{\omega} = \omega + \varepsilon E$, for $0 < \varepsilon \ll 1$.

Step 3. By the induction on the dimension, $\mathcal{O}_{\mathrm{Nqklt}(X,\omega)}(mL)$ is π-generated for $m \gg 0$. Note that $\pi_*\mathcal{O}_X(mL) \to \pi_*\mathcal{O}_{\mathrm{Nqklt}(X,\omega)}(mL)$ is surjective for $m \geq q$ by the vanishing theorem (see Theorem 3.39 (ii)). Then $\mathcal{O}_{\mathrm{Nqklt}(X,\widetilde{\omega})}(mL)$ is π-generated for $m \gg 0$ by the above lifting result and by Lemma 4.3. In particular, $\mathcal{O}_{\widetilde{X}_{-\infty}}(mL)$ is π-generated for $m \gg 0$. We note that $qL-\widetilde{\omega} \sim_{\mathbb{R}} (1-\varepsilon)(qL-\omega)+\varepsilon A$ is π-ample. Therefore, by Theorem 3.66, we obtain that $\mathcal{O}_X(mL)$ is π-generated for $m \gg 0$.

We finish the proof. □

key-le **Lemma 4.3.** *Let $[X, \omega]$ be a quasi-log pair with $X_{-\infty} = \emptyset$. Let E be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then $[X, \omega + \varepsilon E]$ is a quasi-log pair with the following properties for $0 < \varepsilon \ll 1$.*

(i) *We put $[X, \widetilde{\omega}] = [X, \omega + \varepsilon E]$. Then $[X, \widetilde{\omega}]$ is a quasi-log pair and $\mathrm{Nqklt}(X, \widetilde{\omega}) = \mathrm{Nqklt}(X, \omega)$ as closed subsets of X.*

(ii) *There exist natural surjective homomorphisms $\mathcal{O}_{\mathrm{Nqklt}(X,\widetilde{\omega})} \to \mathcal{O}_{\mathrm{Nqklt}(X,\omega)} \to 0$ and $\mathcal{O}_{\mathrm{Nqklt}(X,\widetilde{\omega})} \to \mathcal{O}_{\widetilde{X}_{-\infty}} \to 0$, that is, $\mathrm{Nqklt}(X, \omega)$ and $\widetilde{X}_{-\infty}$ are closed subschemes of $\mathrm{Nqklt}(X, \widetilde{\omega})$, where $\widetilde{X}_{-\infty}$ is the non-qlc locus of $[X, \widetilde{\omega}]$.*

Proof. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution of $[X, \omega]$, where (Y, B_Y) is a global embedded simple normal crossing pair. We can assume that the union of all strata of (Y, B_Y) mapped into $\mathrm{Nqklt}(X, \omega)$, which we denote by Y', is a union of irreducible components of Y. We put $Y'' = Y - Y'$. Then we obtain that $f_*\mathcal{O}_{Y''}(A - Y'|_{Y''})$ is $\mathcal{I}_{\mathrm{Nqklt}(X,\omega)}$, that is, the defining ideal sheaf of $\mathrm{Nqklt}(X, \omega)$ on X, where $A = \ulcorner -(B_Y^{<1})\urcorner$. For the details, see the proof of Theorem adj-th 3.39 (i). Let M be the ambient space of Y and $B_Y = D|_Y$.

Claim. *By modifying M birationally, we can assume that there exists a simple normal crossing divisor F on M such that $\mathrm{Supp}(Y + D + F)$ is simple normal crossing, F and Y'' have no common irreducible components, and $F|_{Y''} = (f'')^*E$, where $f'' = f|_{Y''}$. Of course, $(f'')^*E + B_{Y''}$ has a simple normal crossing support on Y'', where $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$. In general, F may have common irreducible components with D and Y'.*

Proof of Claim. First, we note that $(f'')^*E$ contains no strata of Y''. We can construct a proper birational morphism $h : \widetilde{M} \to M$ from a smooth variety $\widetilde{M}$ such that $K_{\widetilde{M}} + D_{\widetilde{M}} = h^*(K_M + Y + D)$, $h^{-1}((f'')^*E)$ is a divisor on $\widetilde{M}$, and $\mathrm{Exc}(h) \cup \mathrm{Supp} h_*^{-1}(Y + D) \cup h^{-1}((f'')^*E)$ is a simple normal crossing on $\widetilde{M}$ as in the proof of Proposition tai4 3.54. We note that we can assume that h is an isomorphism outside $h^{-1}((f'')^*E)$ by Szabó's resolution lemma. Let $\widetilde{Y}$ be the union of the irreducible components of $D_{\widetilde{M}}^{=1}$ that are mapped into Y. By Proposition taisetsu 3.50, we can replace M, Y, and D with $\widetilde{M}$, $\widetilde{Y}$, and $\widetilde{D} = D_{\widetilde{M}} - \widetilde{Y}$. We finish the proof. □

Let us go back to the proof of Lemma key-le 4.3. Let Y_2 be the union of all the irreducible components of Y that are contained in $\mathrm{Supp} F$. We put $Y_1 = Y -$

Y_2 and $\widetilde{B} = F|_{Y_1}$. We consider $f_1 : (Y_1, B_{Y_1} + \varepsilon\widetilde{B}) \to X$ for $0 < \varepsilon \ll 1$, where $K_{Y_1} + B_{Y_1} = (K_Y + B_Y)|_{Y_1}$ and $f_1 = f|_{Y_1}$. Then, we have $f_1^*(\omega + \varepsilon E) \sim_{\mathbb{R}} K_{Y_1} + B_{Y_1} + \varepsilon\widetilde{B}$. Moreover, the natural inclusion $\mathcal{O}_X \to f_{1*}\mathcal{O}_{Y_1}(\ulcorner -((B_{Y_1} + \varepsilon\widetilde{B})^{<1})\urcorner)$ defines an ideal $\mathcal{I}_{\widetilde{X}_{-\infty}} = f_{1*}\mathcal{O}_{Y_1}(\ulcorner -((B_{Y_1} + \varepsilon\widetilde{B})^{<1})\urcorner - \llcorner (B_{Y_1} + \varepsilon\widetilde{B})^{>1}\lrcorner)$. It is because

$$f_{1*}\mathcal{O}_{Y_1}(\ulcorner -((B_{Y_1} + \varepsilon\widetilde{B})^{<1})\urcorner - \llcorner (B_{Y_1} + \varepsilon\widetilde{B})^{>1}\lrcorner) \subset f_*\mathcal{O}_Y(\ulcorner -(B_Y)^{<1}\urcorner) \simeq \mathcal{O}_X$$

when $0 < \varepsilon \ll 1$. We note that $\llcorner (B_{Y_1} + \varepsilon\widetilde{B})^{>1}\lrcorner \geq Y_2|_{Y_1}$. Namely, the pair $[X, \widetilde{\omega}]$ has a quasi-log structure with a quasi-log resolution $f_1 : (Y_1, B_{Y_1} + \varepsilon\widetilde{B}) \to X$. By the construction and the definition, it is obvious that there exist surjective homomorphisms $\mathcal{O}_{\mathrm{Nqklt}(X,\widetilde{\omega})} \to \mathcal{O}_{\mathrm{Nqklt}(X,\omega)} \to 0$ and $\mathcal{O}_{\mathrm{Nqklt}(X,\widetilde{\omega})} \to \mathcal{O}_{\widetilde{X}_{-\infty}} \to 0$. It is not difficult to see that $\mathrm{Nqklt}(X, \omega) = \mathrm{Nqklt}(X, \widetilde{\omega})$ as closed subsets of X for $0 < \varepsilon \ll 1$. We finish the proof. □

As a special case, we obtain the following base point free theorem of Reid–Fukuda type for log canonical pairs.

bpf-rdlc-th **Theorem 4.4.** (Base point free theorem of Reid–Fukuda type for lc pairs). *Let (X, B) be an lc pair. Let L be a π-nef Cartier divisor on X, where $\pi : X \to S$ is a projective morphism. Assume that $qL - (K_X + B)$ is π-nef and π-log big for some positive real number q. Then $\mathcal{O}_X(mL)$ is π-generated for $m \gg 0$.*

We believe that the above theorem holds under the assumption that π is only *proper*. However, our proof needs projectivity of π.

Remark 4.5. In Theorem bpf-rdlc-th 4.4, if $\mathrm{Nklt}(X, B)$ is projective over S, then we can prove Theorem bpf-rdlc-th 4.4 under the weaker assumption that $\pi : X \to S$ is only *proper*. It is because we can apply Theorem bpf-rd-th 4.1 to $\mathrm{Nklt}(X, B)$. So, we can assume that $\mathcal{O}_X(mL)$ is π-generated on a non-empty open subset containing $\mathrm{Nklt}(X, B)$. In this case, we can prove Theorem bpf-rdlc-th 4.4 by applying the usual X-method to L on (X, B). We note that $\mathrm{Nklt}(X, B)$ is always projective over S when $\dim \mathrm{Nklt}(X, B) \leq 1$. The reader can find a different proof in fukuda [Fk1] when (X, B) is a log canonical surface, where Fukuda used the LMMP with scaling for dlt surfaces.

Finally, we explain the reason why we assumed that $X_{-\infty} = \emptyset$ and π is projective in Theorem bpf-rd-th 4.1.

emp-re **Remark 4.6 (Why $X_{-\infty}$ is empty?).** Let C be a qlc center of $[X, \omega]$. Then we have to consider a quasi-log variety $X' = C \cup X_{-\infty}$ for the inductive arguments. In general, X' is reducible. It sometimes happens that $\dim C < \dim X_{-\infty}$. We do not know how to apply Kodaira's lemma to reducible varieties. So, we assume that $X_{-\infty} = \emptyset$ in Theorem 4.1.

proj-re **Remark 4.7 (Why π is projective?).** We assume that S is a point in Theorem 4.1 for simplicity. If $X_{-\infty} = \emptyset$, then it is enough to treat irreducible quasi-log varieties by Step 1. Thus, we can assume that X is irreducible. Let $f : Y \to X$ be a proper birational morphism from a smooth projective variety. If X is normal, then $H^0(X, \mathcal{O}_X(mL)) \simeq H^0(Y, \mathcal{O}_Y(mf^*L))$ for any $m \geq 0$. However, X is not always normal (see Example 4.8 below). So, it sometimes happens that $\mathcal{O}_Y(mf^*L)$ has many global sections but $\mathcal{O}_X(mL)$ has only a few global sections. Therefore, we can not easily reduce the problem to the case when X is projective. This is the reason why we assume that $\pi : X \to S$ is projective. See also Proposition 3.65.

rd-ex **Example 4.8.** Let $M = \mathbb{P}^2$ and let X be a nodal curve on M. Then (M, X) is an lc pair. By Example 3.31, $[X, K_X]$ is a quasi-log variety with only qlc singularities. In this case, X is irreducible, but it is not normal.

4.2 Basic properties of dlt pairs

b-dlt-sec In this section, we prove supplementary results on dlt pairs. First, let us reprove the following well-known theorem.

