NOTES ON THE LOG MINIMAL MODEL PROGRAM

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Abstract. We discuss the log minimal model program for log canonical pairs. We prove the existence of fourfold log canonical flips. This paper is also a guide to the theory of quasi-log varieties by Ambro. The notion of quasi-log varieties is indispensable for investigating log canonical pairs. We also give a proof to the base point free theorem of Reid–Fukuda type for log canonical pairs.

Contents

1. Introduction

In this paper, we discuss the log minimal model program (LMMP, for short) for log canonical pairs. In Section 2, we will explicitly state the LMMP for lc pairs. It is because we can not find it in the standard literature. The cone and contraction theorems for lc pairs are buried in [A, Section 5]. Therefore, we state them explicitly for lc pairs with the additional estimate of lengths of extremal rays. We also write the flip conjectures for lc pairs. We note that the flip conjecture I (existence of an lc flip) is still open and that the flip conjecture II (termination of a sequence of lc flips) follows from the termination of klt flips. We give a proof of the flip conjecture I in dimension four.

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Theorem 1.1 (cf. Theorem 2.8)**.** *Log canonical flips exist in dimension four.*

Section 3 is a quick review of the theory of *quasi-log varieties* according to Sections 4 and 5 of [A]. We think that the notion of quasi-log varieties is indispensable for investigating lc pairs. Unfortunately, [A] is not accessible and is not reader-friendly. In this section, we will explain how to read [A] (see Remark 3.23). The reader can find that the key points of the theory of quasi-log varieties in [A] are adjunction and the vanishing theorem (see [A, Theorem 4.4]). Adjunction and the vanishing theorems for quasi-log varieties follow from [A, 3. Vanishing Theorems]. However, Section 3 of [A] contains various troubles and is hard to read. Now we have [F6], which gives us sufficiently powerful vanishing and torsion-free theorems for the theory of quasi-log varieties. We succeed in removing all the troublesome problems for the foundation of the theory of quasi-log varieties. It is one of the main contributions of this paper and [F11]. In Section 4, we give some supplementary results on the *quasi-log resolutions*. They are very useful and seems to be indispensable for some applications, but missing in [A]. We will need them in Section 5. By the results in Section 4, the theory of quasi-log varieties becomes a useful theory. As a byproduct, we have the following new definition of quasi-log varieties.

Definition 1.2 (Quasi-log varieties)**.** A *quasi-log variety* is a scheme X endowed with an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor ω, a proper closed subscheme $X_{-\infty} \subset X$, and a finite collection $\{C\}$ of reduced and irreducible subvarieties of X such that there is a proper morphism $f : Y \to X$ from a simple normal crossing divisor Y on a smooth variety M satisfying the following properties:

(0) there exists an $\mathbb{R}$-divisor D on M such that $\mathrm{Supp}(D+Y)$ is simple normal crossing on M and that D and Y have no common irreducible components.
(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$, where $B_Y = D|_Y$.
(2) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner)$ induces an isomorphism
$$\mathcal{I}_{X_{-\infty}} \to f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1}\lrcorner),$$
where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$.
(3) The collection of subvarieties $\{C\}$ coincides with the image of (Y, B_Y)-strata that are not included in $X_{-\infty}$.

Definition 1.2 is equivalent to Ambro's original definition (see [A, Definition 4.1]). However, we think Definition 1.2 is much better than

Ambro's. Once we adopt Definition 1.2, we do not need the notion of *normal crossing pairs* to define quasi-log varieties and get flexibility in the choice of *quasi-log resolutions* $f : Y \to X$ by Proposition 4.6. In Section 5, we will try to prove the base point free theorem of Reid–Fukuda type for quasi-log varieties. It is stated in [A, Theorem 7.2] without proof. We will give a proof under some extra assumptions. As a corollary, we obtain the following almost satisfactory statement.

Theorem 1.3 (cf. Theorem 5.4)**.** *Let (X, B) be an lc pair. Let L be a π-nef Cartier divisor on X, where $\pi : X \to S$ is a projective morphism. Assume that $qL - (K_X + B)$ is π-nef and π-log big for some positive real number q. Then $\mathcal{O}_X(mL)$ is π-generated for $m \gg 0$.*

In [F8], we obtain an effective version of Theorem 1.3. We note that we need Theorem 1.3 to prove the main theorem of [F8]. The reader can find Angehrn–Siu type effective base point freeness for lc pairs in [F9]. Note that this paper does not cover all the results in [A]. The paper [F11] is a gentle introduction to the log minimal model program for lc pairs. It can be read without referring [A] and this paper.

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We will work over the complex number field $\mathbb{C}$ throughout this paper. But we note that by using the Lefschetz principle, we can extend everything to the case where the base field is an algebraically closed field of characteristic zero. We will use the following notation and the notation in [KM] freely.

Notation. For an $\mathbb{R}$-Weil divisor $D = \sum_{j=1}^{r} d_j D_j$ such that $D_i \neq D_j$ for $i \neq j$, we define the *round-up* $\ulcorner D \urcorner = \sum_{j=1}^{r} \ulcorner d_j \urcorner D_j$ (resp. the *round-down* $\llcorner D \lrcorner = \sum_{j=1}^{r} \llcorner d_j \lrcorner D_j$), where for any real number x, $\ulcorner x \urcorner$ (resp. $\llcorner x \lrcorner$) is the integer defined by $x \leq \llcorner x \lrcorner < x + 1$ (resp. $x - 1 < \llcorner x \lrcorner \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \llcorner D \lrcorner$. We define

$$D^{=1} = \sum_{d_j=1} D_j, \quad D^{\leq 1} = \sum_{d_j \leq 1} d_j D_j,$$

$$D^{<1} = \sum_{d_j<1} d_j D_j, \quad \text{and} \quad D^{>1} = \sum_{d_j>1} d_j D_j.$$

We call D a *boundary* (resp. *subboundary*) $\mathbb{R}$-divisor if $0 \le d_j \le 1$ (resp. $d_j \le 1$) for any j. $\mathbb{Q}$-*linear equivalence* (resp. $\mathbb{R}$-*linear equivalence* of two $\mathbb{Q}$-divisors (resp. $\mathbb{R}$-*divisors*) B_1 and B_2 is denoted by $B_1 \sim_{\mathbb{Q}} B_2$ (resp. $B_1 \sim_{\mathbb{R}} B_2$). For a proper birational morphism $f : X \to Y$, the *exceptional locus* $\mathrm{Exc}(f) \subset X$ is the locus where f is not an isomorphism. Let X be a normal variety and let B be an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a resolution such that $\mathrm{Exc}(f) \cup f_*^{-1}B$ has a simple normal crossing support, where $f_*^{-1}B$ is the strict transform of B on Y. We write $K_Y = f^*(K_X + B) + \sum_i a_i E_i$ and $a(E_i, X, B) = a_i$. We say that (X, B) is *log canonical* (resp. *Kawamata log terminal*) (*lc* (resp. *klt*), for short) if and only if $a_i \ge -1$ (resp. $a_i > -1$) for any i. Note that the *discrepancy* $a(E, X, B) \in \mathbb{R}$ can be defined for any prime divisor E *over* X. Let (X, B) be an lc pair. If E is a prime divisor over X such that $a(E, X, B) = -1$, then the center $c_X(E)$ is called an *lc center* of (X, B).

2. LMMP for log canonical pairs

In this section, we explicitly state the log minimal model program (LMMP, for short) for log canonical pairs. It is known to some experts but we can not find it in the standard literature. The following cone theorem is a consequence of Ambro's cone theorem for quasi-log varieties (see Theorem 5.10 in [A]). We will discuss the estimate of lengths of extremal rays in the subsection 2.1. We think that the paper [F11] may help the reader to understand Theorem 2.1.

Theorem 2.1 (Cone and contraction theorems)**.** *Let (X, B) be an lc pair, B an $\mathbb{R}$-divisor, and $f : X \to Y$ a projective morphism between algebraic varieties. Then we have*

(i) *There are* (*countably many*) *rational curves $C_j \subset X$ such that $f(C_j)$ =point, $0 < -(K_X + B) \cdot C_j \le 2 \dim X$, and*
$$\overline{NE}(X/Y) = \overline{NE}(X/Y)_{(K_X+B)\ge 0} + \sum \mathbb{R}_{\ge 0}[C_j].$$
(ii) *For any $\varepsilon > 0$ and f-ample $\mathbb{R}$-divisor H,*
$$\overline{NE}(X/Y) = \overline{NE}(X/Y)_{(K_X+B+\varepsilon H)\ge 0} + \sum_{finite} \mathbb{R}_{\ge 0}[C_j].$$
(iii) *Let $F \subset \overline{NE}(X/Y)$ be a (K_X+B)-negative extremal face. Then there is a unique morphism $\varphi_F : X \to Z$ over Y such that $(\varphi_F)_*\mathcal{O}_X \simeq \mathcal{O}_Z$, Z is projective over Y, and an irreducible curve $C \subset X$ is mapped to a point by φ_F if and only if $[C] \in F$. The map φ_F is called the contraction of F.*

(iv) *Let F and φ_F be as in* (iii)*. Let L be a line bundle on X such that $(L \cdot C) = 0$ for every curve C with $[C] \in F$. Then there is a line bundle L_Z on Z such that $L \simeq \varphi_F^* L_Z$.*

Remark 2.2 (Lengths of extremal rays)**.** In Theorem 2.1 (i), the estimate $-(K_X+B)\cdot C_j \leq 2\dim X$ should be replaced by $-(K_X+B)\cdot C_j \leq \dim X + 1$. For toric varieties, this conjectural estimate and some generalizations were obtained in [F3] and [F4].

