BIRKAR–CASCINI–HACON–M^{C}KERNAN

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1. Birkar–Cascini–Hacon–M^{c}Kernan

[1]

In this short section, we quickly explain some results in [BCHM]. For the details, see the original paper: [BCHM]. Let us recall the definition of *log terminal models* in [BCHM].

Definition 1.1 (cf. [BCHM, Definition 3.6.7])**.** Let $f : X \to S$ be a projective morphism of normal quasi-projective varieties. Assume that (X, Δ) is lc and $\phi : X \dashrightarrow X'$ a birational map of normal quasi-projective varieties over S, where X' is projective over S. We put $\Delta' = \phi_*\Delta$. The pair (X', Δ') is called a *log terminal model* over S if

(1) ϕ^{-1} contracts no divisors,
(2) X' is $\mathbb{Q}$-factorial,
(3) (X', Δ') is dlt,
(4) $K_{X'} + \Delta'$ is f'-nef, where $f' : X' \to S$, and
(5) $a(E, X, \Delta) < a(E, X', \Delta')$ for every ϕ-exceptional divisor $E \subset X$.

We note that (X', Δ') is automatically klt if (X, Δ) is klt by (4), (5), and the negativity lemma.

One of the main theorems of [BCHM] is as follows.

thm-ccc **Theorem 1.2** (cf. [BCHM, Theorems C and D])**.** *Let $f : X \to S$ be a projective morphism of normal quasi-projective varieties and Δ an effective $\mathbb{R}$-divisor on X such that (X, Δ) is klt. Assume that Δ is f-big and $K_X + \Delta$ is f-pseudo-effective. Then (X, Δ) has a log terminal model over S.*

Remark 1.3. Let $f : X \to S$ be a projective morphism of normal quasi-projective varieties and D an $\mathbb{R}$-Cartier divisor on X. Then D is f-pseudo-effective if and only if $D + \varepsilon H$ is f-big for every f-ample divisor H and every $\varepsilon > 0$.

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This note will be contained in my book.

[1]We have to replace McKernan with M^{c}Kernan. We have to define terminal singularities.

By combining it with fujino-mori [?, Theorem 5.2], we obtain the following important result.

thm-bbb **Theorem 1.4** (cf. bchm [BCHM, Corollary 1.1.2])**.** *Let $f : X \to S$ be a projective morphism of quasi-projective varieties and Δ an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is klt. Then*

$$R(X/S, K_X + \Delta) = \bigoplus_{m \geq 0} f_* \mathcal{O}_X(\llcorner m(K_X + \Delta) \lrcorner)$$

is a finitely generated $\mathcal{O}_X$-algebra.

Theorem thm-bbb 1.4 implies the existence of log flips for $\mathbb{Q}$-factorial dlt pairs: Theorem z-flip ??. The following corollary is a special case of Theorem thm-bbb 1.4.

Corollary 1.5. *Let X be a smooth projective variety. Then the canonical ring*

$$R(X, K_X) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$$

is a finitely generated $\mathbb{C}$-algebra.

Furthermore, if X is of general type. Then X has a canonical model

$$X' \simeq \mathrm{Proj} \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X)).$$

The next theorem is one of the most important results in bchm [BCHM]. The log minimal model program with scaling is sufficient for many applications.

thm-aaa **Theorem 1.6** (cf. bchm [BCHM, Corollary 1.4.2])**.** *Let $f : X \to S$ be a projective morphism of normal quasi-projective varieties. Let (X, Δ) be a $\mathbb{Q}$-factorial klt pair such that Δ is f-big and C an effective $\mathbb{R}$-divisor. If $K_X + \Delta + C$ is klt and f-nef, then we can run the log minimal model program over S with scaling of C.*

We explain Theorem thm-aaa 1.6 more explicitly.

1.7 (MMP with scaling for $\mathbb{Q}$-factorial klt pairs with big boundary divisors)**.** We use the same notation as in 126126 ??. Assume that $K_{X_i} + \Delta_i$ is not f_i-nef. Let C_i denote the transform of C on X_i. Then we can find a $(K_{X_i} + \Delta_i)$-negative extremal ray R_i such that $(K_{X_i} + \Delta_i + \lambda_i C_i) \cdot R_i = 0$, where

$$\lambda_i = \inf\{t \geq 0 \,|\, K_{X_i} + \Delta_i + tC_i \text{ is } f_i\text{-nef}\}.$$

Apply the contraction theorem with respect to R_i. Then the above log minimal model program for (X, Δ) works and terminates.

We close this section with the following important result.

Corollary 1.8. *Let X be a smooth projective variety of general type. Then X has a minimal model. This means that there exists a normal projective variety X' with only $\mathbb{Q}$-factorial terminal singularities such that X' is birationally equivalent to X and $K_{X'}$ is nef.*

References

bchm [BCHM]

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