49-dlt **Theorem 4.9.** *Let (X, D) be a dlt pair. Then X has only rational singularities.*

Proof. (cf. [N, Chapter VII, 1.1.Theorem]). By the definition of dlt, we can take a resolution $f : Y \to X$ such that $\mathrm{Exc}(f)$ and $\mathrm{Exc}(f) \cup \mathrm{Supp} f_*^{-1}D$ are both simple normal crossing divisors on Y and that $K_Y + f_*^{-1}D = f^*(K_X + D) + E$ with $\ulcorner E \urcorner \geq 0$. We can take an effective f-exceptional divisor A on Y such $-A$ is f-ample (see, for example, [F7, Proposition 3.7.7]). Then $\ulcorner E \urcorner - (K_Y + f_*^{-1}D + \{-E\} + \varepsilon A) = -f^*(K_X + D) - \varepsilon A$ is f-ample for $\varepsilon > 0$. If $0 < \varepsilon \ll 1$, then $(Y, f_*^{-1}D + \{-E\} + \varepsilon A)$ is dlt. Therefore, $R^i f_* \mathcal{O}_Y(\ulcorner E \urcorner) = 0$ for $i > 0$ (see [KMM, Theorem 1-2-5], Theorem 2.42, or Lemma 4.10 below) and $f_* \mathcal{O}_Y(\ulcorner E \urcorner) \simeq \mathcal{O}_X$. Note that $\ulcorner E \urcorner$ is effective and f-exceptional. Thus, the composition $\mathcal{O}_X \to Rf_* \mathcal{O}_Y \to Rf_* \mathcal{O}_Y(\ulcorner E \urcorner) \simeq \mathcal{O}_X$ is

a quasi-isomorphism. So, X has only rational singularities by [Kv3, Theorem 1]. □

In the above proof, we used the next lemma.

vani-rf-le **Lemma 4.10 (Vanishing lemma of Reid–Fukuda type).** *Let V be a smooth variety and let B be a boundary $\mathbb{R}$-divisor on V such that* Supp$B$ *is a simple normal crossing divisor. Let $f : V \to W$ be a proper morphism onto a variety W. Assume that D is a Cartier divisor on V such that $D - (K_V + B)$ is f-nef and f-log big. Then $R^i f_* \mathcal{O}_V(D) = 0$ for any $i > 0$.*

Proof. We use the induction on the number of irreducible components of $\llcorner B \lrcorner$ and on the dimension of V. If $\llcorner B \lrcorner = 0$, then the lemma follows from the Kawamata–Viehweg vanishing theorem. Therefore, we can assume that there is an irreducible divisor $S \subset \llcorner B \lrcorner$. We consider the following short exact sequence

$$0 \to \mathcal{O}_V(D - S) \to \mathcal{O}_V(D) \to \mathcal{O}_S(D) \to 0.$$

By induction, we see that $R^i f_* \mathcal{O}_V(D - S) = 0$ and $R^i f_* \mathcal{O}_S(D) = 0$ for any $i > 0$. Thus, we have $R^i f_* \mathcal{O}_V(D) = 0$ for $i > 0$. □

4.11 (Weak log-terminal singularities). The proof of Theorem 4.9 works for *weak log-terminal* singularities in the sense of [KMM]. For the definition, see [KMM, Definition 0-2-10]. Thus, we can recover [KMM, Theorem 1-3-6], that is, we obtain the following statement.

Theorem 4.12 (cf. [KMM, Theorem 1-3-6]). *All weak log-terminal singularities are rational.*

We think that this theorem is one of the most difficult results in [KMM]. We do not need the difficult vanishing theorem due to Elkik and Fujita (see [KMM, Theorem 1-3-1]) to obtain the above theorem. In Theorem 4.9, if we assume that (X, D) is only weak log-terminal, then we can not necessarily make Exc(f) and Exc$(f) \cup$ Supp$f_*^{-1}D$ *simple* normal crossing divisors. We can only make them *normal crossing divisors*. However, [KMM, Theorem 1-2-5] and Theorem 2.42 work in this setting. Thus, the proof of Theorem 4.9 works for weak log-terminal. Anyway, the notion of weak log-terminal singularities is not useful in the recent log minimal model program. So, we do not discuss weak log-terminal singularities here.

Remark 4.13. The proofs of Theorem 4.14 and Theorem 4.17 also work for weak log-terminal pairs once we adopt suitable vanishing theorems such as Theorem 2.42 and Theorem 2.54.

The following theorem generalizes [FA, 17.5 Corollary], where it was only proved that S is semi-normal and satisfies Serre's S_2 condition. We use Lemma 2.33 in the proof.

cm-th **Theorem 4.14.** *Let X be a normal variety and $S+B$ a boundary $\mathbb{R}$-divisor such that $(X, S+B)$ is dlt, S is reduced, and $\llcorner B \lrcorner = 0$. Let $S = S_1 + \cdots + S_k$ be the irreducible decomposition and $T = S_1 + \cdots + S_l$ for $1 \leq l \leq k$. Then T is semi-normal, Cohen–Macaulay, and has only Du Bois singularities.*

Proof. Let $f : Y \to X$ be a resolution such that $K_Y + S' + B' = f^*(K_X + S + B) + E$ with the following properties: (i) S' (resp. B') is the strict transform of S (resp. B), (ii) $\mathrm{Supp}(S' + B') \cup \mathrm{Exc}(f)$ and $\mathrm{Exc}(f)$ are simple normal crossing divisors on Y, (iii) f is an isomorphism over the generic point of any lc center of $(X, S + B)$, and (iv) $\ulcorner E \urcorner \geq 0$. We write $S = T + U$. Let T' (resp. U') be the strict transform of T (resp. U) on Y. We consider the following short exact sequence

$$0 \to \mathcal{O}_Y(-T' + \ulcorner E \urcorner) \to \mathcal{O}_Y(\ulcorner E \urcorner) \to \mathcal{O}_{T'}(\ulcorner E|_{T'} \urcorner) \to 0.$$

Since $-T' + E \sim_{\mathbb{R},f} K_Y + U' + B'$ and $E \sim_{\mathbb{R},f} K_Y + S' + B'$, we have $-T' + \ulcorner E \urcorner \sim_{\mathbb{R},f} K_Y + U' + B' + \{-E\}$ and $\ulcorner E \urcorner \sim_{\mathbb{R},f} K_Y + S' + B' + \{-E\}$. By the vanishing theorem, $R^i f_* \mathcal{O}_Y(-T' + \ulcorner E \urcorner) = R^i f_* \mathcal{O}_Y(\ulcorner E \urcorner) = 0$ for any $i > 0$. Note that we used the vanishing lemma of Reid–Fukuda type (see Lemma 4.10). Therefore, we have

$$0 \to f_* \mathcal{O}_Y(-T' + \ulcorner E \urcorner) \to \mathcal{O}_X \to f_* \mathcal{O}_{T'}(\ulcorner E|_{T'} \urcorner) \to 0$$

and $R^i f_* \mathcal{O}_{T'}(\ulcorner E|_{T'} \urcorner) = 0$ for all $i > 0$. Note that $\ulcorner E \urcorner$ is effective and f-exceptional. Thus, $\mathcal{O}_T \simeq f_* \mathcal{O}_{T'} \simeq f_* \mathcal{O}_{T'}(\ulcorner E|_{T'} \urcorner)$. Since T' is a simple normal crossing divisor, T is semi-normal. By the above vanishing result, we obtain $Rf_* \mathcal{O}_{T'}(\ulcorner E|_{T'} \urcorner) \simeq \mathcal{O}_T$ in the derived category. Therefore, the composition $\mathcal{O}_T \to Rf_* \mathcal{O}_{T'} \to Rf_* \mathcal{O}_{T'}(\ulcorner E|_{T'} \urcorner) \simeq \mathcal{O}_T$ is a quasi-isomorphism. Apply $R\mathcal{H}om_T(\_\_, \omega_T^\bullet)$ to the quasi-isomorphism $\mathcal{O}_T \to Rf_* \mathcal{O}_{T'} \to \mathcal{O}_T$. Then the composition $\omega_T^\bullet \to Rf_* \omega_{T'}^\bullet \to \omega_T^\bullet$ is a quasi-isomorphism by the Grothendieck duality. By the vanishing theorem (see, for example, Lemma 2.33), $R^i f_* \omega_{T'} = 0$ for $i > 0$. Hence, $h^i(\omega_T^\bullet) \subseteq R^i f_* \omega_{T'}^\bullet \simeq R^{i+d} f_* \omega_{T'}$, where $d = \dim T =$

$\dim T'$. Therefore, $h^i(\omega_T^\bullet) = 0$ for $i \neq -d$. Thus, T is Cohen–Macaulay. This argument is the same as the proof of Theorem 1 in [Kv3]. Since T' is a simple normal crossing divisor, T' has only Du Bois singularities. The quasi-isomorphism $\mathcal{O}_T \to Rf_*\mathcal{O}_{T'} \to \mathcal{O}_T$ implies that T has only Du Bois singularities (cf. [Kv1, Corollary 2.4]). Since the composition $\omega_T \to f_*\omega_{T'} \to \omega_T$ is an isomorphism, we obtain $f_*\omega_{T'} \simeq \omega_T$. By the Grothendieck duality, $Rf_*\mathcal{O}_{T'} \simeq R\mathcal{H}om_T(Rf_*\omega_{T'}^\bullet, \omega_T^\bullet) \simeq R\mathcal{H}om_T(\omega_T^\bullet, \omega_T^\bullet) \simeq \mathcal{O}_T$. So, $R^i f_*\mathcal{O}_{T'} = 0$ for all $i > 0$. □

We obtained the following vanishing theorem in the proof of Theorem 4.14.

35 **Corollary 4.15.** *Under the notation in the proof of* Theorem 4.14, *$R^i f_*\mathcal{O}_{T'} = 0$ for any $i > 0$ and $f_*\mathcal{O}_{T'} \simeq \mathcal{O}_T$.*

We close this section with a non-trivial example.

416ex **Example 4.16 (cf. [KMM, Remark 0-2-11. (4)]).** We consider the $\mathbb{P}^2$-bundle

$$\pi : V = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)) \to \mathbb{P}^2.$$

Let $F_1 = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ and $F_2 = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ be two hypersurfaces of V which correspond to projections $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$ given by $(x, y, z) \mapsto (x, y)$ and $(x, y, z) \mapsto (x, z)$. Let $\Phi : V \to W$ be the flipping contraction that contracts the negative section of $\pi : V \to \mathbb{P}^2$, that is, the section corresponding to the projection $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_{\mathbb{P}^2} \to 0$. Let $C \subset \mathbb{P}^2$ be an elliptic curve. We put $Y = \pi^{-1}(C)$, $D_1 = F_1|_Y$, and $D_2 = F_2|_Y$. Let $f : Y \to X$ be the Stein factorization of $\Phi|_Y : Y \to \Phi(Y)$. Then the exceptional locus of f is $E = D_1 \cap D_2$. We note that Y is smooth, $D_1 + D_2$ is a simple normal crossing divisor, and $E \simeq C$ is an elliptic curve. Let $g : Z \to Y$ be the blow-up along E. Then

$$K_Z + D_1' + D_2' + D = g^*(K_Y + D_1 + D_2),$$

where D_1' (resp. D_2') is the strict transform of D_1 (resp. D_2) and D is the exceptional divisor of g. Note that $D \simeq C \times \mathbb{P}^1$. Since

$$-D + (K_Z + D_1' + D_2' + D) - (K_Z + D_1' + D_2') = 0,$$

we obtain that $R^i f_*(g_*\mathcal{O}_Z(-D + K_Z + D_1' + D_2' + D)) = 0$ for any $i > 0$ by Theorem 2.47 or Theorem 3.38. We note that $f \circ g$ is an isomorphism

outside D. We consider the following short exact sequence

$$0 \to \mathcal{I}_E \to \mathcal{O}_X \to \mathcal{O}_E \to 0,$$

where $\mathcal{I}_E$ is the defining ideal sheaf of E. Since $\mathcal{I}_E = g_*\mathcal{O}_Z(-D)$, we obtain that

$$\begin{aligned} 0 \to f_*(\mathcal{I}_E \otimes \mathcal{O}_Y(K_Y + D_1 + D_2)) \to f_*\mathcal{O}_Y(K_Y + D_1 + D_2) \\ \to f_*\mathcal{O}_E(K_Y + D_1 + D_2) \to 0 \end{aligned}$$

by $R^1f_*(\mathcal{I}_E \otimes \mathcal{O}_Y(K_Y + D_1 + D_2)) = 0$. By adjunction, $\mathcal{O}_E(K_Y + D_1 + D_2) \simeq \mathcal{O}_E$. Therefore, $\mathcal{O}_Y(K_Y + D_1 + D_2)$ is f-free. In particular, $K_Y + D_1 + D_2 = f^*(K_X + B_1 + B_2)$, where $B_1 = f_*D_1$ and $B_2 = f_*D_2$. Thus, $-D - (K_Z + D_1' + D_2') \sim_{f,\mathbb{Q}} 0$. So, we have $R^if_*\mathcal{I}_E = R^if_*(g_*\mathcal{O}_Z(-D)) = 0$ for any $i > 0$ by Theorem 2.47 or Theorem 3.38. This implies that $R^if_*\mathcal{O}_Y \simeq R^if_*\mathcal{O}_E$ for every $i > 0$. Thus, $R^1f_*\mathcal{O}_Y \simeq \mathbb{C}(P)$, where $P = f(E)$. We consider the following spectral sequence

$$E^{pq} = H^p(X, R^qf_*\mathcal{O}_Y \otimes \mathcal{O}_X(-mA)) \Rightarrow H^{p+q}(Y, \mathcal{O}_Y(-mA)),$$

where A is an ample Cartier divisor on X and m is any positive integer. Since $H^1(Y, \mathcal{O}_Y(-mf^*A)) = H^2(Y, \mathcal{O}_Y(-mf^*A)) = 0$ by the Kawamata–Viehweg vanishing theorem, we have

$$H^0(X, R^1f_*\mathcal{O}_Y \otimes \mathcal{O}_X(-mA)) \simeq H^2(X, \mathcal{O}_X(-mA)).$$