By the above cone and contraction theorems, we can easily see that the LMMP, that is, a recursive procedure explained in [KM, 3.31], works for log canonical pairs if the following two conjectures (Flip Conjectures I and II) hold.

Conjecture 2.3. ((Log) Flip Conjecture I: The existence of a (log) flip)*. Let $\varphi\colon (X, B) \to W$ be an extremal flipping contraction of an n-dimensional pair, that is,*

(1) *(X, B) is lc, B is an $\mathbb{R}$-divisor,*
(2) *φ is small projective and φ has only connected fibers,*
(3) *$-(K_X + B)$ is φ-ample,*
(4) *$\rho(X/W) = 1$, and*
(5) *X is $\mathbb{Q}$-factorial.*

Then there should be a diagram:

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow \quad \swarrow & \\ & W & \end{array}$$

which satisfies the following conditions:

(i) *X^+ is a normal variety,*
(ii) *$\varphi^+\colon X^+ \to W$ is small projective, and*
(iii) *$K_{X^+} + B^+$ is φ^+-ample, where B^+ is the strict transform of B.*

*We call $\varphi^+ : (X^+, B^+) \to W$ a $(K_X + B)$-*flip *of φ.*

Note that to prove Conjecture 2.3 we can assume that B is a $\mathbb{Q}$-divisor, by perturbing B slightly. It is known that Conjecture 2.3 holds when $\dim X = 3$ (see [FA, Chapter 8]). Moreover, if there exists an $\mathbb{R}$-divisor B' on X such that $K_X + B'$ is klt and $-(K_X + B')$ is φ-ample, then Conjecture 2.3 is true by [BCHM]. The following famous conjecture is stronger than Conjecture 2.3. We will see it in Lemma 2.5.

Conjecture 2.4 (Finite generation)**.** *Let X be an n-dimensional smooth projective variety and B a boundary $\mathbb{Q}$-divisor on X such that* Supp$B$

is a simple normal crossing divisor on X. Assume that K_X+B is big. Then the log canonical ring

$$R(X, K_X+B)=\bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(\llcorner m(K_X+B)\lrcorner))$$

is a finitely generated $\mathbb{C}$-algebra.

Note that if there exists a $\mathbb{Q}$-divisor B' on X such that K_X+B' is klt and $K_X+B'\sim_{\mathbb{Q}} K_X+B$, then Conjecture 2.4 holds by [BCHM]. See Remark 2.6.

Lemma 2.5. *Let $f: X\to S$ be a proper surjective morphism between normal varieties with connected fibers. We assume $\dim X=n$. Let B be a $\mathbb{Q}$-divisor on X such that (X, B) is lc. Assume that K_X+B is f-big. Then the relative log canonical ring*

$$R(X/S, K_X+B)=\bigoplus_{m\geq 0} f_*\mathcal{O}_X(\llcorner m(K_X+B)\lrcorner)$$

is a finitely generated $\mathcal{O}_S$-algebra if Conjecture 2.4 *holds. In particular,* Conjecture 2.4 *implies* Conjecture 2.3.

Before we go to the proof of Lemma 2.5, we note one easy remark.

Remark 2.6. For a graded ring $R=\bigoplus_{m\geq 0} R_m$ and a positive integer k, the truncated ring $R^{(k)}$ is defined by $R^{(k)}=\bigoplus_{m\geq 0} R_{km}$. Then R is finitely generated if and only if so is $R^{(k)}$. We consider $\mathrm{Proj} R$ when R is finitely generated. We note that $\mathrm{Proj} R^{(k)}=\mathrm{Proj} R$.

The following argument is well known to the experts.

Proof of Lemma 2.5. Since the problem is local, we can shrink S and assume that S is affine. By compactifying X and S and by the desingularization theorem, we can further assume that X and S are projective, X is smooth, B is effective, and $\mathrm{Supp} B$ is a simple normal crossing divisor. Let A be a very ample divisor on S and $H\in |rA|$ a general member for $r\gg 0$. Note that $K_X+B+(r-1)f^*A$ is big for $r\gg 0$ (cf. [KMM, Corollary 0-3-4]). Let m_0 be a positive integer such that $m_0(K_X+B+f^*H)$ is Cartier. By Conjecture 2.4, $\bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(mm_0(K_X+B+f^*H)))$ is finitely generated. Thus, the relative log canonical model X' over S exists. Indeed, by assuming that m_0 is sufficiently large and divisible, $R(X, K_X+B+f^*H)^{(m_0)}$ is generated by $R(X, K_X+B+f^*H)_{m_0}$ and $|m_0(K_X+B+(r-1)f^*A)|\neq \emptyset$. Then $X'=\mathrm{Proj}\bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(mm_0(K_X+B+f^*H)))$ and X' is the closure of the image of X by the rational map defined by the complete linear system $|m_0(K_X + B + rf^*A)|$. More precisely, let $g : X'' \to X$ be the elimination of the indeterminacy of the rational map defined by $|m_0(K_X + B + rf^*A)|$. Let $g' : X'' \to X'$ be the induced morphism and $h : X'' \to S$ the morphism defined by the complete linear system $|m_0 g^* f^* A|$. Then it is not difficult to see that h factors through X'. Therefore, $\bigoplus_{m\geq 0} f_*\mathcal{O}_X(mm_0(K_X+B))$ is a finitely generated $\mathcal{O}_S$-algebra. We finish the proof. □

The next theorem is an easy consequence of [BCHM], [AHK], [F1], and [F2].

Theorem 2.7. *Let (X, B) be a proper 4-dimensional lc pair such that B is a $\mathbb{Q}$-divisor and $K_X + B$ is big. Then the log canonical ring $\bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(\llcorner m(K_X + B)\lrcorner))$ is finitely generated.*

Proof. Without loss of generality, we can assume that X is smooth projective and Supp$B$ is simple normal crossing. Run a $(K_X + B)$-LMMP. Then we obtain a log minimal model (X', B') by [BCHM] and [AHK] with the aid of the special termination theorem (cf. [F5, Theorem 4.2.1]). By [F2, Theorem 3.1], which is a consequence of the main theorem in [F1], $K_{X'} + B'$ is semi-ample. In particular, $\bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(\llcorner m(K_X + B)\lrcorner)) \simeq \bigoplus_{m\geq 0} H^0(X', \mathcal{O}_{X'}(\llcorner m(K_{X'} + B')\lrcorner))$ is finitely generated. □

As a corollary, we obtain the next theorem by Lemma 2.5.

Theorem 2.8. Conjecture 2.3 *is true if* $\dim X \leq 4$.

Let us go to the flip conjecture II.

Conjecture 2.9. ((Log) Flip Conjecture II: Termination of a sequence of (log) flips). *A sequence of* (*log*) *flips*

$$(X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow (X_2, B_2) \dashrightarrow \cdots$$

terminates after finitely many steps. Namely, there does not exist an infinite sequence of (*log*) *flips.*

Note that it is sufficient to prove Conjecture 2.9 for any sequence of klt flips. The termination of dlt flips with dimension $\leq n-1$ implies the special termination in dimension n. Note that we use the formulation in [F5, Theorem 4.2.1]. The special termination and the termination of klt flips in dimension n implies the termination of dlt flips in dimension n. The termination of dlt flips in dimension n implies the termination of lc flips in dimension n. It is because we can use the LMMP for $\mathbb{Q}$-factorial dlt pairs in full generality by [BCHM] once we obtain the termination of dlt flips. The reader can find all the necessary arguments in [F5, 4.2, 4.4].

Remark 2.10 (Analytic spaces)**.** The proofs of the vanishing theorems in [F6] only work for algebraic varieties. Therefore, the cone, contraction, and base point free theorems stated here and in [F11] for lc pairs hold only for algebraic varieties. Of course, all the results should be proved for complex analytic spaces that are projective over any fixed analytic spaces.

2.1. **Lengths of extremal rays.** In this subsection, we consider the estimate of lengths of extremal rays. Related topics are in [BCHM, 3.8 and 3.9]. Let us recall the following easy lemma.

Lemma 2.11 (cf. [S, Lemma 1])**.** *Let (X, B) be an lc pair, where B is an $\mathbb{R}$-divisor. Then there are positive real numbers r_i and effective $\mathbb{Q}$-divisors B_i for $1 \le i \le l$ and a positive integer m such that $\sum_{i=1}^{l} r_i = 1$, $K_X + B = \sum_{i=1}^{l} r_i(K_X + B_i)$, (X, B_i) is lc, and $m(K_X + B_i)$ is Cartier for any i.*

The next result is essentially due to [K] and [S, Proposition 1].