If we assume that X is Cohen–Macaulay, then we have $H^2(X, \mathcal{O}_X(-mA)) = 0$ for $m \gg 0$ by the Serre duality and the Serre vanishing theorem. On the other hand, $H^0(X, R^1f_*\mathcal{O}_Y \otimes \mathcal{O}_X(-mA)) \simeq \mathbb{C}(P)$ because $R^1f_*\mathcal{O}_Y \simeq \mathbb{C}(P)$. It is a contradiction. Thus, X is not Cohen–Macaulay. In particular, $(X, B_1 + B_2)$ is lc but not dlt. We note that $\mathrm{Exc}(f) = E$ is not a divisor on Y. See Definition 11.7.

Let us recall that $\Phi : V \to W$ is a flipping contraction. Let $\Phi^+ : V^+ \to W$ be the flip of Φ. We can check that $V^+ = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ and the flipped curve $E^+ \simeq \mathbb{P}^1$ is the negative section of $\pi^+ : V^+ \to \mathbb{P}^1$, that is, the section corresponding to the projection $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1} \to 0$. Let Y^+ be the strict transform of Y on V^+. Then Y^+ is Gorenstein, lc along $E^+ \subset Y^+$, and smooth outside E^+. Let D_1^+ (resp. D_2^+) be the strict transform of D_1 (resp. D_2) on Y^+. If we take a Cartier divisor B on Y suitably, then $(Y, D_1 + D_2) \dashrightarrow (Y^+, D_1^+ + D_2^+)$ is the B-flop of $f : Y \to X$. We note that $(Y, D_1 + D_2)$ is dlt. However, $(Y^+, D_1^+ + D_2^+)$ is lc but not dlt.

4.2.1 Appendix: Rational singularities

In this subsection, we give a proof to the following well-known theorem again (see Theorem 4.9).

z-rational **Theorem 4.17.** *Let (X, D) be a dlt pair. Then X has only rational singularities.*

Our proof is a combination of the proofs in [KM, Theorem 5.22] and [Ko4, Section 11]. We need no difficult duality theorems. The arguments here will be used in Section 4.3. First, let us recall the definition of the rational singularities.

Definition 4.18 (Rational singularities). A variety X has *rational singularities* if there is a resolution $f : Y \to X$ such that $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$ and $R^i f_*\mathcal{O}_Y = 0$ for all $i > 0$.

Next, we give a dual form of the Grauert–Riemenschneider vanishing theorem.

lem-gr **Lemma 4.19.** *Let $f : Y \to X$ be a proper birational morphism from a smooth variety Y to a variety X. Let $x \in X$ be a closed point. We put $F = f^{-1}(x)$. Then we have*

$$H^i_F(Y, \mathcal{O}_Y) = 0$$

for any $i < n = \dim X$.

Proof. We take a proper birational morphism $g : Z \to Y$ from a smooth variety Z such that $f \circ g$ is projective. We consider the following spectral sequence

$$E_2^{pq} = H^p_F(Y, R^q g_*\mathcal{O}_Z) \Rightarrow H^{p+q}_E(Z, \mathcal{O}_Z),$$

where $E = g^{-1}(F) = (f \circ g)^{-1}(x)$. Since $R^q g_*\mathcal{O}_Z = 0$ for $q > 0$ and $g_*\mathcal{O}_Z \simeq \mathcal{O}_Y$, we have $H^p_F(Y, \mathcal{O}_Y) \simeq H^p_E(Z, \mathcal{O}_Z)$ for any p. Therefore, we can replace Y with Z and assume that $f : Y \to X$ is projective. Without loss of generality, we can assume that X is affine. Then we compactify X and assume that X and Y are projective. It is well known that

$$H^i_F(Y, \mathcal{O}_Y) \simeq \varinjlim_m \mathrm{Ext}^i(\mathcal{O}_{mF}, \mathcal{O}_Y)$$

(see [G, Theorem 2.8]) and that

$$\mathrm{Hom}(\mathrm{Ext}^i(\mathcal{O}_{mF}, \mathcal{O}_Y), \mathbb{C}) \simeq H^{n-i}(Y, \mathcal{O}_{mF} \otimes \omega_Y)$$

by duality on a smooth projective variety Y (see [H2, Theorem 7.6 (a)]). Therefore,

$$\begin{aligned}\mathrm{Hom}(H^i_F(Y, \mathcal{O}_Y), \mathbb{C}) &\simeq \mathrm{Hom}(\varinjlim_m \mathrm{Ext}^i(\mathcal{O}_{mF}, \mathcal{O}_Y), \mathbb{C}) \\ &\simeq \varprojlim_m H^{n-i}(Y, \mathcal{O}_{mF} \otimes \omega_Y) \\ &\simeq (R^{n-i} f_* \omega_Y)^\wedge_x\end{aligned}$$

by the theorem on formal functions (see [H2, Theorem 11.1]), where $(R^{n-i} f_* \omega_Y)^\wedge_x$ is the completion of $R^{n-i} f_* \omega_Y$ at $x \in X$. On the other hand, $R^{n-i} f_* \omega_Y = 0$ for $i < n$ by the Grauert–Riemenschneider vanishing theorem. Thus, $H^i_F(Y, \mathcal{O}_Y) = 0$ for $i < n$. □

lem-gr-ho **Remark 4.20.** Lemma 4.19 holds true even when Y has rational singularities. It is because $R^q g_* \mathcal{O}_Z = 0$ for $q > 0$ and $g_* \mathcal{O}_Z \simeq \mathcal{O}_Y$ holds in the proof of Lemma 4.19.

Let us go to the proof of Theorem 4.17.

Proof of Theorem 4.17. Without loss of generality, we can assume that X is affine. Moreover, by taking generic hyperplane sections of X, we can also assume that X has only rational singularities outside a closed point $x \in X$. By the definition of dlt, we can take a resolution $f : Y \to X$ such that $\mathrm{Exc}(f)$ and $\mathrm{Exc}(f) \cup \mathrm{Supp} f^{-1}_* D$ are both simple normal crossing divisors on Y, $K_Y + f^{-1}_* D = f^*(K_X + D) + E$ with $\ulcorner E \urcorner \geq 0$, and that f is projective. Moreover, we can make f an isomorphism over the generic point of any lc center of (X, D). Therefore, by Lemma 4.10, we can check that $R^i f_* \mathcal{O}_Y(\ulcorner E \urcorner) = 0$ for any $i > 0$. See also the proof of Theorem 4.9. We note that $f_* \mathcal{O}_Y(\ulcorner E \urcorner) \simeq \mathcal{O}_X$ since $\ulcorner E \urcorner$ is effective and f-exceptional. For any $i > 0$, by the above assumption, $R^i f_* \mathcal{O}_Y$ is supported at a point $x \in X$ if it ever has a non-empty support at all. We put $F = f^{-1}(x)$. Then we have a spectral sequence

$$E_2^{ij} = H^i_x(X, R^j f_* \mathcal{O}_Y(\ulcorner E \urcorner)) \Rightarrow H^{i+j}_F(Y, \mathcal{O}_Y(\ulcorner E \urcorner)).$$

By the above vanishing result, we have

$$H^i_x(X, \mathcal{O}_X) \simeq H^i_F(Y, \mathcal{O}_Y(\ulcorner E \urcorner))$$

for every $i \geq 0$. We obtain a commutative diagram

$$\begin{array}{ccc} H^i_F(Y, \mathcal{O}_Y) & \longrightarrow & H^i_F(Y, \mathcal{O}_Y(\ulcorner E \urcorner)) \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\beta} \\ H^i_x(X, \mathcal{O}_X) & = & H^i_x(X, \mathcal{O}_X). \end{array}$$

We have already checked that β is an isomorphism for every i and that $H^i_F(Y, \mathcal{O}_Y) = 0$ for $i < n$ (see Lemma 4.19). Therefore, $H^i_x(X, \mathcal{O}_X) = 0$ for any $i < n = \dim X$. Thus, X is Cohen–Macaulay. For $i = n$, we obtain that

$$\alpha : H^n_x(X, \mathcal{O}_X) \to H^n_F(Y, \mathcal{O}_Y)$$

is injective. We consider the following spectral sequence

$$E_2^{ij} = H^i_x(X, R^j f_* \mathcal{O}_Y) \Rightarrow H^{i+j}_F(Y, \mathcal{O}_Y).$$

We note that $H^i_x(X, R^j f_* \mathcal{O}_Y) = 0$ for any $i > 0$ and $j > 0$ since $\mathrm{Supp} R^j f_* \mathcal{O}_Y \subset \{x\}$ for $j > 0$. On the other hand, $E_2^{i0} = H^i_x(X, \mathcal{O}_X) = 0$ for any $i < n$. Therefore, $H^0_x(X, R^j f_* \mathcal{O}_Y) \simeq H^j_x(X, \mathcal{O}_X) = 0$ for all $j \leq n-2$. Thus, $R^j f_* \mathcal{O}_Y = 0$ for $1 \leq j \leq n-2$. Since $H^{n-1}_x(X, \mathcal{O}_X) = 0$, we obtain that

$$0 \to H^0_x(X, R^{n-1} f_* \mathcal{O}_Y) \to H^n_x(X, \mathcal{O}_X) \xrightarrow{\alpha} H^n_F(Y, \mathcal{O}_Y) \to 0$$

is exact. We have already checked that α is injective. So, we obtain that $H^0_x(X, R^{n-1} f_* \mathcal{O}_Y) = 0$. This means that $R^{n-1} f_* \mathcal{O}_Y = 0$. Thus, we have $R^i f_* \mathcal{O}_Y = 0$ for any $i > 0$. We complete the proof. □

4.3 Alexeev's criterion for S_3 condition

sec-alex

In this section, we explain Alexeev's criterion for Serre's S_3 condition (see Theorem 4.21). It is a clever application of Theorem 2.39 (i). In general, log canonical singularities are not Cohen–Macaulay. So, the results in this section will be useful for the study of lc pairs.

alex-cri **Theorem 4.21 (cf. [Al, Lemma 3.2]).** *Let (X, B) be an lc pair with $\dim X = n \geq 3$ and let $P \in X$ be a scheme theoretic point such that $\dim \overline{\{P\}} \leq n-3$. Assume that $\overline{\{P\}}$ is not an lc center of (X, B). Then the local ring $\mathcal{O}_{X,P}$ satisfies Serre's S_3 condition.*

We slightly changed the original formulation. The following proof is essentially the same as Alexeev's. We use local cohomologies to calculate depths.