Proposition 2.12. *We use the notation in* Lemma 2.11*. Let (X, B) be an lc pair, B an $\mathbb{R}$-divisor, and $f : X \to Y$ a projective morphism between algebraic varieteis. Let R be a $(K_X + B)$-negative extremal ray of $\overline{NE}(X/Y)$. Then we can find a rational curve C on X such that $[C] \in R$ and $-(K_X + B_i) \cdot C \le 2 \dim X$ for any i. In particular, $-(K_X + B) \cdot C \le 2 \dim X$. More precisely, we can write $-(K_X + B) \cdot C = \sum_{i=1}^{l} \frac{r_i n_i}{m}$, where $n_i \in \mathbb{Z}$ and $n_i \le 2m \dim X$ for any i.*

Proof. By replacing $f : X \to Y$ with the extremal contraction $\varphi_R : X \to W$ over Y, we can assume that the relative Picard number $\rho(X/Y) = 1$. In particular, $-(K_X + B)$ is f-ample. Therefore, we can assume that $-(K_X + B_1)$ is f-ample and $-(K_X + B_i) = -s_i(K_X + B_1)$ in $N^1(X/Y)$ with $s_i \le 1$ for any $i \ge 2$. Thus, it is sufficient to find a rational curve C such that $f(C)$ is a point and that $-(K_X + B_1) \cdot C \le 2 \dim X$. So, we can assume that $K_X + B$ is $\mathbb{Q}$-Cartier and lc. By [BCHM], there is a birational morphism $g : (W, B_W) \to (X, B)$ such that $K_W + B_W = g^*(K_X + B)$, W is $\mathbb{Q}$-factorial, B_W is effective, and $(W, \{B_W\})$ is klt. By [K, Theorem 1], we can find a rational curve C' on W such that $-(K_W + B_W) \cdot C' \le 2 \dim W = 2 \dim X$ and that C' spans a $(K_W + B_W)$-negative extremal ray. Note that Kawamata's proof works in the above situation with only small modifications. See the proof of Theorem 10-2-1 in [M] and Remark 2.13 below. By the projection formula, the g-image of C' is a desired rational curve. So, we finish the proof. □

Remark 2.13. Let (X, D) be an lc pair, D an $\mathbb{R}$-divisor. Let $\phi : X \to Y$ be a projective morphism and H a Cartier divisor on X. Assume that $H - (K_X + D)$ is f-ample. By Theorem 3.12 (ii) below, $R^q\phi_*\mathcal{O}_X(H) = 0$ for any $q > 0$ if X and Y are *algebraic* varieties. If this vanishing theorem holds for analytic spaces X and Y, then Kawamata's original argument in [K] works directly for lc pairs. In that case, we do not need the results in [BCHM] in the proof of Proposition 2.12. We consider the proof of [M, Theorem 10-2-1] when (X, D) is lc such that $(X, \{D\})$ is klt. We need $R^1\phi_*\mathcal{O}_X(H) = 0$ after shrinking X and Y analytically. In our situation, $(X, D - \varepsilon \llcorner D \lrcorner)$ is klt for $0 < \varepsilon \ll 1$. Therefore, $H - (K_X + D - \varepsilon \llcorner D \lrcorner)$ is ϕ-ample and $(X, D - \varepsilon \llcorner D \lrcorner)$ is klt for $0 < \varepsilon \ll 1$. Thus, we can apply the analytic version of the relative Kawamata–Viehweg vanishing theorem. So, we do not need the analytic version of Theorem 3.12 (ii).

By Proposition 2.12, Lemma 2.6 in [B] holds for lc pairs. For the proof, see [B, Lemma 2.6]. It may be useful for the MMP with scaling.

Proposition 2.14. *Let (X, B) be an lc pair, B an $\mathbb{R}$-divisor, and $f : X \to Y$ a projective morphism between algebraic varieties. Let C be an effective $\mathbb{R}$-Cartier divisor on X such that $K_X + B + C$ is f-nef and $(X, B + C)$ is lc. Then, either $K_X + B$ is also f-nef or there is a $(K_X + B)$-negative extremal ray R such that $(K_X + B + \lambda C) \cdot R = 0$, where $\lambda := \inf\{\, t \geq 0 \mid K_X + B + tC$ is f-nef$\}$. Of course, $K_X + B + \lambda C$ is f-nef.*

3. Quasi-log varieties

In this section, we quickly review the theory of quasi-log varieties by Ambro according to [A, Section 4]. This formulation is indispensable for the proofs of the cone, contraction, and base point free theorems for lc pairs. This section will help the reader to understand Ambro's ideas. If the reader is interested only in Theorem 2.1, then it may be better to see [F11]. The following definition is due to Ambro (see [A, Definition 4.1]).

Definition 3.1 (Quasi-log varieties)**.** A *quasi-log variety* is a scheme X endowed with an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor ω, a proper closed subscheme $X_{-\infty} \subset X$, and a finite collection $\{C\}$ of reduced and irreducible subvarieties of X such that there is a proper morphism $f : (Y, B_Y) \to X$

from an embedded *simple* normal crossing pair satisfying the following properties:

(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.
(2) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner)$ induces an isomorphism
$$\mathcal{I}_{X_{-\infty}} \to f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1}\lrcorner),$$
where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$.
(3) The collection of subvarieties $\{C\}$ coincides with the image of (Y, B_Y)-strata that are not included in $X_{-\infty}$.

For the definition of *simple normal crossing pairs*, see Definition 3.8 below. We sometimes simply say that $[X, \omega]$ is a *quasi-log pair*. We use the following terminology according to Ambro. The subvarieties C are the *qlc centers* of X, $X_{-\infty}$ is the *non-qlc locus* of X, and $f : (Y, B_Y) \to X$ is a *quasi-log resolution* of X. We say that X has *qlc singularities* if $X_{-\infty} = \emptyset$. Note that a quasi-log variety X is the union of its qlc centers and $X_{-\infty}$. A *relative quasi-log variety* X/S is a quasi-log variety X endowed with a proper morphism $\pi : X \to S$.

Remark 3.2. In [A, Definition 4.1], Ambro only required that $f : (Y, B_Y) \to X$ is a proper morphism from an embedded normal crossing pair with the conditions (1), (2), and (3) in Definition 3.1. For the definition of *normal crossing pairs*, see [A, Definition 2.3]. However, we can always construct an embedded *simple* normal crossing pair $(Y', B_{Y'})$ and a proper morphism $f' : (Y', B_{Y'}) \to X$ with the above conditions (1), (2), and (3) by blowing up M suitably, where M is the ambient space of Y (see [A, p.218, embedded log transformations, and Remark 4.2.(iv)]). Therefore, there are no differences between Definition 3.1 and [A, Definition 4.1]. If we adopt Ambro's original definition, then we often have to write "We can assume that (Y, B_Y) is an embedded *simple* normal crossing pair by taking suitable blow-ups of the ambient space" in various arguments. We note that the proofs of the vanishing and injectivity theorems on normal crossing pairs are much harder than on *simple* normal crossing pairs (see [F6]). Therefore, there are no advantages to adopt *normal crossing pairs*. See also Remark 3.9.

Remark 3.3. In Definition 3.1, we assume that ω is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. However, it may be better to see $\omega \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. It is because the *quasi-log canonical class* ω is defined up to $\mathbb{R}$-linear equivalence and we often restrict ω to a subvariety of X.

Remark 3.4. By Definition 3.1, X has only qlc singularities if and only if B_Y is a subboundary. We can easily see that $X \setminus X_{-\infty}$ is semi-normal.

Remark 3.5. To prove the cone and contraction theorems for lc pairs, it is enough to treat quasi-log varieties with only qlc singularities. For the details, see [F11].

To understand Definition 3.1, we have to recall the definitions of *simple normal crossing varieties* and *simple normal crossing pairs.*

Definition 3.6 (Simple normal crossing varieties)**.** A variety X has *normal crossing* singularities if, for every closed point $x \in X$,

$$\widehat{\mathcal{O}}_{X,x} \simeq \frac{\mathbb{C}[[x_0, \cdots, x_N]]}{(x_0 \cdots x_k)}$$

for some $0 \le k \le N$, where $N = \dim X$. Furthermore, if each irreducible component of X is smooth, X is called a *simple normal crossing* variety. If X is a normal crossing variety, then X has only Gorenstein singularities. Thus, it has an invertible dualizing sheaf ω_X. So, we can define the *canonical divisor* K_X such that $\omega_X \simeq \mathcal{O}_X(K_X)$. It is a Cartier divisor on X and is well defined up to linear equivalence.

Definition 3.7 (Simple normal crossing divisors)**.** Let X be a simple normal crossing variety. We say that a Cartier divisor D is a *normal crossing* divisor on X if, in the notation of Definition 3.6, we have

$$\widehat{\mathcal{O}}_{D,x} \simeq \frac{\mathbb{C}[[x_0, \cdots, x_N]]}{(x_0 \cdots x_k, x_{i_1} \cdots x_{i_l})}$$

for some $\{i_1, \cdots, i_l\} \subset \{k+1, \cdots, N\}$. Let $\varepsilon_0 : X_0 \to X$ be the normalization and D a normal crossing divisor on X. If $D^0 = \varepsilon_0^* D$ is a simple normal crossing divisor on X^0 in the usual sense, then D is called a *simple normal crossing divisor* on X.

Definition 3.8 (Simple normal crossing pairs)**.** We say that the pair (X, B) is a *simple normal crossing pair* if the following conditions are satisfied.

(1) X is a simple normal crossing variety, and
(2) B is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor whose support is a simple normal crossing divisor on X.

We say that a simple normal crossing pair (X, B) is *embedded* if there exists a closed embedding $\iota : X \to M$, where M is a smooth variety of dimension $\dim X + 1$. We put $K_{X^0} + \Theta = \varepsilon_0^*(K_X + B)$, where $\varepsilon_0 : X^0 \to X$ is the normalization of X. A *stratum* of (X, B) is an irreducible component of X or the image of some lc center of $(X^0, \Theta^{\le 1})$ on X. A Cartier divisor D on a simple normal crossing pair (X, B) is called *permissible with respect to* (X, B) if D contains no strata of the pair (X, B).

Remark 3.9. By Proposition 4.2 below, we can assume that there exists an $\mathbb{R}$-divisor D on M, where M is the ambient space of Y, such that $\mathrm{Supp}(D+Y)$ is simple normal crossing on M and that $B_Y = D|_Y$ in Definition 3.1. This fact, which is missing in [A] and is non-trivial, is very useful and seems to be indispensable for some applications. We will discuss the details in Section 4.

Remark 3.10 (Multicrossing vs simple normal crossing)**.** In [A, Section 2], Ambro discussed *multicrossing singularities* and *multicrossing pairs*. However, we think that *simple normal crossing varieties* and *simple normal crossing divisors* on them are sufficient for the later arguments in [A]. Therefore, we did not introduce the notion of *multicrossing singularities* and their simplicial resolutions in [F6]. For the theory of quasi-log varieties, we may not even need the notion of *simple normal crossing pairs*. See Remark 3.9, Remark 4.9, and [F11].

Let us recall the definition of *nef and log big divisors* for the vanishing theorem.

Definition 3.11 (Nef and log big divisors)**.** Let $f : (Y, B_Y) \to X$ be a proper morphism from an embedded simple normal crossing pair (Y, B_Y). Let $\pi : X \to V$ be a proper morphism and H an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. We say that H is *nef and log big* over V if and only if $H|_C$ is nef and big over V for any C, where

(i) C is a qlc center when X is a quasi-log variety and $f : (Y, B_Y) \to X$ is a quasi-log resolution, or
(ii) C is the image of a stratum of (Y, B_Y) when B_Y is a subboundary.

If X is a quasi-log variety with only qlc singularities and $f : (Y, B_Y) \to X$ is a quasi-log resolution, then the above two cases (i) and (ii) coincide. When (X, B_X) is an lc pair, we choose a log resolution of (X, B_X) to be $f : (Y, B_Y) \to X$, where $K_Y + B_Y = f^*(K_X + B_X)$. We note that if H is ample over V then it is obvious that H is nef and log big over V.

The following theorem is one of the key results in the theory of quasi-log varieties. It is a combination of Theorem 4.4 and Theorem 7.3 in [A]. **From now on, we adopt Definition 1.2 for the definition of quasi-log varieties.** We will see that Definition 1.2 is equivalent to Definition 3.1 in Section 4.

Theorem 3.12 (Adjunction and vanishing theorem)**.** *Let $[X, \omega]$ be a quasi-log pair and X' the union of $X_{-\infty}$ with a* (*possibly empty*) *union of some qlc centers of $[X, \omega]$.*

(i) *Assume that* $X' \neq X_{-\infty}$. *Then* X' *is a quasi-log variety, with* $\omega' = \omega|_{X'}$ *and* $X'_{-\infty} = X_{-\infty}$. *Moreover, the qlc centers of* $[X', \omega']$ *are exactly the qlc centers of* $[X, \omega]$ *that are included in* X'.

(ii) *Assume that* $\pi : X \to S$ *is proper. Let* L *be a Cartier divisor on* X *such that* $L - \omega$ *is nef and log big over* S. *Then* $\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)$ *is* π_**-acyclic, where* $\mathcal{I}_{X'}$ *is the defining ideal sheaf of* X' *on* X.

Theorem 3.12 is the hardest part to prove in the theory of quasi-log varieties. It is because it depends on the non-trivial injectivity and vanishing theorems for simple normal crossing pairs. The adjunction for normal divisors on normal varieties is investigated in [F10].

Proof. By blowing up the ambient space M of Y, we can assume that the union of all strata of (Y, B_Y) mapped to X', which is denoted by Y', is a union of irreducible components of Y. We put $K_{Y'} + B_{Y'} = (K_Y + B_Y)|_{Y'}$ and $Y'' = Y - Y'$. We claim that $[X', \omega']$ is a quasi-log pair and that $f : (Y', B_{Y'}) \to X'$ is a quasi-log resolution. By the construction, $f^*\omega' \sim_{\mathbb{R}} K_{Y'} + B_{Y'}$ on Y' is obvious. We put $A = \ulcorner -(B_Y^{<1}) \urcorner$ and $N = \llcorner B_Y^{>1} \lrcorner$. We consider the following short exact sequence

$$0 \to \mathcal{O}_{Y''}(-Y') \to \mathcal{O}_Y \to \mathcal{O}_{Y'} \to 0.$$

By applying $\otimes \mathcal{O}_Y(A - N)$, we have

$$0 \to \mathcal{O}_{Y''}(A - N - Y') \to \mathcal{O}_Y(A - N) \to \mathcal{O}_{Y'}(A - N) \to 0.$$

By applying f_*, we obtain

$$\begin{aligned} 0 \to f_*\mathcal{O}_{Y''}(A - N - Y') \to f_*\mathcal{O}_Y(A - N) \to f_*\mathcal{O}_{Y'}(A - N) \\ \to R^1 f_*\mathcal{O}_{Y''}(A - N - Y') \to \cdots . \end{aligned}$$

By Theorem 3.13 (i), the support of any non-zero local section of $R^1 f_*\mathcal{O}_{Y''}(A - N - Y')$ can not be contained in $X' = f(Y')$. Therefore, the connecting homomorphism $f_*\mathcal{O}_{Y'}(A - N) \to R^1 f_*\mathcal{O}_{Y''}(A - N - Y')$ is a zero map. Thus,

$$0 \to f_*\mathcal{O}_{Y''}(A - N - Y') \to \mathcal{I}_{X_{-\infty}} \to f_*\mathcal{O}_{Y'}(A - N) \to 0$$

is exact. We put $\mathcal{I}_{X'} = f_*\mathcal{O}_{Y''}(A - N - Y')$. Then $\mathcal{I}_{X'}$ defines a scheme structure on X'. We define $\mathcal{I}_{X'_{-\infty}} = \mathcal{I}_{X_{-\infty}} / \mathcal{I}_{X'}$. Then $\mathcal{I}_{X'_{-\infty}} \simeq$

$f_*\mathcal{O}_{Y'}(A-N)$ by the above exact sequence. By the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & f_*\mathcal{O}_{Y''}(A-N-Y') & \longrightarrow & f_*\mathcal{O}_{Y}(A-N) & \longrightarrow & f_*\mathcal{O}_{Y'}(A-N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & f_*\mathcal{O}_{Y''}(A-Y') & \longrightarrow & f_*\mathcal{O}_{Y}(A) & \longrightarrow & f_*\mathcal{O}_{Y'}(A) & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{I}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & 0,
\end{array}$$

we can see that $\mathcal{O}_{X'} \to f_*\mathcal{O}_{Y'}(\ulcorner -(B^{<1}_{Y'})\urcorner)$ induces an isomorphism $\mathcal{I}_{X'_{-\infty}} \to f_*\mathcal{O}_{Y'}(\ulcorner -(B^{<1}_{Y'})\urcorner - \llcorner B^{>1}_{Y'}\lrcorner)$. Therefore, $[X', \omega']$ is a quasi-log pair such that $X'_{-\infty} = X_{-\infty}$. By the construction, the property about qlc centers are obvious. So, we finish the proof of (i).

Let $f : (Y, B_Y) \to X$ be a quasi-log resolution as in the proof of (i). Then $f^*(L-\omega) \sim_{\mathbb{R}} f^*L - (K_{Y''} + B_{Y''})$ on Y'', where $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$. Note that

$$f^*L-(K_{Y''}+B_{Y''}) = (f^*L+A-N-Y')|_{Y''}-(K_{Y''}+\{B_{Y''}\}+B^{=1}_{Y''}-Y'|_{Y''})$$

and that any stratum of $(Y'', B^{=1}_{Y''} - Y'|_{Y''})$ is not mapped to $X_{-\infty} = X'_{-\infty}$. Then by Theorem 3.13 (ii),

$$R^p\pi_*(f_*\mathcal{O}_{Y''}(f^*L + A - N - Y')) = R^p\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)) = 0$$

for any $p > 0$. Thus, we finish the proof of (ii). $\square$

Theorem 3.13 (cf. [A, Theorems 3.2, 7.4])**.** *Let $(Y, S+B)$ be an embedded simple normal crossing pair such that $S+B$ is a boundary $\mathbb{R}$-divisor, S is reduced, and $\llcorner B\lrcorner = 0$. Let $f : Y \to X$ be a proper morphism and L a Cartier divisor on Y.*

(i) *Assume that $H \sim_{\mathbb{R}} L-(K_Y+S+B)$ is f-semi-ample. Then every non-zero local section of $R^qf_*\mathcal{O}_Y(L)$ contains in its support the f-image of some strata of $(Y, S+B)$.*
(ii) *Let $\pi : X \to V$ be a proper morphism and assume that $H \sim_{\mathbb{R}} f^*H'$ for some π-nef and π-log big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor H' on X. Then, every non-zero local section of $R^qf_*\mathcal{O}_Y(L)$ contains in its support the f-image of some strata of $(Y, S+B)$, and $R^qf_*\mathcal{O}_Y(L)$ is π_*-acyclic, that is, $R^p\pi_*R^qf_*\mathcal{O}_Y(L) = 0$ for any $p > 0$.*

For the rigorous proof of Theorem 3.13, see [F6, Theorems 5.7 and 5.16]. Ambro's original proof of Theorem 3.13 in [A, Section 3] contains various errors. It was a serious obstacle in the theory of quasi-log varieties. The next example shows that the definition of quasi-log varieties is reasonable.