Proof. We note that $\mathcal{O}_{X,P}$ satisfies Serre's S_2 condition because X is normal. Since the assertion is local, we can assume that X is affine. Let $f : Y \to X$ be a resolution of X such that $\mathrm{Exc}(f) \cup \mathrm{Supp} f_*^{-1}B$ is a simple normal crossing divisor on Y. Then we can write

$$K_Y + B_Y = f^*(K_X + B)$$

such that $\mathrm{Supp} B_Y$ is a simple normal crossing divisor on Y. We put $A = \ulcorner -(B_Y^{<1}) \urcorner \geq 0$. Then we obtain

$$A = K_Y + B_Y^{=1} + \{B_Y\} - f^*(K_X + B).$$

Therefore, by Theorem 2.39 (i), the support of every non-zero local section of the sheaf $R^1 f_* \mathcal{O}_Y(A)$ contains some lc centers of (X, B). Thus, P is not an associated point of $R^1 f_* \mathcal{O}_Y(A)$.

We put $X_P = \mathrm{Spec} \mathcal{O}_{X,P}$ and $Y_P = Y \times_X X_P$. Then P is a closed point of X_P and it is sufficient to prove that $H^2_P(X_P, \mathcal{O}_{X_P}) = 0$. We put $F = f^{-1}(P)$, where $f : Y_P \to X_P$. Then we have the following vanishing theorem. It is nothing but Lemma 4.19 when P is a closed point of X.

lem-gr2 **Lemma 4.22 (cf. Lemma 4.19).** *We have $H^i_F(Y_P, \mathcal{O}_{Y_P}) = 0$ for $i < n - \dim \overline{\{P\}}$.*

Proof of Lemma 4.22. Let I denote an injective hull of $\mathcal{O}_{X_P}/m_P$ as an $\mathcal{O}_{X_P}$-module, where m_P is the maximal ideal corresponding to P. We have

$$\begin{aligned} R\Gamma_F \mathcal{O}_{Y_P} &\simeq R\Gamma_P(Rf_* \mathcal{O}_{Y_P}) \\ &\simeq \mathrm{Hom}(R\mathcal{H}om(Rf_* \mathcal{O}_{Y_P}, \omega^{\bullet}_{X_P}), I) \\ &\simeq \mathrm{Hom}(Rf_* \mathcal{O}_Y(K_Y) \otimes \mathcal{O}_{X_P}[n-m], I), \end{aligned}$$

where $m = \dim \overline{\{P\}}$, by the local duality theorem ([H1, Chapter V, Theorem 6.2]) and the Grothendieck duality theorem ([H1, Chapter VII, Theorem

3.3]). We note the shift that normalize the dualizing complex $\omega^{\bullet}_{X_P}$. Therefore, we obtain $H^i_F(Y_P, \mathcal{O}_{Y_P}) = 0$ for $i < n - m$ because $R^j f_* \mathcal{O}_Y(K_Y) = 0$ for any $j > 0$ by the Grauert–Riemenschneider vanishing theorem. □

Let us go back to the proof of the theorem. We consider the following spectral sequences

$$E_2^{pq} = H^p_P(X_P, R^q f_* \mathcal{O}_{Y_P}(A)) \Rightarrow H^{p+q}_F(Y_P, \mathcal{O}_{Y_P}(A)),$$

and

$$'E_2^{pq} = H^p_P(X_P, R^q f_* \mathcal{O}_{Y_P}) \Rightarrow H^{p+q}_F(Y_P, \mathcal{O}_{Y_P}).$$

By the above spectral sequences, we have the next commutative diagram.

$$\begin{array}{ccc} H^2_F(Y_P, \mathcal{O}_{Y_P}) & \longrightarrow & H^2_F(Y_P, \mathcal{O}_{Y_P}(A)) \\ \uparrow & & \uparrow \phi \\ H^2_P(X_P, f_* \mathcal{O}_{Y_P}) & & H^2_P(X_P, f_* \mathcal{O}_{Y_P}(A)) \\ \| & & \| \\ H^2_P\big(X_P, \mathcal{O}_{X_P}\big) & = & H^2_P\big(X_P, \mathcal{O}_{X_P}\big) \end{array}$$

Since P is not an associated point of $R^1 f_* \mathcal{O}_Y(A)$, we have

$$E_2^{0,1} = H^0_P(X_P, R^1 f_* \mathcal{O}_{Y_P}(A)) = 0.$$

By the edge sequence

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to E_2^{2,0} \xrightarrow{\phi} E^2 \to \cdots,$$

we know that $\phi : E_2^{2,0} \to E^2$ is injective. Therefore, $H^2_P(X_P, \mathcal{O}_{X_P}) \to H^2_F(Y_P, \mathcal{O}_{Y_P})$ is injective by the above big commutative diagram. Thus, we obtain $H^2_P(X_P, \mathcal{O}_{X_P}) = 0$ since $H^2_F(Y_P, \mathcal{O}_{Y_P}) = 0$ by Lemma 4.22. □

comp-prob **Remark 4.23.** The original argument in the proof of [A1, Lemma 3.2] has some compactification problems when X is not projective. Our proof does not need any compactifications of X.

As an easy application of Theorem 4.21, we have the following result. It is [A1, Theorem 3.4].

alex-cri2 **Theorem 4.24 (cf. [alex] [Al, Theorem 3.4]).** *Let (X, B) be an lc pair and let D be an effective Cartier divisor. Assume that the pair $(X, B+\varepsilon D)$ is lc for some $\varepsilon > 0$. Then D is S_2.*

Proof. Without loss of generality, we can assume that $\dim X = n \geq 3$. Let $P \in D \subset X$ be a scheme theoretic point such that $\dim \overline{\{P\}} \leq n-3$. We localize X at P and assume that $X = \mathrm{Spec}\mathcal{O}_{X,P}$. By the assumption, $\overline{\{P\}}$ is not an lc center of (X, B). By Theorem [alex-cri] 4.21, we obtain that $H^i_P(X, \mathcal{O}_X) = 0$ for $i < 3$. Therefore, $H^i_P(D, \mathcal{O}_D) = 0$ for $i < 2$ by the long exact sequence

$$\cdots \to H^i_P(X, \mathcal{O}_X(-D)) \to H^i_P(X, \mathcal{O}_X) \to H^i_P(D, \mathcal{O}_D) \to \cdots .$$

We note that $H^i_P(X, \mathcal{O}_X(-D)) \simeq H^i_P(X, \mathcal{O}_X) = 0$ for $i < 3$. Thus, D satisfies Serre's S_2 condition. □

We give a supplement to adjunction (see Theorem [adj-th] 3.39 (i)). It may be useful for the study of limits of stable pairs (see [alex] [Al]).

tu-thm1 **Theorem 4.25 (Adjunction for Cartier divisors on lc pairs).** *Let (X, B) be an lc pair and let D be an effective Cartier divisor on X such that $(X, B+D)$ is log canonical. Let V be a union of lc centers of (X, B). We consider V as a reduced closed subscheme of X. We define a scheme structure on $V \cap D$ by the following short exact sequence*

$$0 \to \mathcal{O}_V(-D) \to \mathcal{O}_V \to \mathcal{O}_{V \cap D} \to 0.$$

Then, $\mathcal{O}_{V \cap D}$ is reduced and semi-normal.

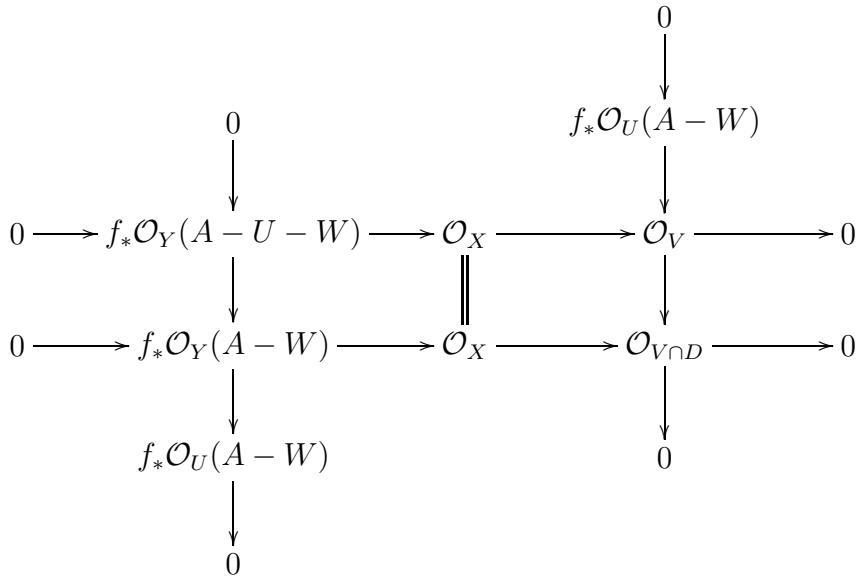
Proof. First, we note that $V \cap D$ is a union of lc centers of $(X, B+D)$ (see Theorem [a1] 3.46). Let $f : Y \to X$ be a resolution such that $\mathrm{Exc}(f) \cup \mathrm{Supp} f^{-1}_*(B+D)$ is a simple normal crossing divisor on Y. We can write

$$K_Y + B_Y = f^*(K_X + B + D)$$

such that $\mathrm{Supp} B_Y$ is a simple normal crossing divisor on Y. We take more blow-ups and can assume that $f^{-1}(V \cap D)$ and $f^{-1}(V)$ are simple normal crossing divisors. Then the union of all strata of $B^{=1}_Y$ mapped to $V \cap D$ (resp. V), which is denoted by W (resp. $U+W$), is a divisor on Y. We put $A = \ulcorner -(B^{<1}_Y) \urcorner \geq 0$ and consider the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_Y(A-U-W) & \longrightarrow & \mathcal{O}_Y(A) & \longrightarrow & \mathcal{O}_{U+W}(A) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_Y(A-W) & \longrightarrow & \mathcal{O}_Y(A) & \longrightarrow & \mathcal{O}_W(A) & \longrightarrow & 0
\end{array}$$

By applying f_*, we obtain the next big diagram by Theorem 2.39 (i) and Theorem 3.39 (i).



A key point is that the connecting homomorphism

$$f_*\mathcal{O}_U(A-W) \to R^1 f_*\mathcal{O}_Y(A-U-W)$$

is a zero map by Theorem 2.39 (i). We note that $\mathcal{O}_V$ and $\mathcal{O}_{V\cap D}$ in the above diagram are the structure sheaves of qlc pairs V and $V\cap D$ induced by $(X, B+D)$. In particular, $\mathcal{O}_V \simeq f_*\mathcal{O}_{U+W}$ and $\mathcal{O}_{V\cap D} \simeq f_*\mathcal{O}_W$. So, $\mathcal{O}_V$ and $\mathcal{O}_{V\cap D}$ are reduced and semi-normal since W and $U+W$ are simple normal crossing divisors on Y.