Example 3.14. Let (X, B_X) be an lc pair. Let $f : Y \to (X, B_X)$ be a log resolution such that $K_Y + S + B = f^*(K_X + B_X)$, where $\mathrm{Supp}(S + B)$ is simple normal crossing, S is reduced, and $\llcorner B \lrcorner \leq 0$. We put $K_S + B_S = (K_X + S + B)|_S$ and consider the short exact sequence $0 \to \mathcal{O}_Y(\ulcorner -B \urcorner - S) \to \mathcal{O}_Y(\ulcorner -B \urcorner) \to \mathcal{O}_S(\ulcorner -B_S \urcorner) \to 0$. Note that $B_S = B|_S$ since Y is smooth. By the Kawamata–Viehweg vanishing theorem, $R^1 f_* \mathcal{O}_Y(\ulcorner -B \urcorner - S) = 0$. This implies that $f_* \mathcal{O}_S(\ulcorner -B_S \urcorner) \simeq \mathcal{O}_{f(S)}$ since $f_* \mathcal{O}_Y(\ulcorner -B \urcorner) \simeq \mathcal{O}_X$. This argument is well known as the proof of the connectedness lemma. We put $W = f(S)$ and $\omega = (K_X + B_X)|_W$. Then $[W, \omega]$ is a quasi-log pair with only qlc singularities and $f : (S, B_S) \to W$ is a quasi-log resolution.

Example 3.14 is a very special case of Theorem 3.12 (i), that is, adjunction from $[X, K_X + B_X]$ to $[W, \omega]$. For other examples, see [F7, §5], where we treat toric polyhedra as quasi-log varieties. In the proof of Theorem 3.12 (i), we used Theorem 3.13 (i), which is a generalization of Kollár's injectivity theorem, instead of the Kawamata–Viehweg vanishing theorem. The notion of *lcs locus* is important for X-method on quasi-log varieties.

Definition 3.15 (LCS locus)**.** The *LCS locus* of a quasi-log pair $[X, \omega]$, denoted by $\mathrm{LCS}(X)$ or $\mathrm{LCS}(X, \omega)$, is the union of $X_{-\infty}$ with all qlc centers of X that are not maximal with respect to the inclusion. The subscheme structure is defined in Theorem 3.12 (i), and we have a natural embedding $X_{-\infty} \subseteq \mathrm{LCS}(X)$. In [F11], $\mathrm{LCS}(X, \omega)$ is denoted by $\mathrm{Nqklt}(X, \omega)$.

The next proposition is easy to prove. However, in some applications, it may be useful. For the proof, see [A, Proposition 4.7].

Proposition 3.16. *Let X be a quasi-log variety whose LCS locus is empty. Then X is normal.*

The following result is an easy consequence of the vanishing theorem: Theorem 3.13. For much more general results, see [A, Section 6].

Proposition 3.17. *Let (X, B) be a proper lc pair. Assume that $-(K_X + B)$ is nef and log big and that (X, B) is not klt. Then there exists a unique minimal lc center C_0 such that every lc center contains C_0. In particular,* $\mathrm{LCS}(X) = \mathrm{LCS}(X, K_X + B)$ *is connected.*

The next theorem easily follows from [F1, Section 2].

Theorem 3.18. *Let (X, B) be a projective lc pair. Assume that $K_X + B$ is numerically trivial. Then* $\mathrm{LCS}(X) = \mathrm{LCS}(X, K_X + B)$ *has at most two connected components.*

Proof. By [BCHM], there is a birational morphism $f : (Y, B_Y) \to (X, B)$ such that $K_Y + B_Y = f^*(K_X + B)$, Y is projective and $\mathbb{Q}$-factorial, B_Y is effective, and $(Y, \{B_Y\})$ is klt. Therefore, it is sufficient to prove that $\llcorner B_Y \lrcorner$ has at most two connected components. We assume that $\llcorner B_Y \lrcorner \neq 0$. Then $K_Y + \{B_Y\}$ is $\mathbb{Q}$-factorial klt and is not pseudo-effective. Apply the arguments in [F1, Proposition 2.1] with using the MMP with scaling (see [BCHM]). Then we obtain that $\llcorner B_Y \lrcorner$ and $\mathrm{LCS}(X)$ have at most two connected components. □

The first benefit of the theory of quasi-log varieties is the following base point free theorem. The proof in [A] is not difficult. Note that the vanishing theorem (see Theorem 3.12 (ii)), which plays important roles in that proof, is non-trivial and is very deep.

Theorem 3.19 (Base point free theorem for quasi-log varieties)**.** *Assume that X/S is a projective quasi-log variety. Let L be a π-nef Cartier divisor on X such that*

(i) *$qL - \omega$ is π-ample for some positive real number q, and*
(ii) *$\mathcal{O}_{X_{-\infty}}(mL)$ is $\pi|_{X_{-\infty}}$-generated for $m \gg 0$.*

Then $\mathcal{O}_X(mL)$ is π-generated for $m \gg 0$.

Proof. We note that we can assume that S is affine without loss of generality. The original proof consists of four steps. We repeat Step 3 in the proof of [A, Theorem 5.1] with slight modifications for the later usage. For the other steps, see [A, Theorem 5.1]. When S is a point and $X_{-\infty} = \emptyset$, the proof of Theorem 3.19 is described in [F11].

Step 3. In this step, we prove that $\mathrm{Bs}_\pi|mL|$ is not contained in $\mathrm{Bs}_\pi|m'L|$ for $m' \gg 0$ under the assumption that $\mathcal{O}_X(mL)$ is π-generated on a non-empty subset containing $\mathrm{LCS}(X, \omega)$. Note that $\mathrm{Bs}_\pi|mL|$ is the locus on X where $\mathcal{O}_X(mL)$ is not π-generated. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution. For a general member $D \in |mL|$, we may assume that $(Y, B_Y + f^*D)$ is a global embedded simple normal crossing pair by Proposition 4.8 below (see also Definition 4.5). Let c be maximal such that $B'_Y = B_Y + cf^*D$ is a subboundary above $X \setminus X_{-\infty}$. Then $f : (Y, B'_Y) \to X$ is a quasi-log resolution of a quasi-log variety X with $\omega' = \omega + cD$ and $X'_{-\infty} = X_{-\infty}$. Moreover, $[X, \omega']$ has a qlc center C that intersects $\mathrm{Bs}_\pi|mL|$. (In [A], Ambro claims that C is included in $\mathrm{Bs}_\pi|mL|$. However, in general, it is not true.) Applying Step 1 with $q' = q + cm$, we infer that $\mathcal{O}_X(m'L)$ is π-generated on C for $m' \gg 0$.

We note that $q'L - \omega' = qL - \omega$ is π-ample. □

The following result is a corollary of Theorem 3.19. It is enough powerful.

Theorem 3.20 (Base point free theorem for lc pairs)**.** *Let (X, B) be an lc pair. Let L be a π-nef Cartier divisor on X, where $\pi : X \to S$ is a projective morphism. Assume that $qL - (K_X + B)$ is π-ample for some positive real number q. Then $\mathcal{O}_X(mL)$ is π-generated for $m \gg 0$.*

In [A, Section 5], Ambro proved the cone theorem for quasi-log varieties, whose proof is essentially the same as the usual one for klt pairs. The paper [F11] will help the reader to understand it. So, we do not repeat it here. For the precise statement, see [A, Theorem 5.10]. Here, we discuss the following non-trivial example. The construction is the same as Kollár's construction of reducible surfaces whose log canonical rings are not finitely generated.

Example 3.21. We consider the first projection $p : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. We take a blow-up $\mu : Z \to \mathbb{P}^1 \times \mathbb{P}^1$ at $(0, \infty)$. Let A_∞ (resp. A_0) be the strict transform of $\mathbb{P}^1 \times \{\infty\}$ (resp. $\mathbb{P}^1 \times \{0\}$) on Z. We define $M = \mathbb{P}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z(A_0))$ and X is the restriction of M on $(p \circ \mu)^{-1}(0)$. Then X is a simple normal crossing divisor on M. More explicitly, X is a $\mathbb{P}^1$-bundle over $(p \circ \mu)^{-1}(0)$ and is obtained by gluing $X_1 = \mathbb{P}^1 \times \mathbb{P}^1$ and $X_2 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ along a fiber. In particular, $[X, K_X]$ is a quasi-log pair with only qlc singularities. By the construction, $M \to Z$ has two sections. Let D^+ (resp. D^-) be the restriction of the section of $M \to Z$ corresponding to $\mathcal{O}_Z \oplus \mathcal{O}_Z(A_0) \to \mathcal{O}_Z(A_0) \to 0$ (resp. $\mathcal{O}_Z \oplus \mathcal{O}_Z(A_0) \to \mathcal{O}_Z \to 0$). Then it is easy to see that D^+ is a nef Cartier divisor on X and $D^+ - K_X$ is nef and log big with respect to $[X, K_X]$. Therefore, the linear system $|mD^+|$ is free for $m \gg 0$ by Theorem 5.1 below (see also Remark 3.22). We take a general member $B_0 \in |mD^+|$ with $m \geq 3$. We consider $K_X + B$ with $B = D^- + B_0 + B_1 + B_2$, where B_1 and B_2 are general fibers of $X_1 = \mathbb{P}^1 \times \mathbb{P}^1 \subset X$. We note that B_0 does not intersect D^-. Then (X, B) is an embedded simple normal crossing pair. In particular, $[X, K_X + B]$ is a quasi-log pair with $X_{-\infty} = \emptyset$. It is easy to see that there exists only one curve C on $X_2 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \subset X$ such that $C \cdot (K_X + B) < 0$. Note that $(K_X + B)|_{X_1}$ is ample on X_1. By Ambro's cone theorem (cf. [A, Theorem 5.10]), we obtain

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+B)\geq 0} + \mathbb{R}_{\geq 0}[C].$$

By the contraction theorem, we have $\varphi : X \to W$ which contracts C. We can easily see that W is a simple normal crossing surface but $K_W + B_W$, where $B_W = \varphi_* B$, is not $\mathbb{Q}$-Cartier. Therefore, we can not run the LMMP for reducible varieties.

The above example implies that the cone and contraction theorems for quasi-log varieties do not directly produce the LMMP for quasi-log varieties.

Remark 3.22. In Example 3.21, M is a projective toric variety. Let E be the section of $M \to Z$ corresponding to $\mathcal{O}_Z \oplus \mathcal{O}_Z(A_0) \to \mathcal{O}_Z(A_0) \to 0$. Then, it is easy to see that E is a nef Cartier divisor on M. Therefore, the linear system $|E|$ is free. In particular, $|D^+|$ is free on X. Note that $D^+ = E|_X$. So, $|mD^+|$ is free for any $m \geq 0$.

We close this section with my advice.

Remark 3.23 (How to read [A])**.** In my opinion, the main contributions of [A] are the definition of quasi-log varieties (see [A, Definition 4.1]), and adjunction and the vanishing theorem for quasi-log varieties (see [A, Theorem 4.4]). To grasp the definition of quasi-log varieties, we have to read various remarks in [A, Remark 4.2] and consider some examples (see [A, Example 4.3]). We also recommend the reader to see Section 4 below. The reader will find that it is better to adopt Definition 1.2 instead of [A, Definition 4.1]. To justify the vanishing theorem, the reader has to consult [F6]. It is technically the hardest part to understand in the theory of quasi-log varieties. We do not have to read the latter half of [A, Section 4] to understand Sections 5, 6, and 7 in [A]. The arguments in [A, Section 5] seem to be more or less well known to the experts. So, we did not touch Section 5 in [A] here. The paper [F11] will help the reader to understand [A, Section 5].

4. Fundamental lemmas

We will discuss some fundamental results on the quasi-log resolutions. They are missing in [A]. By the results in this section, the theory of quasi-log varieties becomes much more useful. First, we treat an elementary result on discrepancies.

Proposition 4.1. *Let $f : Z \to Y$ be a proper birational morphism between smooth varieties and let B_Y be an $\mathbb{R}$-divisor on Y such that* $\mathrm{Supp} B_Y$ *is simple normal crossing. Assume that $K_Z + B_Z = f^*(K_Y + B_Y)$ and that* $\mathrm{Supp} B_Z$ *is simple normal crossing. Then we have*

$$f_*\mathcal{O}_Z(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner) \simeq \mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1}\lrcorner).$$

Furthermore, let S be a simple normal crossing divisor on Y such that $S \subset \mathrm{Supp} B_Y^{=1}$. Let T be the union of the irreducible components of $B_Z^{=1}$ that are mapped into S by f. Assume that $\mathrm{Supp} f_*^{-1}B_Y \cup \mathrm{Exc}(f)$ *is simple normal crossing on Z. Then we have*

$$f_*\mathcal{O}_T(\ulcorner -(B_T^{<1})\urcorner - \llcorner B_T^{>1}\lrcorner) \simeq \mathcal{O}_S(\ulcorner -(B_S^{<1})\urcorner - \llcorner B_S^{>1}\lrcorner),$$

where $(K_Z+B_Z)|_T = K_T+B_T$ *and* $(K_Y+B_Y)|_S = K_S+B_S$.

Proof. By $K_Z+B_Z = f^*(K_Y+B_Y)$, we obtain

$$\begin{aligned} K_Z =& f^*(K_Y+B_Y^{=1}+\{B_Y\}) \\ &+ f^*(\llcorner B_Y^{<1}\lrcorner + \llcorner B_Y^{>1}\lrcorner) - (\llcorner B_Z^{<1}\lrcorner + \llcorner B_Z^{>1}\lrcorner) - B_Z^{=1} - \{B_Z\}. \end{aligned}$$

If $a(\nu, Y, B_Y^{=1}+\{B_Y\}) = -1$ for a prime divisor ν over Y, then we can check that $a(\nu, Y, B_Y) = -1$ by using [KM, Lemma 2.45]. Since $f^*(\llcorner B_Y^{<1}\lrcorner + \llcorner B_Y^{>1}\lrcorner) - (\llcorner B_Z^{<1}\lrcorner + \llcorner B_Z^{>1}\lrcorner)$ is Cartier, we can easily see that $f^*(\llcorner B_Y^{<1}\lrcorner + \llcorner B_Y^{>1}\lrcorner) = \llcorner B_Z^{<1}\lrcorner + \llcorner B_Z^{>1}\lrcorner + E$, where E is an effective f-exceptional divisor. Thus, we obtain

$$f_*\mathcal{O}_Z(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner) \simeq \mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1}\lrcorner).$$

Next, we consider

$$\begin{aligned} 0 &\to \mathcal{O}_Z(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner - T) \\ &\to \mathcal{O}_Z(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner) \to \mathcal{O}_T(\ulcorner -(B_T^{<1})\urcorner - \llcorner B_T^{>1}\lrcorner) \to 0. \end{aligned}$$

Since $T = f^*S - F$, where F is an effective f-exceptional divisor, we can easily see that

$$f_*\mathcal{O}_Z(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner - T) \simeq \mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1}\lrcorner - S).$$

We note that

$$\begin{aligned} (\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner - T) - (K_Z + \{B_Z\} + (B_Z^{=1} - T)) \\ = -f^*(K_Y+B_Y). \end{aligned}$$

Therefore, every local section of $R^1f_*\mathcal{O}_Z(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner - T)$ contains in its support the f-image of some strata of $(Z, \{B_Z\}+B_Z^{=1}-T)$ by Theorem 3.13 (i).

Claim. *No strata of* $(Z, \{B_Z\}+B_Z^{=1}-T)$ *are mapped into* S *by* f.

Proof of Claim. Assume that there is a stratum C of $(Z, \{B_Z\}+B_Z^{=1}-T)$ such that $f(C) \subset S$. Note that $\mathrm{Supp} f^*S \subset \mathrm{Supp} f_*^{-1}B_Y \cup \mathrm{Exc}(f)$ and $\mathrm{Supp} B_Z^{=1} \subset \mathrm{Supp} f_*^{-1}B_Y \cup \mathrm{Exc}(f)$. Since C is also a stratum of $(Z, B_Z^{=1})$ and $C \subset \mathrm{Supp} f^*S$, there exists an irreducible component G of $B_Z^{=1}$ such that $C \subset G \subset \mathrm{Supp} f^*S$. Therefore, by the definition of T, G is an irreducible component of T because $f(G) \subset S$ and G is an irreducible component of $B_Z^{=1}$. So, C is not a stratum of $(Z, \{B_Z\} + B_Z^{=1}-T)$. It is a contradiction. $\square$

On the other hand, $f(T) \subset S$. Therefore,

$$f_*\mathcal{O}_T(\ulcorner -(B_T^{<1})\urcorner - \llcorner B_T^{>1}\lrcorner) \to R^1f_*\mathcal{O}_S(\ulcorner -(B_T^{<1})\urcorner - \llcorner B_T^{>1}\lrcorner - T)$$

is a zero map by the assumption on the strata of $(Z, B_Z^{=1} - T)$. Thus,

$$f_*\mathcal{O}_T(\ulcorner -(B_T^{<1})\urcorner - \llcorner B_T^{>1}\lrcorner) \simeq \mathcal{O}_S(\ulcorner -(B_S^{<1})\urcorner - \llcorner B_S^{>1}\lrcorner).$$

We finish the proof. □

The next proposition is the main result in this section. Proposition 4.1 becomes very powerful if it is combined with Proposition 4.2. See Proposition 4.6 below.

Proposition 4.2. *In* Definition 3.1*, let M be the ambient space of Y. We can assume that there exists an $\mathbb{R}$-divisor D on M such that* $\mathrm{Supp}(D+Y)$ *is simple normal crossing and $B_Y = D|_Y$.*

Proof. We can construct a sequence of blow-ups $M_k \to M_{k-1} \to \cdots \to M_0 = M$ with the following properties.

(i) $\sigma_{i+1} : M_{i+1} \to M_i$ is the blow-up along a smooth irreducible component of $\mathrm{Supp} B_{Y_i}$ for any $i \geq 0$,
(ii) we put $Y_0 = Y$, $B_{Y_0} = B_Y$, and Y_{i+1} is the strict transform of Y_i for any $i \geq 0$,
(iii) we define $K_{Y_{i+1}} + B_{Y_{i+1}} = \sigma_{i+1}^*(K_{Y_i} + B_{Y_i})$ for any $i \geq 0$,
(iv) there exists an $\mathbb{R}$-divisor D on M_k such that $\mathrm{Supp}(Y_k + D)$ is simple normal crossing on M_k and that $D|_{Y_k} = B_{Y_k}$, and
(v) $\sigma_*\mathcal{O}_{Y_k}(\ulcorner -(B_{Y_k}^{<1})\urcorner - \llcorner B_{Y_k}^{>1}\lrcorner) \simeq \mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1}\lrcorner)$, where $\sigma : M_k \to M_{k-1} \to \cdots \to M_0 = M$.