Therefore, to prove this theorem, it is sufficient to see that $f_*\mathcal{O}_U(A-W) \simeq \mathcal{O}_V(-D)$. We can write

$$A = K_Y + B_Y^{=1} + \{B_Y\} - f^*(K_X + B + D)$$

and

$$f^*D = W + E + f_*^{-1}D,$$

where E is an effective f-exceptional divisor. We note that $f_*^{-1}D \cap U = \emptyset$. Since $A - W = A - f^*D + E + f_*^{-1}D$, it is enough to see that $f_*\mathcal{O}_U(A + E + f_*^{-1}D) \simeq f_*\mathcal{O}_U(A+E) \simeq \mathcal{O}_V$. We consider the following short exact sequence

$$0 \to \mathcal{O}_Y(A+E-U) \to \mathcal{O}_Y(A+E) \to \mathcal{O}_U(A+E) \to 0.$$

Note that

$$A + E - U = K_Y + B_Y^{=1} - f_*^{-1}D - U - W + \{B_Y\} - f^*(K_X + B).$$

Thus, the connecting homomorphism $f_*\mathcal{O}_U(A+E) \to R^1 f_*\mathcal{O}_Y(A+E-U)$ is a zero map by Theorem 2.39 (i). Therefore, we obtain that

$$0 \to f_*\mathcal{O}_Y(A+E-U) \to \mathcal{O}_X \to f_*\mathcal{O}_U(A+E) \to 0.$$

So, we have $f_*\mathcal{O}_U(A+E) \simeq \mathcal{O}_V$. We finish the proof of this theorem. □

The next corollary is one of the main results in [AI]. The original proof in [AI] depends on the S_2-fication. Our proof uses adjunction (see Theorem 4.25). As a result, we obtain the semi-normality of $\llcorner B \lrcorner \cap D$.

Corollary 4.26 (cf. [AI, Theorem 4.1]). *Let (X, B) be an lc pair and let D be an effective Cartier divisor on D such that $(X, B+D)$ is lc. Then D is S_2 and the scheme $\llcorner B \lrcorner \cap D$ is reduced and semi-normal.*

Proof. By Theorem 4.24, D satisfies Serre's S_2 condition. By Theorem 4.25, $\llcorner B \lrcorner \cap D$ is reduced and semi-normal. □

The following proposition may be useful. So, we contain it here. It is [AI, Lemma 3.1] with slight modifications as Theorem 4.21.

w-alex **Proposition 4.27 (cf. [AI, Lemma 3.1]).** *Let X be a normal variety with $\dim X = n \geq 3$ and let $f : Y \to X$ be a resolution of singularities. Let $P \in X$ be a scheme theoretic point such that $\dim \overline{\{P\}} \leq n-3$. Then the local ring $\mathcal{O}_{X,P}$ is S_3 if and only if P is not an associated point of $R^1 f_*\mathcal{O}_Y$.*

Proof. We put $X_P = \mathrm{Spec}\mathcal{O}_{X,P}$, $Y_P = Y \times_X X_P$, and $F = f^{-1}(P)$, where $f : Y_P \to X_P$. We consider the following spectral sequence

$$E_2^{ij} = H_P^i(X, R^j f_*\mathcal{O}_{Y_P}) \Rightarrow H_F^{i+j}(Y_P, \mathcal{O}_{Y_P}).$$

Since $H_F^1(Y_P, \mathcal{O}_{Y_P}) = H_F^2(Y_P, \mathcal{O}_{Y_P}) = 0$ by Lemma 4.22, we have an isomorphism $H_P^0(X_P, R^1 f_*\mathcal{O}_{Y_P}) \simeq H_P^2(X_P, \mathcal{O}_{X_P})$. Therefore, the depth of $\mathcal{O}_{X,P}$ is ≥ 3 if and only if $H_P^2(X_P, \mathcal{O}_{X_P}) = H_P^0(X_P, R^1 f_*\mathcal{O}_{Y_P}) = 0$. It is equivalent to the condition that P is not an associated point of $R^1 f_*\mathcal{O}_Y$. □

4.28 (Supplements). Here, we give a slight generalization of [AI, Theorem 3.5]. We can prove it by a similar method to the proof of Theorem 4.21.

ale-18 **Theorem 4.29 (cf. [alex][AI, Theorem 3.5]).** *Let (X, B) be an lc pair and D an effective Cartier divisor on X such that $(X, B+\varepsilon D)$ is lc for some $\varepsilon > 0$. Let V be a union of some lc centers of (X, B). We consider V as a reduced closed subscheme of X. We can define a scheme structure on $V \cap D$ by the following exact sequence*

$$0 \to \mathcal{O}_V(-D) \to \mathcal{O}_V \to \mathcal{O}_{V\cap D} \to 0.$$

Then the scheme $V \cap D$ satisfies Serre's S_1 condition. In particular, $\llcorner B \lrcorner \cap D$ has no embedded point.

Proof. Without loss of generality, we can assume that X is affine. We take a resolution $f : Y \to X$ such that $\mathrm{Exc}(f) \cup \mathrm{Supp} f_*^{-1}B$ is a simple normal crossing divisor on Y. Then we can write

$$K_Y + B_Y = f^*(K_X + B)$$

such that $\mathrm{Supp} B_Y$ is a simple normal crossing divisor on Y. We take more blow-ups and can assume that the union of all strata of $B_Y^{=1}$ mapped to V, which is denoted by W, is a divisor on Y. Moreover, for any lc center C of (X, B) contained in V, we can assume that $f^{-1}(C)$ is a divisor on Y. We consider the following short exact sequence

$$0 \to \mathcal{O}_Y(A - W) \to \mathcal{O}_Y(A) \to \mathcal{O}_W(A) \to 0,$$

where $A = \ulcorner -(B_Y^{<1}) \urcorner \geq 0$. By taking higher direct images, we obtain

$$0 \to f_*\mathcal{O}_Y(A - W) \to \mathcal{O}_X \to f_*\mathcal{O}_W(A) \to R^1 f_*\mathcal{O}_Y(A - W) \to \cdots .$$

By Theorem [8]2.39 (i), we have that $f_*\mathcal{O}_W(A) \to R^1 f_*\mathcal{O}_Y(A - W)$ is a zero map, $f_*\mathcal{O}_W(A) \simeq \mathcal{O}_V$, and $f_*\mathcal{O}_Y(A - W) \simeq \mathcal{I}_V$, the defining ideal sheaf of V on X. We note that $f_*\mathcal{O}_W \simeq \mathcal{O}_V$. In particular, $\mathcal{O}_V$ is reduced and semi-normal. For the details, see Theorem [adj-th]3.39 (i).

Let $P \in V \cap D$ be a scheme theoretic point such that the height of P in $\mathcal{O}_{V\cap D}$ is ≥ 1. We can assume that $\dim V \geq 2$ around P. Otherwise, the theorem is trivial. We put $V_P = \mathrm{Spec}\mathcal{O}_{V,P}$, $W_P = W \times_V V_P$, and $F = f^{-1}(P)$, where $f : W_P \to V_P$. We denote the pull back of D on V_P by D for simplicity. To check this theorem, it is sufficient to see that $H^0_P(V_P \cap D, \mathcal{O}_{V_P\cap D}) = 0$. First, we note that $H^0_P(V_P, \mathcal{O}_{V_P}) = H^0_F(W_P, \mathcal{O}_{W_P}) = 0$ by $f_*\mathcal{O}_W \simeq \mathcal{O}_V$.

Next, as in the proof of Lemma 4.22, we have

$$\begin{aligned} R\Gamma_F\mathcal{O}_{W_P} &\simeq R\Gamma_P(Rf_*\mathcal{O}_{W_P}) \\ &\simeq \mathrm{Hom}(R\mathcal{H}om(Rf_*\mathcal{O}_{W_P}, \omega^{\bullet}_{V_P}), I) \\ &\simeq \mathrm{Hom}(Rf_*\mathcal{O}_W(K_W) \otimes \mathcal{O}_{V_P}[n-1-m], I), \end{aligned}$$

where $n = \dim X$, $m = \dim \overline{\{P\}}$, and I is an injective hull of $\mathcal{O}_{V_P}/m_P$ as an $\mathcal{O}_{V_P}$-module such that m_P is the maximal ideal corresponding to P. Once we obtain $R^{n-m-2}f_*\mathcal{O}_W(K_W) \otimes \mathcal{O}_{V_P} = 0$, then $H^1_F(W_P, \mathcal{O}_{W_P}) = 0$. It implies that $H^1_P(V_P, \mathcal{O}_{V_P}) = 0$ since $H^1_P(V_P, \mathcal{O}_{V_P}) \subset H^1_F(W_P, \mathcal{O}_{W_P})$. By the long exact sequence

$$\begin{aligned} \cdots \to H^0_P(V_P, \mathcal{O}_{V_P}) \to H^0_P(V_P \cap D, \mathcal{O}_{V_P \cap D}) \\ \to H^1_P(V_P, \mathcal{O}_{V_P}(-D)) \to \cdots, \end{aligned}$$

we obtain $H^0_P(V_P \cap D, \mathcal{O}_{V_P \cap D}) = 0$. It is because $H^0_P(V_P, \mathcal{O}_{V_P}) = 0$ and $H^1_P(V_P, \mathcal{O}_{V_P}(-D)) \simeq H^1_P(V_P, \mathcal{O}_{V_P}) = 0$. So, it is sufficient to see that $R^{n-m-2}f_*\mathcal{O}_W(K_W) \otimes \mathcal{O}_{V_P} = 0$.

To check the vanishing of $R^{n-m-2}f_*\mathcal{O}_W(K_W) \otimes \mathcal{O}_{V_P}$, by taking general hyperplane cuts m times, we can assume that $m = 0$ and $P \in X$ is a closed point. We note that the dimension of any irreducible component of V passing through P is ≥ 2 since P is not an lc center of (X, B) (see Theorem 3.46).

On the other hand, we can write $W = U_1 + U_2$ such that U_2 is the union of all the irreducible components of W whose images by f have dimensions ≥ 2 and $U_1 = W - U_2$. We note that the dimension of the image of any stratum of U_2 by f is ≥ 2 by the construction of $f : Y \to X$. We consider the following exact sequence

$$\begin{aligned} \cdots \to R^{n-2}f_*\mathcal{O}_{U_2}(K_{U_2}) \to R^{n-2}f_*\mathcal{O}_W(K_W) \\ \to R^{n-2}f_*\mathcal{O}_{U_1}(K_{U_1} + U_2|_{U_1}) \to R^{n-1}f_*\mathcal{O}_{U_2}(K_{U_2}) \to \cdots. \end{aligned}$$

We have $R^{n-2}f_*\mathcal{O}_{U_2}(K_{U_2}) = R^{n-1}f_*\mathcal{O}_{U_2}(K_{U_2}) = 0$ around P since the dimension of general fibers of $f : U_2 \to f(U_2)$ is $\leq n-3$. Thus, we obtain $R^{n-2}f_*\mathcal{O}_W(K_W) \simeq R^{n-2}f_*\mathcal{O}_{U_1}(K_{U_1} + U_2|_{U_1})$ around P. Therefore, the support of $R^{n-2}f_*\mathcal{O}_W(K_W)$ around P is contained in one-dimensional lc centers of (X, B) in V and $R^{n-2}f_*\mathcal{O}_W(K_W)$ has no zero-dimensional associated point around P by Theorem 2.39 (i). By taking a general hyperplane cut of X again, we have the vanishing of $R^{n-2}f_*\mathcal{O}_W(K_W)$ around P by Lemma 4.30 below. So, we finish the proof. □

We used the following lemma in the proof of Theorem 4.29.

lem-ele-vani **Lemma 4.30.** *Let (Z, Δ) be an n-dimensional lc pair and let $x \in Z$ be a closed point such that x is an lc center of (Z, Δ). Let V be a union of some lc centers of (X, B) such that $\dim V > 0$, $x \in V$, and x is not isolated in V. Let $f : Y \to Z$ be a resolution such that $f^{-1}(x)$ and $f^{-1}(V)$ are divisors on Y and that $\mathrm{Exc}(f) \cup \mathrm{Supp} f_*^{-1}\Delta$ is a simple normal crossing divisor on Y. We can write*

$$K_Y + B_Y = f^*(K_Z + \Delta)$$

such that $\mathrm{Supp} B_Y$ is a simple normal crossing divisor on Y. Let W be the union of all the irreducible components of $B_Y^{=1}$ mapped to V. Then $R^{n-1}f_*\mathcal{O}_W(K_W) = 0$ around x.

Proof. We can write $W = W_1 + W_2$, where W_2 is the union of all the irreducible components of W mapped to x by f and $W_1 = W - W_2$. We consider the following short exact sequence

$$0 \to \mathcal{O}_Y(K_Y) \to \mathcal{O}_Y(K_Y + W) \to \mathcal{O}_W(K_W) \to 0.$$

By the Grauert–Riemenschneider vanishing theorem, we obtain that

$$R^{n-1}f_*\mathcal{O}_Y(K_Y + W) \simeq R^{n-1}f_*\mathcal{O}_W(K_W).$$

Next, we consider the short exact sequence

$$0 \to \mathcal{O}_Y(K_Y + W_1) \to \mathcal{O}_Y(K_Y + W) \to \mathcal{O}_{W_2}(K_{W_2} + W_1|_{W_2}) \to 0.$$

Around x, the image of any irreducible component of W_1 by f is positive dimensional. Therefore, $R^{n-1}f_*\mathcal{O}_Y(K_Y + W_1) = 0$ near x. It can be checked by the induction on the number of irreducible components using the following exact sequence

$$\begin{aligned}\cdots \to R^{n-1}f_*\mathcal{O}_Y(K_Y + W_1 - S) &\to R^{n-1}f_*\mathcal{O}_Y(K_Y + W_1) \\ &\to R^{n-1}f_*\mathcal{O}_S(K_S + (W_1 - S)|_S) \to \cdots,\end{aligned}$$

where S is an irreducible component of W_1. On the other hand, we have

$$R^{n-1}f_*\mathcal{O}_{W_2}(K_{W_2} + W_1|_{W_2}) \simeq H^{n-1}(W_2, \mathcal{O}_{W_2}(K_{W_2} + W_1|_{W_2}))$$

and $H^{n-1}(W_2, \mathcal{O}_{W_2}(K_{W_2} + W_1|_{W_2}))$ is dual to $H^0(W_2, \mathcal{O}_{W_2}(-W_1|_{W_2}))$. Note that $f_*\mathcal{O}_{W_2} \simeq \mathcal{O}_x$ and $f_*\mathcal{O}_W \simeq \mathcal{O}_V$ by the usual argument on adjunction (see Theorem 3.39 (1)). Since W_2 and $W = W_1 + W_2$ are connected over x, $H^0(W_2, \mathcal{O}_{W_2}(-W_1|_{W_2})) = 0$. We note that $W_1|_{W_2} \neq 0$ since x is not isolated in V. This means that $R^{n-1}f_*\mathcal{O}_W(K_W) = 0$ around x by the above arguments. □

4.3.1 Appendix: Cone singularities

431sss

In this subsection, we collect some basic facts on cone singularities for the reader's convenience. First, we give two lemmas which can be proved by the same method as in Section 4.3. We think that these lemmas will be useful for the study of log canonical singularities.

w-lem1 **Lemma 4.31.** *Let X be an n-dimensional normal variety and let $f : Y \to X$ be a resolution of singularities. Assume that $R^i f_* \mathcal{O}_Y = 0$ for $1 \leq i \leq n-2$. Then X is Cohen–Macaulay.*

Proof. We can assume that $n \geq 3$. Since $\mathrm{Supp} R^{n-1} f_* \mathcal{O}_Y$ is zero-dimensional, we can assume that there exists a closed point $x \in X$ such that X has only rational singularities outside x by shrinking X around x. Therefore, it is sufficient to see that the depth of $\mathcal{O}_{X,x}$ is $\geq n = \dim X$. We consider the following spectral sequence

$$E_2^{ij} = H^i_x(X, R^j f_* \mathcal{O}_Y) \Rightarrow H^{i+j}_F(Y, \mathcal{O}_Y),$$

where $F = f^{-1}(x)$. Then $H^i_x(X, \mathcal{O}_X) = E_2^{i0} \simeq E_\infty^{i0} = 0$ for $i \leq n-1$. It is because $H^i_F(Y, \mathcal{O}_Y) = 0$ for $i \leq n-1$ by Lemma 4.19. This means that the depth of $\mathcal{O}_{X,x}$ is $\geq n$. So, we have that X is Cohen–Macaulay. □

w-lem2 **Lemma 4.32.** *Let X be an n-dimensional normal variety and let $f : Y \to X$ be a resolution of singularities. Let $x \in X$ be a closed point. Assume that X is Cohen-Macaulay and that X has only rational singularities outside x. Then $R^i f_* \mathcal{O}_Y = 0$ for $1 \leq i \leq n-2$.*

Proof. We can assume that $n \geq 3$. By the assumption, $\mathrm{Supp} R^i f_* \mathcal{O}_Y \subset \{x\}$ for $1 \leq i \leq n-1$. We consider the following spectral sequence

$$E_2^{ij} = H^i_x(X, R^j f_* \mathcal{O}_Y) \Rightarrow H^{i+j}_F(Y, \mathcal{O}_Y),$$

where $F = f^{-1}(x)$. Then $H^0_x(X, R^j f_* \mathcal{O}_Y) = E_2^{0j} \simeq E_\infty^{0j} = 0$ for $j \leq n-2$ since $E_2^{ij} = 0$ for $i > 0$ and $j > 0$, $E_2^{i0} = 0$ for $i \leq n-1$, and $H^j_F(Y, \mathcal{O}_Y) = 0$ for $j < n$. Therefore, $R^i f_* \mathcal{O}_Y = 0$ for $1 \leq i \leq n-2$. □

We point out the following fact explicitly for the reader's convenience. It is [Ko4, 11.2 Theorem. (11.2.5)].

Lemma 4.33. *Let $f : Y \to X$ be a proper morphism, $x \in X$ a closed point, $F = f^{-1}(x)$ and G a sheaf on Y. If $\mathrm{Supp} R^j f_* G \subset \{x\}$ for $1 \leq i < k$ and $H^i_F(Y, G) = 0$ for $i \leq k$, then $R^j f_* G \simeq H^{j+1}_x(X, f_* G)$ for $j = 1, \cdots, k-1$.*

The assumptions in Lemma 4.32 are satisfied for n-dimensional isolated Cohen–Macaulay singularities. Therefore, we have the following corollary of Lemmas 4.31 and 4.32.

Corollary 4.34. *Let $x \in X$ be an n-dimensional normal isolated singularity. Then $x \in X$ is Cohen–Macaulay if and only if $R^i f_* \mathcal{O}_Y = 0$ for $1 \leq i \leq n-2$, where $f : Y \to X$ is a resolution of singularities.*

We note the following easy example.

Example 4.35. Let V be a cone over a smooth plane cubic curve and let $\varphi : W \to V$ be the blow-up at the vertex. Then W is smooth and $K_W = \varphi^* K_V - E$, where E is an elliptic curve. In particular, V is log canonical. Let C be a smooth curve. We put $Y = W \times C$, $X = V \times C$, and $f = \varphi \times \mathrm{id}_C : Y \to X$, where id_C is the identity map of C. By the construction, X is a log canonical threefold. We can check that X is Cohen–Macaulay by Theorem 4.21 or Proposition 4.27. We note that $R^1 f_* \mathcal{O}_Y \neq 0$ and that $R^1 f_* \mathcal{O}_Y$ has no zero-dimensional associated components. Therefore, the Cohen–Macaulayness of X does not necessarily imply the vanishing of $R^1 f_* \mathcal{O}_Y$.

Let us go to cone singularities (cf. [Ko4, 3.8 Example] and [Ko5, Exercises 70, 71]).

Lemma 4.36 (Projective normality). *Let $X \subset \mathbb{P}^N$ be a normal projective irreducible variety and $V \subset \mathbb{A}^{N+1}$ the cone over X. Then V is normal if and only if $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) \to H^0(X, \mathcal{O}_X(m))$ is surjective for any $m \geq 0$. In this case, $X \subset \mathbb{P}^N$ is said to be* projectively normal.

Proof. Without loss of generality, we can assume that $\dim X \geq 1$. Let $P \in V$ be the vertex of V. By the construction, we have $H^0_P(V, \mathcal{O}_V) = 0$. We consider the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\mathbb{A}^{N+1}, \mathcal{O}_{\mathbb{A}^{N+1}}) & \longrightarrow & H^0(\mathbb{A}^{N+1} \setminus P, \mathcal{O}_{\mathbb{A}^{N+1}}) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & H^0(V, \mathcal{O}_V) & \longrightarrow & H^0(V \setminus P, \mathcal{O}_V) & \longrightarrow & H^1_P(V, \mathcal{O}_V) & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

We note that $H^i(V, \mathcal{O}_V) = 0$ for any $i > 0$ since V is affine. By the above commutative diagram, it is easy to see that the following conditions are equivalent.

(a) V is normal.

(b) the depth of $\mathcal{O}_{V,P}$ is ≥ 2.

(c) $H^1_P(V, \mathcal{O}_V) = 0$.

(d) $H^0(\mathbb{A}^{N+1} \setminus P, \mathcal{O}_{\mathbb{A}^{N+1}}) \to H^0(V \setminus P, \mathcal{O}_V)$ is surjective.

The condition (d) is equivalent to the condition that $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) \to H^0(X, \mathcal{O}_X(m))$ is surjective for any $m \geq 0$. We note that

$$H^0(\mathbb{A}^{N+1} \setminus P, \mathcal{O}_{\mathbb{A}^{N+1}}) \simeq \bigoplus_{m \geq 0} H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))$$

and

$$H^0(V \setminus P, \mathcal{O}_V) \simeq \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(m)).$$

So, we finish the proof. □

The next lemma is more or less well known to the experts.

437lem **Lemma 4.37.** *Let $X \subset \mathbb{P}^N$ be a normal projective irreducible variety and $V \subset \mathbb{A}^{N+1}$ the cone over X. Assume that X is projectively normal and that X has only rational singularities. Then we have the following properties.*

(1) *V is Cohen–Macaulay if and only if $H^i(X, \mathcal{O}_X(m)) = 0$ for any $0 < i < \dim X$ and $m \geq 0$.*

(2) *V has only rational singularities if and only if $H^i(X, \mathcal{O}_X(m)) = 0$ for any $i > 0$ and $m \geq 0$.*

Proof. We put $n = \dim X$ and can assume $n \geq 1$. For (1), it is sufficient to prove that $H^i_P(V, \mathcal{O}_V) = 0$ for $2 \leq i \leq n$ if and only if $H^i(X, \mathcal{O}_X(m)) = 0$ for any $0 < i < n$ and $m \geq 0$ since V is normal, where $P \in V$ is the vertex of V. Let $f : W \to V$ be the blow-up at P and $E \simeq X$ the exceptional divisor of f. We note that W is the total space of $\mathcal{O}_X(-1)$ over $E \simeq X$ and that W has only rational singularities. Since V is affine, we obtain $H^i(V \setminus P, \mathcal{O}_V) \simeq H^{i+1}_P(V, \mathcal{O}_V)$ for any $i \geq 1$. Since W has only rational singularities, we have $H^i_E(W, \mathcal{O}_W) = 0$ for $i < n + 1$ (cf. Lemma 4.19 and Remark 4.20). Therefore,

$$H^i(V \setminus P, \mathcal{O}_V) \simeq H^i(W \setminus E, \mathcal{O}_W) \simeq H^i(W, \mathcal{O}_W)$$

for $i \leq n-1$. Thus,

$$H^i_P(V, \mathcal{O}_V) \simeq H^{i-1}(V \setminus P, \mathcal{O}_V) \simeq H^{i-1}(W, \mathcal{O}_W) \simeq \bigoplus_{m \geq 0} H^{i-1}(X, \mathcal{O}_X(m))$$

for $2 \leq i \leq n$. So, we obtain the desired equivalence.