We note that we can directly check $\sigma_{i+1*}\mathcal{O}_{Y_{i+1}}(\ulcorner -(B_{Y_{i+1}}^{<1})\urcorner - \llcorner B_{Y_{i+1}}^{>1}\lrcorner) \simeq \mathcal{O}_{Y_i}(\ulcorner -(B_{Y_i}^{<1})\urcorner - \llcorner B_{Y_i}^{>1}\lrcorner)$ for any $i \geq 0$ by computations similar to the proof of Proposition 4.1. We replace M and (Y, B_Y) with M_k and (Y_k, B_{Y_k}). □

Remark 4.3. In the proof of Proposition 4.2, M_k and (Y_k, B_{Y_k}) depend on the order of blow-ups. If we change the order of blow-ups, we have another tower of blow-ups $\sigma' : M'_k \to M'_{k-1} \to \cdots \to M'_0 = M$, D', Y'_k on M'_k, and $D'|_{Y'_k} = B_{Y'_k}$ with the desired properties. The relationship between M_k, Y_k, D and M'_k, Y'_k, D' is not clear.

The following corollary is obvious by Proposition 4.1.

Corollary 4.4. *Let X be a normal variety and let B be an $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. Let $f_i : Y_i \to X$ be a log resolution of (X, B) for $i = 1, 2$. We put $K_{Y_i} + B_{Y_i} = f_i^*(K_X + B)$. Then $f_i : (Y_i, B_{Y_i}) \to X$ defines a quasi-log structure on $[X, K_X + B]$ for $i = 1, 2$. By taking a common log resolution of (Y_1, B_{Y_1}) and (Y_2, B_{Y_2}) suitably and applying* Proposition 4.1*, we can see that these two quasi-log structures coincide. Moreover, let X' be the union of $X_{-\infty}$ with a union of some qlc centers of $[X, K_X + B]$. Then we can see that*

$f_1 : (Y_1, B_{Y_1}) \to X$ and $f_2 : (Y_2, B_{Y_2}) \to X$ induce the same quasi-log structure on $[X', (K_X + B)|_{X'}]$ by Proposition 4.1.

Before we go further, let us introduce the notion of *global embedded simple normal crossing pairs.* A global embedded simple normal crossing pair is a special case of Ambro's embedded simple normal crossing pairs (see Definition 3.8).

Definition 4.5 (Global embedded simple normal crossing pairs)**.** Let Y be a simple normal crossing divisor on a smooth variety M and let D be an $\mathbb{R}$-divisor on M such that $\mathrm{Supp}(D+Y)$ is simple normal crossing and that D and Y have no common irreducible components. We put $B_Y = D|_Y$ and consider the pair (Y, B_Y). We call (Y, B_Y) a *global embedded simple normal crossing pair.*

The final results in this section are very useful and indispensable for some applications.

Proposition 4.6. *Let $[X, \omega]$ be a quasi-log pair and let $f : (Y, B_Y) \to X$ be a quasi-log resolution. Assume that (Y, B_Y) is a global embedded simple normal crossing pair as in* Definition 4.5*. Let $\sigma : N \to M$ be a proper birational morphism from a smooth variety N. We define $K_N + D_N = \sigma^*(K_M + D + Y)$ and assume that* $\mathrm{Supp}\sigma_*^{-1}(D + Y) \cup \mathrm{Exc}(\sigma)$ *is simple normal crossing on N. Let Z be the union of the irreducible components of $D_N^{=1}$ that are mapped into Y by σ. Then $f \circ \sigma : (Z, B_Z) \to X$ is a quasi-log resolution of $[X, \omega]$, where $K_Z + B_Z = (K_N + D_N)|_Z$.*

The proof of Proposition 4.6 is obvious by Proposition 4.1.

Remark 4.7. In Proposition 4.6, $\sigma : (Z, B_Z) \to (Y, B_Y)$ is not necessarily a composition of *embedded log transformations* and blow-ups whose centers contain no "strata"(see [A, Section 2]). Compare Proposition 4.6 with [A, Remark 4.2.(iv)].

Proposition 4.8. *Let $f : (Y, B_Y) \to X$ be a quasi-log resolution of a quasi-log pair $[X, \omega]$, where (Y, B_Y) is a global embedded simple normal crossing pair as in* Definition 4.5*. Let E be a Cartier divisor on X such that* $\mathrm{Supp}E$ *contains no qlc centers of $[X, \omega]$. By blowing up M, the ambient space of Y, inside* $\mathrm{Supp}f^*E$*, we can assume that $(Y, B_Y + f^*E)$ is a global embedded simple normal crossing pair.*

Proof. First, we take a blow-up of M along f^*E and apply Hironaka's resolution theorem to M. Then we can assume that there exists a Cartier divisor F on M such that $\mathrm{Supp}(F \cap Y) = \mathrm{Supp}f^*E$. Next, we apply Szabó's resolution lemma to $\mathrm{Supp}(D + Y + F)$ on M. Thus, we obtain the desired properties by Proposition 4.1. □

Remark 4.9. By Remark 4.3, Propositions 4.1, 4.6, and 4.8, it is better to adopt Definition 1.2 instead of Definition 3.1. If we start with Definition 1.2, we do not have to introduce the notion of *simple normal crossing pairs.* Definition 4.5 is sufficient for our purposes.

5. Base point free theorem of Reid–Fukuda type

One of my motivations to study [A] is to understand [A, Theorem 7.2], which is a complete generalization of [F2]. The following theorem is a special case of Theorem 7.2 in [A], which was stated without proof. Here, we will reduce it to Theorem 3.19 by using Kodaira's lemma.

Theorem 5.1. (Base point free theorem of Reid–Fukuda type). *Let $[X, \omega]$ be a quasi-log pair with $X_{-\infty} = \emptyset$, $\pi : X \to S$ a* projective *morphism, and L a π-nef Cartier divisor on X such that $qL - \omega$ is nef and log big over S for some positive real number q. Then $\mathcal{O}_X(mL)$ is π-generated for $m \gg 0$.*

Remark 5.2. In [A, Section 7], Ambro said that the proof of [A, Theorem 7.2] is parallel to [A, Theorem 5.1]. However, I could not check it. Steps 1, 2, and 4 in the proof of [A, Theorem 5.1] work without any modifications. In Step 3 (see the proof of Theorem 3.19), $q'L - \omega'$ is π-nef, but I think that $q'L - \omega' = qL - \omega$ is not always log big over S with respect to $[X, \omega']$. So, we can not directly apply the argument in Step 1 to this new quasi-log pair $[X, \omega']$.

Proof. We divide the proof into three steps.

Step 1. We take an irreducible component X' of X. Then X' has a natural quasi-log structure induced by X (see Theorem 3.12 (i)). By the vanishing theorem (see Theorem 3.12 (ii)), we have $R^1\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(mL)) = 0$ for $m \geq q$. Therefore, we obtain that $\pi_*\mathcal{O}_X(mL) \to \pi_*\mathcal{O}_{X'}(mL)$ is surjective for $m \geq q$. Thus, we can assume that X is irreducible for the proof of this theorem by the following commutative diagram.

$$\begin{array}{ccccc} \pi^*\pi_*\mathcal{O}_X(mL) & \longrightarrow & \pi^*\pi_*\mathcal{O}_{X'}(mL) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathcal{O}_X(mL) & \longrightarrow & \mathcal{O}_{X'}(mL) & \longrightarrow & 0 \end{array}$$

Step 2. Without loss of generality, we can assume that S is affine. Since $qL - \omega$ is nef and big over S, we can write $qL - \omega \sim_{\mathbb{R}} A + E$ by Kodaira's lemma, where A is a π-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X and E is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. We note that X is projective over S and that X is not necessarily normal. By Lemma 5.3

below, we have a new quasi-log structure on $[X, \widetilde{\omega}]$, where $\widetilde{\omega} = \omega + \varepsilon E$, for $0 < \varepsilon \ll 1$.

Step 3. By the induction on the dimension, $\mathcal{O}_{\mathrm{LCS}(X,\omega)}(mL)$ is π-generated for $m \gg 0$. Note that $\pi_*\mathcal{O}_X(mL) \to \pi_*\mathcal{O}_{\mathrm{LCS}(X,\omega)}(mL)$ is surjective for $m \geq q$ by the vanishing theorem (see Theorem 3.12 (ii)). Then $\mathcal{O}_{\mathrm{LCS}(X,\widetilde{\omega})}(mL)$ is π-generated for $m \gg 0$ by the above lifting result and by Lemma 5.3. In particular, $\mathcal{O}_{\widetilde{X}_{-\infty}}(mL)$ is π-generated for $m \gg 0$. We note that $qL - \widetilde{\omega} \sim_{\mathbb{R}} (1-\varepsilon)(qL - \omega) + \varepsilon A$ is π-ample. Therefore, by Theorem 3.19, we obtain that $\mathcal{O}_X(mL)$ is π-generated for $m \gg 0$.