For (2), we consider the following commutative diagram.

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & H^n(V \setminus P, \mathcal{O}_V) & \longrightarrow & H^{n+1}_P(V, \mathcal{O}_V) & \longrightarrow & 0 \\ & & & & \downarrow \simeq & & \downarrow \alpha & & \\ 0 & \longrightarrow & H^n(W, \mathcal{O}_W) & \longrightarrow & H^n(W \setminus E, \mathcal{O}_W) & \longrightarrow & H^{n+1}_E(W, \mathcal{O}_W) & & \end{array}$$

We note that V is Cohen–Macaulay if and only if $R^i f_* \mathcal{O}_W = 0$ for $1 \leq i \leq n-1$ (cf. Lemmas 4.31 and 4.32) since W has only rational singularities. From now on, we assume that V is Cohen–Macaulay. Then, V has only rational singularities if and only if $R^n f_* \mathcal{O}_W = 0$. By the same argument as in the proof of Theorem 4.17, the kernel of α is $H^0_P(V, R^n f_* \mathcal{O}_W)$. Thus, $R^n f_* \mathcal{O}_W = 0$ if and only if $H^n(W, \mathcal{O}_W) \simeq \bigoplus_{m \geq 0} H^n(X, \mathcal{O}_X(m)) = 0$ by the above commutative diagram. So, we obtain the statement (2). □

The following proposition is very useful when we construct some examples. We have already used it in this book.

438p **Proposition 4.38.** *Let $X \subset \mathbb{P}^N$ be a normal projective irreducible variety and $V \subset \mathbb{A}^{N+1}$ the cone over X. Assume that X is projectively normal. Let Δ be an effective $\mathbb{R}$-divisor on X and B the cone over Δ. Then, we have the following properties.*

(1) *$K_V + B$ is $\mathbb{R}$-Cartier if and only if $K_X + \Delta \sim_{\mathbb{R}} rH$ for some $r \in \mathbb{R}$, where $H \subset X$ is the hyperplane divisor on $X \subset \mathbb{P}^N$.*

(2) *If $K_X + \Delta \sim_{\mathbb{R}} rH$, then (V, B) is*

   (a) *terminal if and only if $r < -1$ and (X, Δ) is terminal,*

   (b) *canonical if and only if $r \leq -1$ and (X, Δ) is canonical,*

   (c) *klt if and only if $r < 0$ and (X, Δ) is klt, and*

   (d) *lc if and only if $r \leq 0$ and (X, Δ) is lc.*

Proof. Let $f : W \to V$ be the blow-up at the vertex $P \in V$ and $E \simeq X$ the exceptional divisor of f. If $K_V + B$ is $\mathbb{R}$-Cartier, then $K_W + f_*^{-1}B \sim_{\mathbb{R}} f^*(K_V + B) + aE$ for some $a \in \mathbb{R}$. By restricting it to E, we obtain that $K_X + \Delta \sim_{\mathbb{R}} -(a+1)H$. On the other hand, if $K_X + \Delta \sim_{\mathbb{R}} rH$, then $K_W + f_*^{-1}B \sim_{\mathbb{R}} -(r+1)E$. Therefore, $K_V + B \sim_{\mathbb{R}} 0$ on V. Thus, we have the statement (1). For (2), it is sufficient to note that

$$K_W + f_*^{-1}B = f^*(K_X + B) - (r+1)E$$

and that W is the total space of $\mathcal{O}_X(-1)$ over $E \simeq X$. □

4.4 Toric Polyhedron

to-sec

In this section, we freely use the basic notation of the toric geometry. See, for example, [Fulton, Fl].

Definition 4.39. For a subset Φ of a fan Δ, we say that Φ is *star closed* if $\sigma \in \Phi, \tau \in \Delta$ and $\sigma \prec \tau$ imply $\tau \in \Phi$.

Definition 4.40 (Toric Polyhedron). For a star closed subset Φ of a fan Δ, we denote by $Y = Y(\Phi)$ the subscheme $\bigcup_{\sigma \in \Phi} V(\sigma)$ of $X = X(\Delta)$, and we call it the *toric polyhedron associated to* Φ.

Let $X = X(\Delta)$ be a toric variety and let D be the complement of the big torus. Then the following property is well known.

Proposition 4.41. *The pair (X, D) is log canonical and $K_X + D \sim 0$. Let W be a closed subvariety of X. Then, W is an lc center of (X, D) if and only if $W = V(\sigma)$ for some $\sigma \in \Delta \setminus \{0\}$.*

Therefore, we have the next theorem by adjunction (see Theorem adj-th 3.39 (i)).

topo-th **Theorem 4.42.** *Let $Y = Y(\Phi)$ be a toric polyhedron on $X = X(\Delta)$. Then, the log canonical pair (X, D) induces a natural quasi-log structure on $[Y, 0]$. Note that $[Y, 0]$ has only qlc singularities. Let W be a closed subvariety of Y. Then, W is a qlc center of $[Y, 0]$ if and only if $W = V(\sigma)$ for some $\sigma \in \Phi$.*

Thus, we can use the theory of quasi-log varieties to investigate toric varieties and toric polyhedra. For example, we have the following result as a special case of Theorem adj-th 3.39 (ii).

Corollary 4.43. *We use the same notation as in* Theorem 4.42. *Assume that X is projective and L is an ample Cartier divisor. Then $H^i(X, \mathcal{I}_Y \otimes \mathcal{O}_X(L)) = 0$ for any $i > 0$, where $\mathcal{I}_Y$ is the defining ideal sheaf of Y on X. In particular, $H^0(X, \mathcal{O}_X(L)) \to H^0(Y, \mathcal{O}_Y(L))$ is surjective.*

We can prove various vanishing theorems for toric varieties and toric polyhedra without appealing the results in Chapter 2. For the details, see [F12].

4.5 Non-lc ideal sheaves

43-sec

In [F15], we introduced the notion of *non-lc ideal sheaves* and proved the restriction theorem. In this section, we quickly review the results in [F15].

nlc-def **Definition 4.44 (Non-lc ideal sheaf).** Let X be a normal variety and let Δ be an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a resolution with $K_Y + \Delta_Y = f^*(K_X + \Delta)$ such that $\mathrm{Supp}\Delta_Y$ is simple normal crossing. Then we put

$$\mathcal{J}_{NLC}(X, \Delta) = f_*\mathcal{O}_Y(\ulcorner -(\Delta_Y^{<1})\urcorner - \llcorner \Delta_Y^{>1} \lrcorner) = f_*\mathcal{O}_Y(-\llcorner \Delta_Y \lrcorner + \Delta_Y^{=1})$$

and call it the *non-lc ideal sheaf associated to* (X, Δ).

In Definition 4.44, the ideal $\mathcal{J}_{NLC}(X, \Delta)$ coincides with $\mathcal{I}_{X_{-\infty}}$ for the quasi-log pair $[X, K_X + \Delta]$ when Δ is effective.

Remark 4.45. In the same notation as in Definition 4.44, we put

$$\mathcal{J}(X, \Delta) = f_*\mathcal{O}_Y(-\llcorner \Delta_Y \lrcorner) = f_*\mathcal{O}_Y(K_Y - \llcorner f^*(K_X + \Delta) \lrcorner).$$

It is nothing but the well-known *multiplier ideal sheaf*. It is obvious that $\mathcal{J}(X, \Delta) \subseteq \mathcal{J}_{NLC}(X, \Delta)$.

The following theorem is the main theorem of [F15]. We hope that it will have many applications. For the proof, see [F15].

rest-th **Theorem 4.46 (Restriction Theorem).** *Let X be a normal variety and let $S + B$ be an effective $\mathbb{R}$-divisor on X such that S is reduced and normal and that S and B have no common irreducible components. Assume that $K_X + S + B$ is $\mathbb{R}$-Cartier. We put $K_S + B_S = (K_X + S + B)|_S$. Then we obtain that*

$$\mathcal{J}_{NLC}(S, B_S) = \mathcal{J}_{NLC}(X, S + B)|_S.$$

Theorem 4.46 is a generalization of the inversion of adjunction on log canonicity in some sense.

Corollary 4.47 (Inversion of Adjunction). *We use the notation as in* Theorem 4.46. *Then, (S, B_S) is lc if and only if $(X, S + B)$ is lc around S.*

In [Kw], Kawakita proved the inversion of adjunction on log canonicity without assuming that S is normal.

4.6 Effective Base Point Free Theorems

In this section, we state effective base point free theorems for log canonical pairs without proof. First, we state Angehrn–Siu type effective base point free theorems (see [AS] and [Ko4]). For the details of Theorems 4.48 and 4.49, see [F14].

Theorem 4.48 (Effective Freeness). *Let (X, Δ) be a projective* log canonical *pair such that Δ is an effective $\mathbb{Q}$-divisor and let M be a line bundle on X. Assume that $M \equiv K_X + \Delta + N$, where N is an ample $\mathbb{Q}$-divisor on X. Let $x \in X$ be a closed point and assume that there are positive numbers $c(k)$ with the following properties:*

(1) *If $x \in Z \subset X$ is an irreducible (positive dimensional) subvariety, then*

$$(N^{\dim Z} \cdot Z) > c(\dim Z)^{\dim Z}.$$

(2) *The numbers $c(k)$ satisfy the inequality*

$$\sum_{k=1}^{\dim X} \frac{k}{c(k)} \leq 1.$$

Then M has a global section not vanishing at x.

Theorem 4.49 (Effective Point Separation). *Let (X, Δ) be a projective* log canonical *pair such that Δ is an effective $\mathbb{Q}$-divisor and let M be a line bundle on X. Assume that $M \equiv K_X + \Delta + N$, where N is an ample $\mathbb{Q}$-divisor on X. Let $x_1, x_2 \in X$ be closed points and assume that there are positive numbers $c(k)$ with the following properties:*

(1) *If $Z \subset X$ is an irreducible (positive dimensional) subvariety which contains x_1 or x_2, then*

$$(N^{\dim Z} \cdot Z) > c(\dim Z)^{\dim Z}.$$

(2) *The numbers $c(k)$ satisfy the inequality*

$$\sum_{k=1}^{\dim X} \sqrt[k]{2}\frac{k}{c(k)} \leq 1.$$

Then global sections of M separates x_1 and x_2.

The key points of the proofs of Theorems 4.48 and 4.49 are the vanishing theorem (see Theorem 3.39 (ii)) and the inversion of adjunction on log canonicity (see Corollary 4.47 and [Kw]).

The final theorem in this book is a generalization of Kollár's effective base point freeness (see [Ko2]). The proof is essentially the same as Kollár's once we adopt Theorem 3.39 (ii) and Theorem 4.4. For the details, see [F13].

eff-th3 **Theorem 4.50.** *Let (X, Δ) be a* log canonical *pair with $\dim X = n$ and let $\pi : X \to V$ be a* projective *surjective morphism. Note that Δ is an effective $\mathbb{Q}$-divisor on X. Let L be a π-nef Cartier divisor on X. Assume that $aL - (K_X + \Delta)$ is π-nef and π-log big for some $a \geq 0$. Then there exists a positive integer $m = m(n, a)$, which only depends on n and a, such that $\mathcal{O}_X(mL)$ is π-generated.*

Chapter 5

Appendix

chap5

In this final chapter, we will explain some sample computations of flips. We use the toric geometry to construct explicit examples here.