We finish the proof. □

Lemma 5.3. *Let $[X, \omega]$ be a quasi-log pair with $X_{-\infty} = \emptyset$. Let E be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then $[X, \omega + \varepsilon E]$ is a quasi-log pair with the following properties for $0 < \varepsilon \ll 1$.*

(i) *We put $[X, \widetilde{\omega}] = [X, \omega + \varepsilon E]$. Then $[X, \widetilde{\omega}]$ is a quasi-log pair and $\mathrm{LCS}(X, \widetilde{\omega}) = \mathrm{LCS}(X, \omega)$ as closed subsets of X.*

(ii) *There exist natural surjective homomorphisms $\mathcal{O}_{\mathrm{LCS}(X,\widetilde{\omega})} \to \mathcal{O}_{\mathrm{LCS}(X,\omega)} \to 0$ and $\mathcal{O}_{\mathrm{LCS}(X,\widetilde{\omega})} \to \mathcal{O}_{\widetilde{X}_{-\infty}} \to 0$, that is, $\mathrm{LCS}(X, \omega)$ and $\widetilde{X}_{-\infty}$ are closed subschemes of $\mathrm{LCS}(X, \widetilde{\omega})$, where $\widetilde{X}_{-\infty}$ is the non-qlc locus of $[X, \widetilde{\omega}]$.*

Proof. Let $f : (Y, B_Y) \to X$ be a quasi-log resolution of $[X, \omega]$. We can assume that (Y, B_Y) is a global embedded simple normal crossing pair as in Definition 4.5 and that the union of all strata of (Y, B_Y) mapped into $\mathrm{LCS}(X, \omega)$, which we denote by Y', is a union of irreducible components of Y. We put $Y'' = Y - Y'$. Then $f_*\mathcal{O}_{Y''}(A - Y'|_{Y''})$ is $\mathcal{I}_{\mathrm{LCS}(X,\omega)}$, that is, the defining ideal sheaf of $\mathrm{LCS}(X, \omega)$ on X, where $A = \ulcorner -(B_Y^{<1}) \urcorner$. For the details, see the proof of Theorem 3.12 (i).

Claim (cf. Proposition 4.8)**.** *By modifying M birationally, we can assume that $(f'')^*E + B_{Y''}$ has a simple normal crossing support on Y'', where $f'' = f|_{Y''}$ and $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$.*

Proof of Claim. First, we note that $(f'')^*E$ contains no strata of Y''. We can construct a proper birational morphism $h : \widetilde{M} \to M$ from a smooth variety $\widetilde{M}$ such that $K_{\widetilde{M}} + D_{\widetilde{M}} = h^*(K_M + Y + D)$, $h^{-1}((f'')^*E)$ is a divisor on $\widetilde{M}$, and $\mathrm{Exc}(h) \cup \mathrm{Supp} h_*^{-1}(Y + D) \cup h^{-1}((f'')^*E)$ is simple normal crossing on $\widetilde{M}$. By Szabó's resolution lemma, we can assume that h is an isomorphism outside $h^{-1}((f'')^*E)$. Let $\widetilde{Y}$ be the union of the irreducible components of $D_{\widetilde{M}}^{=1}$ that are mapped into Y. By Proposition 4.1, we can replace M, Y, and D with $\widetilde{M}$, $\widetilde{Y}$, and $\widetilde{D} = D_{\widetilde{M}} - \widetilde{Y}$. □

We put $\widetilde{B} = (f'')^*E$ and consider $f'' : (Y'', B_{Y''} + \varepsilon\widetilde{B}) \to X$, where $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$. Then we have $(f'')^*(\omega + \varepsilon E) \sim_{\mathbb{R}} K_{Y''} + B_{Y''} + \varepsilon\widetilde{B}$ and this gives a desired quasi-log structure on $[X, \widetilde{\omega}]$, with $\widetilde{\omega} = \omega + \varepsilon E$, if $0 < \varepsilon \ll 1$. □

As a special case, we obtain the following base point free theorem of Reid–Fukuda type for log canonical pairs.

Theorem 5.4. (Base point free theorem of Reid–Fukuda type for lc pairs). *Let (X, B) be an lc pair. Let L be a π-nef Cartier divisor on X, where $\pi : X \to S$ is a projective morphism. Assume that $qL-(K_X+B)$ is π-nef and π-log big for some positive real number q. Then $\mathcal{O}_X(mL)$ is π-generated for $m \gg 0$.*

We believe that the above theorem holds under the assumption that π is only *proper*. However, our proof needs projectivity of π.

Remark 5.5. In Theorem 5.4, if $\mathrm{LCS}(X, \omega)$, where $\omega = K_X + B$, is projective over S, then we can prove Theorem 5.4 under the weaker assumption that $\pi : X \to S$ is only *proper*. It is because we can apply Theorem 5.1 to $\mathrm{LCS}(X, \omega)$. So, we can assume that $\mathcal{O}_X(mL)$ is π-generated on a non-empty open subset containing $\mathrm{LCS}(X, \omega)$. In this case, we can prove Theorem 5.4 by applying the usual X-method to L on (X, B). We note that $\mathrm{LCS}(X, \omega)$ is always projective over S when $\dim \mathrm{LCS}(X, \omega) \leq 1$. The reader can find a different proof in [Fk] when (X, B) is a log canonical surface, where Fukuda used the MMP with scaling for dlt surfaces.

Finally, we explain the reason why we assumed that $X_{-\infty} = \emptyset$ and π is projective in Theorem 5.1.

Remark 5.6 (Why $X_{-\infty}$ is empty?)**.** Let C be a qlc center of $[X, \omega]$. Then we have to consider a quasi-log variety $X' = C \cup X_{-\infty}$ for the inductive arguments. In general, X' is reducible. It sometimes happens that $\dim C < \dim X_{-\infty}$. We do not know how to apply Kodaira's lemma to reducible varieties. So, we assume that $X_{-\infty} = \emptyset$ in Theorem 5.1.

Remark 5.7 (Why π is projective?)**.** We assume that S is a point in Theorem 5.1 for simplicity. If $X_{-\infty} = \emptyset$, then it is enough to treat irreducible quasi-log varieties by Step 1. Thus, we can assume that X is irreducible. Let $f : Y \to X$ be a proper birational morphism from a smooth projective varieties. If X is normal, then $H^0(X, \mathcal{O}_X(mL)) \simeq H^0(Y, \mathcal{O}_Y(mf^*L))$ for any $m \geq 0$. However, X is not always normal (see Example 5.8 below). So, it sometimes happens that $\mathcal{O}_Y(mf^*L)$

has many global sections but $\mathcal{O}_X(mL)$ has only a few global sections. Therefore, we can not easily reduce the problem to the case when X is projective. This is the reason why we assume that $\pi : X \to S$ is projective.

Example 5.8. Let $M = \mathbb{P}^2$ and let X be a nodal curve on M. Then (M, X) is an lc pair. By Example 3.14, $[X, K_X]$ is a quasi-log variety with only qlc singularities. In this case, X is irreducible, but it is not normal.

References

[AHK] V. Alexeev, C. Hacon, and Y. Kawamata, Termination of (many) 4-dimensional log flips, Invent. Math. **168** (2007), no. 2, 433–448.

[A] F. Ambro, Quasi-log varieties, Tr. Mat. Inst. Steklova **240** (2003), Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 220–239; translation in Proc. Steklov Inst. Math. 2003, no. 1 (240), 214–233.

[B] C. Birkar, On existence of log minimal models, arXiv:0706.1792v1 [math.AG], preprint, 2007.

[BCHM] C. Birkar, P. Cascini, C. Hacon, and J. McKernan, Existence of minimal models for varieties of log general type, arXiv:math/0610203.

[F1] O. Fujino, Abundance theorem for semi log canonical threefolds, Duke Math. J. **102** (2000), no. 3, 513–532.

[F2] O. Fujino, Base point free theorem of Reid–Fukuda type, J. Math. Sci. Univ. Tokyo **7** (2000), no. 1, 1–5.

[F3] O. Fujino, Notes on toric varieties from Mori theoretic viewpoint, Tohoku Math. J. (2) **55** (2003), no. 4, 551–564.

[F4] O. Fujino, Equivariant completions of toric contraction morphisms, With an appendix by Fujino and Hiroshi Sato, Tohoku Math. J. (2) **58** (2006), no. 3, 303–321.

[F5] O. Fujino, Special termination and reduction to pl flips, in *Flips for 3-folds and 4-folds* (Alessio Corti, ed.), 63–75, Oxford University Press, 2007.

[F6] O. Fujino, Vanishing and injectivity theorems for LMMP, preprint 2007.

[F7] O. Fujino, Vanishing theorems for toric polyhedra, preprint 2007.

[F8] O. Fujino, Effective base point free theorem for log canonical pairs, preprint 2007.

[F9] O. Fujino, Effective base point free theorem for log canonical pairs II, preprint 2007.

[F10] O. Fujino, Theory of non-lc ideal sheaves, preprint 2007.

[F11] O. Fujino, Lectures on the log minimal model program, preprint 2007.

[Fk] S. Fukuda, A base point free theorem for log canonical surfaces, Osaka J. Math. **36** (1999), no. 2, 337–341.

[K] Y. Kawamata, On the length of an extremal rational curve, Invent. Math. **105** (1991), no. 3, 609–611.

[KMM] Y. Kawamata, K. Matsuda, and K. Matsuki, Introduction to the Minimal Model Problem, in *Algebraic Geometry, Sendai 1985,* Advanced Studies in Pure Math. **10**, (1987) Kinokuniya and North-Holland, 283–360.

[FA] J. Kollár et al., *Flips and Abundance for algebraic threefolds*, Astérisque **211**, (1992).
[KM] J. Kollár, S. Mori, *Birational geometry of algebraic varieties,* Cambridge Tracts in Mathematics, Vol. **134**, 1998.
[M] K. Matsuki, *Introduction to the Mori program, Universitext,* Springer-Verlag, New York, 2002.
[S] V. V. Shokurov, Letters of a Bi-Rationalist: VII. Ordered termination, arXiv:math/0607822v2 [math.AG], preprint, 2006.

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