5.1 Francia's flip revisited

We give an example of Francia's flip on a projective toric variety explicitly. It is a monumental example (see francia [Fr]). So, we contain it here. Our description looks slightly different from the usual one because we use the toric geometry.

fran11 **Example 5.1.** We fix a lattice $N \simeq \mathbb{Z}^3$ and consider the lattice points $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$, $e_4 = (1,1,-2)$, and $e_5 = (-1,-1,1)$. First, we consider the complete fan Δ_1 spanned by e_1, e_2, e_4, and e_5. Since $e_1+e_2+e_4+2e_5 = 0$, $X_1 = X(\Delta_1)$ is $\mathbb{P}(1,1,1,2)$. Next, we take the blow-up $f : X_2 = X(\Delta_2) \to X_1$ along the ray $e_3 = (0,0,1)$. Then X_2 is a projective $\mathbb{Q}$-factorial toric variety with only one $\frac{1}{2}(1,1,1)$-singular point. Since $\rho(X_2) = 2$, we have one more contraction morphism $\varphi : X_2 \to X_3 = X(\Delta_3)$. This contraction morphism φ corresponds to the removal of the wall $\langle e_1, e_2\rangle$ from Δ_2. We can easily check that φ is a flipping contraction. By adding the wall $\langle e_3, e_4\rangle$ to Δ_3, we obtain a flipping diagram.

$$\begin{array}{ccc} X_2 & \dashrightarrow & X_4 \\ & \searrow \quad \swarrow & \\ & X_3 & \end{array}$$

It is an example of Francia's flip. We can easily check that $X_4 \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ and that the flipped curve $C \simeq \mathbb{P}^1$ is the section of π :

$\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \to \mathbb{P}^1$ defined by the projection $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \to \mathcal{O}_{\mathbb{P}^1} \to 0$.

By taking double covers, we have an interesting example (cf. [Fr]).

Example 5.2. We use the same notation as in Example 5.1. Let $g : X_5 \to X_2$ be the blow-up along the ray $e_6 = (1, 1, -1)$. Then X_5 is a smooth projective toric variety. Let $\mathcal{O}_{X_4}(1)$ be the tautological line bundle of the $\mathbb{P}^2$-bundle $\pi : X_4 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \to \mathbb{P}^1$. It is easy to see that $\mathcal{O}_{X_4}(1)$ is nef and $\mathcal{O}_{X_4}(1) \cdot C = 0$. Therefore, there exists a line bundle $\mathcal{L}$ on X_3 such that $\mathcal{O}_{X_4}(1) \simeq \psi^*\mathcal{L}$, where $\psi : X_4 \to X_3$. We take a general member $D \in |\mathcal{L}^{\otimes 8}|$. We note that $|\mathcal{L}|$ is free since $\mathcal{L}$ is nef. We take a double cover $X \to X_4$ (resp. $Y \to X_5$) ramifying along $\mathrm{Supp}\psi^{-1}D$ (resp. $\mathrm{Supp}(\varphi \circ g)^{-1}D$). Then X is a smooth projective variety such that K_X is ample. It is obvious that Y is a smooth projective variety and is birational to X. So, X is the unique minimal model of Y. We need flips to obtain the minimal model X from Y by running the MMP.

5.2 A sample computation of a log flip

Here, we treat an example of threefold log flips. In general, it is difficult to know what happens around the flipping curve. Therefore, the following nontrivial example is valuable because we can see the behavior of the flip explicitly. It helps us understand the proof of the special termination in [F8].

Example 5.3. We fix a lattice $N = \mathbb{Z}^3$. We put $e_1 = (1, 0, 0)$, $e_2 = (-1, 2, 0)$, $e_3 = (0, 0, 1)$, and $e_4 = (-1, 3, -3)$. We consider the fan

$$\Delta = \{\langle e_1, e_3, e_4\rangle, \langle e_2, e_3, e_4\rangle, \text{and their faces}\}.$$

We put $X = X(\Delta)$, that is, X is the toric variety associated to the fan Δ. We define torus invariant prime divisors $D_i = V(e_i)$ for $1 \leq i \leq 4$. We can easily check the following claim.

Claim. *The pair $(X, D_1 + D_3)$ is a $\mathbb{Q}$-factorial dlt pair.*

We put $C = V(\langle e_3, e_4\rangle) \simeq \mathbb{P}^1$, which is a torus invariant irreducible curve on X. Since $\langle e_2, e_3, e_4\rangle$ is a non-singular cone, the intersection number

$D_2 \cdot C = 1$. Therefore, $C \cdot D_4 = -\frac{2}{3}$ and $-(K_X + D_1 + D_3) \cdot C = \frac{1}{3}$. We note the linear relation $e_1 + 3e_2 - 6e_3 - 2e_4 = 0$. We put $Y = X(\langle e_1, e_2, e_3, e_4 \rangle)$, that is, Y is the affine toric variety associated to the cone $\langle e_1, e_2, e_3, e_4 \rangle$. Then we have the next claim.

Claim. *The birational map $f : X \to Y$ is an elementary pl flipping contraction with respect to $K_X + D_1 + D_3$.*

For the definition of pl flipping contractions, see [F8, Definition 4.3.1]. We note the intersection numbers $C \cdot D_1 = \frac{1}{3}$ and $D_3 \cdot C = -2$. Let $\varphi : X \dashrightarrow X^+$ be the flip of f. We note that the flip φ is an isomorphism around any generic points of lc centers of $(X, D_1 + D_3)$. We restrict the flipping diagram

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

to D_3. Then we have the following diagram.

$$\begin{array}{ccc} D_3 & \dashrightarrow & D_3^+ \\ & \searrow \quad \swarrow & \\ & f(D_3) & \end{array}$$

It is not difficult to see that $D_3^+ \to f(D_3)$ is an isomorphism. We put $(K_X + D_1 + D_3)|_{D_3} = K_{D_3} + B$. Then $f : D_3 \to f(D_3)$ is an extremal divisorial contraction with respect to $K_{D_3} + B$. We note that $B = D_1|_{D_3}$.

Claim. *The birational morphism $f : D_3 \to f(D_3)$ contracts $E \simeq \mathbb{P}^1$ to a point Q on $D_3^+ \simeq f(D_3)$ and Q is a $\frac{1}{2}(1,1)$-singular point on $D_3^+ \simeq f(D_3)$. The surface D_3 has a $\frac{1}{3}(1,1)$-singular point P, which is the intersection of E and B. We also have the adjunction formula $(K_{D_3} + B)|_B = K_B + \frac{2}{3}P$.*

Let D_i^+ be the torus invariant prime divisor $V(e_i)$ on X^+ for all i and B^+ the strict transform of B on D_3^+.

Claim. *We have*

$$(K_{X^+} + D_1^+ + D_3^+)|_{D_3^+} = K_{D_3^+} + B^+$$

and

$$(K_{D_3^+} + B^+)|_{B^+} = K_{B^+} + \frac{1}{2}Q.$$

We note that $f^+ : D_3^+ \to f(D_3)$ is an isomorphism. In particular,

$$\begin{array}{ccc} D_3 & \dashrightarrow & D_3^+ \\ & \searrow \quad \swarrow & \\ & f(D_3) & \end{array}$$

is of type (DS) in the sense of [F8, Definition 4.2.6]. Moreover, $f : B \to B^+$ is an isomorphism but $f : (B, \frac{2}{3}P) \to (B^+, \frac{1}{2}Q)$ is not an isomorphism of pairs (see [F8, Definition 4.2.5]). We note that B is an lc center of $(X, D_1 + D_3)$. So, we need [F8, Lemma 4.2.15]. Next, we restrict the flipping diagram to D_1. Then we obtain the diagram.

$$\begin{array}{ccc} D_1 & \dashrightarrow & D_1^+ \\ & \searrow \quad \swarrow & \\ & f(D_1) & \end{array}$$

In this case, $f : D_1 \to f(D_1)$ is an isomorphism.

Claim. *The surfaces D_1 and D_1^+ are smooth.*

It can be directly checked. Moreover, we obtain the following adjunction formulas.

Claim. *We have*

$$(K_X + D_1 + D_3)|_{D_1} = K_{D_1} + B + \frac{2}{3}B',$$

where B (resp. B') comes from the intersection of D_1 and D_3 (resp. D_4). We also obtain

$$(K_{X^+} + D_1^+ + D_3^+)|_{D_1^+} = K_{D_1^+} + B^+ + \frac{2}{3}B'^+ + \frac{1}{2}F,$$

where B^+ (resp. B'^+) is the strict transform of B (resp. B') and F is the exceptional curve of $f^+ : D_1^+ \to f(D_1)$.

Claim. *The birational morphism $f^+ : D_1^+ \to f(D_1) \simeq D_1$ is the blow-up at $P = B \cap B'$.*

We can easily check that

$$K_{D_1^+} + B^+ + \frac{2}{3}B'^+ + \frac{1}{2}F = f^{+*}(K_{D_1} + B + \frac{2}{3}B') - \frac{1}{6}F.$$

It is obvious that $K_{D_1^+} + B^+ + \frac{2}{3}B'^+ + \frac{1}{2}F$ is f^+-ample. Note that F comes from the intersection of D_1^+ and D_2^+. Note that the diagram

$$\begin{array}{ccc} D_1 & \dashrightarrow & D_1^+ \\ & \searrow \quad \swarrow & \\ & f(D_1) & \end{array}$$

is of type (SD) in the sense of [F8, Definition 4.2.6].

5.3 A non-$\mathbb{Q}$-factorial flip

I apologize for the mistake in [F7, Example 4.4.2]. We give an example of a three-dimensional non-$\mathbb{Q}$-factorial canonical Gorenstein toric flip. See also [FSTU]. We think that it is difficult to construct such examples without using the toric geometry.

Example 5.4 (Non-$\mathbb{Q}$-factorial canonical Gorenstein toric flip). We fix a lattice $N = \mathbb{Z}^3$. Let n be a positive integer with $n \geq 2$. We take lattice points $e_0 = (0, -1, 0)$, $e_i = (n+1-i, \sum_{k=n+1-i}^{n-1} k, 1)$ for $1 \leq i \leq n+1$, and $e_{n+2} = (-1, 0, 1)$. We consider the following fans.

$$\begin{aligned} \Delta_X &= \{\langle e_0, e_1, e_{n+2}\rangle, \langle e_1, e_2, \cdots, e_{n+1}, e_{n+2}\rangle, \text{and their faces}\}, \\ \Delta_W &= \{\langle e_0, e_1, \cdots, e_{n+1}, e_{n+2}\rangle, \text{and its faces}\}, \text{and} \\ \Delta_{X^+} &= \{\langle e_0, e_i, e_{i+1}\rangle, \text{for } i = 1, \cdots, n+1, \text{and their faces}\}. \end{aligned}$$

We define $X = X(\Delta_X), X^+ = X(\Delta_{X^+})$, and $W = X(\Delta_W)$. Then we have a diagram of toric varieties.

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow \quad \swarrow & \\ & W & \end{array}$$

We can easily check the following properties.

(i) X has only canonical Gorenstein singularities.

(ii) X is not $\mathbb{Q}$-factorial.

(iii) X^+ is smooth.

(iv) $-K_X$ is φ-ample and K_{X^+} is φ^+-ample.

(v) $\varphi : X \to W$ and $\varphi^+ : X^+ \to W$ are small projective toric morphisms.

(vi) $\rho(X/W) = 1$ and $\rho(X^+/W) = n$.

Therefore, the above diagram is a desired flipping diagram. We note that $e_i + e_{i+2} = 2e_{i+1} + e_0$ for $i = 1, \cdots, n-1$ and $e_n + e_{n+2} = 2e_{n+1} + \frac{n(n-1)}{2}e_0$. We recommend the reader to draw pictures of Δ_X and Δ_{X^+}.

By this example, we see that a flip sometimes increases the Picard number when the variety is not $\mathbb{Q}$-factorial.

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